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THE EXISTENCE AND UNIQUENESS OF UNSTABLE EIGENVALUES FOR STRIPE PATTERNS IN THE GIERER-MEINHARDT SYSTEM

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Abstract. The Gierer-Meinhardt system is a mathematical model describing the process of hydra regeneration. This system has a stationary solution with a stripe pattern on a rectangular domain, but numerical results suggest that such stripe pattern is unstable. In [8], Kolokolnikov et al. proved the existence of a positive eigenvalue, which is called an *unstable eigenvalue*, for a stationary solution with a stripe pattern by the NLEP method, which implies the instability of the stripe pattern. In addition, the uniqueness of the unstable eigenvalue was shown under some technical assumptions in [8]. In this paper, we prove the existence and uniqueness of an unstable eigenvalue by using the SLEP method without any extra conditions. We also prove the existence of a single-spike solution in one-dimension.

1. **Introduction.** The development of an organism is a complex phenomenon involving elementary processes such as gene regulation, alteration of cell shapes and cell to cell interaction, cell proliferation, growth and cell movement. One of these elementary processes is the formation of a spatial pattern of tissue structures. In embryology and developmental biology, such a spatial pattern is believed to follow a ground plan and result from a primary pattern of morphogen concentrations or other physical factors which are distributed in space with gradients (see [5]).

The fundamental question in morphology is how the developmental ground plan is established and what is the mechanism which produces the spatial pattern necessary for specifying the various organs ([10]). Recently it is reported in [14] that the interactions between two pigment cells, called melanophores and xanthophores are observed on spotted and stripe patterns in the skin of zebrafish. Using modern genetic and molecular techniques, the authors identify putative elements of interactive networks that fulfill the criteria of short-range positive feedback and long-range negative feedback, and state that the pigment pattern is generated in the skin by interactions between pigment cells, that is, reaction-diffusion (Turing) mechanism. Thus reaction-diffusion equations are fundamental models in morphology.

Also, the positive and negative feedback mechanism is observed in the regulatory system of *HyWnt3* which is a putative master Wnt ligand in Hydra axial patterning and is expressed at the earliest phase of head regeneration. According to [15],

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HyWnt3 regulatory region consists of activator and repressor modules, which is essential for the maintenance of spatial pattern by Turing mechanism. Although the identity and expression pattern of the repressor molecule are not known yet, the reaction-diffusion mechanism plays an important role of pattern formation in head regeneration of hydra.

These biological evidences based on molecular technique clarify the existence of the positive and negative feedback loops in biological systems and suggest that pattern formation processes adopt the reaction-diffusion mechanism. In spite of the current development of modern molecular technique, it is still difficult to predict what kinds of spatial pattern is generated, which is called *pattern selection problem.* As well as the work [14] for the pattern formation in the skin of zebrafish, the observation in [15] does not point out any nonlinear effect in the reaction process so that the we cannot answer which spatial pattern emerges from an initial homogeneous state. Thus theoretical analysis is necessary to study the pattern selection problem.

In 1972, A. Gierer and H. Meinhardt proposed a reaction-diffusion system to explain hydra regeneration with local concentration of some chemical products (see [5]). This system is given by

$$\begin{cases} \dfrac{\partial A}{\partial t} = \varepsilon^2 \Delta A - A + \dfrac{A^p}{H^q}, \\ \tau \dfrac{\partial H}{\partial t} = d\Delta H - \mu H + \dfrac{A^r}{H^s} \end{cases}$$

and extensively applied to biological pattern formation. In this system, A and H represent the activator concentration and inhibitor concentration, respectively. As seen in [11], [12] and long literature, this system generates spiky patterns in a wide range of parameters (see Figure 1). Some mathematicians succeeded in proving the existence and stability of a stationary solution with spiky pattern, called a *spike solution.* For example, see [2], [3], [7], [19], [24], [25], [26].

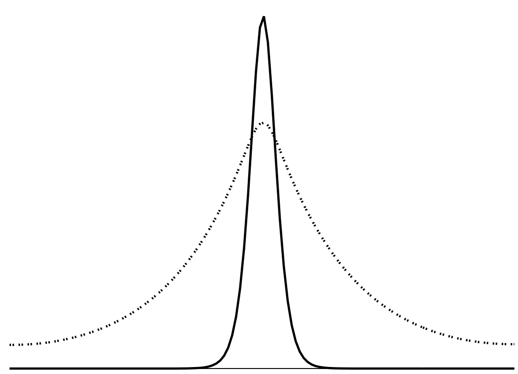

FIGURE 1. The graph of the spike solution (a, h) of (1). The solid line is the graph of a while the dotted line is that of h. In the numerical simulation, the parameter values were given by $\varepsilon = 0.223607, d = 1.0, \tau = 0.1, \mu = 1.0, p = 2.0, q = 1.0, r = 2.0, s = 0.0$.

It is known that other patterns, e.g. stripe patterns, hardly appear in this system (Figure 2). The author of [13] stated that stripe patterns are unlikely to appear in this system. However other reaction diffusion models may have possibilities to generate stripe patterns, which was pointed out in [9], [18]. In [18], the authors considered how the choice of the reaction terms which satisfy conditions to derive the Turing's instability affects the tendency to generate either striped, spotted (spotted is equivalent to spiky), or reversed spotted pattern, and they concluded as follows: It is because the reaction term of the first equation in the Gierer-Meinhardt system has some constraint at $A = 0$ that spotted pattern are generated, and if we add more constraints to the reaction terms, the new system may generate stripe pattern. From their results, we understand that constraints of reaction terms play an important role in pattern selection. Since the results in [18] are based on numerical technique, we would like to show that their results are mathematically correct. As the first step, we will prove that some stripe solution is unstable in the Gierer-Meinhardt system in a two-dimensional rectangular region.

Now we formulate our problem. Since a stationary solution with spiky pattern tends to ∞ and approaches the Dirac's δ-function as $\varepsilon \to 0$, one needs to do some rescaling of A, H such as $A = \varepsilon^{-q/(qr-(p-1)(s+1))}a$ and $H = \varepsilon^{-(p-1)/(qr-(p-1)(s+1))}h$. Then the new functions a and h are solutions of

$$\text{(GM)} \qquad \begin{cases} \dfrac{\partial a}{\partial t} = \varepsilon^2 \Delta a - a + \dfrac{a^p}{h^q}, & (x,y) \in (-1,1) \times (-L,L),\ t > 0, \\ \tau \dfrac{\partial h}{\partial t} = d\Delta h - \mu h + \dfrac{1}{\varepsilon}\dfrac{a^r}{h^s}, & (x,y) \in (-1,1) \times (-L,L),\ t > 0, \\ \dfrac{\partial a}{\partial x} = \dfrac{\partial h}{\partial x} = 0, & x = \pm 1, y \in (-L,L),\ t > 0, \\ \dfrac{\partial a}{\partial y} = \dfrac{\partial h}{\partial y} = 0, & y = \pm L, x \in (-1,1),\ t > 0, \end{cases}$$

where ε, d, μ and τ are positive parameters, and the exponents p, q, r and s satisfy

$$p > 1, q > 0, r > 0, s \geq 0 \text{ and } \frac{qr}{(p-1)(s+1)} > 1.$$

The conditions for these exponents imply that the Gierer-Meinhardt system exhibits Turing's instability, which means that a homogeneous state becomes unstable by the presence of diffusion (see [23]). At first we consider the following one-dimensional steady state problem:

$$\begin{cases} \varepsilon^2 a'' - a + \dfrac{a^p}{h^q} = 0, & x \in (-1,1), \\ dh'' - \mu h + \dfrac{1}{\varepsilon}\dfrac{a^r}{h^s} = 0, & x \in (-1,1), \\ a' = h' = 0, & x = \pm 1. \end{cases} \tag{1}$$

In [19], [25], it is shown that there exists some stationary solution $(a(x), h(x))$ with a single spike pattern at the origin. Thus we call (a, h) a *spike solution* for (1) (see Figure 1).

In this paper, we consider the stationary solution (a, h) for (1) with the following properties. Setting $(\tilde{a}(y), \tilde{h}(y)) = (a(\varepsilon y), h(\varepsilon y))$ for $-1/\varepsilon < y < 1/\varepsilon$, we assume that there exists a constant $c > 0$ independent of $\varepsilon > 0$ such that $(\tilde{a}, \tilde{h})$ satisfies

$$\|\tilde{a} - \zeta^{q/(p-1)} w\|_{H^2(-1/\varepsilon, 1/\varepsilon)} \leq c\sqrt{\varepsilon} \tag{2}$$

and for each $R > 0$,

$$\sup_{-R<y<R} |\tilde{h}(y) - \zeta| \to 0 \tag{3}$$

as $\varepsilon \to 0$. Here w is a unique solution of

$$\begin{cases} w'' - w + w^p = 0, & y \in (-\infty, \infty), \\ w \to 0, & |y| \to \infty, \\ w(0) = \max w, \end{cases}$$

which is called *ground state solution* and is explicitly written by

$$w(y) = \left(\frac{p+1}{2\cosh^2 (p-1)y/2} \right)^{1/(p-1)}$$

and ζ is a positive constant given by

$$\zeta = \left(\frac{2\sqrt{d\mu}}{\int_{-\infty}^{\infty} w^r dy} \tanh \sqrt{\frac{\mu}{d}} \right)^{(p-1)/(qr-(p-1)(s+1))}.$$

These properties imply that $(\tilde{a}, \tilde{h})$ approaches a spiky pattern $(\zeta^{q/(p-1)} w, \zeta)$ as $\varepsilon \to 0$. It seems that the existence of a solution satisfying (2) and (3) has not been shown so far. So we shall prove it in Section 4.2 (see Theorem 4.3). Since (a, h) is independent of y-variable and the region of (GM) is a rectangle, it is clear that (a, h) also satisfies (GM). Thus we call (a, h) a *stripe solution* for (GM) (see Figure 2). In this paper, we study the stability of (a, h) in (GM).

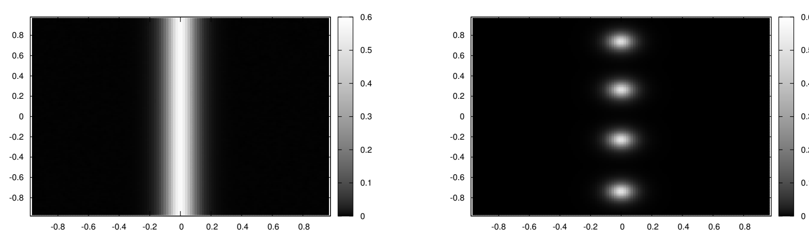

FIGURE 2. The breakup of a stripe pattern and convergence to a spot pattern in a two-dimensional region $(-1,1) \times (-1,1)$ in (GM). The initial state with a stripe pattern (left figure) breaks into a four-spike pattern (right figure, which is a snapshot at $t = 1000.0$). In this numerical simulation, the parameter values were given by $\varepsilon = 0.223607, d = 1.0, \tau = 0.1, \mu = 1.0, p = 2.0, q = 1.0, r = 2.0, s = 0.0$.

In order to study the stability of the stripe solution it suffices to consider the following two-dimensional eigenvalue problem associated with the linearized system of (GM):

$$\begin{cases} \lambda\phi = \varepsilon^2 \Delta\phi - \phi + p\dfrac{a^{p-1}}{h^q}\phi - q\dfrac{a^p}{h^{q+1}}\eta, & (x, y) \in (-1, 1) \times (-L, L), \\ \tau\lambda\eta = d\Delta\eta - \mu\eta + \dfrac{1}{\varepsilon}\left(r\dfrac{a^{r-1}}{h^s}\phi - s\dfrac{a^r}{h^{s+1}}\eta \right), & (x, y) \in (-1, 1) \times (-L, L), \\ \dfrac{\partial\phi}{\partial x} = \dfrac{\partial\eta}{\partial x} = 0, & x = \pm 1, y \in (-L, L), \\ \dfrac{\partial\phi}{\partial y} = \dfrac{\partial\eta}{\partial y} = 0, & y = \pm L, x \in (-1, 1), \end{cases}$$

since (GM) is a semilinear system. Due to the lack of y-direction with (a, h), without loss of generality, we can assume that the eigenfunction (ϕ, η) satisfies $(\phi(x, y), \eta(x, y)) = (\phi(x)\psi_k(y), \eta(x)\psi_k(y))$, where ψ_k is defined by $\psi_k(y) = \cos k\pi y/2L$ (resp. $\sin k\pi y/2L$) if k is even (resp. odd). Hence we rewrite the above system to the following one-dimensional eigenvalue problem:

$$\text{(P)} \qquad \begin{cases} \lambda\phi = \varepsilon^2\phi'' - (1 + \varepsilon^2 l^2)\phi + p\dfrac{a^{p-1}}{h^q}\phi - q\dfrac{a^p}{h^{q+1}}\eta & \text{in } (-1, 1), \\ \tau\lambda\eta = d\eta'' - (\mu + dl^2)\eta + \dfrac{1}{\varepsilon}\left(r\dfrac{a^{r-1}}{h^s}\phi - s\dfrac{a^r}{h^{s+1}}\eta\right) & \text{in } (-1, 1), \\ \phi'(\pm 1) = \eta'(\pm 1) = 0, \end{cases}$$

where $l = k\pi/2L$. If (P) has an eigenvalue λ and $\lim_{\varepsilon\to 0} Re\lambda > 0$, we call it an unstable eigenvalue in this paper, where "Re" represents the real part of a complex value. Although there may be a positive eigenvalue $\tilde{\lambda}$ of (P) such as $\lim_{\varepsilon\to 0} Re\lambda = 0$, we do not call it an unstable eigenvalue.

