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WEIGHTED ENERGY METHOD AND LONG WAVE SHORT WAVE DECOMPOSITION ON THE LINEARIZED COMPRESSIBLE NAVIER-STOKES EQUATION

Sun-Ho Choi

Department of Mathematics
National University of Singapore
Singapore 117543, Singapore

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Abstract. The purpose of this paper is to study asymptotic behaviors of the Green function of the linearized compressible Navier-Stokes equation. Liu, T.-P. and Zeng, Y. obtained a point-wise estimate for the Green function of the linearized compressible Navier-Stokes equation in [Comm. Pure Appl. Math. **47**, 1053–1082 (1994)] and [Mem. Amer. Math. Soc. **125** (1997), no. 599]. In this paper, we propose a new methodology to investigate point-wise behavior of the Green function of the compressible Navier-Stokes equation. This methodology consists of complex analysis method and weighted energy estimate which was originally proposed by Liu, T.-P. and Yu, S.-H. in [Comm. Pure Appl. Math., **57**, 1543–1608 (2004)] for the Boltzmann equation. We will apply this methodology to get a point-wise estimate of the Green function for large $t > 0$.

1. Introduction.

In this paper, we will study a new scheme to obtain the large-time behavior of the Green function of the p-system which is the linearized compressible Navier-Stokes equation. This scheme was originally proposed by Liu, T.-P. and Yu, S.-H. for kinetic theory (See [6]). This method consists of complex analysis and weighted energy estimate. We will obtain a point-wise estimate of the Green function for the p-system by using this method.

The p-system is a linearized equation of the compressible Navier-Stokes equation. To derive the p-system, consider the one-dimensional, isentropic compressible Navier-Stokes equation:

$$\rho_t + (\rho u)_x = 0, \qquad x \in \mathbb{R},\ t > 0,$$
$$(\rho u)_t + (\rho u^2 + p(\rho))_x = (\mu u_x)_x,$$

where $\rho(x,t)$ is the density, $u(x,t)$ is flow velocity and $p(\rho)$ is pressure. For notational convenience, we normalize the viscosity μ and the sound speed $\sqrt{p'(\rho_0)}$ to be unity, and take a constant state $(\rho_0, u_0) = (1, 0)$. Then we can linearize the

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Navier-Stokes equations around a constant state (ρ_0, u_0) to obtain the linearized system:

$$\rho_t + u_x = 0, \quad u_t + \rho_x = u_{xx}, \qquad (p\text{-system}) \quad (1)$$

where $x \in \mathbb{R}$ and $t \in \mathbb{R}_+ \cup \{0\}$ are spatial and time variable respectively. The Green function G is a unique solution to the p-system (1) which is a matrix valued function and initial data is matrix valued measure $\delta(x)I$ where I is the 2×2 identity matrix and $\delta(x)$ is the dirac delta measure.

The main result of this paper is as follows:

Theorem 1.1. *Let $G(x,t)$ be the 2×2 matrix valued solution to the p-system (1) with initial data $\delta(x)I$ here $\delta(x)$ is the dirac delta measure. For a sufficiently large $t > 0$, we have a point-wise estimate for the Green function $G(x,t)$ as follows:*

$$\left| G(x,t) - e^{-t}\delta(x)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right| \leq \frac{\mathcal{O}(1)}{\sqrt{t}} e^{-\frac{(x-t)^2}{2ct}} + \frac{\mathcal{O}(1)}{\sqrt{t}} e^{-\frac{(x+t)^2}{2ct}} + Ce^{-(|x|+t)/c},$$

where C and c are positive constants.

The rest of this paper is organized as follows. In Section 2, we will survey the previous results related to the large-time behavior of the Green function G to the p-system (1) and provide an explicit formula by the Fourier transform and an inverse Fourier transform formula. To investigate the large-time behavior of the Green function G, we will have to calculate this inverse Fourier transform. In Section 3 and 4, we get an estimate of this inverse Fourier transform. The key idea to calculate the inverse Fourier transform is to subtract dirac delta distribution from the Green function G. Then we can observe the exponential structure of the remainder part. We will divide a time-spatial variable plane (x,t) into two parts which are the finite Mach number region and the outside finite Mach number region. On the finite Mach number region, we will consider the long-wave component and the short-wave component respectively. For the long-wave component, we adopt the complex analysis via analytic property of $\hat{G}(\xi,t)$ at $\xi = 0$, here $\hat{G}$ is the Fourier transform of G. For the short-wave component, we will use a property of the Fourier transform and substraction of low-order term in the Taylor series of $\hat{G}$ to obtain the exponential structure. For the outside finite Mach number region, the weighted energy estimate allows us to obtain a point-wise estimate with the Sobolev embedding theorem. Finally, we will complete the proof of Theorem 1.1 in Section 5.

Notation. Throughout this paper, we will use simplified notation as follows:

$$\begin{aligned} ||f||_{L^p(\mathbb{R})} &:= \left(\int_{\mathbb{R}} |f(x,t)|^p dx \right)^{1/p}, \quad 1 \leq p < \infty, \\ ||f||_{L^\infty(\mathbb{R})} &:= \sup_{\mathbb{R}} |f(x,t)|, \\ ||f||_{H^1(\mathbb{R})} &:= \left(\int_{\mathbb{R}} |f(x,t)|^2 + \left| \frac{d}{dx} f(x,t) \right|^2 dx \right)^{1/2}. \end{aligned}$$

2. **Preliminary.**

2.1. **Review of previous results.** In this section, we provide brief review of previous results related to large-time behaviors of the Green function G for the p-system (1).

In [3], the author presented an L^2-decay rate of the Green function G for the p-system (1). In [9], authors obtain large-time behavior of solutions to a 3×3 hyperbolic-parabolic system in L^1-scheme. In [7] and [10], authors obtain a point-wise estimate for the Green function G of the p-system (1). In these papers, to obtain a point-wise estimate, they divide x-t plane into two parts which are the finite Mach number region and the outside finite Mach number region. For the finite Mach number region, they consider the Green function G^* of a computative equation as follows:

$$\rho_t + u_x = \frac{\rho_{xx}}{2}, \qquad u_t + \rho_x = \frac{u_{xx}}{2}$$

and they calculate an estimate of $(G - G^*)^\wedge$ for the long-wave component by using complex analysis because $\hat{G}$ has analytic property. For the short-wave component, they calculate estimate of

$$G - e^{-t}\delta(x)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2}$$

by using direct calculation. For the outside finite Mach number region, they calculate an estimate of the above equation (2) by using the Cauchy theorem in complex analysis. For other related results for the Green function G of the p-system (1), we refer to [1, 2, 4, 5, 8].

2.2. **The explicit formula for the Green function G.** In this part, we will obtain an explicit formula for the Green function G. It is the first step to observe the asymptotic behavior of the Green function G of the p-system (1) by using Fourier analysis. Because this step is a well-known fact, we just provide the sketch. For a detailed computation, see [7, 10].

