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TRAVELING FRONTS OF PYRAMIDAL SHAPES IN COMPETITION-DIFFUSION SYSTEMS

Wei-Ming Ni

Center for Partial Differential Equations
East China Normal University
Minhang, Shanghai, 200241, China
and
School of Mathematics, University of Minnesota
Minneapolis, MN 55455, USA

Masaharu Taniguchi

Department of Mathematical and Computing Sciences
Tokyo Institute of Technology
2-12-1-W8-38 Ookayama, Meguro-ku, Tokyo 152-8552, Japan

Abstract. It is well known that a competition-diffusion system has a one-dimensional traveling front. This paper studies traveling front solutions of pyramidal shapes in a competition-diffusion system in $\mathbb{R}^N$ with $N \geq 2$. By using a multi-scale method, we construct a suitable pair of a supersolution and a subsolution, and find a pyramidal traveling front solution between them.

1. **Introduction.** In this paper we consider a competition-diffusion equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + u(1 - u - c_0 v) \\ &\frac{\partial v}{\partial t} = d\Delta v + v(a_0 - b_0 u - v) \end{aligned} \qquad \boldsymbol{x} \in \mathbb{R}^N,\ t > 0, \tag{1}$$

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad v(\boldsymbol{x}, 0) = v_0(\boldsymbol{x}) \qquad \boldsymbol{x} \in \mathbb{R}^N.$$

Here $N \geq 2$ and $\boldsymbol{x} = {}^t(x_1, \ldots, x_N)$, and we are interested in the "strong competition" case; i.e. a_0, b_0, c_0 and d are positive constants with

$$\frac{1}{c_0} < a_0 < b_0, \tag{2}$$

and u_0 and v_0 are given bounded and uniformly continuous functions with $0 \leq u_0 \leq 1$ and $0 \leq v_0 \leq a_0$. This competition system is called the Lotka-Volterra system, and its traveling fronts have been studied. See [17], Kan-on [7], Kan-on and Fang [8], Hosono [6], Guo and Liang [1] for instance. It is well known that there exists $(U(y), V(y))$ that satisfies

$$\begin{aligned} &U''(y) + sU'(y) + U(y)(1 - U(y) - c_0 V(y)) = 0 \\ &dV''(y) + sV'(y) + V(y)(a_0 - b_0 U(y) - V(y)) = 0 \end{aligned} \qquad y \in \mathbb{R}, \tag{3}$$

$$\begin{pmatrix} U(-\infty) \\ V(-\infty) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} U(+\infty) \\ V(+\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ a_0 \end{pmatrix}, \tag{4}$$

$$-U'(y) > 0, \quad V'(y) > 0 \qquad y \in \mathbb{R}.$$

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Here $s = s(a_0, b_0, c_0, d)$ is the speed of a traveling front (U, V). By Kan-on [7], $s(a_0, b_0, c_0, d)$ is a function of class C^1 with

$$\frac{\partial s}{\partial a_0} < 0, \quad \frac{\partial s}{\partial b_0} > 0, \quad \frac{\partial s}{\partial c_0} < 0,$$

which implies that $s \neq 0$ is generic for given (a_0, b_0, c_0, d).

We now introduce a general setting that includes the Lotka-Volterra system. Let $m \geq 2$ and let $u(\boldsymbol{x}, t) = {}^t(u_1(\boldsymbol{x}, t), \ldots, u_m(\boldsymbol{x}, t))$ satisfy

$$\frac{\partial \boldsymbol{u}}{\partial t} = D\Delta \boldsymbol{u} + \boldsymbol{f}(\boldsymbol{u}) \qquad \boldsymbol{x} \in \mathbb{R}^N,\ t > 0, \tag{5}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}) \qquad \boldsymbol{x} \in \mathbb{R}^N. \tag{6}$$

Here

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_m \end{pmatrix}, \quad d_j > 0 \ (1 \leq j \leq m), \tag{7}$$

and $\boldsymbol{f}(\boldsymbol{u}) = {}^t(f_1(\boldsymbol{u}), \ldots, f_m(\boldsymbol{u}))$ is defined on $[0, 2]^m$ and is of class C^1. We denote the solution $\boldsymbol{u}(\boldsymbol{x}, t; \boldsymbol{u}_0) = {}^t(u_1(\boldsymbol{x}, t; \boldsymbol{u}_0), \ldots, u_m(\boldsymbol{x}, t; \boldsymbol{u}_0))$. For any $\boldsymbol{v} = {}^t(v_1, \ldots, v_m) \in \mathbb{R}^m$ and $\boldsymbol{w} = {}^t(w_1, \ldots, w_m) \in \mathbb{R}^m$ we define $\boldsymbol{v} \leq \boldsymbol{w}$ if and only if $v_j \leq w_j$ for all $1 \leq j \leq m$, and define $\boldsymbol{v} < \boldsymbol{w}$ if and only if $v_j < w_j$ for all $1 \leq j \leq m$. We define $\mathbf{0} = {}^t(0, \ldots, 0) \in \mathbb{R}^m$ and $\mathbf{1} = {}^t(1, \ldots, 1) \in \mathbb{R}^m$. Here $\boldsymbol{u}_0$ is a bounded and uniformly continuous function with $\mathbf{0} \leq \boldsymbol{u}_0(\boldsymbol{x}) \leq \mathbf{1}$.

We state the following standing assumptions:

(A1) One has $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$, $\boldsymbol{f}(\mathbf{1}) = \mathbf{0}$ and

$$\min_{[0,1]^m} \frac{\partial f_i}{\partial u_j} \geq 0 \qquad \text{for } i \neq j.$$

For each $i \neq j$ there exists a constant $L_{ij} \geq 0$ with

$$\frac{\partial f_i}{\partial u_j} + L_{ij} \max\{0, u_i - 1\} \geq 0 \qquad \text{for } \boldsymbol{u} \in [0, 2]^m.$$

(A2) There exist $k > 0$ and $\boldsymbol{\Phi} = {}^t(\Phi_1, \ldots, \Phi_m)$ such that one has

$$D\boldsymbol{\Phi}''(y) + k\boldsymbol{\Phi}'(y) + \boldsymbol{f}(\boldsymbol{\Phi}(y)) = \mathbf{0} \qquad \text{for all } y \in \mathbb{R}, \tag{8}$$

$$\boldsymbol{\Phi}(-\infty) = \mathbf{1},\ \boldsymbol{\Phi}(+\infty) = \mathbf{0}, \tag{9}$$

$$-\boldsymbol{\Phi}'(y) \geq \mathbf{0} \qquad \text{for all } y \in \mathbb{R}.$$

(A3) There exist $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^m$ with $\boldsymbol{p} > \mathbf{0}$, $\boldsymbol{q} > \mathbf{0}$ such that $\boldsymbol{f}'(\mathbf{1})\boldsymbol{p} < \mathbf{0}$ and $\boldsymbol{f}'(\mathbf{0})\boldsymbol{q} < \mathbf{0}$.

The Lotka-Volterra system (1) does satisfy the assumptions (A1), (A2) and (A3) stated above when $s \neq 0$. Indeed, when $s > 0$, we put

$$\boldsymbol{u}(\boldsymbol{x}, t) = {}^t\left(u(\boldsymbol{x}, t), 1 - \frac{v(\boldsymbol{x}, t)}{a_0}\right),$$

and

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \quad \boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix} f_1(\boldsymbol{u}) \\ f_2(\boldsymbol{u}) \end{pmatrix} = \begin{pmatrix} u_1(1 - c_0 a_0 - u_1 + c_0 a_0 u_2) \\ (1 - u_2)(b_0 u_1 - a_0 u_2) \end{pmatrix}.$$

Then $\boldsymbol{f}$ satisfies (A1) and $\boldsymbol{u}$ satisfies (5) and (6) with

$$\mathbf{0} \leq \boldsymbol{u}_0(\boldsymbol{x}) = \left(u_0(\boldsymbol{x}), 1 - \frac{v_0(\boldsymbol{x})}{a_0}\right) \leq \mathbf{1}.$$

Using

$$\boldsymbol{f}'(\mathbf{0}) = \begin{pmatrix} 1-c_0a_0 & 0 \\ b_0 & -a_0 \end{pmatrix}, \quad \boldsymbol{f}'(\mathbf{1}) = \begin{pmatrix} -1 & c_0a_0 \\ 0 & a_0-b_0 \end{pmatrix},$$

we can find $\boldsymbol{p}$ and $\boldsymbol{q}$ to satisfy (A3). Since $(s, {}^t(U,V))$ satisfies (3) and (4), we put

$$k = s, \quad \mathbf{\Phi}(y) = {}^t\left(U(y), 1 - \frac{V(y)}{a_0}\right)$$

and $(k, \mathbf{\Phi})$ satisfies (A2). When $s < 0$, we put

$$\boldsymbol{u}(\boldsymbol{x}, t) = {}^t\left(\frac{v(-\boldsymbol{x}, t)}{a_0}, 1 - u(-\boldsymbol{x}, t)\right),$$

and

$$D=\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{f}(\boldsymbol{u})=\begin{pmatrix} f_1(\boldsymbol{u}) \\ f_2(\boldsymbol{u}) \end{pmatrix} = \begin{pmatrix} u_1(a_0 - b_0 - a_0u_1 + b_0u_2) \\ (1-u_2)(c_0a_0u_1 - u_2) \end{pmatrix}.$$

Then $\boldsymbol{f}$ satisfies (A1) and $\boldsymbol{u}$ satisfies (5) and (6) with

$$\mathbf{0} \leq \boldsymbol{u}_0(\boldsymbol{x}) = \left(\frac{v_0(-\boldsymbol{x})}{a_0}, 1 - u_0(-\boldsymbol{x})\right) \leq \mathbf{1}.$$

Using

$$\boldsymbol{f}'(\mathbf{0}) = \begin{pmatrix} a_0-b_0 & 0 \\ c_0a_0 & -1 \end{pmatrix}, \quad \boldsymbol{f}'(\mathbf{1}) = \begin{pmatrix} -a_0 & b_0 \\ 0 & 1-c_0a_0 \end{pmatrix},$$

we can find $\boldsymbol{p}$ and $\boldsymbol{q}$ to satisfy (A3). Since $(s, {}^t(U,V))$ satisfies (3) and (4), we put

$$k = -s, \quad \mathbf{\Phi}(y) = \left(\frac{V(-y)}{a_0}, 1 - U(-y)\right)$$

and $(k, \mathbf{\Phi})$ satisfies (A2).