Now we describe our result, from which we know that (P) has exactly one real and unstable eigenvalue. In other words, the stripe solution is unstable.

Theorem 1.1. *Let (a, h) be a solution of (1) satisfying (2) and (3). Fix $l \neq 0$. Then there exist $d_0 > 0$ and $\varepsilon_0 > 0$ such that for each $d > d_0$ and $\varepsilon < \varepsilon_0$, (P) has exactly one real eigenvalue λ satisfying $\lim_{\varepsilon\to 0} \lambda > 0$. More precisely, if $\hat{\lambda}$ is an eigenvalue of (P) in $\{z \in \mathbb{C} \mid Rez \geq 0\}$, one has either $\hat{\lambda} = \lambda$ or $\lim_{\varepsilon\to 0} \hat{\lambda} = 0$.*

Theorem 1.1 implies not only the existence a positive eigenvalue λ but also the uniqueness of unstable eigenvalues in some sense. The study of (P) is based on a limiting eigenvalue problem (9), given in Section 2.1. Generally speaking, however, there is a possibility of an unstable eigenvalue in (P) close to 0 because 0 is a solution of a limiting eigenvalue problem of (8) as $\varepsilon \to 0$. In other words, the eigenvalue problem (8) may have an eigenvalue $\hat{\lambda}$ with $Re\hat{\lambda} > 0$ and $\lim_{\varepsilon\to 0} \hat{\lambda} = 0$. In the end, we do not completely know the uniqueness of unstable eigenvalues from our result. Note that we can show the uniqueness of an eigenvalue in (8) which does not approach 0 as $\varepsilon \to 0$.

Here we remark two related results on the instability of stripe patterns in the Gierer-Meinhardt system. One is the work of Doelman and van der Ploeg [3], and another is that of Kolokolnikov et al. [8]. In both of these works, stationary solutions with the stripe pattern depending on only one spatial variable are considered for 2-dimensional Gierer-Meinhardt system. Hence, as described as above, eigenvalue problems on one dimensional space with a parameter l arise naturally. In [3], the eigenvalue problem is studied on the whole line by using the theory of Evans functions, which does not seem to be powerful in the case of bounded interval. On the other hand, in [8], the eigenvalue problem is considered on a bounded interval as our formulation. In order to analyze the eigenvalue problem, Kolokolnikov et al. adopt the NLEP (Non-Local Eigenvalue Problem) method. The results in [8] implies the existence of an unstable eigenvalue, which is the same as in Theorem 1.1. On the other hand, to obtain the uniqueness of an unstable eigenvalue in the sense of Theorem 1.1, Kolokolnikov et al. needed some extra conditions in [8]; p, r satisfy either $p = r = 2$, or $r = p + 1$ and $1 < p \leq 5$. In our work, we will use the SLEP (Singular Limit Eigenvalue Problem) method without assuming any extra conditions on the exponents.

The SLEP method, introduced in [17], has been used to consider the stability of stationary solutions or traveling wave solutions of a reaction-diffusion system with thin layers (see [16], [20]–[22]). In particular, the author of [21], [22] considered the stability of planar interfaces on a rectangle. Hence eigenvalue problems on one-dimensional with a parameter l naturally arise. However, since the stationary solution (a, h) has a spiky pattern on one-dimension and g_a^ε contains the term $1/\varepsilon$, the eigenvalue problem (P) is also quite different from theirs. This paper is the first case that the SLEP method is used to consider the stability of a stationary solution with a spiky pattern.

Now we give the summary of this paper. In order to explain how we use the SLEP method, we first carry out formal calculations in Section 2.1. Then in Section 2.2, we give some key lemmas, in particular for the stripe solution (a, h). By using the lemmas, we describe the proof of our theorem in Section 3. In Section 4.1, we shall prove Lemmas 2.1 and 2.2. In Section 4.2, we shall prove the existence of a spike solution of (1) satisfying (2), (3).

We use the following useful notations throughout our paper. First we define several Banach spaces for each open interval I of $\mathbb{R}$ and notations associated with their Banach spaces:

$$\begin{aligned} L^p(I) &= \text{the usual Lebesgue space of order } p(\geq 1), \\ H^p(I) &= \text{the usual Sobolev space of order } p \text{ in } L^2(I)\text{–framework}, \\ (H^1(I))' &= \text{the dual space of } H^1(I), \\ C(\overline{I}) &= \text{the space of continuous functions on } \overline{I}, \\ \|\cdot\|_X &= \text{the norm in the Banach space } X, \\ (\cdot,\cdot)_{L^2(I)} &= \text{the inner product in } L^2(I), \\ \langle\cdot,\cdot\rangle &= \text{the pairing between } (H^1(-1,1))' \text{ and } H^1(-1,1). \end{aligned}$$

Secondly we introduce a new coordinate y. For each $x \in (-1, 1)$, we define $y = x/\varepsilon$. Note that $-1/\varepsilon < y < 1/\varepsilon$. We call this *stretched-coordinate*. For some function ψ, we denote the stretched function of ψ by $\tilde{\psi}(y) = \psi(\varepsilon y)$. Furthermore we define $\hat{\psi}(y) = \sqrt{\varepsilon}\tilde{\psi}(y)$ if $\psi \in L^2(-1, 1)$ satisfies $\|\psi\|_{L^2(-1,1)} = 1$. We note that

$$\|\hat{\psi}\|^2_{L^2(-1/\varepsilon,1/\varepsilon)} = \varepsilon \int_{-1/\varepsilon}^{1/\varepsilon} \psi(\varepsilon y)^2 dy = \int_{-1}^{1} \psi(x)^2 dx \ = \ 1.$$

Thirdly we prepare some definitions and notations for simplicity. We define $D \equiv qr - (p-1)(s+1) > 0$. For each function $\phi(y)$ on $(-1/\varepsilon, 1/\varepsilon)$, we extend it to the whole line by a natural way, that is, $\phi \equiv 0$ for sufficiently large $|y|$. Then we do not distinguish the extended function from the original function, and we write the extended function as ϕ.

2. **Preliminaries.**

2.1. **Formal calculations.**

In this subsection, we carry out formal calculations to describe the outline of the proof of our theorem. We rewrite $(\cdot,\cdot)_{L^2(-1,1)}$ as $(\cdot,\cdot)$ for simplicity throughout this subsection. At first, we rewrite the eigenvalue problem

(P) to the following form:

$$\lambda \begin{pmatrix} \phi \\ \eta \end{pmatrix} = \begin{pmatrix} L_\varepsilon & f_h^\varepsilon \\ g_a^\varepsilon & M_\varepsilon \end{pmatrix} \begin{pmatrix} \phi \\ \eta \end{pmatrix},$$

where L_ε and M_ε are differential operators defined by

$$L_\varepsilon \equiv \varepsilon^2 \frac{d^2}{dx^2} + f_a^\varepsilon, \quad M_\varepsilon \equiv \frac{d}{\tau}\frac{d^2}{dx^2} + g_h^\varepsilon,$$

and f_a^ε, f_h^ε, g_a^ε and g_h^ε are given by

$$f_a^\varepsilon = -(1+\varepsilon^2 l^2) + p\frac{a^{p-1}}{h^q}, \quad f_h^\varepsilon = -q\frac{a^p}{h^{q+1}},$$
$$g_a^\varepsilon = \frac{r}{\tau\varepsilon}\frac{a^{r-1}}{h^s}, \qquad g_h^\varepsilon = -\frac{dl^2+\mu}{\tau} - \frac{s}{\tau\varepsilon}\frac{a^r}{h^{s+1}}.$$

Since we consider an unstable eigenvalue of (P), it suffices to restrict λ to the set of $\mathbb{C}$, $\Lambda_+ = \{\lambda \in \mathbb{C} \mid Re\lambda \geq 0\ \}$. As we will see, if $\lambda \in \Lambda_+$ is an eigenvalue in (P) and that of L_ε, λ must be close to 0.

Lemma 2.1. *Let $\lambda \in \Lambda_+$ be an eigenvalue of (P). Then λ is close to 0 or does not correspond to any eigenvalue of L_ε for sufficiently small $\varepsilon > 0$.*

We shall prove this lemma in Section 4.1. From Lemma 2.1 we can solve the first equation of (P) with respect to ϕ to obtain

$$\phi = (L_\varepsilon - \lambda)^{-1}(-f_h^\varepsilon \eta), \tag{4}$$

where $(L_\varepsilon - \lambda)^{-1}$ is the invertible operator of $L_\varepsilon - \lambda$. We can decompose $(L_\varepsilon - \lambda)^{-1}$ to three parts by using pairs of eigenvalues and eigenfunctions of L_ε, denoted by $\{\xi_i^\varepsilon, \varphi_i^\varepsilon\}_{i=0}^\infty$ satisfying that $\xi_0^\varepsilon > \xi_1^\varepsilon > \xi_2^\varepsilon > \cdots$ and $\|\varphi_i^\varepsilon\|_{L^2(-1,1)} = 1$ for each $i \geq 0$ as follows:

$$(L_\varepsilon - \lambda)^{-1} = \frac{(\cdot, \varphi_0^\varepsilon)}{\xi_0^\varepsilon - \lambda}\varphi_0^\varepsilon + \frac{(\cdot, \varphi_1^\varepsilon)}{\xi_1^\varepsilon - \lambda}\varphi_1^\varepsilon + R_{\varepsilon,\lambda},$$

where $R_{\varepsilon,\lambda} : L^2(-1,1) \to L^2(-1,1)$ is defined by

$$R_{\varepsilon,\lambda} \equiv \sum_{i=2}^\infty \frac{(\cdot, \varphi_i^\varepsilon)}{\xi_i^\varepsilon - \lambda}\varphi_i^\varepsilon.$$

Substituting (4) into the second equation of (P), we have

$$\begin{aligned}(M_\varepsilon - \lambda)\eta &= -g_a^\varepsilon \phi = -g_a^\varepsilon (L_\varepsilon - \lambda)^{-1}(-f_h^\varepsilon \eta) \\ &= -g_a^\varepsilon \left\{ \frac{(-f_h^\varepsilon \eta, \varphi_0^\varepsilon)}{\xi_0^\varepsilon - \lambda}\varphi_0^\varepsilon + \frac{(-f_h^\varepsilon \eta, \varphi_1^\varepsilon)}{\xi_1^\varepsilon - \lambda}\varphi_1^\varepsilon + R_{\varepsilon,\lambda}(-f_h^\varepsilon \eta) \right\},\end{aligned}$$

or equivalently

$$(-M_\varepsilon - g_a^\varepsilon R_{\varepsilon,\lambda}(-f_h^\varepsilon \cdot) + \lambda)\eta = \frac{(-f_h^\varepsilon \eta, \varphi_0^\varepsilon)}{\xi_0^\varepsilon - \lambda} g_a^\varepsilon \varphi_0^\varepsilon + \frac{(-f_h^\varepsilon \eta, \varphi_1^\varepsilon)}{\xi_1^\varepsilon - \lambda} g_a^\varepsilon \varphi_1^\varepsilon. \tag{5}$$

We see that the invertible operator of $(-M_\varepsilon - g_a^\varepsilon R_{\varepsilon,\lambda}(-f_h^\varepsilon \cdot) + \lambda)$ exists, and is denoted by $K_{\varepsilon,\lambda} : (H^1(-1,1))' \to H^1(-1,1)$ (see Lemma 3.2). Applying $K_{\varepsilon,\lambda}$ to the both sides of (5), we have

$$\eta = \frac{(-f_h^\varepsilon \eta, \varphi_0^\varepsilon)}{\xi_0^\varepsilon - \lambda} K_{\varepsilon,\lambda}(g_a^\varepsilon \varphi_0^\varepsilon) + \frac{(-f_h^\varepsilon \eta, \varphi_1^\varepsilon)}{\xi_1^\varepsilon - \lambda} K_{\varepsilon,\lambda}(g_a^\varepsilon \varphi_1^\varepsilon). \tag{6}$$

This implies that η must be written as

$$\eta = \alpha K_{\varepsilon,\lambda}(g_a^\varepsilon \varphi_0^\varepsilon) + \beta K_{\varepsilon,\lambda}(g_a^\varepsilon \varphi_1^\varepsilon), \tag{7}$$

where α, β are constants. Substituting (7) into (6), we have

$$\alpha K_{\varepsilon,\lambda}(g_a^\varepsilon \varphi_0^\varepsilon) + \beta K_{\varepsilon,\lambda}(g_a^\varepsilon \varphi_1^\varepsilon) = \alpha A_{00} K_{\varepsilon,\lambda}(g_a^\varepsilon \varphi_0^\varepsilon) + \beta A_{01} K_{\varepsilon,\lambda}(g_a^\varepsilon \varphi_0^\varepsilon) + \alpha A_{10} K_{\varepsilon,\lambda}(g_a^\varepsilon \varphi_1^\varepsilon) + \beta A_{11} K_{\varepsilon,\lambda}(g_a^\varepsilon \varphi_1^\varepsilon),$$

where $A_{ij} = (K_{\varepsilon,\lambda}(g_a^\varepsilon \varphi_j^\varepsilon),\ -f_h^\varepsilon \varphi_i^\varepsilon)/(\xi_i^\varepsilon - \lambda)$ for $i, j = 0, 1$. Since $K_{\varepsilon,\lambda}(g_a^\varepsilon \varphi_0^\varepsilon)$ and $K_{\varepsilon,\lambda}(g_a^\varepsilon \varphi_1^\varepsilon)$ are linearly independent, α, β satisfy