Consider $f(x)$ which is an $L^2(\mathbb{R})$-function. Then the Fourier transform is defined as follows:

$$\mathcal{F}_f(\xi) = \hat{f}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dx.$$

For this Fourier operator $\mathcal{F}$ as the above, we have an inverse operator $\mathcal{F}^{-1}$ in $L^2(\mathbb{R})$-function space, i.e., the inverse Fourier transform defined by

$$\mathcal{F}_f^{-1}(x) = \check{f}(x) = \int_{-\infty}^{\infty} f(\xi)e^{i\xi x}d\xi.$$

Since the p-system (1) is a linear system, we can obtain a linear system for the transformed Green function $\hat{G}$ of the p-system (1) by using the Fourier transform as follows:

$$\hat{\rho}_t + i\xi\hat{u} = 0, \qquad \hat{u}_t + i\xi\hat{\rho} = -\xi^2\hat{u}. \tag{3}$$

Initial data $\delta(x)I$ of the Green function G corresponds to a constant 2×2 matrix I, i.e., identity matrix.

We can easily solve (3) by using linear algebra. The solution matrix is as follows:

$$\hat{G}(\xi, t) = e^{\lambda_+(\xi)t}P_+(\xi) + e^{\lambda_-(\xi)t}P_-(\xi) \tag{4}$$

where

$$\lambda_\pm(\xi) = -\frac{1}{2}\xi(\xi \pm \sqrt{\xi^2-4}) \tag{5}$$

and

$$P_\pm(\xi) = \begin{pmatrix} \frac{1}{2} \mp \frac{\xi}{2\sqrt{\xi^2-4}} & \pm\frac{i}{\sqrt{\xi^2-4}} \\ \pm\frac{i}{\sqrt{\xi^2-4}} & \frac{1}{2} \pm \frac{\xi}{2\sqrt{\xi^2-4}} \end{pmatrix}.$$

The function $\hat{G}$ has differentiable property as follows. The following lemma will be used in next section.

Lemma 2.1. ([7, 10]) *Let $\hat{G}$ be a Fourier transformed function of the Green function G and $\lambda_\pm$ is defined by equation (5). Then $\hat{G}(\xi,t)$ is an entire function of complex variable ξ and for nonzero real number ξ, $Re\{\lambda_\pm(\xi)t\}$ has a negative value.*

Proof. For the detailed proof of this lemma, see [7, 10]. □

By the inverse Fourier transform $\mathcal{F}^{-1}$, we have an explicit formula for the Green function G as follows:

$$G(x,t) = \int_{\mathbb{R}} \hat{G}(\xi,t)e^{ix\xi}d\xi, \tag{6}$$

where $\hat{G}$ is the explicitly defined function in (4).

In the sequel, we will produce a point-wise estimate for the Green function G of (6). For the estimate of $G(x,t)$ in (6), we will apply two kinds of different methods, i.e., complex analysis and weighed energy method. Let us divide $(x,t) \in \mathbb{R}\times\mathbb{R}^+$ into the finite Mach number region $\{|x| < 2t\}$ and the outside finite Mach number region $\{|x| \geq 2t\}$. For the finite Mach number region, we will use the long wave-short wave decomposition to use complex analysis. For the outside finite Mach number region, we will adapt the weighted energy estimate method to obtain a point-wise estimate of the Green function G.

3. A point-wise estimate in the finite Mach number region.

In this section, we will focus on a point-wise estimate for the Green function G in the finite Mach number region $\{|x| < 2t\}$. The strategy to obtain the estimate is to divide the Fourier variable region into two parts which are the long-wave component $\{|\xi| < \epsilon\}$ and the short-wave component $\{|\xi| \geq \epsilon\}$, i.e.,

$$G(x,t) = \int_{\mathbb{R}} \hat{G}(\xi,t)e^{ix\xi}d\xi = \int_{|\xi|<\epsilon} \hat{G}(\xi,t)e^{ix\xi}d\xi + \int_{|\xi|\geq\epsilon} \hat{G}(\xi,t)e^{ix\xi}d\xi.$$

For the long-wave component, we can use analytic property of $\hat{G}$, i.e., $\hat{G}(\xi)$ is analytic on $|\xi| < \epsilon$, if $\epsilon > 0$ is sufficiently small. Therefore, we will use complex analysis, i.e., the Cauchy Theorem to calculate directly (see Section 3.1). For the short-wave part, we will obtain the point-wise estimate of G by using a property of the Fourier Transform (see Section 3.2).

3.1. The long-wave component.

The crucial idea for the long-wave component region is to use complex analysis. The following lemma plays a key role to obtain a point-wise estimate for the Green function G of the p-system (1).

Lemma 3.1. *Assume that (x,t) is on the finite Mach number region $\{|x| < 2t\}$. For a sufficiently small $\epsilon > 0$, we have the following estimate:*

$$\left|\int_{|\xi|<\epsilon} \hat{G}(\xi,t)e^{i\xi x}d\xi\right| \leq \frac{\mathcal{O}(1)}{\sqrt{t}}e^{-\frac{(x-t)^2}{2ct}} + \frac{\mathcal{O}(1)}{\sqrt{t}}e^{-\frac{(x+t)^2}{2ct}} + Ce^{-t/c},$$

where C and c are positive constants.

Proof. We now consider ξ as a complex variable. Since $\sqrt{\xi^2-4}$ is analytic at $\xi=0$, the follows are also analytic at $\xi=0$:

$$\begin{aligned} e^{i\xi x-\frac{1}{2}\xi(\xi+\sqrt{\xi^2-4})t}P_+(\xi) &= \mathcal{O}(1)e^{i\xi x-\frac{1}{2}\xi^2 t-i\xi t+\mathcal{O}(1)\xi^3 t} \\ &= \mathcal{O}(1)e^{-\frac{(x-t)^2}{2t}-\frac{t}{2}(\xi-i\frac{x-t}{t})^2+\mathcal{O}(1)\xi^3 t}. \end{aligned} \tag{7}$$

Finite Mach number assumption implies $\left|\dfrac{x-t}{t}\right|<3$. This inequality and the analytic property of the above function (7) allow us to use the Cauchy theorem to obtain the following change of integral path:

$$\begin{aligned} &\int_{|\xi|<\epsilon} e^{i\xi x-\frac{1}{2}\xi(\xi+\sqrt{\xi^2-4})t}P_+(\xi)d\xi \\ &\qquad = \left\{\int_{-\epsilon+i\alpha\frac{x-t}{t}}^{\epsilon+i\alpha\frac{x-t}{t}}+\int_{c_1}+\int_{c_2}\right\}e^{i\xi x-\frac{1}{2}\xi(\xi+\sqrt{\xi^2-4})t}P_+(\xi)d\xi \\ &\qquad =: \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3, \end{aligned}$$

where $c_1=\{\xi=-\epsilon+i\omega : 0<\omega<\alpha\dfrac{x-t}{t}$ for $x>t$, $\alpha\dfrac{x-t}{t}<\omega<0$ for $x<t\}$ and $c_2=\{\xi=\epsilon+i\omega : 0<\omega<\alpha\dfrac{x-t}{t}$ for $x>t$, $\alpha\dfrac{x-t}{t}<\omega<0$ for $x<t\}$. Here α is sufficiently small that will be determined later.