Multi-dimensional traveling fronts in the Allen-Cahn equation have been studied by [10, 3, 11, 4, 5, 14, 15, 16, 9]. In particular, N-dimensional traveling fronts of pyramidal shapes in the Allen-Cahn equations are studied in [9], and two-dimensional V-form fronts for competition-diffusion systems are studied by [18]. The purpose of this paper is to prove the existence of N-dimensional traveling fronts of pyramidal shapes for (5). In fact, a comparison principle for systems as well as vector notation enable us to present the existence proof in a similar way as in the scalar case of the Allen-Cahn equation.

In order to study traveling front solutions, we adopt the moving coordinate of speed c towards the x_N-axis without loss of generality. We write $\boldsymbol{x} = (x_1, ..., x_N) \in \mathbb{R}^N$ and $\boldsymbol{x}' = (x_1, ..., x_{N-1}) \in \mathbb{R}^{N-1}$, and put $s = x_N - ct$ and $\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{w}(\boldsymbol{x}', s, t)$. We denote $w(\boldsymbol{x}', s, t)$ by $w(\boldsymbol{x}, t)$ for simplicity. Then we have

$$\frac{\partial \boldsymbol{w}}{\partial t} - D\Delta\boldsymbol{w} - c\frac{\partial \boldsymbol{w}}{\partial x_N} - \boldsymbol{f}(\boldsymbol{w}) = \mathbf{0} \quad \text{in } \mathbb{R}^N, t > 0, \tag{10}$$

$$\boldsymbol{w}|_{t=0} = \boldsymbol{u}_0 \quad \text{in } \mathbb{R}^N.$$

We denote the solution of this equation by $\boldsymbol{w}(\boldsymbol{x}, t; \boldsymbol{u}_0)$. If $\boldsymbol{v}$ is a traveling wave with speed c, it satisfies the following profile equation

$$-D\Delta\boldsymbol{v} - c\frac{\partial \boldsymbol{v}}{\partial x_N} - \boldsymbol{f}(\boldsymbol{v}) = \mathbf{0} \text{ in } \mathbb{R}^N. \tag{11}$$

To study this equation, we introduce a nonlinear operator

$$\mathcal{L}[\boldsymbol{v}] = -D\Delta\boldsymbol{v} - c\frac{\partial \boldsymbol{v}}{\partial x_N} - \boldsymbol{f}(\boldsymbol{v}) \tag{12}$$

for a function $\boldsymbol{v} \in C^2(\mathbb{R}^N)$. We assume $c > k$ throughout this paper, because the curvature effect is expected to accelerate the speed.

Let $n \geq 2$ be a given integer, and

$$m_* = \frac{\sqrt{c^2 - k^2}}{k} > 0. \tag{13}$$

Let $\{\boldsymbol{A}_j\}_{j=1}^n$ be a set of unit vectors in $\mathbb{R}^{N-1}$ such that $\boldsymbol{A}_i \neq \boldsymbol{A}_j$ for $i \neq j$. Then $\boldsymbol{A}_j = (A_{1,j}, \ldots, A_{N-1,j}) \in \mathbb{R}^{N-1}$ satisfies

$$|\boldsymbol{A}_j|^2 = \sum_{i=1}^{N-1} (A_{i,j})^2 = 1 \qquad \text{for } j = 1, \ldots, n. \tag{14}$$

Now $(-m_* \boldsymbol{A}_j, 1) \in \mathbb{R}^N$ is the normal vector of $\left\{ \boldsymbol{x} \in \mathbb{R}^N \mid x_N = m_*(\boldsymbol{A}_j, \boldsymbol{x}') \right\}$, where $(\boldsymbol{A}_j, \boldsymbol{x}')$ is the inner product of $\boldsymbol{A}_j$ and $\boldsymbol{x}'$ given by

$$(\boldsymbol{A}_j, \boldsymbol{x}') = \sum_{i=1}^{N-1} A_{i,j} x_i.$$

We put

$$\begin{aligned} h_j(\boldsymbol{x}') &= m_*(\boldsymbol{A}_j, \boldsymbol{x}'), \\ h(\boldsymbol{x}') &= \max_{1 \leq j \leq n} h_j(\boldsymbol{x}') = m_* \max_{1 \leq j \leq n} (\boldsymbol{A}_j, \boldsymbol{x}') \end{aligned} \tag{15}$$

for $\boldsymbol{x}' \in \mathbb{R}^{N-1}$. We call $\left\{ \boldsymbol{x} = (\boldsymbol{x}', x_{N-1}) \in \mathbb{R}^N \,|\, x_N = h(\boldsymbol{x}') \right\}$ an N-dimensional pyramid in $\mathbb{R}^N$. Setting

$$\Omega_j = \left\{ \boldsymbol{x}' \in \mathbb{R}^{N-1} \mid h(\boldsymbol{x}') = h_j(\boldsymbol{x}') \right\}$$

for $j = 1, \ldots, n$, we have

$$\mathbb{R}^{N-1} = \cup_{j=1}^n \Omega_j. \tag{16}$$

We denote the boundary of Ω_j by $\partial\Omega_j$. Then we put

$$S_j = \left\{ \boldsymbol{x} \in \mathbb{R}^N \,|\, x_N = h_j(\boldsymbol{x}') \quad \text{for } \boldsymbol{x}' \in \Omega_j \right\}$$

for each j, and call $\cup_{j=1}^n S_j \subset \mathbb{R}^N$ the lateral faces of a pyramid. Finally we set

$$\Gamma_j = \left\{ \boldsymbol{x} \in \mathbb{R}^N \,|\, x_N = h_j(\boldsymbol{x}') \quad \text{for } \boldsymbol{x}' \in \partial\Omega_j \right\}$$

for $j = 1, ..., n$, then $\cup_{j=1}^n \Gamma_j$ represents the set of all edges of a pyramid.

For every $\boldsymbol{A}_j$ with (14), (11) has a planar front solution $\boldsymbol{\Phi}((k/c)(x_N - h_j(\boldsymbol{x}')))$. We define $\underline{\boldsymbol{v}}(\boldsymbol{x}) = {}^t(v_1(\boldsymbol{x}), \ldots, v_m(\boldsymbol{x}))$ as

$$\underline{\boldsymbol{v}}(\boldsymbol{x}) = \boldsymbol{\Phi}\left(\frac{k}{c}(x_N - h(\boldsymbol{x}')) \right) = \max_{1 \leq j \leq n} \boldsymbol{\Phi}\left(\frac{k}{c}(x_N - h_j(\boldsymbol{x}')) \right) \tag{17}$$

Then $\underline{\boldsymbol{v}}$ satisfies

$$\underline{\boldsymbol{v}}(\boldsymbol{x}) \leq \boldsymbol{w}(\boldsymbol{x}, t; \underline{\boldsymbol{v}}) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^N,\ t \geq 0,$$

and becomes a weak subsolution to (11). For $\gamma > 0$ we define

$$D(\gamma) = \left\{ \boldsymbol{x} \in \mathbb{R}^N \mid \operatorname{dist}(\boldsymbol{x}, \cup_{j=1}^n \Gamma_j) > \gamma \right\} \tag{18}$$

and put

$$\widehat{\mu} = \frac{\alpha x_N - \varphi(\alpha \boldsymbol{x}')}{\alpha \sqrt{1 + |\nabla \varphi(\alpha \boldsymbol{x}')|^2}}, \tag{19}$$

and show that

$$\overline{\boldsymbol{v}}(\boldsymbol{x})=\boldsymbol{\Phi}\left(\widehat{\mu}\right)+\varepsilon S(\alpha\boldsymbol{x}')\left(\boldsymbol{p}+\chi\left(\widehat{\mu}\right)(\boldsymbol{q}-\boldsymbol{p})\right) \tag{20}$$

is a supersolution with $\underline{\boldsymbol{v}} < \overline{\boldsymbol{v}}$. Here φ, S and χ are functions given by (31), (32) and (29), respectively, and ε and α are positive constants in (41) and (42), respectively. Then a solution of (11) between $\underline{\boldsymbol{v}}$ and $\overline{\boldsymbol{v}}$ is guaranteed by the comparison principle, and this solution has a contour surface of a pyramidal front shape.

The following theorem is the main result in this paper.

Theorem 1.1. *Let $c > k$, and let $\underline{\boldsymbol{v}}(\boldsymbol{x})$ be given by (17). There exists a solution $\boldsymbol{V}(\boldsymbol{x}) = {}^t(V_1(\boldsymbol{x}),\dots,V_m(\boldsymbol{x}))$ to (11) with*

$$\lim_{\gamma\to\infty}\sup_{\boldsymbol{x}\in D(\gamma)}|\boldsymbol{V}(\boldsymbol{x})-\underline{\boldsymbol{v}}(\boldsymbol{x})|=0. \tag{21}$$

Moreover one has

$$\mathbf{0}<\underline{v}(\boldsymbol{x})<\boldsymbol{V}(\boldsymbol{x})<\mathbf{1},\quad \frac{\partial\boldsymbol{V}}{\partial x_N}(\boldsymbol{x})<\mathbf{0}$$

for all $\boldsymbol{x}\in\mathbb{R}^N$.