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

From $(\alpha, \beta)^t \neq 0$, it follows that $(A_{00} - 1)(A_{11} - 1) - A_{10}A_{01} = 0$. Hence we obtain

$$\begin{aligned} &((K_{\varepsilon,\lambda}(g_a^\varepsilon \varphi_0^\varepsilon), -f_h^\varepsilon \varphi_0^\varepsilon) - \xi_0^\varepsilon + \lambda)((K_{\varepsilon,\lambda}(g_a^\varepsilon \varphi_1^\varepsilon), -f_h^\varepsilon \varphi_1^\varepsilon) - \xi_1^\varepsilon + \lambda) \\ &= (K_{\varepsilon,\lambda}(g_a^\varepsilon \varphi_0^\varepsilon), -f_h^\varepsilon \varphi_1^\varepsilon)(K_{\varepsilon,\lambda}(g_a^\varepsilon \varphi_1^\varepsilon), -f_h^\varepsilon \varphi_0^\varepsilon). \end{aligned} \tag{8}$$

It is known that $\xi_0^\varepsilon \to \xi_0^* > 0$ and $\xi_1^\varepsilon \to 0$ as $\varepsilon \to 0$, as described in Section 2.3. Moreover, we shall show in Section 3 that

$$-\frac{1}{\sqrt{\varepsilon}} f_h^\varepsilon \varphi_0^\varepsilon \to c_1 \delta, \quad -\frac{1}{\sqrt{\varepsilon}} f_h^\varepsilon \varphi_1^\varepsilon \to 0, \quad \sqrt{\varepsilon} g_a^\varepsilon \varphi_0^\varepsilon \to c_2 \delta, \quad \sqrt{\varepsilon} g_a^\varepsilon \varphi_1^\varepsilon \to 0$$

in $(H^1(-1,1))'$ as $\varepsilon \to 0$, where c_1, c_2 are some positive constants and uniformly bounded in large $d > 0$, and δ is the usual Dirac's δ-function at the origin (see Lemma 3.1). Furthermore we see that there exists an operator $K_{*,\lambda} : (H^1(-1,1))' \to H^1(-1,1)$ such that $K_{\varepsilon,\lambda} \to K_{*,\lambda}$ as $\varepsilon \to 0$ in a certain sense (see Lemmas 3.5, 3.6). Finally we have in the limit of (8) as $\varepsilon \to 0$

$$\lambda(\lambda - \xi_0^* + c_1 c_2 \langle \delta, K_{*,\lambda} \delta \rangle) = 0.$$

Since we are interested in an unstable eigenvalue satisfying $\lim_{\varepsilon \to 0} \lambda \neq 0$, λ must satisfy

$$\lambda = \xi_0^* - c_1 c_2 \langle \delta, K_{*,\lambda} \delta \rangle. \tag{9}$$

Since $\langle \delta, K_{*,\lambda} \delta \rangle$ is small if d is large, as shown in Section 3, the implicit function theorem shows that there exists an unstable eigenvalue satisfying (8).

The rest of the pater is organized as follows. In Section 2.2, we shall introduce some known results and show lemmas, which imply that a has some exponentially decaying property and that h is bounded away from zero on $[-1, 1]$ if d is large. In Section 2.3, we shall show that the eigenfunction $\hat{\varphi}_0^\varepsilon$ (resp. $\hat{\varphi}_1^\varepsilon$) corresponding to ξ_0^ε (resp. ξ_1^ε) approaches $\hat{\varphi}_0^*$ (resp. $\hat{\varphi}_1^*$) in $H^1(\mathbb{R})$, where $\hat{\varphi}_0^*$, $\hat{\varphi}_1^*$ are eigenfunctions for

$$\begin{cases} \lambda \varphi = \varphi'' - \varphi + p w^{p-1} \varphi, & y \in (-\infty, \infty), \\ \varphi \to 0, & |y| \to \infty, \end{cases} \tag{10}$$

corresponding to the unique positive eigenvalue $\xi_0^*, 0$ and satisfying $\|\hat{\varphi}_0^*\|_{L^2(\mathbb{R})} = 1$ and $\|\hat{\varphi}_1^*\|_{L^2(\mathbb{R})} = 1$, respectively. Also ξ_0^*, $\hat{\varphi}_0^*$ and $\hat{\varphi}_1^*$ can be expressed explicitly as $\xi_0^* = (p+1)^2/4 - 1 > 0$, $\hat{\varphi}_0^* = w^{(p+1)/2}/\|w^{(p+1)/2}\|_{L^2(\mathbb{R})}$ and $\hat{\varphi}_1^* = w'/\|w'\|_{L^2(\mathbb{R})}$, respectively. These explicit expressions can be obtained by using the hypergeometric functions (see [2]). In Section 3, we will first determine c_1 and c_2 explicitly. Their explicit representations imply that they are uniformly bounded in large d. Secondly we define the operators $K_{\varepsilon,\lambda}$ and $K_{*,\lambda}$. In order to define them, we shall use some bilinear forms, $B_{\varepsilon,\lambda}$ and $B_{*,\lambda}$, and apply Lax-Milgram's theorem to $B_{\varepsilon,\lambda}$ and $B_{*,\lambda}$. Then we complete the proof of Theorem 1.1 by using the implicit function theorem. In the last section, we shall prove Lemmas 2.1 and 2.2. In Lemma 2.2, we consider a Sturm-Liouville type problem and show that a solution for the problem has some exponentially decaying property. In Section 4.2, we also prove the existence of a single-spike solution satisfying (2), (3).

Remark 1. From the explicit expression of w, we see that there exists a constant $c > 0$ such that

$$\left|\frac{d^k}{dy^k} w\right| \leq c e^{-|y|} \tag{11}$$

for each $y \in \mathbb{R}$ and $k = 0, 1, 2$. Furthermore there exists a positive constant $\tilde{c}$ such that $w \geq \tilde{c} e^{-|y|}$.

2.2. **Known results and some lemmas.** We first consider an ordinary differential equation of an inhomogeneous Sturm-Liouville type. We expect that if the coefficient functions have exponentially decaying properties, the solution of the problem also has the same property.

Lemma 2.2. *Let $U(y), V(y)$ be continuous functions satisfying*

$$|U(y)| \leq c_u e^{-u|y|}, \quad |V(y)| \leq c_v e^{-v|y|} \tag{12}$$

for $y \in (-1/\varepsilon, 1/\varepsilon)$, where u, v, c_u and c_v are positive constants independent of ε. Suppose that there exists a positive constant c_0 such that the solution, denoted by ϕ, of

$$\phi'' - (\alpha^2 + U(y))\phi = V(y), \quad y \in (-1/\varepsilon, 1/\varepsilon), \tag{13}$$

$$\phi' = 0, \quad y = \pm 1/\varepsilon, \tag{14}$$

satisfies $\|\phi\|_{H^1(-1/\varepsilon,1/\varepsilon)} \leq c_0$ for sufficiently small ε, where $\alpha = \alpha_1 + \alpha_2 i$ is a complex number satisfying $\alpha_1 \geq \gamma$ for a constant $\gamma > u$ independent of ε. Assume that c_0 is independent of ε, α. If γ is not equal to v, then ϕ satisfies

$$\left|\frac{d^k}{dy^k} \phi\right| \leq c|\alpha|^k e^{-\rho|y|} \tag{15}$$

for any $y \in (-1/\varepsilon, 1/\varepsilon)$, sufficiently small ε and $k = 0, 1, 2$, where $\rho = \min\{\gamma, v\}$ and c is a constant independent of ε, α. In particular, if $\alpha_1 < v$ and α_1 is independent of ε, one can take $\rho = \alpha_1$ in (15).

Note that the decaying rate ρ of ϕ is independent of the decay rate u of U. Since the proof of Lemma 2.2 needs length argument, we shall describe it in Section 4.1. Throughout this paper, we often use the above lemma.

Next we shall obtain some properties of the stationary solution of (a, h). First we show that $h(x)$ does not approach 0 as $\varepsilon \to 0$ at any point on $[-1, 1]$. If this is the case, $1/h$ is uniformly bounded on $[-1, 1]$ as $\varepsilon \to 0$.

Lemma 2.3 ([19])**.** *The following inequality holds:*

$$\max h < e^{\sqrt{\mu/d}} \min h.$$

The property (3) and Lemma 2.3 imply that if d is sufficiently large and ε is sufficiently small, h is close to ζ on $[-1, 1]$.

Next we show that $\tilde{a}(y)$ satisfies a similar inequality to (11). Since $\tilde{a}$ approaches $\zeta^{q/(p-1)} w$ as $\varepsilon \to 0$, we expect that $\tilde{a}$ has similarities to w.

Lemma 2.4. *There exists a constant $c > 0$ independent of ε such that*

$$\left|\frac{d^k}{dy^k} \tilde{a}\right| \leq c e^{-|y|} \tag{16}$$

for $y \in (-1/\varepsilon, 1/\varepsilon)$ and $k = 0, 1, 2$.

Proof. By (2) and the fact that w decays exponentially as $|y| \to \infty$, as mentioned in Remark 1, we show that $\tilde{a}$ satisfies

$$\tilde{a}(y) \le ce^{-\alpha|y|}, \quad \text{for any } y \in (-1/\varepsilon, 1/\varepsilon) \tag{17}$$

for a constant $c > 0$ and an exponent $\alpha \in (0, 1)$ by using the same argument as in the proof of Lemma 3.11 in [16]. Hence we omit the details of the proof of (17).

Next we show that (16) holds by using (17) and Lemma 2.2. We should note that $\phi = \tilde{a}$ is a solution of the problem

$$\begin{cases} \phi'' - (1 - \dfrac{\tilde{a}^{p-1}}{\tilde{h}^q})\phi = 0, & y \in (-1/\varepsilon, 1/\varepsilon), \\ \phi' = 0, & y = \pm 1/\varepsilon, \end{cases}$$

and satisfies $\|\phi\|_{H^1(-1/\varepsilon,1/\varepsilon)} \le c$. Since there exists a positive constant c such that $\tilde{a}^{p-1}/\tilde{h}^q \le ce^{-(p-1)\alpha|y|}$ for $y \in (-1/\varepsilon, 1/\varepsilon)$, Lemma 2.2 implies that $\tilde{a} \le ce^{-|y|}$ for $y \in (-1/\varepsilon, 1/\varepsilon)$. Using this and the above differential equation for $\tilde{a}$, we can easily verify that (16) holds for $k = 1, 2$. □

Thus it is shown that $\tilde{a}$ has some exponentially decaying property as w. In fact, $\tilde{a}$ has another similarity. By Remark 1, we have $w \ge \tilde{c}e^{-|y|}$ for $y \in \mathbb{R}$ with a constant $\tilde{c}$. Then $\tilde{a}$ also has this property, at least in the region where $|y|$ is sufficiently large.

Lemma 2.5. *There exist $R_0 > 0$ and $\varepsilon_0 > 0$ such that $\tilde{a}(y) \ge \tilde{c}e^{-|y|}$ for $\varepsilon < \varepsilon_0$, $R > R_0$ and $R < |y| < 1/\varepsilon$, where $\tilde{c}$ is independent of ε, R, d.*

Proof. In what follows, we consider only the case of $y \in (R, 1/\varepsilon)$. The case of $y \in (-1/\varepsilon, -R)$ can be proved in the same way.

Using (2) and Remark 1, we have $\tilde{a}(R) \ge ce^{-R}$, where c is independent of ε, R, d if ε is sufficiently small for large $R > 0$. Here we set $\psi(y) = \tilde{a}(y)/\tilde{a}(R)$. Then ψ must be a solution of

$$\begin{cases} -\psi'' + \psi = \dfrac{1}{\tilde{a}(R)} \dfrac{\tilde{a}^p}{\tilde{h}^q}, & y \in (R, 1/\varepsilon), \\ \psi(R) = 1, \quad \psi'(1/\varepsilon) = 0. \end{cases} \tag{18}$$