We use the Taylor series expansion (7) to obtain the following for the $\mathcal{I}_1$ case.

$$\begin{aligned} &\int_{-\epsilon+i\alpha\frac{x-t}{t}}^{\epsilon+i\alpha\frac{x-t}{t}} e^{i\xi x-\frac{1}{2}\xi(\xi+\sqrt{\xi^2-4})t}P_+(\xi)d\xi \\ &\qquad = \int_{-\epsilon+i\alpha\frac{x-t}{t}}^{\epsilon+i\alpha\frac{x-t}{t}} \mathcal{O}(1)e^{-\frac{(x-t)^2}{2t}-\frac{t}{2}(\xi-i\frac{x-t}{t})^2+\mathcal{O}(1)\xi^3 t}d\xi \\ &\qquad = \int_{-\epsilon}^{\epsilon}\mathcal{O}(1)e^{-\frac{(x-t)^2}{2t}-\frac{t}{2}(\xi-i(1-\alpha)\frac{x-t}{t})^2+\mathcal{O}(1)(\xi+i\alpha\frac{x-t}{t})^3 t}d\xi. \end{aligned} \tag{8}$$

We have the following estimate by (8):

$$\begin{aligned} &\left|\int_{-\epsilon+i\alpha\frac{x-t}{t}}^{\epsilon+i\alpha\frac{x-t}{t}} e^{i\xi x-\frac{1}{2}\xi(\xi+\sqrt{\xi^2-4})t}P_+(\xi)d\xi\right| \\ &\qquad = \mathcal{O}(1)e^{-\frac{(x-t)^2}{2t}}\left|\int_{-\epsilon}^{\epsilon} e^{-\frac{t}{2}(\xi-i(1-\alpha)\frac{x-t}{t})^2+\mathcal{O}(1)(\xi+i\alpha\frac{x-t}{t})^3 t}d\xi\right| \\ &\qquad \le \mathcal{O}(1)e^{-\frac{(x-t)^2}{2t}}\int_{-\epsilon}^{\epsilon}\left|e^{-\frac{t}{2}(\xi-i(1-\alpha)\frac{x-t}{t})^2+\mathcal{O}(1)(\xi+i\alpha\frac{x-t}{t})^3 t}\right|d\xi. \end{aligned} \tag{9}$$

Since we will take $\alpha,\epsilon>0$ sufficiently small and $\left|\dfrac{x-t}{t}\right|<3$, we can obtain the following estimate for the right hand side of (9):

$$\begin{aligned} &\mathcal{O}(1)e^{-\frac{(x-t)^2}{2t}}e^{\frac{t}{2}((1-\alpha)\frac{x-t}{t})^2}\int_{-\epsilon}^{\epsilon}\left|e^{-\frac{t}{2}\xi^2+\mathcal{O}(1)(\xi+i\alpha\frac{x-t}{t})^3 t}\right|d\xi \\ &\qquad \le \mathcal{O}(1)e^{-(1-(1-\alpha)^2)\frac{(x-t)^2}{2t}}\int_{-\epsilon}^{\epsilon}e^{-\frac{t}{2}\xi^2+\mathcal{O}(1)(|\xi|+\alpha\frac{|x-t|}{t})^3 t}d\xi \end{aligned}$$

$$
\begin{aligned}
&\leq \mathcal{O}(1)e^{-(1-(1-\alpha)^2)\frac{(x-t)^2}{2t}}\int_{-\epsilon}^{\epsilon} e^{-\frac{t}{4}\xi^2+\mathcal{O}(1)(|\xi|\alpha^2\frac{|x-t|^2}{t}+\alpha^3\frac{|x-t|^3}{t^2})}d\xi \\
&\leq \mathcal{O}(1)e^{-(1-(1-\alpha)^2)\frac{(x-t)^2}{2t}+\mathcal{O}(1)(\epsilon\alpha^2\frac{|x-t|^2}{t}+3\alpha^3\frac{|x-t|^2}{t})}\int_{-\epsilon}^{\epsilon} e^{-\frac{t}{4}\xi^2}d\xi \\
&\leq \frac{\mathcal{O}(1)}{\sqrt{t}}e^{-\frac{(x-t)^2}{2ct}}.
\end{aligned}
$$

To estimate $\mathcal{I}_2$ and $\mathcal{I}_3$ term, we will use the Taylor series expansion (7) and smallness assumption of $\alpha, \epsilon > 0$. Because $\mathcal{I}_3$ case is exactly same as in $\mathcal{I}_2$ case, we just show for the $\mathcal{I}_2$ case.

We assume that $x > t$. $x \leq t$ case is exactly the same as $x > t$ case. Therefore, we will omit its proof. The above argument and $x > t$ lead to the following:

$$
\begin{aligned}
&\int_{c_1} e^{i\xi x-\frac{1}{2}\xi(\xi+\sqrt{\xi^2-4})t}P_+(\xi)d\xi \\
&\qquad = \int_{c_1}\mathcal{O}(1)e^{i\xi x-\frac{1}{2}\xi^2 t-i\xi t+\mathcal{O}(1)\xi^3 t}d\xi \\
&\qquad = \int_0^{\alpha\frac{x-t}{t}}\mathcal{O}(1)e^{i(-\epsilon+i\omega)(x-t)-\frac{1}{2}(-\epsilon+i\omega)^2 t+\mathcal{O}(1)(-\epsilon+i\omega)^3 t}d\omega.
\end{aligned}
$$

Therefore, we can obtain the estimate for $\mathcal{I}_2$ term as follows:

$$
\begin{aligned}
&\left|\int_{c_1} e^{i\xi x-\frac{1}{2}\xi(\xi+\sqrt{\xi^2-4})t}P_+(\xi)d\xi\right| \\
&\qquad = \left|\int_0^{\alpha\frac{x-t}{t}}\mathcal{O}(1)e^{i(-\epsilon+i\omega)(x-t)-\frac{1}{2}(-\epsilon+i\omega)^2 t+\mathcal{O}(1)(-\epsilon+i\omega)^3 t}d\omega\right| \\
&\qquad \leq C\int_0^{\alpha\frac{x-t}{t}}\left|e^{i(-\epsilon+i\omega)(x-t)-\frac{1}{2}(-\epsilon+i\omega)^2 t+\mathcal{O}(1)(-\epsilon+i\omega)^3 t}\right|d\omega \\
&\qquad \leq C\int_0^{\alpha\frac{x-t}{t}} e^{-\omega(x-t)-\frac{1}{2}(\epsilon^2-\omega^2)t+\mathcal{O}(1)(|\epsilon|+|\omega|)^3 t}d\omega.
\end{aligned}
\tag{10}
$$

We can obtain an estimate for the right hand side of the inequality (10) by direct calculation as follows:

$$
\begin{aligned}
&C\int_0^{\alpha\frac{x-t}{t}} e^{-\frac{t}{2}\epsilon^2-\omega(x-t)+\frac{t}{2}\omega^2+\mathcal{O}(1)(|\epsilon|+|\omega|)^3 t}d\omega \leq C\int_0^{\alpha\frac{x-t}{t}} e^{-\frac{t}{2}\epsilon^2+\mathcal{O}(1)(|\epsilon|+|\omega|)^3 t}d\omega \\
&\leq C\int_0^{\alpha\frac{x-t}{t}} e^{-\frac{t}{4}\epsilon^2}d\omega \leq C\alpha\frac{x-t}{t}e^{-\frac{t}{4}\epsilon^2} \;\leq\; Ce^{-t/c}.
\end{aligned}
$$

As mentioned above, for $x \leq t$ case, we also obtain the same result by the same method as in $x > t$ case. In summary, we have the following:

$$
\left|\int_{|\xi|<\epsilon} e^{i\xi x-\frac{1}{2}\xi(\xi+\sqrt{\xi^2-4})t}P_+(\xi)d\xi\right| \leq \frac{\mathcal{O}(1)}{\sqrt{t}}e^{-\frac{(x-t)^2}{2ct}}+Ce^{-t/c}. \tag{11}
$$

We can also establish the same estimate for the $P_-(\xi)$ part by the same method as in the $P_+(\xi)$ case and we can obtain the following result for $P_-(\xi)$:

$$
\left|\int_{|\xi|<\epsilon} e^{i\xi x-\frac{1}{2}\xi(\xi-\sqrt{\xi^2-4})t}P_-(\xi)d\xi\right| \leq \frac{\mathcal{O}(1)}{\sqrt{t}}e^{-\frac{(x+t)^2}{2ct}}+Ce^{-t/c}. \tag{12}
$$

Therefore, we prove this lemma with the above two estimates (11) and (12). □

In the rest of this section, we will establish a point-wise estimate of the Green function G for the compact interval $\{\epsilon < |\xi| < N\}$ on the finite Mach number region $\{|x| < 2t\}$.