The main part of the proof of Theorem 1.1 is devoted to the construction of a supersolution $\overline{\boldsymbol{v}}$. The idea is as follows. Taking $\alpha > 0$ very small, we make a very flat surface $\{x_N = \alpha^{-1}\varphi(\alpha\boldsymbol{x}')\}$ over a pyramid, and then $\widehat{\mu}$ represents the signed length of the perpendicular from $\boldsymbol{x}\in\mathbb{R}^N$ onto the tangent plane of $\{x_N = \alpha^{-1}\varphi(\alpha\boldsymbol{x}')\}$ at $(\boldsymbol{x}',\alpha^{-1}\varphi(\alpha\boldsymbol{x}'))$. Then the first term of $\overline{\boldsymbol{v}}$ converges to a planar front traveling with a slower speed $k\in(0,c)$ as $\alpha\to+0$. If we use the moving coordinate of a speed c, we expect that $w(\boldsymbol{x},t;\overline{\boldsymbol{v}})$ is monotone decreasing in $t>0$. This suggests that $\overline{\boldsymbol{v}}$ is a supersolution of (11). We will show that this argument applies to system (5) with the help of a comparison principle for systems as well as vector notation. Theorem 1.1 is established in Section 3.

2. Preliminaries.

For $\boldsymbol{u}=(u_1,\dots,u_m)\in\mathbb{R}^m$ and an $n\times n$ real matrix A we put $|\boldsymbol{u}|=\sqrt{\sum_{j=1}^m u_j^2}$ and $|A|=\sup_{|\boldsymbol{u}|\le 1}|A\boldsymbol{u}|$. We state the maximum principle and comparison principle needed in this paper. Let $G(\boldsymbol{x},t)=(g_{ij}(\boldsymbol{x},t))_{1\le i,j\le m}$ be given and satisfies

$$\sup_{\boldsymbol{x}\in\mathbb{R}^N,0\le t\le T}|g_{ij}(\boldsymbol{x},t)|<\infty \qquad \text{for } 1\le i,j\le m,$$
$$\inf_{\boldsymbol{x}\in\mathbb{R}^N,0\le t\le T}g_{ij}(\boldsymbol{x},t)\ge 0 \qquad \text{if } i\ne j$$

for given $T>0$.

The maximum principle is as follows. See [12] for the proof.

Lemma 2.1. *Let $G(\boldsymbol{x},t)=(g_{ij}(\boldsymbol{x},t))_{1\le i,j\le m}$ satisfy the assumption stated above, and let $u(\boldsymbol{x},t)={}^t(u_1(\boldsymbol{x},t),\dots,u_m(\boldsymbol{x},t))$ satisfy*

$$\frac{\partial\boldsymbol{u}}{\partial t}-D\Delta\boldsymbol{u}\le G(\boldsymbol{x},t)\boldsymbol{u}\qquad \boldsymbol{x}\in\mathbb{R}^N,\ 0<t<T,$$
$$\sup_{\boldsymbol{x}\in\mathbb{R}^N,0\le t\le T}|\boldsymbol{u}(\boldsymbol{x},t)|<\infty,$$

where $T>0$ is a given number, and D is as in (7). If $\boldsymbol{u}(\boldsymbol{x},0)\le\mathbf{0}$ for $\boldsymbol{x}\in\mathbb{R}^N$, one has $\boldsymbol{u}(\boldsymbol{x},t)\le\mathbf{0}$ for $\boldsymbol{x}\in\mathbb{R}^N$ and $0<t<T$. Moreover, if $u_j(\cdot,0)\not\equiv 0$ for some j in addition, one has $u_j(\boldsymbol{x},t)<0$ for all $\boldsymbol{x}\in\mathbb{R}^N$ and $0<t<T$.

We introduce a function $\widetilde{\boldsymbol{f}}$ following to [18]. We define $\widetilde{\boldsymbol{f}} = (\widetilde{f}_1, \ldots, \widetilde{f}_m)$ by

$$\widetilde{f}_i(\boldsymbol{u}) = f_i(\boldsymbol{u}) + \sum_{j \neq i} L_{ij}(u_j - 1) \max\{0, u_i - 1\} \qquad \text{for all } \boldsymbol{u} \in [0, 2]^m,$$

where L_{ij} is as in (A1). Then $\widetilde{\boldsymbol{f}}$ satisfies $\widetilde{\boldsymbol{f}}'(\mathbf{1}) = \boldsymbol{f}'(\mathbf{1})$, $\widetilde{\boldsymbol{f}}(\boldsymbol{u}) = \boldsymbol{f}(\boldsymbol{u})$ for all $\boldsymbol{u} \in [0, 1]^m$, and

$$\inf_{[0,2]^m} \frac{\partial \widetilde{f}_i}{\partial u_j} \geq 0 \qquad \text{for } i \neq j.$$

Hereafter we write $\widetilde{\boldsymbol{f}}$ simply as $\boldsymbol{f}$. We put $\mathbf{2}={}^t(2, \ldots, 2) \in \mathbb{R}^m$. By the lemma stated above, the comparison principle follows immediately:

If $\overline{\boldsymbol{u}}(\boldsymbol{x}, t) = {}^t(\overline{u}_1(\boldsymbol{x}, t), \ldots, \overline{u}_m(\boldsymbol{x}, t))$ and $\underline{\boldsymbol{u}}(\boldsymbol{x}, t) = {}^t(\underline{u}_1(\boldsymbol{x}, t), \ldots, \underline{u}_m(\boldsymbol{x}, t))$ satisfy

$$(D_t - D\Delta)\overline{\boldsymbol{u}} - \boldsymbol{f}(\overline{\boldsymbol{u}}) \geq \mathbf{0}, \ (D_t - D\Delta)\underline{\boldsymbol{u}} - \boldsymbol{f}(\underline{\boldsymbol{u}}) \leq \mathbf{0} \qquad \boldsymbol{x} \in \mathbb{R}^N, \ 0 < t < T,$$
$$\mathbf{0} \leq \underline{\boldsymbol{u}}(\boldsymbol{x}, 0) \leq \overline{\boldsymbol{u}}(\boldsymbol{x}, 0) \leq \mathbf{2} \qquad \boldsymbol{x} \in \mathbb{R}^N,$$

then one has $\underline{\boldsymbol{u}}(\boldsymbol{x}, t) \leq \overline{\boldsymbol{u}}(\boldsymbol{x}, t)$ for $\boldsymbol{x} \in \mathbb{R}^N$ and $0 < t < T$. (Here $D_t = \partial/\partial t$.) Moreover, if $\underline{u}_j(\,\cdot\,, 0) \not\equiv \overline{u}_j(\,\cdot\,, 0)$ for some j in addition, one has $u_j(\boldsymbol{x}, t) < 0$ for all $\boldsymbol{x} \in \mathbb{R}^N$ and $0 < t < T$.

Since $\mathbf{0}$ and $\mathbf{1}$ are equilibrium solutions to (5), we have

$$\mathbf{0} \leq \boldsymbol{u}(\boldsymbol{x}, t; \boldsymbol{u}_0) \leq \mathbf{1} \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^N \text{ and } t > 0$$

from the comparison principle, (A1) and $\mathbf{0} \leq \boldsymbol{u}_0(\boldsymbol{x}) \leq \mathbf{1}$. From the comparison principle and (A2) we have

$$-\boldsymbol{\Phi}'(y) > \mathbf{0} \qquad \text{for all } y \in \mathbb{R}. \tag{22}$$

Without loss of generality, we can assume $|\boldsymbol{p}| = 1$ and $|\boldsymbol{q}| = 1$. From the assumption (A3), $\boldsymbol{p} = {}^t(p_1, \ldots, p_m)$ and $\boldsymbol{q} = {}^t(q_1, \ldots, q_m)$ satisfy

$$\boldsymbol{f}'(\mathbf{1})\boldsymbol{p} \leq -2\beta\mathbf{1}, \quad \boldsymbol{f}'(\mathbf{0})\boldsymbol{q} \leq -2\beta\mathbf{1}$$

for a constant $\beta > 0$. Then there exists a positive constant $\delta*$ with $0 < \delta_* < \frac{1}{4}$ and

$$-\boldsymbol{f}'(\boldsymbol{v})\boldsymbol{p} \geq \beta\mathbf{1} \quad \text{if} \quad |\boldsymbol{v} - \mathbf{1}| < 2\delta_*, \quad -\boldsymbol{f}'(\boldsymbol{v})\boldsymbol{q} \geq \beta\mathbf{1} \quad \text{if} \quad |\boldsymbol{v}| < 2\delta_*.$$

We put

$$r_0 = \min\left\{\min_{1 \leq j \leq m} p_j, \min_{1 \leq j \leq m} q_j\right\} > 0.$$

Next we choose a constant $\mu_* > 1$ such that

$$\sup_{y \geq \mu_*} |\boldsymbol{\Phi}(y)| < \delta_*, \quad \sup_{y \leq -\mu_*} |\boldsymbol{\Phi}(y) - \mathbf{1}| < \delta_*,$$

and we put

$$\lambda_* = \min\left\{-\Phi_j'(y) \;\middle|\; |y| \leq \mu_*, 1 \leq j \leq m\right\} > 0, \tag{23}$$
$$M = \max_{\mathbf{0} \leq \boldsymbol{u} \leq \mathbf{2}} |\boldsymbol{f}'(\boldsymbol{u})|. \tag{24}$$

Lemma 2.2. *One has*

$$\max\{|\boldsymbol{\Phi}'(y)|, |\boldsymbol{\Phi}''(y)|\} \leq K_0 \exp(-\kappa_0|y|) \qquad \text{for all } y \in \mathbb{R}, \tag{25}$$

where K_0 and κ_0 are positive constants.