Since (18) is a Sturm-Liouville type problem, we can express ψ as

$$\begin{aligned} \psi(y) &= c_+\phi_+(y) + c_-\phi_-(y) \\ &- \frac{1}{W}\left(\phi_-(y)\int_R^y \phi_+ \frac{1}{\tilde{a}(R)}\frac{\tilde{a}^p}{\tilde{h}^q} d\zeta + \phi_+(y)\int_y^{1/\varepsilon} \phi_- \frac{1}{\tilde{a}(R)}\frac{\tilde{a}^p}{\tilde{h}^q} d\zeta\right), \end{aligned} \tag{19}$$

where c_+, c_- are constants, ϕ_+, ϕ_- are positive functions given by

$$\phi_+(y) = \frac{e^{-1/\varepsilon+R}}{1 + e^{-2/\varepsilon+2R}}(e^{y-R} + e^{-(y-R)}),$$

$$\phi_-(y) = \frac{1}{1 + e^{-2/\varepsilon+2R}}(e^{-(y-R)} + e^{-2/\varepsilon+R+y}),$$

respectively, and W is a constant defined by

$$W = \phi_-'(R)\phi_+(R) - \phi_-(R)\phi_+'(R).$$

We should note that ϕ_+, ϕ_- are solutions of

$$-\phi'' + \phi = 0, \quad y \in (R, 1/\varepsilon),$$

satisfying

$$\phi_+'(R) = 0, \phi_+(1/\varepsilon) = 1, \quad \phi_-(R) = 1, \phi_-'(1/\varepsilon) = 0,$$

respectively. From the boundary condition of ψ and ϕ_- at $y = 1/\varepsilon$, we can easily show $c_+ = 0$. Furthermore, setting $y = R$ in (19) and using another boundary conditions of ϕ_- and ψ at $y = R$, we see that

$$c_- \geq 1 - \frac{\phi_+(R)}{|W|}\int_R^{1/\varepsilon} \phi_- \frac{1}{\tilde{a}(R)}\frac{\tilde{a}^p}{\tilde{h}^q}d\zeta \geq 1 - ce^{-(p-1)R} \geq \frac{1}{2}$$

if R is sufficiently large. Hence ψ satisfies

$$\psi(y) \geq c_-\phi_- - ce^{-y-(p-1)R+R} \geq \frac{1}{4}e^{-y+R},$$

which implies that $\tilde{a} \geq \tilde{c}e^{-y}$ on $(R, 1/\varepsilon)$. □

2.3. **Properties of $\{\xi_i^\varepsilon, \phi_i^\varepsilon\}$.** In this subsection we consider pairs of the eigenvalues and eigenfunctions of L_ε, denoted by $\{\xi_i^\varepsilon, \varphi_i^\varepsilon\}_{i=0}^\infty$. In particular, we will study the properties of $\{\xi_0^\varepsilon, \varphi_0^\varepsilon\}$ and $\{\xi_1^\varepsilon, \varphi_1^\varepsilon\}$. Without loss of generality, we may assume that $\{\varphi_i^\varepsilon\}_{i=0}^\infty$ is an orthonormal system in $L^2(-1,1)$. It is not difficult to see that there exists a positive constant γ independent of ε such that

$$\xi_0^\varepsilon > \gamma > \xi_1^\varepsilon > -\gamma > \xi_2^\varepsilon > \xi_3^\varepsilon > \cdots, \tag{20}$$

which can be shown in the same way as in the proof of Lemma 3.10 in [16]. We also see that $\xi_0^\varepsilon \to \xi_0^*$ and $\xi_1^\varepsilon \to 0$ as $\varepsilon \to 0$, where ξ_0^* is a unique positive eigenvalue of the eigenvalue problem (10). We denote the eigenfunctions corresponding to the eigenvalues $\xi_0^*, 0$ of (10) by $\hat{\varphi}_0^*$, $\hat{\varphi}_1^*$ satisfying $\|\hat{\varphi}_0^*\|_{L^2(\mathbb{R})} = 1$ and $\|\hat{\varphi}_1^*\|_{L^2(\mathbb{R})} = 1$, respectively. As mentioned in Section 1, $\hat{\varphi}_0^*$ and $\hat{\varphi}_1^*$ can be represented explicitly. Those explicit expressions imply that $\hat{\varphi}_0^*$ and $\hat{\varphi}_1^*$ approach 0 exponentially as $|y| \to \infty$. Since the eigenvalue problem (10) is expected to be the limiting problem of

$$\begin{cases} \phi'' - (1+\varepsilon^2 l^2)\phi + p\dfrac{\tilde{a}^{p-1}}{\tilde{h}^q}\phi = \lambda\phi, & y \in (-1/\varepsilon, 1/\varepsilon), \\ \phi' = 0, & y = \pm 1/\varepsilon, \end{cases} \tag{21}$$

we expect that $\hat{\varphi}_0^\varepsilon$ and $\hat{\varphi}_1^\varepsilon$ have some similarities to $\hat{\varphi}_0^*$ and $\hat{\varphi}_1^*$, respectively.

Lemma 2.6. *Fix $0 < \theta < 1$ arbitrarily independent of ε. Then there exist a positive constant c independent of ε and $\varepsilon_0 > 0$ such that*

$$\left|\frac{d^k}{dy^k}\hat{\varphi}_0^\varepsilon\right| \leq ce^{-\sqrt{1+\gamma}|y|}, \quad \left|\frac{d^k}{dy^k}\hat{\varphi}_1^\varepsilon\right| \leq ce^{-\theta|y|} \tag{22}$$

for $\varepsilon < \varepsilon_0$, $y \in (-1/\varepsilon, 1/\varepsilon)$ and $k = 0, 1, 2$. Furthermore $\hat{\varphi}_0^\varepsilon$, $\hat{\varphi}_1^\varepsilon$ converge to $\hat{\varphi}_0^*$, $\hat{\varphi}_1^*$ in $C^2(\mathbb{R})$ as $\varepsilon \to 0$, respectively.

Proof. Since $\hat{\varphi}_0^\varepsilon$ is the solution of the problem (21), we can easily show that $\hat{\varphi}_0^\varepsilon$ satisfies $\|\hat{\varphi}_0^\varepsilon\|_{H^1(-1/\varepsilon, 1/\varepsilon)} \leq c$. In addition, we have $p\tilde{a}^{p-1}/\tilde{h}^q \leq ce^{-(p-1)|y|}$ for each $y \in (-1/\varepsilon, 1/\varepsilon)$. Hence $\hat{\varphi}_0^\varepsilon$ satisfies (22) by $\gamma < \xi_0^\varepsilon$ and Lemma 2.2. Also it is not difficult to show that $\hat{\varphi}_1^\varepsilon$ satisfies (22), by the same argument of φ_0^ε, because it holds that $1 + \varepsilon^2 l^2 + \xi_1^\varepsilon \geq \theta$ for sufficiently small $\varepsilon > 0$.

Next we show the last part. By $\|\hat{\varphi}_0^\varepsilon\|_{H^1(-1/\varepsilon, 1/\varepsilon)} \leq c$ for $\varepsilon > 0$, there exist $\phi_0 \in H^1(\mathbb{R})$ and a subsequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ and $\hat{\varphi}_0^{\varepsilon_n} \to \phi_0$ in $(H^1(\mathbb{R}))'$ as $n \to \infty$. Then ϕ_0 must satisfy the problem (10) for $\lambda = \xi_0^*$. Using the differential equation (10), we have

$$\left|\frac{d^k}{dy^k}\phi_0\right| \leq ce^{-\sqrt{1+\xi_0^*}|y|}$$

for any $y \in \mathbb{R}$ and $k = 0, 1, 2$. Hence, by (22), we have $\|\hat{\varphi}_0^{\varepsilon_n} - \phi_0\|_{L^2(\mathbb{R})} \to 0$ as $n \to \infty$, which implies $\|\phi_0\|_{L^2(\mathbb{R})} = 1$. Since the eigenfunction of (10) corresponding to ξ_0^* is unique, ϕ_0 must be $\hat{\varphi}_0^*$. Hence $\hat{\varphi}_0^\varepsilon$ converges to $\hat{\varphi}_0^*$ in $(H^1(\mathbb{R}))'$ as $\varepsilon \to 0$. Here we do not need to choose a subsequence of ε. Furthermore, taking account of (10), (21), we can easily verify that $\hat{\varphi}_0^\varepsilon$ converges to $\hat{\varphi}_0^*$ in $C^2(\mathbb{R})$ as $\varepsilon \to 0$. On the other hand, we can show in the same way as in the proof of $\hat{\varphi}_0^\varepsilon$ that $\hat{\varphi}_1^\varepsilon$ converges to $\hat{\varphi}_1^*$ in $C^2(\mathbb{R})$ as $\varepsilon \to 0$. This completes the proof of the lemma. □

3. Proof of Theorem 1.1.

In this section, we prove Theorem 1.1. In the following, if we use a constant c, we suppose that it is independent of ε, α.

3.1. Asymptotic behavior of $f_h^\varepsilon \varphi_i^\varepsilon/\sqrt{\varepsilon}$ and $\sqrt{\varepsilon} g_a^\varepsilon \varphi_i^\varepsilon$.

In order to prove Theorem 1.1, we consider the convergence of each term in (8). As the first step, we study the asymptotic behaviors of $f_h^\varepsilon \varphi_0^\varepsilon/\sqrt{\varepsilon}$, $f_h^\varepsilon \varphi_1^\varepsilon/\sqrt{\varepsilon}$, $\sqrt{\varepsilon} g_a^\varepsilon \varphi_0^\varepsilon$ and $\sqrt{\varepsilon} g_a^\varepsilon \varphi_1^\varepsilon$ as $\varepsilon \to 0$ in $(H^1(-1,1))'$.

Lemma 3.1. *As $\varepsilon \to 0$, one has*

$$-\frac{1}{\sqrt{\varepsilon}} f_h^\varepsilon \varphi_0^\varepsilon \to c_1 \delta, \quad -\frac{1}{\sqrt{\varepsilon}} f_h^\varepsilon \varphi_1^\varepsilon \to 0, \tag{23}$$

$$\sqrt{\varepsilon} g_a^\varepsilon \varphi_0^\varepsilon \to c_2 \delta, \quad \sqrt{\varepsilon} g_a^\varepsilon \varphi_1^\varepsilon \to 0 \tag{24}$$

in $(H^1(-1,1))'$, where c_1 and c_2 are positive constants, respectively defined by

$$c_1 \equiv q\zeta^{(q-p+1)/(p-1)} \int_{-\infty}^{\infty} w^p \hat{\varphi}_0^* dy, \quad c_2 \equiv \frac{r}{\tau} \zeta^{(q(r-1)-s(p-1))/(p-1)} \int_{-\infty}^{\infty} w^{r-1} \hat{\varphi}_0^* dy.$$

Proof. For each $z \in H^1(-1,1)$, we have

$$-\int_{-1}^{1} \frac{1}{\sqrt{\varepsilon}} f_h^\varepsilon \varphi_0^\varepsilon z dx = \int_{-1/\varepsilon}^{1/\varepsilon} q \frac{\tilde{a}^p}{\tilde{h}^{q+1}} \hat{\varphi}_0^\varepsilon \tilde{z} dy \to z(0) q \zeta^{(q-p+1)/(p-1)} \int_{-\infty}^{\infty} w^p \hat{\varphi}_0^* dy.$$

On the other hand, we have

$$-\int_{-1}^{1} \frac{1}{\sqrt{\varepsilon}} f_h^\varepsilon \varphi_1^\varepsilon z dx = \int_{-1/\varepsilon}^{1/\varepsilon} q \frac{\tilde{a}^p}{\tilde{h}^{q+1}} \hat{\varphi}_1^\varepsilon \tilde{z} dy \to \frac{z(0) q \zeta^{(q-p+1)/(p-1)}}{\|w'\|_{L^2(\mathbb{R})}} \int_{-\infty}^{\infty} w^p w' dy = 0$$

for each $z \in H^1(-1,1)$ by the Lebesgue dominated convergence theorem. Thus (23) holds.

Next we will show (24). If $r \geq 1$, we can easily show in the same way as in the proof of (23) that (24) holds, so we may assume $r < 1$. At first we have

$$\langle \sqrt{\varepsilon} g_a^\varepsilon \varphi_0^\varepsilon, z \rangle = \int_{-1/\varepsilon}^{1/\varepsilon} \frac{r}{\tau} \frac{\tilde{a}^{r-1}}{\tilde{h}^s} \hat{\varphi}_0^\varepsilon \tilde{z} dy = \left(\int_{-R}^{R} + \int_{R}^{1/\varepsilon} + \int_{-1/\varepsilon}^{-R} \right) \frac{r}{\tau} \frac{\tilde{a}^{r-1}}{\tilde{h}^s} \hat{\varphi}_0^\varepsilon \tilde{z} dy. \tag{25}$$

From (2), (3) and Lemmas 2.3, 2.6, the first term of the right-hand side of (25) converges to

$$\frac{r}{\tau} \zeta^{(q(r-1)-s(p-1))/(p-1)} z(0) \int_{-R}^{R} w^{r-1} \hat{\varphi}_0^* dy$$

as $\varepsilon \to 0$. Next we estimate the second and the third terms of the right-hand side of (25) from above such as

$$\begin{aligned} \left(\int_{R}^{1/\varepsilon} + \int_{-1/\varepsilon}^{-R} \right) \left| \frac{r}{\tau} \frac{\tilde{a}^{r-1}}{\tilde{h}^s} \hat{\varphi}_0^\varepsilon \tilde{z} \right| dy &\leq c\|z\|_{L^\infty(-1,1)} \int_{R}^{1/\varepsilon} e^{-(\sqrt{1+\gamma}-1+r)y} dy \\ &\leq c\|z\|_{L^\infty(-1,1)} e^{-(\sqrt{1+\gamma}-1+r)R}. \end{aligned}$$

Hence we have

$$\left| \lim_{\varepsilon \to 0} \int_{-1}^{1} \sqrt{\varepsilon} g_a^\varepsilon \varphi_0^\varepsilon z dx - \frac{r}{\tau} \zeta^{(q(r-1)-s(p-1))/(p-1)} z(0) \int_{-\infty}^{\infty} w^{r-1} \hat{\varphi}_0^* dy \right|$$
$$\leq c\|z\|_{L^\infty(-1,1)} e^{-(\sqrt{1+\gamma}-1+r)R} \to 0$$

as $R \to \infty$. In the same way, we see that (24) holds. This completes the proof of the lemma. □

3.2. **Asymptotic behavior of $K_{\varepsilon,\lambda}$.** In Section 2.1, we introduced $K_{\varepsilon,\lambda}$ which is the invertible operator of $(-M_\varepsilon - g_a^\varepsilon R_{\varepsilon,\lambda}(-f_h^\varepsilon \cdot) + \lambda)$ and $K_{*,\lambda}$ which is the limiting operator of $K_{\varepsilon,\lambda}$ as $\varepsilon \to 0$ in a certain sense. In this subsection, we give precise definition of these operators. In order to do so, we consider the following bilinear form:

$$B_{\varepsilon,\lambda}(z_1, z_2) = \frac{d}{\tau}(z_1', z_2') - (\{g_a^\varepsilon R_{\varepsilon,\lambda}(-f_h^\varepsilon \cdot) + g_h^\varepsilon - \lambda\} z_1, z_2), \tag{26}$$

where $\lambda \in \Lambda_+,\ z_1, z_2 \in H^1(-1,1)$. Here we abbreviate $(\cdot,\cdot)_{L^2(-1,1)}$ as $(\cdot,\cdot)$ for simplicity. Applying Lax-Milgram's theorem to this bilinear form, we shall define $K_{\varepsilon,\lambda}$.