Lemma 3.2. *For a sufficiently small number $\epsilon > 0$ and a large number $N > 0$, we have*

$$\left| \int_{\epsilon<|\xi|<N} \hat{G}(\xi,t)e^{ix\xi}d\xi \right| \leq Ce^{-t/c},$$

where positive constants C and c depend on ϵ and N.

Proof. In Lemma 2.1, we observed that $Re\{-\frac{1}{2}\xi(\xi \pm \sqrt{\xi^2-4})\} < 0$ and $\hat{G}$ is entire function. Because $\{\epsilon < |\xi| < N\}$ is compact, by the above properties, we have the following:

$$Re\{-\frac{t}{2}\xi(\xi \pm \sqrt{\xi^2-4})\} \leq -t/C,$$

where C is a positive constant. By the above property, we can verify this lemma. □

We are ready to obtain a point-wise estimate for the long-wave component. The following is the main result of this section.

Proposition 1. *On the finite Mach number region $\{|x| < 2t\}$, we can obtain a point-wise estimate of G for the long-wave component as follows:*

$$\left| \int_{|\xi|<N} \hat{G}(\xi,t)e^{ix\xi}d\xi \right| \leq \frac{\mathcal{O}(1)}{\sqrt{t}}e^{-\frac{(x-t)^2}{2ct}} + \frac{\mathcal{O}(1)}{\sqrt{t}}e^{-\frac{(x+t)^2}{2ct}} + Ce^{-t/c}$$

where $N > 0$ is a sufficiently large number, C and c are positive constants depending on N.

Proof. The proof is straightforward by Lemma 3.1 and 3.2. □

To derive the similar result for derivatives of G, we will consider $(i\xi)^k\hat{G}(\xi,t)$ in place of $\hat{G}(\xi,t)$. It suffices because of a property of the Fourier transform, i.e., $(\partial_x^k G)^\wedge = (i\xi)^k\hat{G}$. We summarize this in the following corollary, and its result will be used in next section.

Corollary 1. *For the finite Mach number region $\{|x| < 2t\}$ and $k \in \mathbb{N}$, we can obtain the similar result to Proposition 1 as follows:*

$$\left| \int_{|\xi|<N} (i\xi)^k\hat{G}e^{ix\xi}d\xi \right| \leq \frac{\mathcal{O}(1)}{\sqrt{t}}e^{-\frac{(x-t)^2}{2ct}} + \frac{\mathcal{O}(1)}{\sqrt{t}}e^{-\frac{(x+t)^2}{2ct}} + Ce^{-t/c}$$

where $N > 0$ is a sufficiently large number, C and c are positive constants.

Proof. Because of compactness of interval $|\xi| < N$, $(i\xi)^k$ term does not have influence on the proof of Lemma 3.2 case. For the proof of Lemma 3.1 case, we also easily verify that the method in Lemma 3.1 works well in $(i\xi)^k\hat{G}$ case. □

3.2. **The short-wave component.** In this part, we will obtain an estimate of the Green function G for the short-wave component. Since $e^t\hat{G}$ converges to matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ as $|\xi| \to \infty$, it is natural to consider the following term:

$$e^t\hat{G} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \Big(\xi^{-1} \text{ order term of } e^t\hat{G}\Big).$$

We will obtain an $H^1(\mathbb{R})$-bound for the above expression by using a property of the Fourier transform (See Lemma 3.3). The following lemma is a key step in this part to produce an $H^1(\mathbb{R})$-bound for the Green function G.

Lemma 3.3. *Let $f(x)$ be a function with a well-defined Fourier transformed function $\hat{f}(\xi)$ for a complex variable $\xi = \eta + i\alpha$ with $|\alpha| < \epsilon$ and $\epsilon > 0$ be any fixed number, where if $\hat{f}(\xi)$ has weighed $L^2(\mathbb{R})$-bound for the complex variable ξ as follow:*

$$\int_{\mathbb{R}} (|\xi|^2 + 1)|\hat{f}(\xi)|^2 d\eta \leq K \quad \text{for all } |\alpha| < \epsilon. \tag{13}$$

Then $f(x)$ satisfies $|f(x)| \leq Ce^{-|x|/c}$ where C and c are positive constants depend on K.

Proof. For notational convenience, we denote that $F(x) = f(x)e^{\alpha x}$. Since a Fourier transformed function $\hat{f}$ is well defineding on complex variable $\xi = \eta + i\alpha$ with $|\alpha| < \epsilon$, we have

$$\int_{\mathbb{R}} f(x)e^{-i(\eta+i\alpha)x}dx \quad = \quad \hat{f}(\eta + i\alpha) = \int_{\mathbb{R}} f(x)e^{\alpha x}e^{-i\eta x}dx =: \hat{F}(\eta).$$

The Plancherel theorem implies the following:

$$\int_{\mathbb{R}} |\hat{f}(\eta + i\alpha)|^2 d\eta = \int_{\mathbb{R}} |\hat{F}(\eta)|^2 d\eta = \int_{\mathbb{R}} |F(x)|^2 dx := \int_{\mathbb{R}} |f(x)e^{\alpha x}|^2 dx.$$

Similarly, we can easily verify that

$$\int_{\mathbb{R}} |i\eta + \alpha|^2 |\hat{F}(\eta)|^2 d\eta = \int_{\mathbb{R}} |i\eta + \alpha|^2 |\hat{f}(\eta + i\alpha)|^2 d\eta = \int_{\mathbb{R}} |f'(x)e^{\alpha x}|^2 dx$$

and

$$\begin{aligned} \int_{\mathbb{R}} |f'(x)e^{\alpha x}|^2 dx \quad &= \quad \int_{\mathbb{R}} |(f(x)e^{\alpha x})' - f(x)\alpha e^{\alpha x}|^2 dx \\ &\leq \quad \int_{\mathbb{R}} |\eta|^2 |\hat{F}(\eta)|^2 d\eta + \alpha \int_{\mathbb{R}} |F(x)|^2 dx \end{aligned}$$

These above two relations and assumption (13) imply the following inequality:

$$\int_{\mathbb{R}} |f(x)e^{\alpha x}|^2 + |f'(x)e^{\alpha x}|^2 dx \leq K,$$

for any α satisfying $|\alpha| < \epsilon$. Therefore, by the Sobolev embedding theorem, we have $|f(x)| < Ce^{-|x|/c}$ for some positive constants C and c. □

Remark 1. Lemma 3.3 allows us to control the $\mathcal{O}(1/\xi^2)$-term. As mentioned before, we can control zero-order term by subtraction of Dirac delta distribution.