Proof. First we have $\mathbf{\Phi}'(-\infty) = \mathbf{0}$ and $\mathbf{\Phi}'(+\infty) = \mathbf{0}$. We study an ordinary differential equation

$$\frac{d}{dy}\begin{pmatrix}\boldsymbol{\phi}\\ \boldsymbol{\phi}'\end{pmatrix} = F\begin{pmatrix}\boldsymbol{\phi}\\ \boldsymbol{\phi}'\end{pmatrix} \qquad y \in \mathbb{R}, \tag{26}$$

where $\boldsymbol{\phi} = {}^t(\phi_1(y), \ldots, \phi_m(y))$, $\boldsymbol{\phi}' = {}^t(\phi_1'(y), \ldots, \phi_m'(y))$ and

$$F\begin{pmatrix}\boldsymbol{\phi}\\ \boldsymbol{\phi}'\end{pmatrix} = \begin{pmatrix}\boldsymbol{\phi}'\\ -kD^{-1}\boldsymbol{\phi}' - D^{-1}\boldsymbol{f}(\boldsymbol{\phi}).\end{pmatrix}$$

Then we find

$$F'\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix} = \begin{pmatrix}O_m & I_m\\ -D^{-1}\boldsymbol{f}'(\mathbf{0}) & -kD^{-1}\end{pmatrix}, \quad F'\begin{pmatrix}\mathbf{1}\\ \mathbf{0}\end{pmatrix} = \begin{pmatrix}O_m & I_m\\ -D^{-1}\boldsymbol{f}'(\mathbf{1}) & -kD^{-1}\end{pmatrix}.$$

Here I_m is the $m \times m$ unit matrix and O_m is the $m \times m$ zero matrix, respectively. After a simple calculation we obtain

$$\left(\Pi_{j=1}^m d_j\right) \det\left(\lambda I_{2m} - F'\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}\right) = \det\left(\lambda^2 D + k\lambda I_m + \boldsymbol{f}'(\mathbf{0})\right),$$
$$\left(\Pi_{j=1}^m d_j\right) \det\left(\lambda I_{2m} - F'\begin{pmatrix}\mathbf{1}\\ \mathbf{0}\end{pmatrix}\right) = \det\left(\lambda^2 D + k\lambda I_m + \boldsymbol{f}'(\mathbf{1})\right).$$

From $k > 0$, every root $\lambda \in \mathbb{C}$ of

$$\det(\lambda^2 D + k\lambda I + \boldsymbol{f}'(\mathbf{0})) = 0 \tag{27}$$

has a nonzero real part and every root $\lambda \in \mathbb{C}$ of

$$\det(\lambda^2 D + k\lambda I + \boldsymbol{f}'(\mathbf{1})) = 0 \tag{28}$$

has a nonzero real part. Then we can apply the stable manifold theorem to (26) at $(\mathbf{0}, \mathbf{0})$ and $(\mathbf{1}, \mathbf{0})$. See Hale [2, Theorem 6.1] for the details. Now $(\mathbf{\Phi}, \mathbf{\Phi}')$ belongs to the stable manifold of $(\mathbf{0}, \mathbf{0})$ and belongs to the unstable manifold of $(\mathbf{1}, \mathbf{0})$ simultaneously. The stable manifold theorem says that since $(\mathbf{\Phi}(0), \mathbf{\Phi}'(0))$ belongs to the stable manifold of $(\mathbf{0}, \mathbf{0})$, one has

$$\sqrt{|\mathbf{\Phi}(y)|^2 + |\mathbf{\Phi}'(y)|^2} \le K_0 e^{-\kappa_0 y} \qquad \text{for } y \ge 0$$

for positive constants K_0 and κ_0. Also since $(\mathbf{\Phi}(0), \mathbf{\Phi}'(0))$ belongs to the unstable manifold of $(\mathbf{1}, \mathbf{0})$, one has

$$\sqrt{|\mathbf{1} - \mathbf{\Phi}(y)|^2 + |\mathbf{\Phi}'(y)|^2} \le K_0 e^{\kappa_0 y} \qquad \text{for } y \le 0.$$

Replacing K_0 by a larger constant if necessary, we complete the proof. □

Now we fix a function $\chi \in C^\infty(\mathbb{R})$ with

$$\begin{array}{rcl} \chi(y) \equiv 1 & \text{if} & y \ge 1, \\ 0 < \chi(y) < 1,\ 0 < \chi'(y) < 1 & \text{if} & -1 < y < 1, \\ \chi(y) \equiv 0 & \text{if} & y \le -1. \end{array} \tag{29}$$

Since unit vectors $\{\boldsymbol{A}_j\}_{j=1}^n$ satisfy $\boldsymbol{A}_i \ne \boldsymbol{A}_j$ for $i \ne j$, we have

$$-1 \le (\boldsymbol{A}_i, \boldsymbol{A}_j) < 1 \qquad \text{for } i \ne j.$$

Let $\widetilde{\rho}(r) \in C^{\infty}[0,\infty)$ be a function with the following properties:

$$\begin{aligned}&\widetilde{\rho}(r) > 0,\ \widetilde{\rho}_r(r) \leq 0 \quad \text{for } r \geq 0,\\ &\widetilde{\rho}(r) = 1 \quad \text{if } r > 0 \text{ is small enough},\\ &\widetilde{\rho}(r) = e^{-r} \quad \text{if } r > 0 \text{ is large enough, say } r > R_0,\\ &\int_{\mathbb{R}^{N-1}} \widetilde{\rho}(|\boldsymbol{x}'|)\, d\boldsymbol{x}' = 1.\end{aligned}$$

We assume $R_0 > 1$ without loss of generality. We have

$$\int_{\mathbb{R}^{N-1}} \widetilde{\rho}(|\boldsymbol{x}'|)\, d\boldsymbol{x}' = \frac{(N-1)\pi^{\frac{N-1}{2}}}{\Gamma\left(\dfrac{N+1}{2}\right)} \int_0^{\infty} r^{N-2}\widetilde{\rho}(r)dr,$$

where Γ is the Gamma function. We put $\rho(\boldsymbol{x}')=\widetilde{\rho}(|\boldsymbol{x}'|)$. Then $\rho : \mathbb{R}^{N-1} \to \mathbb{R}$ belongs to $C^{\infty}(\mathbb{R}^{N-1})$ and satisfies

$$\int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{x}')d\boldsymbol{x}' = 1$$

for all $\boldsymbol{x}' \in \mathbb{R}^{N-1}$ and $1 \leq j \leq n$. Let $\rho * h_j$ be the convolution of ρ and h_j given by

$$(\rho * h_j)(\boldsymbol{x}')= \int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{y}')h_j(\boldsymbol{x}' - \boldsymbol{y}')d\boldsymbol{y}'.$$

Using

$$(\rho * x_i)(\boldsymbol{x}') = \int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{y}')(x_i - y_i)\, d\boldsymbol{y}' = x_i \int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{y}')\, d\boldsymbol{y}' - \int_{\mathbb{R}^{N-1}} y_i\rho(\boldsymbol{y}')\, d\boldsymbol{y}' = x_i$$

for $\boldsymbol{x}' \in \mathbb{R}^{N-1}$, $1 \leq i \leq N-1$, we have

$$(\rho * h_j)(\boldsymbol{x}') = h_j(\boldsymbol{x}') \qquad \text{for all } \boldsymbol{x}' \in \mathbb{R}^{N-1}.$$

For all nonnegative integers $j_1, j_2, \dots, j_{N-1}$ with $0 \leq \sum_{p=1}^{N-1} j_p \leq 3$, we have

$$\left| D_1^{j_1} D_2^{j_2} \cdots D_{N-1}^{j_{N-1}} \rho(\boldsymbol{x}') \right| \leq M_1 \rho(\boldsymbol{x}') \qquad \text{for all } \boldsymbol{x}' \in \mathbb{R}^{N-1}, \tag{30}$$

where $M_1 > 0$ is a constant. Here we denote the p-th derivative by the j-th component by D_j^p, that is, $D_j^p \rho(\boldsymbol{x}') = \partial^p \rho/\partial x_j^p(\boldsymbol{x}')$. We put $\varphi=\rho * h$, that is,

$$\varphi(\boldsymbol{x}')= \int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{x}' - \boldsymbol{y}')h(\boldsymbol{y}')d\boldsymbol{y}' = \int_{\mathbb{R}^{N-1}} \rho(\boldsymbol{y}')h(\boldsymbol{x}' - \boldsymbol{y}')d\boldsymbol{y}' \tag{31}$$

for $\boldsymbol{x}' \in \mathbb{R}^{N-1}$. We call $x_N = \varphi(\boldsymbol{x}')$ a mollified pyramid for a pyramid $x_N = h(\boldsymbol{x}')$. We put

$$S(\boldsymbol{x}')= \frac{c}{\sqrt{1 + |\nabla\varphi(\boldsymbol{x}')|^2}} - k, \tag{32}$$

where $\nabla\varphi(\boldsymbol{x}') = {}^t\,(D_1\varphi(\boldsymbol{x}'), \dots, D_{N-1}\varphi(\boldsymbol{x}'))$. We have the following properties.

Lemma 2.3 ([9]). *Let φ and S be as in (31) and (32), respectively. Then one has*

$$\begin{aligned}&h(\boldsymbol{x}') < \varphi(\boldsymbol{x}') \leq h(\boldsymbol{x}') + m_* \int_{\mathbb{R}^{N-1}} |\boldsymbol{y}'|\rho(\boldsymbol{y}')\, d\boldsymbol{y}', \qquad \text{for all } \boldsymbol{x}' \in \mathbb{R}^{N-1},\\ &|\nabla\varphi(\boldsymbol{x}')| < m_*, \quad 0 < S(\boldsymbol{x}') \leq c - k \qquad \text{for all } \boldsymbol{x}' \in \mathbb{R}^{N-1},\end{aligned}$$

and

$$\sup_{\boldsymbol{x}' \in \mathbb{R}^{N-1}} \left| D_1^{i_1} D_2^{i_2} \cdots D_{N-1}^{i_{N-1}} \varphi(\boldsymbol{x}') \right| < \infty$$

for any fixed $(i_1, \dots, i_{N-1}) \neq (0, \dots, 0)$ *with* $i_p \geq 0$ *(*$p = 1, \dots, N-1$*).*

Proposition 1 ([9])**.** *One has*

$$0 < \inf_{\boldsymbol{x}' \in \mathbb{R}^{N-1}} \frac{\varphi(\boldsymbol{x}') - h(\boldsymbol{x}')}{S(\boldsymbol{x}')} \leq \sup_{\boldsymbol{x}' \in \mathbb{R}^{N-1}} \frac{\varphi(\boldsymbol{x}') - h(\boldsymbol{x}')}{S(\boldsymbol{x}')} < \infty.$$

For every integer $j_p \geq 0$ *for* $p = 1, \dots, N-1$ *with* $2 \leq \sum_{p=1}^{N-1} j_p \leq 3$*,*

$$\sup_{\boldsymbol{x}' \in \mathbb{R}^{N-1}} \left| \frac{D_1^{j_1} D_2^{j_2} \cdots D_{N-1}^{j_{N-1}} \varphi(\boldsymbol{x}')}{S(\boldsymbol{x}')} \right| < \infty$$

holds true.