Lemma 3.2. *For each $T \in (H^1(-1,1))'$, there exists a unique $\phi \in H^1(-1,1)$ such that*

$$B_{\varepsilon,\lambda}(\phi, \psi) = \langle T, \psi \rangle \quad \text{for } \psi \in H^1(-1,1), \tag{27}$$

where ε is sufficiently small, d is sufficiently large and $\lambda \in \Lambda_+$. Moreover, the operator

$$K_{\varepsilon,\lambda} : (H^1(-1,1))' \to H^1(-1,1)$$

is well-defined by $\phi = K_{\varepsilon,\lambda} T$, and satisfies

$$\|K_{\varepsilon,\lambda}\| \leq \frac{2\tau}{\min\{d, dl^2 + \mu\}},$$

where $\|\cdot\|$ is the usual norm for operators. In addition, $K_{\varepsilon,\lambda}$ is analytic with respect to λ and continuous with respect to ε.

In order to prove Lemma 3.2, we first study the asymptotic behavior of $g_a^\varepsilon R_{\varepsilon,\lambda}(-f_h^\varepsilon \cdot)$ in the right-hand side of (26).

Lemma 3.3. *For each $z \in H^1(-1,1)$ and $\lambda \in \Lambda_+$, it holds that*

$$g_a^\varepsilon R_{\varepsilon,\lambda}(-f_h^\varepsilon z) \to z(0) F \delta, \quad g_a^\varepsilon \frac{d}{d\lambda} R_{\varepsilon,\lambda}(-f_h^\varepsilon z) \to z(0) \frac{dF}{d\lambda} \delta \tag{28}$$

in $(H^1(-1,1))'$ as $\varepsilon \to 0$. Here $F = F(\lambda)$ is an analytic function in Λ_+, which is real-valued for real λ, defined by

$$F(\lambda) = \frac{r}{\tau} \zeta^{(q(r-1)-s(p-1))/(p-1)} (\phi_0, w^{r-1})_{L^2(\mathbb{R})},$$

where ϕ_0 is some function belonging to $H^1(\mathbb{R})$ independent of ε, z and has the exponentially decaying property such as $|\phi_0| \leq ce^{-|y|}$ for $y \in \mathbb{R}$, with a positive constant c independent of $\lambda \in \Lambda_+$. Furthermore, as $\varepsilon \to 0$, one has

$$\|g_a^\varepsilon R_{\varepsilon,\lambda}(-f_h^\varepsilon z) - z(0) F \delta\|_{(H^1(-1,1))'} \to 0$$

uniformly in $\|z\|_{H^1(-1,1)} \leq M$ and $\lambda \in \Lambda_+$.

Proof. Throughout this proof, we abbreviate $(\cdot,\cdot)_{L^2(-1/\varepsilon,1/\varepsilon)}$, $\|\cdot\|_{L^2(-1/\varepsilon,1/\varepsilon)}$, and $\|\cdot\|_{H^1(-1/\varepsilon,1/\varepsilon)}$ as $(\cdot,\cdot)$, $\|\cdot\|_{L^2}$, and $\|\cdot\|_{H^1}$, respectively, for simplicity. We set $\phi_\varepsilon = R_{\varepsilon,\lambda}(-f_h^\varepsilon z)$.

First, we show that $\|\tilde{\phi}_\varepsilon\|_{H^1}$, $\|d\tilde{\phi}_\varepsilon/d\lambda\|_{H^1}$ are uniformly bounded in $\varepsilon>0$ and $\lambda\in\Lambda_+$. Since $\tilde{\phi}_\varepsilon$ can be expressed as

$$\tilde{\phi}_\varepsilon = \sum_{i=2}^{\infty} \frac{(-f_h^\varepsilon z, \varphi_i^\varepsilon)_{L^2(-1,1)}}{\xi_i^\varepsilon - \lambda}\tilde{\varphi}_i^\varepsilon = \sum_{i=2}^{\infty} \frac{(-\tilde{f}_h^\varepsilon \tilde{z}, \hat{\varphi}_i^\varepsilon)\hat{\varphi}_i^\varepsilon}{\xi_i^\varepsilon - \lambda}, \tag{29}$$

we see that

$$\|\tilde{\phi}_\varepsilon\|_{L^2}^2 = \sum_{i=2}^{\infty}\frac{|(\tilde{f}_h^\varepsilon\tilde{z},\hat{\varphi}_i^\varepsilon)|^2}{|\xi_i^\varepsilon-\lambda|^2} \le \frac{\|\tilde{f}_h^\varepsilon\tilde{z}\|_{L^2}^2}{\gamma^2+|\lambda|^2}$$

by using Parseval's identity and $|\xi_i^\varepsilon-\lambda|^2\ge\gamma^2+|\lambda|^2$ for $i\ge 2$, sufficiently small ε and $\lambda\in\Lambda_+$, which implies that $\|\tilde{\phi}_\varepsilon\|_{L^2}$ is uniformly bounded in $\varepsilon>0$ and $\lambda\in\Lambda_+$.

Next we show that $\|\tilde{\phi}_\varepsilon\|_{H^1}$ is also uniformly bounded in $\varepsilon>0$ and $\lambda\in\Lambda_+$. By (29), $\tilde{\phi}_\varepsilon$ satisfies $(\tilde{\phi}_\varepsilon,\hat{\varphi}_0^\varepsilon)=0$, $(\tilde{\phi}_\varepsilon,\hat{\varphi}_1^\varepsilon)=0$ and the following equation with Neumann boundary condition:

$$\begin{cases} \phi'' + (\tilde{f}_a^\varepsilon - \lambda)\phi = -\tilde{f}_h^\varepsilon\tilde{z} + (\tilde{f}_h^\varepsilon\tilde{z},\hat{\varphi}_0^\varepsilon)\hat{\varphi}_0^\varepsilon + (\tilde{f}_h^\varepsilon\tilde{z},\hat{\varphi}_1^\varepsilon)\hat{\varphi}_1^\varepsilon, & y\in(-1/\varepsilon,1/\varepsilon),\\ \phi' = 0, & y=\pm 1/\varepsilon. \end{cases} \tag{30}$$

Then, multiplying $\overline{\tilde{\phi}_\varepsilon}$ to the first equation of (30) for $\phi=\tilde{\phi}_\varepsilon$ and integrating it over $(-1/\varepsilon,1/\varepsilon)$, we have

$$\begin{aligned} \|\tilde{\phi}_\varepsilon'\|_{L^2}^2 + (1+\varepsilon^2 l^2 + Re\lambda)\|\tilde{\phi}_\varepsilon\|_{L^2}^2 &\le (p\frac{\tilde{a}^{p-1}}{\tilde{h}^q}\tilde{\phi}_\varepsilon,\tilde{\phi}_\varepsilon) + |(\tilde{f}_h^\varepsilon\tilde{z},\tilde{\phi}_\varepsilon)| \\ &\le c(\|\tilde{\phi}_\varepsilon\|_{L^2}^2 + \|z\|_{L^\infty(-1,1)}\|\tilde{\phi}_\varepsilon\|_{L^2}). \end{aligned} \tag{31}$$

Since $\|\tilde{\phi}_\varepsilon\|_{L^2}$ is uniformly bounded in $\varepsilon>0$ and $\lambda\in\Lambda_+$, $\|\tilde{\phi}_\varepsilon\|_{H^1}$ is also uniformly bounded in $\varepsilon>0$ and $\lambda\in\Lambda_+$.

Next we show that $\|d\tilde{\phi}_\varepsilon/d\lambda\|_{H^1}$ is uniformly bounded in $\varepsilon>0$ and $\lambda\in\Lambda_+$. We should note that $d\tilde{\phi}_\varepsilon/d\lambda$ is expressed as

$$\frac{d\tilde{\phi}_\varepsilon}{d\lambda} = \sum_{i=2}^{\infty}\frac{(-\tilde{f}_h^\varepsilon\tilde{z},\hat{\varphi}_i^\varepsilon)\hat{\varphi}_i^\varepsilon}{(\xi_i^\varepsilon-\lambda)^2}.$$

Then we have $(d\tilde{\phi}_\varepsilon/d\lambda,\hat{\varphi}_0^\varepsilon)=0$ and $(d\tilde{\phi}_\varepsilon/d\lambda,\hat{\varphi}_1^\varepsilon)=0$. Using this and Parseval's identity, we obtain $\|d\tilde{\phi}_\varepsilon/d\lambda\|_{L^2}\le\|\tilde{f}_h^\varepsilon\tilde{z}\|_{L^2}/(\gamma^2+|\lambda|^2)$, which implies that $\|d\tilde{\phi}_\varepsilon/d\lambda\|_{L^2}$ is uniformly bounded in $\varepsilon>0$ and $\lambda\in\Lambda_+$. Then we can show in the same way as in the case of $\tilde{\phi}_\varepsilon$ that $\|d\tilde{\phi}_\varepsilon/d\lambda\|_{H^1}$ is also uniformly bounded in $\varepsilon>0$ and $\lambda\in\Lambda_+$. Furthermore, Lemmas 2.2, 2.4 and 2.6 imply that

$$|\tilde{\phi}_\varepsilon| \le ce^{-\theta|y|}, \quad \left|\frac{d\tilde{\phi}_\varepsilon}{d\lambda}\right| \le ce^{-\theta|y|} \tag{32}$$

for $y\in(-1/\varepsilon,1/\varepsilon)$ with $0<\theta<1$ fixed arbitrarily independent of ε,λ.

Next we shall show that there exists $\phi_0\in H^1(\mathbb{R})$ such that

$$\tilde{\phi}_\varepsilon \to z(0)\phi_0, \quad \frac{d\tilde{\phi}_\varepsilon}{d\lambda} \to z(0)\frac{d\phi_0}{d\lambda} \tag{33}$$

in $(H^1(\mathbb{R}))'$ as $\varepsilon \to 0$. By $\|\tilde{\phi}_\varepsilon\|_{H^1(-1/\varepsilon,1/\varepsilon)} \le c$ and $\|d\tilde{\phi}_\varepsilon/d\lambda\|_{H^1(-1/\varepsilon,1/\varepsilon)} \le c$ for $\varepsilon > 0$, there exist $\phi_0, \psi_0 \in H^1(\mathbb{R})$ and a subsequence $\{\varepsilon_n\}_{n=1}^\infty$ such that $\varepsilon_n \to 0$ and

$$\tilde{\phi}_{\varepsilon_n} \to z(0)\phi_0, \quad \frac{d\tilde{\phi}_{\varepsilon_n}}{d\lambda} \to z(0)\psi_0$$

in $(H^1(\mathbb{R}))'$ as $n \to \infty$. We note that ϕ_0 satisfies $(\phi_0, \hat{\varphi}_0^*)_{L^2(\mathbb{R})} = 0$, $(\phi_0, \hat{\varphi}_1^*)_{L^2(\mathbb{R})} = 0$ and

$$\begin{aligned} &\phi_0'' - (1+\lambda)\phi_0 + pw^{p-1}\phi_0 \\ &\quad = q\zeta^{(q-p+1)/(p-1)}(w^p - (w^p, \hat{\varphi}_0^*)_{L^2(\mathbb{R})}\hat{\varphi}_0^* - (w^p, \hat{\varphi}_1^*)_{L^2(\mathbb{R})}\hat{\varphi}_1^*) \\ &\quad = q\zeta^{(q-p+1)/(p-1)}(w^p - (w^p, \hat{\varphi}_0^*)_{L^2(\mathbb{R})}\hat{\varphi}_0^*) \end{aligned} \tag{34}$$

for each $y \in \mathbb{R}$. Here we used $(w^p, \hat{\varphi}_1^*)_{L^2(\mathbb{R})} = (w^p, w'/\|w'\|_{L^2})_{L^2(\mathbb{R})} = 0$. On the other hand, ψ_0 satisfies $(\psi_0, \hat{\varphi}_0^*)_{L^2(\mathbb{R})} = 0$, $(\psi_0, \hat{\varphi}_1^*)_{L^2(\mathbb{R})} = 0$ and

$$\psi_0'' - (1+\lambda)\psi_0 + pw^{p-1}\psi_0 = \phi_0 \tag{35}$$

for each $y \in \mathbb{R}$. Owing to $\lambda \in \Lambda_+$ and the orthogonal conditions $(\phi_0, \hat{\varphi}_0^*)_{L^2(\mathbb{R})} = 0$ and $(\phi_0, \hat{\varphi}_1^*)_{L^2(\mathbb{R})} = 0$, (34) admits only one solution. Moreover, (35) also admits only one solution. These imply that

$$\tilde{\phi}_\varepsilon \to z(0)\phi_0, \quad \frac{d\tilde{\phi}_\varepsilon}{d\lambda} \to z(0)\psi_0$$

in $(H^1(\mathbb{R}))'$ as $\varepsilon \to 0$. Here we do not need to choose a subsequence of ε.