In the next lemma, we can calculate a bound for the $1/\xi$ order term, i.e., $\mathcal{F}^{-1}\Big(\frac{\chi_{\{|\xi|>N\}}}{\xi}\Big)(x)$.

Lemma 3.4. *Let $\mathcal{F}^{-1}$ be the inverse Fourier transform operator, and χ be a characteristic function. Then the following estimate holds for any non-zero real number N:*

$$\left|\mathcal{F}^{-1}\Big(\frac{\chi_{\{|\xi|>N\}}}{\xi}\Big)(x)\right| \leq C, \tag{14}$$

where $C > 0$ is a constant.

Proof. Obviously, (14) holds for $x = 0$ case. Consider the $x \neq 0$ case. Because we take any non-zero x, we have the following equality:

$$\int_0^\infty \frac{\sin x\xi}{\xi} d\xi = \frac{\pi}{2}.$$

Therefore, we can obtain the following relation for any non-zero x:

$$\mathcal{F}^{-1}\Big(\frac{\chi_{\{|\xi|>N\}}}{\xi}\Big) \;=\; \int_{|\xi|>N} e^{ix\xi}\frac{1}{\xi}d\xi = -2\int_N^\infty \frac{i\sin x\xi}{\xi}d\xi = 2i\int_0^N \frac{\sin x\xi}{\xi}d\xi - \pi i.$$

Therefore, the inverse Fourier transformed function is bounded, i.e.,

$$\Big|\mathcal{F}^{-1}\Big(\frac{\chi_{\{|\xi|>N\}}}{\xi}\Big)(x)\Big| \le C,$$

where $C > 0$ is a generic constant. □

We will close this section with the following proposition and corollary, which describe a point-wise estimate of the Green function G for the short-wave component.

Proposition 2. *For sufficiently large number $N > 0$, we have a point-wise estimate of the Green function G for the short-wave component as follows:*

$$\Big|\int_{|\xi|>N}\Big(\hat{G} - e^{-t}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\Big)e^{i\xi x}d\xi\Big| \le Ce^{-t} + Ce^{-|x|/c-t}$$

where C and c are positive constants and $\hat{G}$ is the Fourier transform of the Green function G for the p-system (1).

Proof. For any complex variable ξ, we have

$$e^t\hat{G} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{i}{\xi} \\ -\frac{i}{\xi} & 0 \end{pmatrix} + \mathcal{O}\Big(\frac{1}{\xi^2}\Big).$$

For any real number $N > 0$, $\dfrac{\chi_{|\xi|>N}}{\xi^2}$ satisfies the following relation:

$$\int_{\mathbb{R}}(|\xi|^2+1)\Big|\frac{\chi_{|\xi|>N}}{\xi^2}\Big|^2 d\eta = \int_{|\xi|>N}(|\xi|^2+1)\Big|\frac{1}{\xi^2}\Big|^2 d\eta < C. \tag{15}$$

By Lemma 3.3 and the above equation (15), we obtain the following estimate:

$$\Big|\int_{|\xi|>N}\Big(e^t\hat{G} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{i}{\xi} \\ \frac{i}{\xi} & 0 \end{pmatrix}\Big)e^{ix\xi}d\xi\Big| \le Ce^{-|x|/c}.$$

Therefore, by Lemma 3.4, we have that

$$\begin{aligned}\Big|\int_{|\xi|>N}\Big(e^t\hat{G} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\Big)e^{ix\xi}d\xi\Big| \;&\le\; \Big|\int_{|\xi|>N}\begin{pmatrix} 0 & \frac{i}{\xi} \\ \frac{i}{\xi} & 0 \end{pmatrix}e^{ix\xi}d\xi\Big| + Ce^{-|x|/c} \\ &\le\; C + Ce^{-|x|/c}.\end{aligned}$$

This yields the proof for Proposition 2. □

We next use the higher order Taylor series expansion to obtain the same result for the derivatives of G, i.e., $(i\eta)^k\hat{G}(\eta,t)$.

Corollary 2. *For d^kG/dx^k with $k = 1, 2$, we can obtain point-wise estimate for the short-wave component as follows:*

$$\begin{aligned}\Big|\int_{|\xi|>N}\Big[(i\xi)^k\hat{G} - e^{-t}(i\xi)^k\begin{pmatrix} 1+\frac{1-t}{\xi^2} & -\frac{i}{\xi}+\frac{i(t-2)}{\xi^3} \\ -\frac{i}{\xi}+\frac{i(t-2)}{\xi^3} & -\frac{1}{\xi^2} \end{pmatrix}\Big]e^{i\xi x}d\xi\Big| \\ \le Ce^{-t} + Ce^{-|x|/c-t}\end{aligned}$$

for some constants C and c.

Proof. We have series expansion of $(i\xi)^k\hat{G}$:

$$e^t(i\xi)^k\hat{G} = (i\xi)^k\begin{pmatrix} 1+\frac{1-t}{\xi^2} & -\frac{i}{\xi}+\frac{i(t-2)}{\xi^3} \\ -\frac{i}{\xi}+\frac{i(t-2)}{\xi^3} & -\frac{1}{\xi^2}\end{pmatrix} + \mathcal{O}\Big(\frac{(i\xi)^k}{\xi^4}\Big).$$

By the same method as in Proposition 2, we have

$$\left|\int_{|\xi|>N}\left[(i\xi)^k\hat{G} - e^{-t}(i\xi)^k\begin{pmatrix} 1+\frac{1-t}{\xi} & -\frac{i}{\xi}+\frac{i(t-2)}{\xi^3} \\ -\frac{i}{\xi}+\frac{i(t-2)}{\xi^3} & -\frac{1}{\xi^2}\end{pmatrix}\right]e^{i\xi x}d\xi\right| \le Ce^{-|x|/c}$$

for some constants C and c. Therefore, by Lemma 3.4, we complete the proof of this corollary. □

4. Weighed energy estimate in the outside finite Mach number region.

In this section, we will obtain the weighted energy estimate for the Green function G to the p-system (1) in the outside finite Mach number region $\{|x| \ge 2t\}$. By the weighted energy estimate, we will obtain a point-wise estimate of the Green function G in the outside finite Mach number region.

Initial data of the Green function G is $\delta(x)I$ where I is the 2×2 identity matrix and $\delta(x)$ is Dirac delta measure and the Green function G is matrix valued. However, the weighted energy estimate is a scalar relation related to the vector (ρ, u). Therefore, to obtain the weighted energy estimate for the Green function G, we have to consider two cases which is divided into initial data $(\rho_0, u_0) = (\delta(x), 0)$ case and initial data $(\rho_0, u_0) = (0, \delta(x))$ case. We first deal with initial data $(\rho_0, u_0) = (\delta(x), 0)$ case. For other case, i.e, $(\rho_0, u_0) = (0, \delta(x))$, we can apply the same method as in $(\rho_0, u_0) = (\delta(x), 0)$ case to obtain the same result. Therefore, we will omit the second case for the conciseness of this paper.