We set

$$\widetilde{\varphi}(\boldsymbol{x}') = \varphi(\boldsymbol{x}') - h(\boldsymbol{x}') \qquad \text{for } \boldsymbol{x} \in \mathbb{R}^{N-1},$$

and have $\|\widetilde{\varphi}\|_{L^\infty(\mathbb{R}^{N-1})} < \infty$. For each $1 \leq j \leq n$ we have

$$\widetilde{\varphi}(\boldsymbol{x}') = \varphi(\boldsymbol{x}') - h_j(\boldsymbol{x}') = (\rho * (h - h_j))(\boldsymbol{x}') \qquad \text{for all } \boldsymbol{x}' \in \Omega_j.$$

Since ρ decays exponentially, as $\operatorname{dist}(\boldsymbol{x}, \cup_{j=1}^n \Gamma_j) \to +\infty$, we have either $|x_N - h(\boldsymbol{x}')| \to \infty$ or

$$|\widetilde{\varphi}(\boldsymbol{x}')| \to 0, \quad |\nabla \widetilde{\varphi}(\boldsymbol{x}')| \to 0.$$

Then, as $\operatorname{dist}(\boldsymbol{x}, \cup_{j=1}^n \Gamma_j) \to +\infty$, we have either $|x_N - h(\boldsymbol{x}')| \to \infty$ or

$$|\widetilde{\varphi}(\boldsymbol{x}')| \to 0, \quad |\nabla \widetilde{\varphi}(\boldsymbol{x}')| \to 0. \quad |\nabla \varphi(\boldsymbol{x}')| \to m_*, \quad S(\boldsymbol{x}') \to 0. \tag{33}$$

3. Proof of Theorem 1.1.

In this section we construct a pyramidal traveling front solution and prove Theorem 1.1 by constructing a supersolution and a subsolution with the methods of [10, 14, 9, 18].

We put $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$ and $\boldsymbol{\xi}' = (\xi_1, \dots, \xi_{N-1}) \in \mathbb{R}^{N-1}$. We study $\xi_N = \varphi(\boldsymbol{\xi}')$. For $\alpha \in (0,1)$ we consider the graph of

$$\left\{ \boldsymbol{x} \in \mathbb{R}^N \;\middle|\; x_N = \frac{1}{\alpha} \varphi(\alpha \boldsymbol{x}') \right\}. \tag{34}$$

Later we will choose $\alpha > 0$ to be small enough. We note that

$$\frac{1}{\alpha} h(\alpha \boldsymbol{x}') = h(\boldsymbol{x}').$$

Putting $\boldsymbol{\xi} = \alpha \boldsymbol{x}$, we have $\xi_N = \varphi(\boldsymbol{\xi}')$. For a given constant $\boldsymbol{b}' \in \mathbb{R}^{N-1}$, the tangent plane of (34) at $(\boldsymbol{b}', (1/\alpha)\varphi(\alpha \boldsymbol{b}'))$ is expressed by

$$-\left(\nabla \varphi(\alpha \boldsymbol{b}'), (\boldsymbol{x}' - \boldsymbol{b}')\right) + x_N - \frac{1}{\alpha} \varphi(\alpha \boldsymbol{b}') = 0. \tag{35}$$

We use

$$\nabla \varphi(\boldsymbol{\xi}') = {}^t\left(D_1 \varphi(\boldsymbol{\xi}'), \dots, D_{N-1} \varphi(\boldsymbol{\xi}')\right) = {}^t\left(\frac{\partial \varphi}{\partial \xi_1}(\boldsymbol{\xi}'), \dots \frac{\partial \varphi}{\partial \xi_{N-1}}(\boldsymbol{\xi}') \right).$$

The length of the perpendicular from $\boldsymbol{b} = (\boldsymbol{b}', b_N) \in \mathbb{R}^N$ onto the tangent plane is given by

$$\frac{|b_N - \frac{1}{\alpha} \varphi(\alpha \boldsymbol{b}')|}{\sqrt{1 + |\nabla \varphi(\alpha \boldsymbol{b}')|^2}}.$$

We set

$$\widehat{\mu} = \frac{x_N - \frac{1}{\alpha} \varphi(\alpha \boldsymbol{x}')}{\sqrt{1 + |\nabla \varphi(\alpha \boldsymbol{x}')|^2}} = \frac{1}{\alpha} \frac{\xi_N - \varphi(\boldsymbol{\xi}')}{\sqrt{1 + |\nabla \varphi(\boldsymbol{\xi}')|^2}}, \tag{36}$$

and have

$$\frac{\partial \widehat{\mu}}{\partial x_N} = \frac{1}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}}, \quad \frac{\partial^2 \widehat{\mu}}{\partial x_N^2} = 0.$$

Here $\widehat{\mu}$ is the signed length of the perpendicular from $\boldsymbol{x} \in \mathbb{R}^N$ onto the tangent plane of (34) at $(\boldsymbol{x}', (1/\alpha)\varphi(\alpha\boldsymbol{x}'))$. We have

$$\frac{\partial \widehat{\mu}}{\partial x_i} = -\frac{D_i\varphi(\boldsymbol{\xi}')}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}} + \alpha\widehat{\mu}F_i(\boldsymbol{\xi}'), \quad \frac{\partial^2 \widehat{\mu}}{\partial x_i^2} = \alpha G_i(\boldsymbol{\xi}') + \alpha^2\widehat{\mu}H_i(\boldsymbol{\xi}'),$$

where

$$F_i(\boldsymbol{\xi}')=\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}\frac{\partial}{\partial \xi_i}\left(\frac{1}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}}\right), \tag{37}$$

$$G_i(\boldsymbol{\xi}')=-\frac{\partial}{\partial \xi_i}\left(\frac{D_i\varphi(\boldsymbol{\xi}')}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}}\right) - \frac{F_i(\boldsymbol{\xi}')D_i\varphi(\boldsymbol{\xi}')}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}}, \tag{38}$$

$$H_i(\boldsymbol{\xi}')=D_iF_i(\boldsymbol{\xi}') + (F_i(\boldsymbol{\xi}'))^2 \tag{39}$$

for $i = 1, ..., N-1$. We define

$$\overline{\boldsymbol{v}}(\boldsymbol{x}) = \boldsymbol{\Phi}(\widehat{\mu}) + \boldsymbol{\sigma}(\boldsymbol{x}'), \tag{40}$$

where $\widehat{\mu}$ is as in (36) and

$$\boldsymbol{\sigma}(\boldsymbol{x}')=\varepsilon S(\alpha\boldsymbol{x}')\left(\boldsymbol{p} + \chi(\widehat{\mu})(\boldsymbol{q}-\boldsymbol{p})\right).$$

Here we will fix $\varepsilon > 0$ later. We have

$$\begin{aligned}
\frac{\partial \overline{\boldsymbol{v}}}{\partial x_N} &= \frac{1}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}}\boldsymbol{\Phi}'(\widehat{\mu}) + \varepsilon\frac{1}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}}\chi'(\widehat{\mu})S(\boldsymbol{\xi}')(\boldsymbol{q}-\boldsymbol{p}),\\
\frac{\partial^2 \overline{\boldsymbol{v}}}{\partial x_N^2} &= \frac{1}{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}\boldsymbol{\Phi}''(\widehat{\mu}) + \varepsilon\frac{1}{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}\chi''(\widehat{\mu})S(\boldsymbol{\xi}')(\boldsymbol{q}-\boldsymbol{p}).
\end{aligned}$$

For $1 \leq i \leq N-1$ we have

$$\begin{aligned}
\frac{\partial \boldsymbol{\sigma}}{\partial x_i} &= \varepsilon\alpha D_iS(\boldsymbol{\xi}')(\boldsymbol{p}+\chi(\widehat{\mu})(\boldsymbol{q}-\boldsymbol{p})) + \varepsilon\chi'(\widehat{\mu})\frac{\partial\widehat{\mu}}{\partial x_i}S(\boldsymbol{\xi}')(\boldsymbol{q}-\boldsymbol{p}),\\
\frac{\partial^2 \boldsymbol{\sigma}}{\partial x_i^2} &= \varepsilon\alpha^2 D_i^2S(\boldsymbol{\xi}')(\boldsymbol{p}+\chi(\widehat{\mu})(\boldsymbol{q}-\boldsymbol{p})) + 2\alpha\varepsilon\chi'(\widehat{\mu})\frac{\partial\widehat{\mu}}{\partial x_i}D_iS(\boldsymbol{\xi}')(\boldsymbol{q}-\boldsymbol{p})\\
&\quad +\varepsilon S(\boldsymbol{\xi}')\left(\chi''(\widehat{\mu})\left(\frac{\partial\widehat{\mu}}{\partial x_i}\right)^2 + \chi'(\widehat{\mu})\frac{\partial^2\widehat{\mu}}{\partial x_i^2}\right)(\boldsymbol{q}-\boldsymbol{p}).
\end{aligned}$$