Now let us show that ϕ_0 is analytic with respect to λ and $\psi_0 = d\phi_0/d\lambda$. It is easy to see that ϕ_0 is C^1 with respect to $\lambda \in \Lambda_+$ (see Hartman [6] section V). In order to prove the analyticity, it suffices to verify that ϕ_0 satisfies Cauchy-Riemann conditions. We differentiate ϕ_0 with respect to $\overline{\lambda} = \lambda_1 - \lambda_2 i$ if $\lambda = \lambda_1 + \lambda_2 i$. Then $d\phi_0/d\overline{\lambda}$ satisfies $(d\phi_0/d\overline{\lambda}, \hat{\varphi}_0^*)_{L^2(\mathbb{R})} = 0$, $(d\phi_0/d\overline{\lambda}, \hat{\varphi}_1^*)_{L^2(\mathbb{R})} = 0$ and

$$\phi'' - (1+\lambda)\phi + pw^{p-1}\phi = 0, \quad y \in \mathbb{R}.$$

This problem is the same as equivalent to (10), which admits only two solutions $(\lambda, \phi) = (\xi_0^*, c\hat{\varphi}_0^*)$ or $(0, c\hat{\varphi}_1^*)$ since λ is restricted in Λ_+, where c is an arbitrary constant. From $(d\phi_0/d\overline{\lambda}, \hat{\varphi}_0^*)_{L^2(\mathbb{R})} = 0$ and $(d\phi_0/d\overline{\lambda}, \hat{\varphi}_1^*)_{L^2(\mathbb{R})} = 0$, we see that $d\phi_0/d\overline{\lambda}$ is identically equal to 0. This implies that ϕ_0 satisfies the Cauchy-Riemann conditions so that ϕ_0 is analytic with respect to λ. Now we differentiate both sides of (34) with respect to λ. Then $d\phi_0/d\lambda$ must satisfy (35). The equality (35) with the orthogonal conditions admits only one solution so that $\psi_0 = d\phi_0/d\lambda$. By this, (32), (33), and the Lebesgue dominated convergence theorem, it is easy to verify that (28) holds. Furthermore ϕ_0 satisfies $|\phi_0| \le ce^{-|y|}$ for c independent of $\lambda \in \Lambda_+$.

Next we show the last part of the lemma by contradiction. For each $\kappa > 0$, we may assume that there exist $z_n \in \{z \in H^1(-1,1) \mid \|z\|_{H^1(-1,1)} \le M\}$, $\lambda_n \in \Lambda_+$, and a subsequence $\{\varepsilon_n\}_{n=1}^\infty$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and

$$\|g_a^{\varepsilon_n} R_{\varepsilon_n,\lambda_n}(-f_h^{\varepsilon_n} z_n) - z_n(0)F(\lambda_n)\delta\|_{(H^1(-1,1))'} \ge \kappa.$$

Due to this inequality and (31), λ_n is supposed to satisfy $|\lambda_n| \le c$ for a constant $c > 0$ independent of n without loss of generality. Since $\|z_n\|_{H^1(-1,1)} \le M$ and $|\lambda_n| \le c$, there exist $z_0 \in H^1(-1,1)$ and $\lambda_0 \in \Lambda_+$ such that $z_n \to z_0$ in $(H^1(-1,1))'$ and $\lambda_n \to \lambda_0$ as $n \to \infty$. Here we may replace n with an appropriate subsequence if needed, but we use the same notation. The space $H^1(-1,1)$ is compactly embedded in $C[-1,1]$ so that $z_n \to z_0$ in $C[-1,1]$ as $n \to \infty$. Then we have

$$\begin{aligned}\kappa &\le \|g_a^{\varepsilon_n} R_{\varepsilon_n,\lambda_n}(-f_h^{\varepsilon_n} z_n) - z_n(0)F(\lambda_n)\delta\| \\ &\le \|g_a^{\varepsilon_n} R_{\varepsilon_n,\lambda_n}(-f_h^{\varepsilon_n}(z_n - z_0))\| + \|g_a^{\varepsilon_n} R_{\varepsilon_n,\lambda_0}(-f_h^{\varepsilon_n} z_0) - z_0(0)F(\lambda_0)\delta\| \\ &\quad + \|g_a^{\varepsilon_n} R_{\varepsilon_n,\lambda_n}(-f_h^{\varepsilon_n} z_0) - g_a^{\varepsilon_n} R_{\varepsilon_n,\lambda_0}(-f_h^{\varepsilon_n} z_0)\| \\ &\quad + \|z_0(0)(F(\lambda_0) - F(\lambda_n))\delta\| + \|(z_0(0) - z_n(0))F(\lambda_n)\delta\|,\end{aligned} \tag{36}$$

where we abbreviate $\|\cdot\|_{(H^1(-1,1))'}$ as $\|\cdot\|$. Note that the second term of the right side of the above inequality tends to 0 as $n \to \infty$ by (28).

Next we verify that the first term converges to 0 as $n \to \infty$. For each $\varphi \in H^1(-1,1)$, $\langle g_a^{\varepsilon_n} R_{\varepsilon_n,\lambda}(-f_h^{\varepsilon_n}(z_n - z_0)), \varphi\rangle$ can be represented by $(R_{\varepsilon_n,\lambda}(-f_h^{\varepsilon_n}(z_n - z_0)), g_a^{\varepsilon_n}\overline{\varphi})_{L^2(-1,1)}$. For simplicity, we set $\phi_n = R_{\varepsilon_n,\lambda}(-f_n^{\varepsilon_n}(z_n - z_0))$. Then we see in the same way as in the above proof that there exists a positive constant c independent of n such that $\|\tilde{\phi}_n\|_{H^1(\mathbb{R})} \le c$ and $|\tilde{\phi}_n| \le ce^{-\theta|y|}$ for $y \in (-1/\varepsilon_n, 1/\varepsilon_n)$ with any $\theta \in (0,1)$ independent of n. Using these facts, we see that

$$|(\phi_n, g_a^{\varepsilon_n}\overline{\varphi})_{L^2(-1,1)}| \le c\|\varphi\|_{H^1(-1,1)} \int_{-1/\varepsilon_n}^{1/\varepsilon_n} |\tilde{\phi}_n|\tilde{a}^{r-1} dy \to 0$$

as $n \to \infty$.

It is easy to see that the other terms also tends to 0 as $n \to \infty$. This is a contradiction. Hence we complete the proof. □

Next we consider the continuity and convergence of $a^r z/\varepsilon h^{s+1}$ in $(H^1(-1,1))'$ as $\varepsilon \to 0$, contained in the term $g_h^\varepsilon z$, where $z \in H^1(-1,1)$. In the following we use a constant $D = qr - (p-1)(s+1)$, which was already given in the last part of Section 1.1.

Lemma 3.4. *For any $z \in H^1(-1,1)$, as $\varepsilon \to 0$, one has*

$$\frac{a^r}{\varepsilon h^{s+1}} z \to z(0)\zeta^{D/(p-1)} \int_{\mathbb{R}} w^r dy\delta \tag{37}$$

in $(H^1(-1,1))'$. Furthermore for each $M > 0$,

$$\Big\|\frac{a^r}{\varepsilon h^{s+1}} z - z(0)\zeta^{D/(p-1)} \int_{\mathbb{R}} w^r dy\delta\Big\|_{(H^1(-1,1))'} \to 0$$

as $\varepsilon \to 0$ uniformly in $\|z\|_{H^1(-1,1)} \le M$.

Proof. By the Lebesgue dominated convergence theorem, it is easy to verify that (37) holds. Let us show the second part of the lemma by contradiction. We suppose that for each $\kappa > 0$, there exist $z_n \in \{z \in H^1(-1,1) \mid \|z\|_{H^1(-1,1)} \le M\}$ and a subsequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ and

$$\Big\|\frac{a^r}{\varepsilon_n h^{s+1}} z_n - z_n(0)\zeta^{D/(p-1)} \int_{\mathbb{R}} w^r dy\delta\Big\|_{(H^1(-1,1))'} \ge \kappa.$$

Since $\|z_n\|_{H^1(-1,1)} \le M$ for any n, there exists $z_0 \in H^1(-1,1)$ such that $z_n \to z_0$ in $(H^1(-1,1))'$ as $n \to \infty$. Here we may replace z_n with an appropriate subsequence if needed (we use the same notation for the subsequence). Since $H^1(-1,1)$ is compactly embedded in $C[-1,1]$, we may assume that $z_n \to z_0$ in $C[-1,1]$. Then we can show in the same argument as in the proof of Lemma 3.3 that

$$\kappa \le \|\frac{a^r}{\varepsilon_n h^{s+1}}(z_n - z_0)\| + \|\frac{a^r}{\varepsilon_n h^{s+1}} z_0 - z_0(0)\zeta^{D/(p-1)} \int_{\mathbb{R}} w^r dy \delta\| \\ + \|(z_n(0) - z_0(0))\zeta^{D/(p-1)} \int_{\mathbb{R}} w^r dy \delta\| \to 0$$

as $n \to \infty$, where we abbreviate $\|\cdot\|_{(H^1(-1,1))'}$ as $\|\cdot\|$ for simplicity. This is a contradiction. □

Applying Lemmas 3.3 and 3.4, we can prove Lemma 3.2. In what follows, we abbreviate $(\cdot,\cdot)_{L^2(-1,1)}$, $\|\cdot\|_{L^2(-1,1)}$, $\|\cdot\|_{H^1(-1,1)}$ and $\|\cdot\|_{(H^1(-1,1))'}$ as $(\cdot,\cdot)$, $\|\cdot\|_{L^2}$, $\|\cdot\|_{H^1}$, and $\|\cdot\|_{(H^1)'}$, respectively, for simplicity.

Proof. For any $z_1, z_2 \in H^1(-1,1)$, we have

$$|B_{\varepsilon,\lambda}(z_1,z_2)| \le \frac{d}{\tau}\|z_1'\|_{L^2}\|z_2'\|_{L^2} + \frac{dl^2+\mu+\tau|\lambda|}{\tau}\|z_1\|_{L^2}\|z_2\|_{L^2} + c\|z_1\|_{H^1}\|z_2\|_{H^1} \\ \le \frac{2}{\tau}\max\{d, dl^2+\mu+\tau|\lambda|\}\|z_1\|_{H^1}\|z_2\|_{H^1},$$

where c is independent of ε, d, λ. Next, for any $z \in H^1(-1,1)$, we have

$$|B_{\varepsilon,\lambda}(z,z)| \ge \frac{d}{\tau}\|z'\|^2_{L^2} + \frac{dl^2+\mu+\tau Re\lambda}{\tau}\|z\|^2_{L^2} - c\|z\|^2_{H^1} \\ \ge \frac{\min\{d, dl^2+\mu\}}{2\tau}\|z\|^2_{H^1}, \tag{38}$$

where c is independent of ε, d, λ. By these two inequalities, we can apply Lax-Milgram's theorem to $B_{\varepsilon,\lambda}$, that is, for each $T \in (H^1(-1,1))'$, there exists $\phi \in H^1(-1,1)$ such that

$$B_{\varepsilon,\lambda}(\phi,\psi) = \langle T,\psi\rangle \quad \text{for any } \psi \in H^1(-1,1)$$

for $\varepsilon > 0$ and $\lambda \in \Lambda_+$. Then we define the operator $K_{\varepsilon,\lambda} : (H^1(-1,1))' \to H^1(-1,1)$ by $K_{\varepsilon,\lambda}T = \phi$, which implies that $K_{\varepsilon,\lambda}$ is well-defined. Furthermore we take $\psi = K_{\varepsilon,\lambda}T$ for the above equation and have an estimate such as

$$\frac{\min\{d, dl^2+\mu\}}{2\tau}\|K_{\varepsilon,\lambda}T\|^2_{H^1} \le |\langle T, K_{\varepsilon,\lambda}T\rangle| \le \|T\|_{(H^1)'}\|K_{\varepsilon,\lambda}T\|_{H^1}.$$

This implies that

$$\|K_{\varepsilon,\lambda}\| \le \frac{2\tau}{\min\{d, dl^2+\mu\}}, \tag{39}$$

where $\|\cdot\|$ is the usual norm for operators.

Secondly we shall show that $K_{\varepsilon,\lambda}$ is continuous with respect to ε and analytic with respect to $\lambda \in \Lambda_+$ in a certain sense. We take any $T \in (H^1(-1,1))'$. Then it holds that for each $\psi \in H^1(-1,1)$, $\varepsilon, \varepsilon' > 0$, $\lambda, \lambda' \in \Lambda_+$,

$$B_{\varepsilon,\lambda}(K_{\varepsilon,\lambda}T,\psi) = \langle T,\psi\rangle = B_{\varepsilon',\lambda'}(K_{\varepsilon',\lambda'}T,\psi).$$

Using this, we obtain

$$B_{\varepsilon',\lambda'}((K_{\varepsilon,\lambda} - K_{\varepsilon',\lambda'})T,\psi) = B_{\varepsilon',\lambda'}(K_{\varepsilon,\lambda}T,\psi) - B_{\varepsilon,\lambda}(K_{\varepsilon,\lambda}T,\psi). \tag{40}$$

Substituting $(K_{\varepsilon,\lambda} - K_{\varepsilon',\lambda'})T$ into ψ and using the inequality (38), we obtain

$$|B_{\varepsilon',\lambda'}((K_{\varepsilon,\lambda} - K_{\varepsilon',\lambda'})T, (K_{\varepsilon,\lambda} - K_{\varepsilon',\lambda'})T)| \ge \frac{\min\{d, dl^2+\mu\}}{2\tau}\|(K_{\varepsilon,\lambda} - K_{\varepsilon',\lambda'})T\|^2_{H^1}.$$