To avoid singularity of the Green function G, we define $\bar{\rho} = \rho - e^{-t}\delta(x)$ and $\bar{u} = u$. Then $(\bar{\rho}, \bar{u})$ satisfy the following two equations:

$$\bar{\rho}_t + \bar{u}_x = e^{-t}\delta(x), \qquad \bar{u}_t + \bar{\rho}_x - \Delta\bar{u} = -e^{-t}\delta'(x). \tag{16}$$

The source terms in (16) have no effect to calculate the weighted energy because we will consider integration on interval $\{|x| > \frac{5}{4}t\}$.

We multiply each side of the first and the second of (16) by $e^{\alpha(|x|-\frac{3}{2}t)}\bar{\rho}$ and $e^{\alpha(|x|-\frac{3}{2}t)}\bar{u}$ respectively and integrate with respect to x on $\{|x| > \frac{5}{4}t\}$ to obtain as follows:

$$\begin{aligned}&\int_{|x|>\frac{5}{4}t}\bar{\rho}_t\bar{\rho}e^{\alpha(|x|-\frac{3}{2}t)} + \bar{u}_x\bar{\rho}e^{\alpha(|x|-\frac{3}{2}t)}dx = 0,\\ &\int_{|x|>\frac{5}{4}t}\bar{u}_t\bar{u}e^{\alpha(|x|-\frac{3}{2}t)} + \bar{\rho}_x\bar{u}e^{\alpha(|x|-\frac{3}{2}t)} - \Delta\bar{u}\bar{u}e^{\alpha(|x|-\frac{3}{2}t)}dx = 0.\end{aligned} \tag{17}$$

We next add the above two equations of (17) to get the following equation:

$$\begin{aligned}&\int_{|x|>\frac{5}{4}t}(\bar{\rho}_t\bar{\rho} + \bar{u}_t\bar{u})e^{\alpha(|x|-\frac{3}{2}t)}dx\\ &+\int_{|x|>\frac{5}{4}t}(\bar{u}_x\bar{\rho} + \bar{\rho}_x\bar{u})e^{\alpha(|x|-\frac{3}{2}t)}dx - \int_{|x|>\frac{5}{4}t}\Delta\bar{u}\bar{u}e^{\alpha(|x|-\frac{3}{2}t)}dx = 0.\end{aligned} \tag{18}$$

We will modify three terms in (18) to get the weighted energy estimate by using the integration-by-part. To get this energy estimate, we first define the weighted energy as follow:

$$\mathbf{E}^{w,0}_{\rho,u}(t) =: \int_{|x|>\frac{5}{4}t} e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{\rho}^2+\bar{u}^2}{2}dx.$$

For the first term in (18), we will use the integration-by-part with respect to $t>0$ to form the weighed energy defined as in the above:

$$\begin{aligned}
\int_{|x|>\frac{5}{4}t}(\bar{\rho}_t\bar{\rho}+\bar{u}_t\bar{u})e^{\alpha(|x|-\frac{3}{2}t)}dx &= \int_{|x|>\frac{5}{4}t} e^{\alpha(|x|-\frac{3}{2}t)}\big(\frac{\bar{\rho}^2+\bar{u}^2}{2}\big)_t dx\\
&= \int_{|x|>\frac{5}{4}t}\frac{d}{dt}\big[e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{\rho}^2+\bar{u}^2}{2}\big]dx - \int_{|x|>\frac{5}{4}t}\big(e^{\alpha(|x|-\frac{3}{2}t)}\big)_t\frac{\bar{\rho}^2+\bar{u}^2}{2}dx\\
&= \frac{5}{4}e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{\rho}^2+\bar{u}^2}{2}\Big|_{x=-\frac{5}{4}t} + \frac{5}{4}e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{\rho}^2+\bar{u}^2}{2}\Big|_{x=\frac{5}{4}t}\\
&\quad + \frac{d}{dt}\int_{|x|>\frac{5}{4}t} e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{\rho}^2+\bar{u}^2}{2}dx + \frac{3\alpha}{2}\int_{|x|>\frac{5}{4}t} e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{\rho}^2+\bar{u}^2}{2}dx\\
&= \frac{d}{dt}\mathbf{E}^{w,0}_{\rho,u}(t) + \frac{3\alpha}{2}\mathbf{E}^{w,0}_{\rho,u}(t)\\
&\quad + \frac{5}{4}e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{\rho}^2+\bar{u}^2}{2}\Big|_{x=-\frac{5}{4}t} + \frac{5}{4}e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{\rho}^2+\bar{u}^2}{2}\Big|_{x=\frac{5}{4}t}.
\end{aligned}$$

For the second term of (18), we have the following equality:

$$\begin{aligned}
&\int_{|x|>\frac{5}{4}t}(\bar{u}_x\bar{\rho}+\bar{\rho}_x\bar{u})e^{\alpha(|x|-\frac{3}{2}t)}dx\\
&\qquad = \int_{|x|>\frac{5}{4}t} e^{\alpha(|x|-\frac{3}{2}t)}(\bar{u}\bar{\rho})_x dx\\
&\qquad = \int_{|x|>\frac{5}{4}t}\frac{d}{dx}\big(e^{\alpha(|x|-\frac{3}{2}t)}\bar{u}\bar{\rho}\big)dx - \int_{|x|>\frac{5}{4}t}\big(e^{\alpha(|x|-\frac{3}{2}t)}\big)_x\bar{u}\bar{\rho}dx\\
&\qquad = e^{\alpha(|x|-\frac{3}{2}t)}\bar{u}\bar{\rho}\Big|_{x=\frac{5}{4}t}^{x=-\frac{5}{4}t} - \int_{|x|>\frac{5}{4}t}\frac{\alpha x}{|x|}e^{\alpha(|x|-\frac{3}{2}t)}\bar{u}\bar{\rho}dx.
\end{aligned}$$

For this term, we will make an estimate for the weighed energy estimate by using the inequality of arithmetic and geometric means later.

For the last term, by using the integration-by-part, we can obtain

$$\begin{aligned}
&-\int_{|x|>\frac{5}{4}t}\Delta\bar{u}\bar{u}e^{\alpha(|x|-\frac{3}{2}t)}dx\\
&\quad = \int_{|x|>\frac{5}{4}t}\bar{u}_x^2 e^{\alpha(|x|-\frac{3}{2}t)} + \bar{u}_x\bar{u}(e^{\alpha(|x|-\frac{3}{2}t)})_x - \frac{d}{dx}\big(\bar{u}_x\bar{u}e^{\alpha(|x|-\frac{3}{2}t)}\big)dx \qquad (19)\\
&\quad = \int_{|x|>\frac{5}{4}t}\bar{u}_x^2 e^{\alpha(|x|-\frac{3}{2}t)} + \bar{u}_x\bar{u}\frac{\alpha x}{|x|}e^{\alpha(|x|-\frac{3}{2}t)}dx + \bar{u}_x\bar{u}e^{\alpha(|x|-\frac{3}{2}t)}\Big|_{x=-\frac{5}{4}t}^{x=\frac{5}{4}t}.
\end{aligned}$$