Thus we get

$$\sum_{i=1}^{N-1}\frac{\partial^2\boldsymbol{\sigma}}{\partial x_i^2}=\varepsilon\alpha^2\left(\sum_{i=1}^{N-1}D_i^2S(\boldsymbol{\xi}')\right)(\boldsymbol{p}+\chi(\widehat{\mu})(\boldsymbol{q}-\boldsymbol{p}))$$
$$+2\alpha\varepsilon\chi'(\widehat{\mu})\sum_{i=1}^{N-1}\left(-\frac{D_i\varphi(\boldsymbol{\xi}')}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}}+\alpha\widehat{\mu}F_i(\boldsymbol{\xi}')\right)D_iS(\boldsymbol{\xi}')(\boldsymbol{q}-\boldsymbol{p})$$
$$+\varepsilon S(\boldsymbol{\xi}')\chi''(\widehat{\mu})$$
$$\times\left(\frac{|\nabla\varphi(\boldsymbol{\xi}')|^2}{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}-\frac{2\alpha\widehat{\mu}\sum_{i=1}^{N-1}F_i(\boldsymbol{\xi}')D_i\varphi(\boldsymbol{\xi}')}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}}+\alpha^2(\widehat{\mu})^2\sum_{i=1}^{N-1}F_i(\boldsymbol{\xi}')^2\right)(\boldsymbol{q}-\boldsymbol{p})$$
$$+\varepsilon S(\boldsymbol{\xi}')\chi'(\widehat{\mu})\left(\alpha\sum_{i=1}^{N-1}G_i(\boldsymbol{\xi}')+\alpha^2\widehat{\mu}\sum_{i=1}^{N-1}H_i(\boldsymbol{\xi}')\right)(\boldsymbol{q}-\boldsymbol{p}).$$

Using

$$\frac{\partial\overline{\boldsymbol{v}}}{\partial x_i}=\frac{\partial\widehat{\mu}}{\partial x_i}\boldsymbol{\Phi}'(\widehat{\mu})+\frac{\partial\boldsymbol{\sigma}}{\partial x_i},\quad \frac{\partial^2\overline{\boldsymbol{v}}}{\partial x_i^2}=\left(\frac{\partial\widehat{\mu}}{\partial x_i}\right)^2\boldsymbol{\Phi}''(\widehat{\mu})+\frac{\partial^2\widehat{\mu}}{\partial x_i^2}\boldsymbol{\Phi}'(\widehat{\mu})+\frac{\partial^2\boldsymbol{\sigma}}{\partial x_i^2}$$

for $i=1,\dots,N-1$, we have

$$\sum_{i=1}^{N-1}\left(\frac{\partial\widehat{\mu}}{\partial x_i}\right)^2=\sum_{i=1}^{N-1}\left(-\frac{D_i\varphi(\boldsymbol{\xi}')}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}}+\alpha\widehat{\mu}F_i(\boldsymbol{\xi}')\right)^2$$
$$=\frac{|\nabla\varphi(\boldsymbol{\xi}')|^2}{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}+\sum_{i=1}^{N-1}\left(\frac{-2\alpha\widehat{\mu}F_i(\boldsymbol{\xi}')D_i\varphi(\boldsymbol{\xi}')}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}}+\alpha^2\left(\widehat{\mu}\right)^2F_i(\boldsymbol{\xi}')^2\right)$$

and

$$\sum_{i=1}^{N-1}\frac{\partial^2\overline{\boldsymbol{v}}}{\partial x_i^2}=\boldsymbol{\Phi}'(\widehat{\mu})\sum_{i=1}^{N-1}\frac{\partial^2\widehat{\mu}}{\partial x_i^2}+\boldsymbol{\Phi}''(\widehat{\mu})\sum_{i=1}^{N-1}\left(\frac{\partial\widehat{\mu}}{\partial x_i}\right)^2+\sum_{i=1}^{N-1}\frac{\partial^2\boldsymbol{\sigma}}{\partial x_i^2}$$
$$=\boldsymbol{\Phi}'(\widehat{\mu})\sum_{i=1}^{N-1}(\alpha G_i(\boldsymbol{\xi}')+\alpha^2\widehat{\mu}H_i(\boldsymbol{\xi}'))+\boldsymbol{\Phi}''(\widehat{\mu})\frac{|\nabla\varphi(\boldsymbol{\xi}')|^2}{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}$$
$$+\boldsymbol{\Phi}''(\widehat{\mu})\sum_{i=1}^{N-1}\left(\frac{-2\alpha\widehat{\mu}F_i(\boldsymbol{\xi}')D_i\varphi(\boldsymbol{\xi}')}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}}+\alpha^2\left(\widehat{\mu}\right)^2F_i(\boldsymbol{\xi}')^2\right)+\sum_{i=1}^{N-1}\frac{\partial^2\boldsymbol{\sigma}}{\partial x_i^2}.$$

By taking ε as in (41), we have $\overline{\boldsymbol{v}}(\boldsymbol{x})\in[0,2]^m$ for all $\boldsymbol{x}\in\mathbb{R}^N$, and we can define $\boldsymbol{f}(\overline{\boldsymbol{v}})$. We calculate $\mathcal{L}[\overline{\boldsymbol{v}}]$ as

$$\begin{aligned}\mathcal{L}[\overline{\boldsymbol{v}}] &= -D\sum_{i=1}^{N}\frac{\partial^2\overline{\boldsymbol{v}}}{\partial x_i^2}-c\frac{\partial\overline{\boldsymbol{v}}}{\partial x_N}-\boldsymbol{f}(\overline{\boldsymbol{v}})\\ &= -D\boldsymbol{\Phi}''(\widehat{\mu})-\frac{c}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}}\boldsymbol{\Phi}'(\widehat{\mu})-\boldsymbol{f}(\boldsymbol{\Phi}(\widehat{\mu})+\boldsymbol{\sigma})\\ &\quad +\alpha\boldsymbol{Y}(\boldsymbol{\xi}',\widehat{\mu};\varepsilon,\alpha)+\varepsilon\boldsymbol{Z}(\boldsymbol{\xi}',\widehat{\mu};\varepsilon,\alpha).\end{aligned}$$

Here we set

$$\begin{aligned}\boldsymbol{Y}(\boldsymbol{\xi}', y; \varepsilon, \alpha) =& - D\boldsymbol{\Phi}'(y) \left(\sum_{i=1}^{N-1} G_i(\boldsymbol{\xi}') + \alpha y \sum_{i=1}^{N-1} H_i(\boldsymbol{\xi}') \right) \\ &+ D\boldsymbol{\Phi}''(y) \left(\frac{2y}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}} \sum_{i=1}^{N-1} F_i(\boldsymbol{\xi}') D_i\varphi(\boldsymbol{\xi}') - \alpha y^2 \sum_{i=1}^{N-1} F_i(\boldsymbol{\xi}')^2 \right) \\ &- \varepsilon\alpha \left(\sum_{i=1}^{N-1} D_i^2 S(\boldsymbol{\xi}') \right) D(\boldsymbol{p} + \chi(y)(\boldsymbol{q} - \boldsymbol{p}))\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{Z}(\boldsymbol{\xi}', y; \varepsilon, \alpha) =& - \frac{\chi''(y) S(\boldsymbol{\xi}')}{1+|\nabla\varphi(\boldsymbol{\xi}')|^2} D(\boldsymbol{q} - \boldsymbol{p}) - \frac{c\chi'(y) S(\boldsymbol{\xi}')}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}} (\boldsymbol{q} - \boldsymbol{p}) \\ &+ 2\alpha\chi'(y) \left(\sum_{i=1}^{N-1} \left(\frac{D_i\varphi(\boldsymbol{\xi}')}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}} - \alpha y F_i(\boldsymbol{\xi}') \right) D_i S(\boldsymbol{\xi}') \right) D(\boldsymbol{q} - \boldsymbol{p}) \\ &- \chi''(y) S(\boldsymbol{\xi}') \\ &\times \left(\frac{|\nabla\varphi(\boldsymbol{\xi}')|^2}{1+|\nabla\varphi(\boldsymbol{\xi}')|^2} - \frac{2\alpha y \sum_{i=1}^{N-1} F_i(\boldsymbol{\xi}') D_i\varphi(\boldsymbol{\xi}')}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}} + \alpha^2 y^2 \sum_{i=1}^{N-1} F_i(\boldsymbol{\xi}')^2 \right) D(\boldsymbol{q} - \boldsymbol{p}) \\ &- \chi'(y) S(\boldsymbol{\xi}') \left(\alpha \sum_{i=1}^{N-1} G_i(\boldsymbol{\xi}') + \alpha^2 y \sum_{i=1}^{N-1} H_i(\boldsymbol{\xi}') \right) D(\boldsymbol{q} - \boldsymbol{p}).\end{aligned}$$

Using

$$\int_0^1 \boldsymbol{f}'(\boldsymbol{\Phi}(\widehat{\mu}) + s\boldsymbol{\sigma}) ds\, \boldsymbol{\sigma} = \boldsymbol{f}(\boldsymbol{\Phi}(\widehat{\mu}) + \boldsymbol{\sigma}) - \boldsymbol{f}(\boldsymbol{\Phi}(\widehat{\mu})),$$

the profile equation of Φ and the definitions of S and $\boldsymbol{\sigma}$, we obtain

$$\begin{aligned}\mathcal{L}[\overline{\boldsymbol{v}}] &= -S(\boldsymbol{\xi}')\boldsymbol{\Phi}'(\widehat{\mu}) - \int_0^1 \boldsymbol{f}'(\boldsymbol{\Phi}(\widehat{\mu}) + s\boldsymbol{\sigma}) ds\, \boldsymbol{\sigma} + \alpha \boldsymbol{Y}(\boldsymbol{\xi}', \widehat{\mu}; \varepsilon, \alpha) + \varepsilon \boldsymbol{Z}(\boldsymbol{\xi}', \widehat{\mu}; \varepsilon, \alpha) \\ &= S(\boldsymbol{\xi}') \left(-\boldsymbol{\Phi}'(\widehat{\mu}) - \varepsilon \int_0^1 \boldsymbol{f}'(\boldsymbol{\Phi}(\widehat{\mu}) + s\boldsymbol{\sigma}) ds (\boldsymbol{p} + \chi(\widehat{\mu})(\boldsymbol{q} - \boldsymbol{p})) \right. \\ &\qquad \left. + \alpha \frac{\boldsymbol{Y}(\boldsymbol{\xi}', \widehat{\mu}; \varepsilon, \alpha)}{S(\boldsymbol{\xi}')} + \varepsilon \frac{\boldsymbol{Z}(\boldsymbol{\xi}', \widehat{\mu}; \varepsilon, \alpha)}{S(\boldsymbol{\xi}')} \right).\end{aligned}$$