On the other hand, the right side of (40) is estimated from above as

$$\begin{aligned}
&|B_{\varepsilon',\lambda'}(K_{\varepsilon,\lambda}T,\psi)-B_{\varepsilon,\lambda}(K_{\varepsilon,\lambda}T,\psi)|\\
&\leq |((g_a^{\varepsilon'}R_{\varepsilon',\lambda'}(-f_h^{\varepsilon'}\cdot)-g_a^{\varepsilon}R_{\varepsilon,\lambda}(-f_h^{\varepsilon}\cdot))K_{\varepsilon,\lambda}T,\psi)|\\
&\quad +|((\frac{s}{\tau\varepsilon'}\frac{a_{\varepsilon'}^r}{h_{\varepsilon'}^{s+1}}-\frac{s}{\tau\varepsilon}\frac{a_{\varepsilon}^r}{h_{\varepsilon}^{s+1}})K_{\varepsilon,\lambda}T,\psi)|+|\lambda'-\lambda||(K_{\varepsilon,\lambda}T,\psi)|\\
&\leq \Bigg(\|(g_a^{\varepsilon'}R_{\varepsilon',\lambda'}(-f_h^{\varepsilon'}\cdot)-g_a^{\varepsilon}R_{\varepsilon,\lambda}(-f_h^{\varepsilon}\cdot))K_{\varepsilon,\lambda}T\|_{(H^1)'}\\
&\quad +\|(\frac{s}{\tau\varepsilon'}\frac{a_{\varepsilon'}^r}{h_{\varepsilon'}^{s+1}}-\frac{s}{\tau\varepsilon}\frac{a_{\varepsilon}^r}{h_{\varepsilon}^{s+1}})K_{\varepsilon,\lambda}T\|_{(H^1)'}+|\lambda'-\lambda|\|K_{\varepsilon,\lambda}T\|_{H^1}\Bigg)\|\psi\|_{H^1},
\end{aligned}$$

where we write a,h as $a_\varepsilon,h_\varepsilon$ for ε to distinguish $a_\varepsilon,h_\varepsilon$ from $a_{\varepsilon'},h_{\varepsilon'}$. Taking $\psi=(K_{\varepsilon,\lambda}-K_{\varepsilon',\lambda'})T$ in (40), we obtain from the above two inequalities

$$\begin{aligned}
&\|(K_{\varepsilon,\lambda}-K_{\varepsilon',\lambda'})T\|_{H^1}\\
&\leq \frac{2\tau}{\min\{d,dl^2+\mu\}}\{\|((g_a^{\varepsilon'}R_{\varepsilon',\lambda'}(-f_h^{\varepsilon'}\cdot)-g_a^{\varepsilon}R_{\varepsilon,\lambda}(-f_h^{\varepsilon}\cdot))K_{\varepsilon,\lambda}T\|_{(H^1)'}\\
&\quad +\|(\frac{s}{\tau\varepsilon'}\frac{a_{\varepsilon'}^r}{h_{\varepsilon'}^{s+1}}-\frac{s}{\tau\varepsilon}\frac{a_{\varepsilon}^r}{h_{\varepsilon}^{s+1}})K_{\varepsilon,\lambda}T\|_{(H^1)'}+|\lambda'-\lambda|\|K_{\varepsilon,\lambda}T\|_{H^1}\}.
\end{aligned}\tag{41}$$

Here it follows from (39) that $\|K_{\varepsilon,\lambda}T\|_{H^1}\leq 2\tau\|T\|_{(H^1)'}/\min\{d,dl^2+\mu\}$, which implies $\|K_{\varepsilon,\lambda}T\|_{H^1}$ is uniformly bounded in $\varepsilon>0$ and $\lambda\in\Lambda_+$. By this and Lemmas 3.3, 3.4, three terms of the right-hand side of (41) tend to 0 as $\varepsilon'\to\varepsilon$ and $\lambda'\to\lambda$. Hence $K_{\varepsilon,\lambda}$ is continuous with respect to ε and analytic with respect to λ. □

Next let us define $K_{*,\lambda}$. Let $B_{*,\lambda}$ be a bilinear form which is the limiting form of $B_{\varepsilon,\lambda}$ as $\varepsilon\to 0$. From Lemmas 3.3, 3.4, we have

$$\begin{gathered}
(g_a^\varepsilon R_{\varepsilon,\lambda}(-f_h^\varepsilon z_1),z_2)_{L^2(-1,1)}\to F(\lambda)z_1(0)z_2(0),\\
(\frac{a^r}{\varepsilon h^{s+1}}z_1,z_2)_{L^2(-1,1)}\to\zeta^{D/(p-1)}\int_{-\infty}^{\infty}w^r dy z_1(0)z_2(0)
\end{gathered}$$

as $\varepsilon\to 0$. Hence we define $B_{*,\lambda}$ by

$$B_{*,\lambda}(z_1,z_2)=\frac{d}{\tau}(z_1',z_2')+\frac{dl^2+\mu+\tau\lambda}{\tau}(z_1,z_2)+(E-F(\lambda))z_1(0)z_2(0),$$

where $E=s\zeta^{D/(p-1)}\int_{-\infty}^{\infty}w^r dy/\tau$ and we abbreviate $(\cdot,\cdot)_{L^2(-1,1)}$, as $(\cdot,\cdot)$ for simplicity. We apply Lax-Milgram's theorem to $B_{*,\lambda}$ as well as $B_{\varepsilon,\lambda}$ and define $K_{*,\lambda}$.

Lemma 3.5. *For each $T\in(H^1(-1,1))'$, there exists a unique $\phi\in H^1(-1,1)$ such that*

$$B_{*,\lambda}(\phi,\psi)=\langle T,\psi\rangle\quad \textit{for } \psi\in H^1(-1,1),\tag{42}$$

where d is sufficiently large and $\lambda\in\Lambda_+$. Moreover, the operator

$$K_{*,\lambda}:(H^1(-1,1))'\to H^1(-1,1)$$

is well-defined by $\phi=K_{*,\lambda}T$, and is analytic with respect to λ.

Proof. It is easily shown that $B_{*,\lambda}$ satisfies

$$\begin{gathered}
|B_{*,\lambda}(z_1,z_2)|\leq\frac{2}{\tau}\max\{d,dl^2+\mu+\tau|\lambda|\}\|z_1\|_{H^1}\|z_2\|_{H^1},\\
|B_{*,\lambda}(z,z)|\geq\frac{\min\{d,dl^2+\mu\}}{2\tau}\|z\|_{H^1}^2
\end{gathered}\tag{43}$$

for $z_1, z_2, z \in H^1(-1,1)$. Then the proof of Lemma 3.5 is obtained in the same way as in the proof of Lemma 3.2. So, we omit details of the proof. □

As described previously, $B_{*,\lambda}$ is the limiting form of $B_{\varepsilon,\lambda}$ as $\varepsilon \to 0$. Hence it is expected that $K_{\varepsilon,\lambda}$ converges to $K_{*,\lambda}$ as $\varepsilon \to 0$ in a certain sense. Indeed, the following lemma holds.

Lemma 3.6. *Fix $\lambda \in \Lambda_+$ and $T \in (H^1(-1,1))'$. Let $\varepsilon_n > 0$, $\lambda_n \in \Lambda_+$ and $T_n \in (H^1(-1,1))'$ satisfy $\varepsilon_n \to 0$, $\lambda_n \to \lambda$ and $T_n \to T$ in $(H^1(-1,1))'$ as $n \to \infty$, respectively. Then $K_{\varepsilon_n,\lambda_n}T_n$ converges to $K_{*,\lambda}T$ in $H^1(-1,1)$ as $n \to \infty$.*

Proof. By the same argument as in the proof of Lemma 3.2, we obtain

$$\begin{aligned}
&\|K_{\varepsilon_n,\lambda_n}T_n - K_{*,\lambda}T\|_{H^1} \le \frac{2\tau}{\min\{d, dl^2+\mu\}}\big\{\|T_n - T\|_{(H^1)'} \\
&+ \|g_a^{\varepsilon_n} R_{\varepsilon_n,\lambda_n}(-f_h^{\varepsilon_n} K_{\varepsilon_n,\lambda_n}T) - F(\lambda)(K_{*,\lambda}T)(0)\delta\|_{(H^1)'} \\
&+ \frac{s}{\tau}\Big\|\frac{a_{\varepsilon_n}^r}{h_{\varepsilon_n}^{s+1}} K_{\varepsilon_n,\lambda_n}T - \zeta^{D/(p-1)}\int_{\mathbb{R}} w^r dy (K_{*,\lambda}T)(0)\delta\Big\|_{(H^1)'} + |\lambda_n - \lambda| \|K_{\varepsilon_n,\lambda_n}T\|_{H^1}\big\}.
\end{aligned}$$

This completes the proof. □

3.3. **Proof of Theorem 1.1.** Now we are in a position to prove Theorem 1.1. Since $K_{*,\lambda}\delta$ satisfies $B_{*,\lambda}(K_{*,\lambda}\delta, \psi) = \langle \delta, \psi\rangle$ for any $\psi \in H^1(-1,1)$, we have

$$-\frac{d}{\tau}(K_{*,\lambda}\delta)'' + \frac{dl^2+\mu+\tau\lambda}{\tau}K_{*,\lambda}\delta = \{1 + (F-E)(K_{*,\lambda}\delta)(0)\}\delta.$$

Then we have

$$(K_{*,\lambda}\delta)(x) = \frac{\tau}{dl^2+\mu+\tau\lambda}\{1 + (F-E)(K_{*,\lambda}\delta)(0)\}G_{d,\lambda}(x,0), \tag{44}$$

where $G_{d,\lambda}$ is the Green's function with Neumann boundary condition for the following equation

$$-\frac{d}{dl^2+\mu+\tau\lambda}G(\cdot,z)'' + G(\cdot,z) = \delta_z \text{ on } (-1,1),$$

where $z \in (-1,1)$ and δ_z is Dirac's δ-function at z. Here it follows from (44) that $(K_{*,\lambda}\delta)(0)$ must satisfy a compatibility condition so that we can solve (44) with respect to $(K_{*,\lambda}\delta)(0)$ to obtain

$$(K_{*,\lambda}\delta)(0) = \frac{\tau G_{d,\lambda}(0,0)}{dl^2+\mu+\tau\lambda+\tau G_{d,\lambda}(0,0)(E-F)},$$

where $G_{d,\lambda}(0,0)$ is explicitly given by

$$G_{d,\lambda}(0,0) = \frac{\sqrt{dl^2+\mu+\tau\lambda}}{2\sqrt{d}\tanh(\sqrt{dl^2+\mu+\tau\lambda}/\sqrt{d})}.$$

If d is sufficiently large, $(K_{*,\lambda}\delta)(0)$ is much smaller than ξ_0^*. Furthermore, $d(K_{*,\lambda}\delta)/d\lambda$ is also small if d is sufficiently large. Therefore we obtain an unstable eigenvalue λ by the implicit function theorem.

Next we show that (P) has exactly one unstable eigenvalue and it is a real number. Let λ'_ε be an eigenvalue in Λ_+. Then $\lim_{\varepsilon\to 0}\lambda'_\varepsilon$ exists in Λ_+ by (8) and (39). Here we may replace ε with an appropriate subsequence if needed (we use the same notation for the subsequence). Setting $\lambda' = \lim_{\varepsilon\to 0}\lambda'_\varepsilon$, we have $\lambda' = \xi_0^* - c_1c_2(K_{*,\lambda'}\delta)(0)$.

Since $(K_{*,\lambda'}\delta)(0)$ is small, a neighborhood of ξ_0^* contains λ' if d is sufficiently large. If λ' is not equal to λ, we obtain

$$1 = \frac{(K_{*,\lambda}\delta)(0) - (K_{*,\lambda'}\delta)(0)}{\lambda - \lambda'}.$$

This is a contradiction because $d(K_{*,\lambda}\delta)(0)/d\lambda$ is smaller than 1 if d is sufficiently large. Hence we have the uniqueness of a solution of (9) in Λ_+. Since F is a real continuous function with respect to λ if $\lambda \in \Lambda_+$ is real number, the uniqueness of a solution of (9) in Λ_+ implies that λ must be a real number. This completes the proof.

4. **Proof of Lemmas and Theorem 4.3.**

4.1. **Proof of Lemmas 2.1, 2.2.** We first prove Lemma 2.1. In fact, we can show that an unstable eigenvalue of (P) does not approach ξ_0^* as $\varepsilon \to 0$, which implies that Lemma 2.1 holds. Hence it suffices to show the following lemma.

Lemma 4.1. *There is no eigenvalue of* (P) *which approaches* ξ_0^* *as* $\varepsilon \to 0$.

Proof. We show this lemma by contradiction. Suppose there exists an eigenvalue of (P), denoted by λ_ε, such that $\lambda_\varepsilon \to \xi_0^*$ as $\varepsilon \to 0$. Then two cases can occur: the first case is when λ_ε is equal to ξ_0^ε and the second case is when there exists a sequence $\{\varepsilon_n\}$ such that ε_n tends to 0 as $\varepsilon \to 0$ and λ_{ε_n} is not equal to $\xi_0^{\varepsilon_n}$ for each n. We will denote the eigenfunction of (P) by $(\phi_\varepsilon, \eta_\varepsilon)$.

In the first case, ϕ_ε can be expressed as

$$\phi_\varepsilon = \kappa_\varepsilon \varphi_0^\varepsilon + \frac{(-f_h^\varepsilon \eta_\varepsilon, \varphi_1^\varepsilon)_{L^2(-1,1)}}{\xi_1^\varepsilon - \lambda_\varepsilon} \varphi_1^\varepsilon + R_{\varepsilon,\lambda}(-f_h^\varepsilon \eta_\varepsilon), \tag{45}$$

where κ_ε is an arbitrary constant. Furthermore $f_h^\varepsilon \eta_\varepsilon$ must satisfy the solvability condition $(-f_h^\varepsilon \eta_\varepsilon, \varphi_0^\varepsilon)_{L^2(-1,1)} = 0$. Substituting (45) into the second equation of (P) and carrying out the same calculations as in Section 2.1, we have $\langle \delta, K_{*,\xi_0^*}\delta \rangle = 0$. However this contradicts the positive definiteness of $K_{*,\lambda}$ (see (43)). Hence the first case does not occur.