In the above equation (19), we will apply the integration-by-part for the second term

$$\int_{|x|>\frac{5}{4}t}\bar{u}_x\bar{u}\frac{\alpha x}{|x|}e^{\alpha(|x|-\frac{3}{2}t)}dx$$

to obtain as follows:

$$\begin{aligned}
&\int_{|x|>\frac{5}{4}t} \bar{u}_x \bar{u} \frac{\alpha x}{|x|} e^{\alpha(|x|-\frac{3}{2}t)} dx \\
&= \bar{u}^2 \frac{\alpha x}{|x|} e^{\alpha(|x|-\frac{3}{2}t)} \Big|_{x=\frac{5}{4}t}^{x=-\frac{5}{4}t} - \int_{|x|>\frac{5}{4}t} \bar{u}\bar{u}_x \frac{\alpha x}{|x|} e^{\alpha(|x|-\frac{3}{2}t)} \\
&+\bar{u}^2 \frac{d}{dx}\Big(\frac{\alpha x}{|x|} e^{\alpha(|x|-\frac{3}{2}t)}\Big) dx.
\end{aligned}$$

The above equation implies the following:

$$\int_{|x|>\frac{5}{4}t} \bar{u}_x \bar{u} \frac{\alpha x}{|x|} e^{\alpha(|x|-\frac{3}{2}t)} dx = \frac{\bar{u}^2}{2} \frac{\alpha x}{|x|} e^{\alpha(|x|-\frac{3}{2}t)} \Big|_{x=\frac{5}{4}t}^{x=-\frac{5}{4}t} - \frac{\alpha^2}{2} \int_{|x|>\frac{5}{4}t} \bar{u}^2 e^{\alpha(|x|-\frac{3}{2}t)} dx.$$

Therefore, we have the following equality for the third term:

$$\begin{aligned}
&- \int_{|x|>\frac{5}{4}t} \Delta \bar{u} \bar{u} e^{\alpha(|x|-\frac{3}{2}t)} dx \\
&\quad = -\int_{|x|>\frac{5}{4}t} \bar{u}\bar{u}_x \frac{\alpha x}{|x|} e^{\alpha(|x|-\frac{3}{2}t)} dx - \int_{|x|>\frac{5}{4}t} \bar{u}^2 \frac{d}{dx}\Big(\frac{\alpha x}{|x|} e^{\alpha(|x|-\frac{3}{2}t)}\Big) dx \\
&\quad + \int_{|x|>\frac{5}{4}t} \bar{u}_x^2 e^{\alpha(|x|-\frac{3}{2}t)} dx + \bar{u}^2 \frac{\alpha x}{|x|} e^{\alpha(|x|-\frac{3}{2}t)} \Big|_{x=\frac{5}{4}t}^{x=-\frac{5}{4}t} \\
&\quad + \bar{u}_x \bar{u} e^{\alpha(|x|-\frac{3}{2}t)} \Big|_{x=-\frac{5}{4}t}^{x=\frac{5}{4}t}.
\end{aligned}$$

In summary, we can modify (18) into the following form by the above calculation. For computational convenience, we now classify each term into two groups according to the integration part(I part) and the boundary part(B part) as follows:

$$\begin{aligned}
0 = \;& \left.\begin{aligned}
&\frac{d}{dt}\mathbf{E}^{w,0}_{\rho,u}(t) + \frac{3\alpha}{2}\mathbf{E}^{w,0}_{\rho,u}(t) - \int_{|x|>\frac{5}{4}t} \frac{\alpha x}{|x|} e^{\alpha(|x|-\frac{3}{2}t)} \bar{u}\bar{\rho} \\
&+ \int_{|x|>\frac{5}{4}t} \bar{u}_x^2 e^{\alpha(|x|-\frac{3}{2}t)} dx - \frac{\alpha^2}{2} \int_{|x|>\frac{5}{4}t} \bar{u}^2 e^{\alpha(|x|-\frac{3}{2}t)} dx
\end{aligned}\right\} \text{(I)} \\
& \left.\begin{aligned}
&+\frac{5}{4} e^{\alpha(|x|-\frac{3}{2}t)} \frac{\bar{\rho}^2 + \bar{u}^2}{2}\Big|_{x=-\frac{5}{4}t} + \frac{5}{4} e^{\alpha(|x|-\frac{3}{2}t)} \frac{\bar{\rho}^2 + \bar{u}^2}{2}\Big|_{x=\frac{5}{4}t} \\
&+ e^{\alpha(|x|-\frac{3}{2}t)} \bar{u}\bar{\rho}\Big|_{x=\frac{5}{4}t}^{x=-\frac{5}{4}t} + \frac{\bar{u}^2}{2} \frac{\alpha x}{|x|} e^{\alpha(|x|-\frac{3}{2}t)}\Big|_{x=\frac{5}{4}t}^{x=-\frac{5}{4}t} \\
&- \bar{u}_x \bar{u} e^{\alpha(|x|-\frac{3}{2}t)}\Big|_{x=\frac{5}{4}t}^{x=-\frac{5}{4}t}.
\end{aligned}\right\} \text{(B)}
\end{aligned} \tag{20}$$

For the integration part, we can easily obtain the following inequality by using the inequality of arithmetic and geometric means as mentioned before:

$$\begin{aligned}
&-\int_{|x|>\frac{5}{4}t}\frac{\alpha x}{|x|}e^{\alpha(|x|-\frac{3}{2}t)}\bar{u}\bar{\rho}dx+\int_{|x|>\frac{5}{4}t}\bar{u}_x^2e^{\alpha(|x|-\frac{3}{2}t)}dx\\
&-\frac{\alpha^2}{2}\int_{|x|>\frac{5}{4}t}\bar{u}^2e^{\alpha(|x|-\frac{3}{2}t)}dx\\
&\qquad\geq-\alpha\int_{|x|>\frac{5}{4}t}e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{u}^2+\bar{\rho}^2}{2}dx-\alpha^2\int_{|x|>\frac{5}{4}t}e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{u}^2+\bar{\rho}^2}{2}dx\\
&\qquad=-(\alpha+\alpha^2)\mathbf{E}^{w,0}_{\rho,u}(t)
\end{aligned}\tag{21}$$

and for the boundary part, we can obtain the following estimate by the same inequality and smallness assumption for a real number $\alpha>0$:

$$\begin{aligned}
&\frac{5}{4}e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{\rho}^2+\bar{u}^2}{2}\Big|_{x=-\frac{5}{4}t}+\frac{5}{4}e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{\rho}^2+\bar{u}^2}{2}\Big|_{x=\frac{5}{4}t}\\
&\quad+e^{\alpha(|x|-\frac{3}{2}t)}\bar{u}\bar{\rho}\Big|_{x=-\frac{5}{4}t}-e^{\alpha(|x|-\frac{3}{2}t)}\bar{u}\bar{\rho}\Big|_{x=\frac{5}{4}t}+\frac{\bar{u}^2}{2}\frac{\alpha x}{|x|}e^{\alpha(|x|-\frac{3}{2}t)}\Big|_{x=\frac{5}{4}t}^{x=-\frac{5}{4}t}\\
&\geq\frac{e^{\alpha(|x|-\frac{3}{2}t)}}{4}\frac{\bar{\rho}^2+\bar{u}^2}{2}\Big|_{x=-\frac{5}{4}t}+\frac{e^{\alpha(|x|-\frac{3}{2}t)}}{4}\frac{\bar{\rho}^2+\bar{u}^2}{2}\Big|_{x=\frac{5}{4}t}\\
&\quad+\frac{\bar{u}^2}{2}\frac{\alpha x}{|x|}e^{\alpha(|x|-\frac{3}{2}t)}\Big|_{x=\frac{5}{4}t}^{x=-\frac{5}{4}t}\\
&\geq 0.
\end{aligned}\tag{22}$$