There exists a constant $\nu_0 > 0$ with

$$\begin{aligned}&\sum_{i=1}^{N-1} \left(|F_i(\boldsymbol{\xi}')| + |G_i(\boldsymbol{\xi}')| + |H_i(\boldsymbol{\xi}')| + |D_i S(\boldsymbol{\xi}')| + \left| D_i^2 S(\boldsymbol{\xi}') \right| \right) \\ &\qquad\qquad \le \nu_0 \sum_{2 \le \sum_{p=1}^{N-1} i_p \le 3} \left| D_1^{i_1} D_2^{i_2} \cdots D_{N-1}^{i_{N-1}} \varphi(\boldsymbol{\xi}') \right|\end{aligned}$$

for all $\boldsymbol{\xi}' \in \mathbb{R}^{N-1}$. Using Proposition 1, we have a constant $\nu_* > 0$ and get

$$\frac{|\boldsymbol{Y}(\boldsymbol{\xi}', y; \varepsilon, \alpha)|}{S(\boldsymbol{\xi}')} < \nu_*, \quad \frac{|\boldsymbol{Z}(\boldsymbol{\xi}', y; \varepsilon, \alpha)|}{S(\boldsymbol{\xi}')} < \nu_*,$$

for all $\boldsymbol{\xi}' \in \mathbb{R}^{N-1}$, $y \in \mathbb{R}$, $\varepsilon \in (0,1)$ and $\alpha \in (0,1)$. Constants ν_0 and ν_* are independent of $\boldsymbol{\xi}'$, $\alpha \in (0,1)$ and $\varepsilon \in (0,1)$.

We put

$$\omega = \inf_{\boldsymbol{x}' \in \mathbb{R}^{N-1}} \frac{\varphi(\boldsymbol{x}') - h(\boldsymbol{x}')}{S(\boldsymbol{x}')} \in (0, \infty)$$

by using Proposition 1. Now we choose ε small enough to satisfy

$$0 < \varepsilon < \min \left\{ \frac{1}{2}, \frac{\delta_*}{c}, \frac{\lambda_*}{4 \max\{1, \nu_*\} \max\{1, M\}}, \frac{4K_0}{ek\kappa_0 r_0} \right\}. \tag{41}$$

Then we choose α small enough to satisfy

$$0 < \alpha < \min \left\{ \frac{1}{2}, \frac{\beta\varepsilon}{2\nu_*}, \frac{\lambda_*}{4\nu_*}, \frac{k^2 \omega \kappa_0}{2c \log\left(\dfrac{4K_0}{ek\kappa_0 r_0 \varepsilon} \right)} \right\}. \tag{42}$$

Using (41), the definition of $\boldsymbol{Z}(\boldsymbol{\xi}', y; \varepsilon, \alpha)$ and

$$\boldsymbol{Z}(\boldsymbol{\xi}', y; \varepsilon, \alpha) = \mathbf{0} \qquad \text{if } |y| > 1,$$

we have

$$\varepsilon \frac{|\boldsymbol{Z}(\boldsymbol{\xi}', y; \varepsilon, \alpha)|}{S(\boldsymbol{\xi}')} \leq \frac{1}{4} \min_{1 \leq j \leq m} \left(-\Phi_j'(y) \right) \qquad \text{for all } \boldsymbol{\xi}' \in \mathbb{R}^{N-1},\ y \in \mathbb{R}.$$

Then we continue to calculate $\mathcal{L}[\overline{\boldsymbol{v}}]$ as

$$\begin{aligned} \frac{\mathcal{L}[\overline{\boldsymbol{v}}]}{S(\boldsymbol{\xi}')} &\geq -\frac{1}{2} \boldsymbol{\Phi}'(\widehat{\mu}) - \varepsilon \int_0^1 \boldsymbol{f}'(\boldsymbol{\Phi}(\widehat{\mu}) + s\boldsymbol{\sigma}) ds \, (\boldsymbol{p} + \chi(\widehat{\mu})(\boldsymbol{q} - \boldsymbol{p})) + \alpha \frac{\boldsymbol{Y}(\boldsymbol{\xi}', \widehat{\mu}; \varepsilon, \alpha)}{S(\boldsymbol{\xi}')} \\ &\geq -\frac{1}{2} \boldsymbol{\Phi}'(\widehat{\mu}) - \varepsilon \int_0^1 \boldsymbol{f}'(\boldsymbol{\Phi}(\widehat{\mu}) + s\boldsymbol{\sigma}) ds \, (\boldsymbol{p} + \chi(\widehat{\mu})(\boldsymbol{q} - \boldsymbol{p})) - \alpha \nu_* \mathbf{1} \end{aligned} \tag{43}$$

Using Lemma 2.3, we have

$$|\boldsymbol{\sigma}| \leq \varepsilon(c - k) \max\{|\boldsymbol{p}|, |\boldsymbol{q}|\} \leq \varepsilon c < \delta_*.$$

When $|\widehat{\mu}| \geq \mu_*$ we have

$$-\int_0^1 \boldsymbol{f}'(\boldsymbol{\Phi}(\widehat{\mu}) + s\boldsymbol{\sigma}) ds \, (\boldsymbol{p} + \chi(\widehat{\mu})(\boldsymbol{q} - \boldsymbol{p})) \geq \beta \mathbf{1},$$

and thus

$$\begin{aligned} \frac{\mathcal{L}[\overline{\boldsymbol{v}}]}{S(\boldsymbol{\xi}')} &\geq -\frac{1}{2} \boldsymbol{\Phi}'(\widehat{\mu}) - \varepsilon \int_0^1 \boldsymbol{f}'(\boldsymbol{\Phi}(\widehat{\mu}) + s\boldsymbol{\sigma}) ds \, (\boldsymbol{p} + \chi(\widehat{\mu})(\boldsymbol{q} - \boldsymbol{p})) - \alpha \nu_* \mathbf{1} \\ &\geq -\frac{1}{2} \boldsymbol{\Phi}'(\widehat{\mu}) + (\varepsilon\beta - \alpha\nu_*) \mathbf{1} \geq \mathbf{0}. \end{aligned}$$

When $|\widehat{\mu}| \leq \mu_*$ we have

$$\begin{aligned} \frac{\mathcal{L}[\overline{\boldsymbol{v}}]}{S(\boldsymbol{\xi}')} &\geq -\frac{1}{2} \boldsymbol{\Phi}'(\widehat{\mu}) - \varepsilon \int_0^1 \boldsymbol{f}'(\boldsymbol{\Phi}(\widehat{\mu}) + s\boldsymbol{\sigma}) ds \, (\boldsymbol{p} + \chi(\widehat{\mu})(\boldsymbol{q} - \boldsymbol{p})) - \alpha \nu_* \mathbf{1} \\ &\geq -\frac{1}{2} \boldsymbol{\Phi}'(\widehat{\mu}) - (\varepsilon M + \alpha\nu_*) \mathbf{1} \geq -\frac{1}{4} \boldsymbol{\Phi}'(\widehat{\mu}) - \alpha\nu_* \mathbf{1} \geq \mathbf{0}. \end{aligned}$$

Now we show that $\overline{\boldsymbol{v}}$ is a supersolution and is larger than $\underline{\boldsymbol{v}}$.

Lemma 3.1. *Assume ε and α satisfy (41) and (42), respectively. Let $\underline{\boldsymbol{v}}$ and $\overline{\boldsymbol{v}}$ be as in (17) and (40), respectively. Then*

$$\mathcal{L}[\overline{\boldsymbol{v}}] \geq 0 \ \text{in } \mathbb{R}^N$$

holds true. Moreover one has $\underline{\boldsymbol{v}}(\boldsymbol{x}) \leq \overline{\boldsymbol{v}}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^N$.

Proof. It suffices to prove the latter statement. We put $\overline{\boldsymbol{v}} = {}^t(\overline{v}_1, \ldots, \overline{v}_m)$, that is,

$$\overline{v}_i(\boldsymbol{x}) = \Phi_i(\widehat{\mu}) + \varepsilon(p_i + \chi(\widehat{\mu})(q_i - p_i))S(\alpha \boldsymbol{x}')$$

for $1 \le i \le m$. We prove

$$\Phi_i\left(\frac{k}{c}(x_N - h_j(\boldsymbol{x}'))\right) \le \overline{v}_i(\boldsymbol{x})$$

for all $1 \le j \le n$ and $1 \le i \le m$. Let $1 \le i \le m$ be fixed. If we have

$$\widehat{\mu} \le \frac{k}{c}(x_N - h_j(\boldsymbol{x}')),$$

we get

$$\Phi_i\left(\frac{k}{c}\,(x_N - h_j(\boldsymbol{x}'))\right) \le \Phi_i(\widehat{\mu}) < \overline{v}_i(\boldsymbol{x})$$

for all $1 \le j \le n$. Thus it suffices to consider the case of

$$\frac{k}{c}(x_N - h_j(\boldsymbol{x}')) < \widehat{\mu}.$$

Substituting the definition of $\widehat{\mu}$ into this inequality, we obtain

$$\frac{k}{c}\,(x_N - h_j(\boldsymbol{x}')) < \frac{x_N - \frac{1}{\alpha}\varphi(\boldsymbol{\xi}')}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}')|^2}} = \frac{x_N - h_j(\boldsymbol{x}') + (h_j(\boldsymbol{x}') - \frac{1}{\alpha}\varphi(\boldsymbol{\xi}'))}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}')|^2}}$$

and thus

$$\frac{c}{\alpha}\,\frac{\varphi(\boldsymbol{\xi}') - h_j(\boldsymbol{\xi}')}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}')|^2}} < \left(\frac{c}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}')|^2}} - k\right)(x_N - h_j(\boldsymbol{x}'))\,.$$