Finally, by the inequalities (21) and (22) and equality (20), we have the following inequality:

$$\frac{d}{dt}\mathbf{E}^{w,0}_{\rho,u}(t)+\big(\frac{\alpha}{2}-\alpha^2\big)\mathbf{E}^{w,0}_{\rho,u}(t)\leq-\bar{u}_x\bar{u}e^{\alpha(|x|-\frac{3}{2}t)}\Big|_{x=-\frac{5}{4}t}+\bar{u}_x\bar{u}e^{\alpha(|x|-\frac{3}{2}t)}\Big|_{x=\frac{5}{4}t}.$$

We can apply the same method as in the above for $\bar{\rho}_x$, $\bar{u}_x$ with higher order energy:

$$\mathbf{E}^{w,1}_{\rho,u}(t)=:\int_{|x|>\frac{5}{4}t}e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{\rho}_x^2+\bar{u}_x^2}{2}dx$$

and we can easily obtain the following inequality for higher order energy:

$$\frac{d}{dt}\mathbf{E}^{w,1}_{\rho,u}(t)+\big(\frac{\alpha}{2}-\alpha^2\big)\mathbf{E}^{w,1}_{\rho,u}(t)\leq-\bar{u}_{xx}\bar{u}_xe^{\alpha(|x|-\frac{3}{2}t)}\Big|_{x=-\frac{5}{4}t}+\bar{u}_{xx}\bar{u}_xe^{\alpha(|x|-\frac{3}{2}t)}\Big|_{x=\frac{5}{4}t}.$$

As mentioned above, we can apply the same weighted energy method for initial data $(\rho_0,u_0)=(0,\delta(x))$ case. Therefore, we can obtain the weighted energy estimate for the Green function G of the p-system (1) in the outside finite Mach number region.

5. **The proof of Theorem 1.1.** We choose a number $\alpha>0$ sufficiently small such that $\alpha/2-\alpha^2>0$. By the Corollary 1 and 2, smallness of $\alpha>0$ and the weighted energy estimate for $\mathbf{E}^{w,0}_{\rho,u}$ and $\mathbf{E}^{w,1}_{\rho,u}$ in previous section, we can obtain the following two inequalities:

$$\frac{d}{dt}\int_{|x|>\frac{5}{4}t}e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{\rho}^2+\bar{u}^2}{2}dx\leq\frac{C}{\sqrt{t}}e^{-\frac{1}{4}t}$$

and

$$\frac{d}{dt}\int_{|x|>\frac{5}{4}t} e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{\rho}_x^2+\bar{u}_x^2}{2}dx \le \frac{C}{t}e^{-\frac{1}{4}t},$$

where $C>0$ is a constant. The above two inequalities mean that

$$\Big|\Big|e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{\rho}^2+\bar{u}^2}{2}\Big|\Big|_{W^{1,1}(|x|>\frac{5}{4}t)} \le K,$$

for all sufficiently large time $t>0$. This implies the following by the Sobolev embedding theorem:

$$\Big|\Big|e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{\rho}^2+\bar{u}^2}{2}\Big|\Big|_{L^\infty(|x|>\frac{5}{4}t)} \le K. \tag{23}$$

Since inequality (23) also holds on subset $\{|x|>2t\}$, we have the following:

$$e^{\alpha(|x|-\frac{3}{2}t)}\frac{\bar{\rho}^2+\bar{u}^2}{2} \le K \quad \text{on } \{|x|>2t\}, \tag{24}$$

for all large time $t>0$.

We have the following relation for x, t:

$$|x|-\frac{3}{2}t \;>\; \frac{1}{8}|x|+\frac{7}{8}|x|-\frac{3}{2}t \;>\; \frac{1}{8}|x|+\frac{7}{4}t-\frac{3}{2}t \;>\; \frac{1}{8}|x|+\frac{1}{8}t, \tag{25}$$

because we assume that (x,t) is on the outside finite Mach number region $\{|x|>2t\}$. The estimate (24) and (25) imply that

$$\frac{\bar{\rho}^2+\bar{u}^2}{2} \le Ce^{-(|x|+t)/c} \quad \text{on } \{|x|>2t\},$$

for all large time $t>0$. Therefore, we obtain a point-wise estimate for the Green function G on the outside finite Mach number region $\{|x|>2t\}$. The following proposition is the summary of the above argument.

Proposition 3. *Let $t>0$ be large sufficiently. Then we can obtain a point-wise estimate for the Green function $G(x,t)$ on the outside finite Mach number region $\{|x|>2t\}$ as the following:*

$$|G(x,t)| \le Ce^{-(|x|+t)/c}$$

where C and c are positive constants.

The proof of Theorem 1.1. We are ready to prove the main theorem of this paper. We will integrate results in previous sections to produce a point-wise estimate of the Green function G. For the finite Mach number region $\{|x|<2t\}$, we have the following estimate by Proposition 1 and 2:

$$\begin{aligned}\left|G(x,t)-e^{-t}\delta(x)\begin{pmatrix}1&0\\0&0\end{pmatrix}\right| &= \left|\int_{\mathbb{R}}\left(\hat{G}-e^{-t}\begin{pmatrix}1&0\\0&0\end{pmatrix}\right)e^{i\xi x}d\xi\right|\\ &\le Ce^{-t}+Ce^{-|x|/c-t}+\frac{\mathcal{O}(1)}{\sqrt{t}}e^{-\frac{(x-t)^2}{2ct}}+\frac{\mathcal{O}(1)}{\sqrt{t}}e^{-\frac{(x+t)^2}{2ct}}\\ &\le Ce^{-\frac{|x|+t}{c}}+\frac{\mathcal{O}(1)}{\sqrt{t}}e^{-\frac{(x-t)^2}{2ct}}+\frac{\mathcal{O}(1)}{\sqrt{t}}e^{-\frac{(x+t)^2}{2ct}},\end{aligned} \tag{26}$$

where C and c are positive constants. For the outside finite Mach number region $\{|x|>2t\}$, we have the following by Proposition 3:

$$|G(x,t)| \le Ce^{-\frac{|x|+t}{c}}, \tag{27}$$

where C and c are positive constants. (26) and (27) lead to the proof of Theorem 1.1.

6. **Conclusion.** We studied a new methodology to investigate a large-time behavior of the Green function G of the linearized compressible Navier-Stokes equation, i.e., p-system (1). We established an explicit formula (4) for the Green function G by using the Fourier transform and an inverse Fourier transform formula. To calculate this inverse Fourier transformed formula, we subtract dirac delta distribution from the Green function G and we divided a time-spatial variable plane $(x,t) \in \mathbb{R}^2$ into the finite Mach number region and the outside finite Mach number region. On the finite Mach number region, we apply the long wave-short wave analysis. For the long-wave component, we apply the Cauchy theorem to chage the path of integral and then we calculate it directly. For the short-wave component, we use the property of the Fourier transform, i.e., Lemma 3.3. For the outside finite Mach number region, we adapt the weighted energy estimate to obtain the estimate of the Green function G of p-system (1).

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E-mail address: matcs@nus.edu.sg