By (32), we have

$$\frac{c}{\alpha}\frac{1}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}')|^2}}\frac{\varphi(\boldsymbol{\xi}') - h_j(\boldsymbol{\xi}')}{S(\boldsymbol{\xi}')} < x_N - h_j(\boldsymbol{x}').$$

By the definition of ω and Lemma 2.3 we have

$$\frac{k\omega}{\alpha} < \frac{c}{\alpha}\frac{\omega}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}')|^2}} < x_N - h_j(\boldsymbol{x}'). \tag{44}$$

Since

$$h_j(\boldsymbol{x}') = \frac{1}{\alpha}h_j(\boldsymbol{\xi}') \le \frac{1}{\alpha}h(\boldsymbol{\xi}') \le \frac{1}{\alpha}\varphi(\boldsymbol{\xi}'),$$

we have

$$\begin{aligned}
&\overline{v}_i(\boldsymbol{x}) - \Phi_i\left(\frac{k}{c}\,(x_N - h_j(\boldsymbol{x}'))\right) \\
&= \Phi_i\left(\frac{x_N - \frac{1}{\alpha}\varphi(\boldsymbol{\xi}')}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}')|^2}}\right) + \varepsilon(p_i + \chi(\widehat{\mu})(q_i - p_i))S(\boldsymbol{\xi}') - \Phi_i\left(\frac{k}{c}\,(x_N - h_j(\boldsymbol{x}'))\right) \\
&\ge \Phi_i\left(\frac{x_N - h_j(\boldsymbol{x}')}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}')|^2}}\right) - \Phi_i\left(\frac{k}{c}\,(x_N - h_j(\boldsymbol{x}'))\right) + \varepsilon(p_i + \chi(\widehat{\mu})(q_i - p_i))S(\boldsymbol{\xi}').
\end{aligned}$$

Combining this inequality and

$$\Phi_i\left(\frac{x_N-h_j(\boldsymbol{x}')}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}}\right)-\Phi_i\left(\frac{k}{c}\left(x_N-h_j(\boldsymbol{x}')\right)\right)$$
$$=\frac{(x_N-h_j(\boldsymbol{x}'))S(\boldsymbol{\xi}')}{c}\int_0^1\Phi_i'\left(\left(\frac{\theta}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}}+\frac{k}{c}(1-\theta)\right)(x_N-h_j(\boldsymbol{x}'))\right)d\theta,$$

we find

$$\overline{v}_i(\boldsymbol{x})-\Phi_i\left(\frac{k}{c}\left(x_N-h_j(\boldsymbol{x}')\right)\right)$$
$$\geq\frac{(x_N-h_j(\boldsymbol{x}'))S(\boldsymbol{\xi}')}{c}\int_0^1\Phi_i'\left(\left(\frac{\theta}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}}+\frac{k}{c}(1-\theta)\right)(x_N-h_j(\boldsymbol{x}'))\right)d\theta$$
$$+\varepsilon(p_i+\chi(\widehat{\mu})(q_i-p_i))S(\boldsymbol{\xi}').$$

By Lemma 2.3 we have

$$\frac{k}{c}\leq\frac{\theta}{\sqrt{1+|\nabla\varphi(\boldsymbol{\xi}')|^2}}+\frac{k}{c}(1-\theta)\leq 1.$$

Then using (44), we get

$$\overline{v}_i(\boldsymbol{x})-\Phi_i\left(\frac{k}{c}\left(x_N-h_j(\boldsymbol{x}')\right)\right)\geq-\frac{S(\boldsymbol{\xi}')}{c}\sup_{|y|\geq\frac{k^2\omega}{c\alpha}}\left|\frac{c}{k}y\Phi'(y)\right|+\varepsilon r_0S(\boldsymbol{\xi}')$$
$$=S(\boldsymbol{\xi}')\left(\varepsilon r_0-\frac{1}{k}\sup_{|y|\geq\frac{k^2\omega}{c\alpha}}|y\Phi'(y)|\right).$$

By virtue of (25) we have

$$|y\Phi_i'(y)|\leq\frac{2K_0}{e\kappa_0}\exp\left(-\frac{\kappa_0}{2}|y|\right).$$

Using (42), we get

$$\frac{1}{k}\sup_{|y|\geq\frac{k^2\omega}{c\alpha}}|y\Phi'(y)|<\frac{\varepsilon r_0}{2}.$$

Then we obtain

$$\overline{v}_i(\boldsymbol{x})-\Phi_i\left(\frac{k}{c}(x_N-h_j(\boldsymbol{x}'))\right)>\frac{\varepsilon r_0}{2}S(\boldsymbol{\xi}')>0.$$

This completes the proof. □

Now we prove the main theorem in this paper.

Proof of Theorem 1.1. Recall (17) and (40), and consider solutions of (10) given by $w(\boldsymbol{x},t;\underline{\boldsymbol{v}})$ and $w(\boldsymbol{x},t;\overline{\boldsymbol{v}})$ respectively. Since $\underline{\boldsymbol{v}}$ is a weak subsolution and $\overline{\boldsymbol{v}}$ is a supersolution, we have

$$\underline{\boldsymbol{v}}\leq w(\boldsymbol{x},t;\underline{\boldsymbol{v}})\leq w(\boldsymbol{x},t;\overline{\boldsymbol{v}})\leq\overline{\boldsymbol{v}}$$

for $\boldsymbol{x}\in\mathbb{R}^N$ and $t\geq 0$ by using [13, Theorem 3.4]. Then

$$V(\boldsymbol{x})=\lim_{t\to\infty}w(\boldsymbol{x},t;\underline{\boldsymbol{v}})\tag{45}$$

exists in $L^\infty(\mathbb{R}^N)$ with

$$\underline{\boldsymbol{v}}(\boldsymbol{x})<V(\boldsymbol{x})<\overline{\boldsymbol{v}}(\boldsymbol{x})\qquad\text{for }\boldsymbol{x}\in\mathbb{R}^N.$$

This $V(\boldsymbol{x})$ is a solution of (11). See Sattinger [13, Theorem 3.6] for detailed arguments.

Now we prove (21). For any given $\varepsilon > 0$ we prove

$$\sup_{\boldsymbol{x}\in D(\gamma)} |\overline{\boldsymbol{v}}(\boldsymbol{x}) - \underline{\boldsymbol{v}}(\boldsymbol{x})| < 2c\varepsilon, \tag{46}$$

if $\gamma > 0$ is large enough. Indeed, let assume the contrary. Then there exist sequences $(\gamma_i)_{i=1}^{\infty} \subset \mathbb{R}$ and $(\boldsymbol{x}_i)_{i=1}^{\infty} \subset \mathbb{R}^N$ such that we have

$$\lim_{i\to\infty} \gamma_i = \infty, \qquad \boldsymbol{x}_i \in D(\gamma_i), \tag{47}$$

and

$$\left| \boldsymbol{\Phi}(\widehat{\mu}_i) - \boldsymbol{\Phi}\left(\frac{k}{c}(x_{N,i} - h(\boldsymbol{x}'_i))\right) \right| \geq c\varepsilon, \tag{48}$$

where $\boldsymbol{x}_i = {}^t(x_{1,i}, ..., x_{N,i})$ and $\boldsymbol{x}'_i = {}^t(x_{1,i}, ..., x_{N-1,i})$ for $i \in \mathbb{N}$. Here we put $\boldsymbol{\xi}'_i = \alpha \boldsymbol{x}'_i$ with $\boldsymbol{\xi}' = {}^t(\xi_{1,i}, \ldots, \xi_{N,i})$ and

$$\begin{aligned}\widehat{\mu}_i &= \frac{1}{\alpha}\frac{\xi_{N,i} - \varphi(\boldsymbol{\xi}'_i)}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}'_i)|^2}} = \frac{x_{N,i} - h(\boldsymbol{x}'_i) - \frac{1}{\alpha}(\varphi(\boldsymbol{\xi}'_i) - h(\boldsymbol{\xi}_i))}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}'_i)|^2}} \\ &= \frac{x_{N,i} - h(\boldsymbol{x}'_i) - \frac{1}{\alpha}\widetilde{\varphi}(\boldsymbol{\xi}'_i)}{\sqrt{1 + |\nabla\varphi(\boldsymbol{\xi}'_i)|^2}}.\end{aligned}$$

If $\lim_{i\to\infty} |x_{N,i} - h(\boldsymbol{x}'_i)| = \infty$, we have $\lim_{i\to\infty} |\widehat{\mu}_i| = \infty$ by using $\|\widetilde{\varphi}\|_{L^\infty(\mathbb{R}^{N-1})} < \infty$ and Lemma 2.3, and get a contradiction in (48). Otherwise we can assume that $\sup_{i\to\infty} |x_{N,i} - h(\boldsymbol{x}'_i)| < \infty$, and using

$$\lim_{i\to\infty} |\widetilde{\varphi}(\boldsymbol{\xi}_i)| = 0, \quad \lim_{i\to\infty} |\nabla\varphi(\boldsymbol{\xi}'_i)| = m_*, \quad \lim_{i\to\infty} S(\boldsymbol{\xi}'_i) = 0$$

in (33), we obtain

$$\lim_{i\to\infty} \left| \widehat{\mu}_i - \frac{k}{c}(x_{N,i} - h(\boldsymbol{x}'_i)) \right| = 0.$$

This implies

$$\lim_{i\to\infty} \left| \boldsymbol{\Phi}(\widehat{\mu}_i) - \boldsymbol{\Phi}\left(\frac{k}{c}(x_{N,i} - h(\boldsymbol{x}'_i))\right) \right| = 0.$$

This contradicts (48). Since we can take ε in (41) arbitrarily small, we complete the proof of Theorem 1.1.

The uniqueness and stability of pyramidal traveling fronts in $\mathbb{R}^N$ remain as interesting open problems.

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E-mail address: wmni@umn.edu
E-mail address: masaharu.taniguchi@is.titech.ac.jp