In the second case, we also obtain $\langle \delta, K_{*,\xi_0^*}\delta \rangle = 0$, by carrying out the same calculations as in the first case. Hence the second case does not occur either. This completes the proof. □

Next we prove Lemma 2.2. We denote the solution of (13), (14) by ϕ and suppose that $\|\phi\|_{H^1(-1/\varepsilon,1/\varepsilon)} \le c_0$ for a constant c_0 independent of ε, α. From $\|\phi\|_{H^1(-1/\varepsilon,1/\varepsilon)} \le c_0$, there exists a constant $c > 0$ independent of ε, α such that $\|\phi\|_{L^\infty(-1/\varepsilon,1/\varepsilon)} \le c$ for any $\varepsilon > 0$ by Sobolev's embedding (Theorems 8.6, 8.8 in [1]). Hence in order to prove Lemma 2.2, it suffices to show that ϕ has an exponentially decaying property on $(R, 1/\varepsilon)$, that is, (15) holds on $(R, 1/\varepsilon)$, where R is assumed to be sufficiently large and fixed independently of ε, α.

Since the problem (13) on $(R, 1/\varepsilon)$ is of an inhomogeneous Sturm-Liouville type, it is natural to consider the following homogeneous equation:

$$\phi'' - (\alpha^2 + U(y))\phi = 0, \quad y \in (R, 1/\varepsilon). \tag{46}$$

Taking two linearly independent solutions of (46), we can express ϕ by using them and their Wronskian. To show that ϕ has an exponentially decaying property, it suffices to show that they have such property. In what follows, if we use a constant c, we suppose that it is independent of ε, R, α.

We first consider the problem (46) with the following boundary conditions:

$$\phi(R) = 1, \quad \phi'(1/\varepsilon) = 0. \tag{47}$$

By using (12), it is not difficult to show that there exists a solution of (46), (47), denoted by ϕ_-. Equation (46) can be regarded as a perturbed problem for

$$\phi'' - \alpha^2\phi = 0, \quad y \in (R, 1/\varepsilon). \tag{48}$$

A unique solution of (48), (47), denoted by ϕ_0, can be explicitly written as

$$\phi_0(y) = \frac{1}{1+e^{-2\alpha(1/\varepsilon - R)}}(e^{-\alpha(2/\varepsilon - y - R)} + e^{-\alpha(y-R)})$$

and satisfies

$$\left|\frac{d^k}{dy^k}\phi_0\right| \leq c|\alpha|^k e^{-\alpha_1(y-R)} \tag{49}$$

for $y \in (R, 1/\varepsilon)$ and $k = 0, 1, 2$, where $\alpha = \alpha_1 + \alpha_2 i$ and α_1, α_2 are real constants. We consider that the solution of (46), (47) also satisfies (49). Indeed, the following lemma holds.

Lemma 4.2. *Let ϕ_- be the solution of (46), (47). Suppose that $R > 0$ is sufficiently large and fixed independently of ε, α. Then there exists $\varepsilon_0 > 0$ such that*

$$\left|\frac{d^k}{dy^k}\phi_-\right| \leq c|\alpha|^k e^{-\alpha_1(y-R)} \tag{50}$$

for any $\varepsilon < \varepsilon_0$, $k = 0, 1, 2$ and $y \in (R, 1/\varepsilon)$. Moreover, one has

$$|\phi_-'(R) + \alpha| \leq c|\alpha|e^{-uR} \tag{51}$$

and

$$|\phi_-(1/\varepsilon) - 2e^{-\alpha(1/\varepsilon - R)}| \leq ce^{-\alpha_1(1/\varepsilon - R) - uR/2}. \tag{52}$$

In order to prove Lemma 4.2, we rewrite (46) to the following first order ordinary differential system:

$$\Phi' = A\Phi + U(y)K_0\Phi, \tag{53}$$

where $\Phi = (\phi_-, \phi_-')^t$ and

$$A = \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix}, \quad K_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let

$$P = \begin{pmatrix} 1 & 1 \\ \alpha & -\alpha \end{pmatrix}.$$

Then

$$P^{-1} = \frac{1}{2\alpha}\begin{pmatrix} \alpha & 1 \\ \alpha & -1 \end{pmatrix}, \quad P^{-1}AP = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}.$$

Then $\Psi = e^{-\alpha y}P^{-1}\Phi(\equiv (\Psi_1, \Psi_2)^t)$ satisfies an ordinary differential system

$$\Psi' = B\Psi + U(y)K\Psi, \tag{54}$$

where $B = P^{-1}AP - \alpha I$ and $K = P^{-1}K_0P$ with the identity I on $\mathbb{C}^2$. For each matrix M on $\mathbb{C}^2$, we put

$$\|M\| \equiv \sup_{v\in\mathbb{C}^2}\frac{|Mv|}{|v|}.$$

By using these notation and definition, we shall show Lemma 4.2.

Proof. By (54), we have the integral equation

$$\Psi(y) = e^{B(y-R)}\Psi(R) + \int_R^y e^{B(y-\zeta)}U(\zeta)K\Psi(\zeta)d\zeta. \tag{55}$$

Here, by (55), we obtain

$$\left|\int_R^y e^{B(y-\zeta)}U(y)K\Psi(\zeta)d\zeta\right| \le c \max_{\zeta\in[R,1/\varepsilon]}|\Psi(\zeta)| \int_R^y e^{-u\zeta}d\zeta \le ce^{-uR} \max_{\zeta\in[R,1/\varepsilon]}|\Psi(\zeta)|.$$

By assuming that R is so large that $ce^{-uR} \le 1/2$, it follows from (55) that

$$|\Psi(y)| \le |\Psi(R)| + \frac{1}{2}\max_{\zeta\in[R,1/\varepsilon]}|\Psi(\zeta)|.$$

Hence we have $\max_{\zeta\in[R,1/\varepsilon]}|\Psi(\zeta)| \le 2|\Psi(R)|$ and

$$|\Psi_1(y) - \Psi_1(R)| \le c|\Psi(R)|e^{-uR},$$
$$|\Psi_2(y) - e^{-2\alpha(y-R)}\Psi_2(R)| \le c|\Psi(R)|e^{-uy}.$$

Setting $y = 1/\varepsilon$ in these two inequalities, we have

$$|\alpha\phi_-(1/\varepsilon)e^{-\alpha/\varepsilon} - e^{-\alpha R}(\alpha + \phi_-'(R))| \le c\sqrt{|\alpha|^2 + |\phi_-'(R)|^2}e^{-uR-\alpha_1 R},$$
$$|\alpha\phi_-(1/\varepsilon)e^{-\alpha/\varepsilon} - e^{\alpha(-2/\varepsilon+R)}(\alpha - \phi_-'(R))| \le c\sqrt{|\alpha|^2 + |\phi_-'(R)|^2}e^{-u/\varepsilon-\alpha_1 R}.$$

Then we have

$$|\alpha + \phi_-'(R) - e^{-2\alpha(1/\varepsilon-R)}(\alpha - \phi_-'(R))| \le c\sqrt{|\alpha|^2 + |\phi_-'(R)|^2}e^{-uR}, \tag{56}$$

which implies $|\phi_-'(R)| \le 2|\alpha|$ and

$$|\alpha + \phi_-'(R)| \le c|\alpha|e^{-uR}$$

if R is large and ε is small. Hence (51) holds. Furthermore we have

$$|\Psi(1/\varepsilon)| \le c_1 e^{-u/\varepsilon-\alpha_1 R}, \tag{57}$$

where c_1 is a positive constant independent of ε, R, α.

To prove (50) and (52) we next consider the following integral equation:

$$\Psi(y) = e^{B(y-1/\varepsilon)}\Psi(1/\varepsilon) - \int_y^{1/\varepsilon} e^{B(y-\zeta)}U(\zeta)K\Psi(\zeta)d\zeta. \tag{58}$$

Now we define a positive constant c_2 independent of ε, R by $c_2 = 4c_u/|\alpha|(2\alpha_1 - u)$. Using (57) and (58), we obtain

$$\begin{aligned}|\Psi(y)| &\le e^{-2\alpha_1(y-1/\varepsilon)}|\Psi(1/\varepsilon)| + 2c_u\|K\||\Psi(R)| \int_y^{1/\varepsilon} e^{-2\alpha_1(y-\zeta)-u\zeta}d\zeta \\ &\le (c_1 + c_2)e^{\alpha_1(-2y+2/\varepsilon-R)-u/\varepsilon}.\end{aligned} \tag{59}$$

Note that $\|K\| = 1/|\alpha|$ and $|\Psi(R)| \le 2e^{-\alpha_1 R}$. Then we have

$$\begin{aligned}&|\Psi(y) - e^{B(y-1/\varepsilon)}\Psi(1/\varepsilon)| \\ &\le c_u\|K\|(c_1 + c_2)\int_y^{1/\varepsilon} e^{-2\alpha_1(y-\zeta)-u\zeta+\alpha_1(-2\zeta+2/\varepsilon-R)-u/\varepsilon}d\zeta \\ &\le \frac{c_u(c_1 + c_2)}{|\alpha|u}e^{\alpha_1(-2y+2/\varepsilon-R)-uy-u/\varepsilon}.\end{aligned} \tag{60}$$

Taking $y = R$ in (60), we have

$$\begin{aligned}|\Psi_2(1/\varepsilon)| &\le |\Psi_2(R)|e^{2\alpha_1(R-1/\varepsilon)} + \frac{c_u(c_1+c_2)}{|\alpha|u}e^{-\alpha_1 R - uR - u/\varepsilon} \\ &\le 2e^{\alpha_1(-2/\varepsilon+R)} + \frac{c_u(c_1+c_2)}{|\alpha|u}e^{-\alpha_1 R - uR - u/\varepsilon} \\ &\le \left(2 + \frac{c_u(c_1+c_2)}{|\alpha|u}\right)e^{-\alpha_1 R - uR - u/\varepsilon}.\end{aligned}$$

Note that $2\alpha_1(1/\varepsilon - R) > u(1/\varepsilon + R)$ if ε is small. Here we assume that R is so large that

$$\left(2 + \frac{c_u(c_1+c_2)}{|\alpha|u}\right)e^{-uR/2} \le \min\{\frac{c_1}{\sqrt{2}}, c_2\}. \tag{61}$$

Note that R can be given independently of α, ε. Then we see that

$$|\Psi(1/\varepsilon)| = \sqrt{2}|\Psi_2(1/\varepsilon)| \le c_1 e^{-u/\varepsilon - \alpha_1 R - uR/2}. \tag{62}$$

Next we shall show that

$$|\Psi(1/\varepsilon)| \le c_1 e^{-u/\varepsilon - \alpha_1 R - muR/2}, \tag{63}$$

$$|\Psi(y)| \le (c_1+c_2)e^{\alpha_1(-2y+2/\varepsilon-R) - u/\varepsilon - (m-1)uR/2} \tag{64}$$

if the integer m satisfies

$$2\alpha_1(1/\varepsilon - R) > u/\varepsilon + (m+1)uR/2$$

by using the principle of induction. The case of $m = 1$ was already proved in (59), (62). Next we suppose that (63), (64) hold for m. Then it follows from (58), (61) that

$$\begin{aligned}|\Psi(y)| &\le |\Psi(1/\varepsilon)|e^{-2\alpha_1(y-1/\varepsilon)} + c_u\|K\| \int_y^{1/\varepsilon} e^{-2\alpha_1(y-\zeta) - u\zeta}|\Psi(\zeta)|d\zeta \\ &\le c_1 e^{-2\alpha_1(y-1/\varepsilon) - u/\varepsilon - \alpha_1 R - muR/2} \\ &\quad + \frac{c_u(c_1+c_2)}{|\alpha|u}e^{-2\alpha_1 y + \alpha_1(2/\varepsilon - R) - u/\varepsilon - (m+1)uR/2} \\ &\le (c_1+c_2)e^{-2\alpha_1(y-1/\varepsilon) - u/\varepsilon - \alpha_1 R - muR/2}.\end{aligned}$$

Then we obtain

$$\begin{aligned}&|\Psi(y) - e^{B(y-1/\varepsilon)}\Psi(1/\varepsilon)| \\ &\le c_u\|K\|(c_1+c_2)\int_y^{1/\varepsilon} e^{-2\alpha_1(y-\zeta) - u\zeta - 2\alpha_1(\zeta - 1/\varepsilon) - u/\varepsilon - \alpha_1 R - muR/2}d\zeta \\ &\le \frac{c_u(c_1+c_2)}{|\alpha|u}e^{\alpha_1(-2y+2/\varepsilon-R) - u/\varepsilon - (m+2)uR/2}.\end{aligned} \tag{65}$$

Setting $y = R$ in (65), we have

$$\begin{aligned}|\Psi_2(1/\varepsilon)| &\le |\Psi_2(R)|e^{2\alpha_1(R-1/\varepsilon)} + \frac{c_u(c_1+c_2)}{|\alpha|u}e^{-\alpha_1 R - u/\varepsilon - (m+2)uR/2} \\ &\le 2e^{-\alpha_1(2/\varepsilon - R)} + \frac{c_u(c_1+c_2)}{|\alpha|u}e^{-\alpha_1 R - u/\varepsilon - (m+2)uR/2}.\end{aligned}$$

If $2\alpha_1(1/\varepsilon - R) > u/\varepsilon + (m+2)uR/2$, we obtain

$$|\Psi_2(1/\varepsilon)| \le (2 + \frac{c_u(c_1+c_2)}{|\alpha|u})e^{-\alpha_1 R - u/\varepsilon - (m+2)uR/2} \le \frac{c_1}{\sqrt{2}}e^{-\alpha_1 R - u/\varepsilon - (m+1)uR/2}.$$

This implies that (63) and (64) hold for $m+1$.

From the above argument, there exists an integer n such that

$$u/\varepsilon + (n+2)uR/2 \geq 2\alpha_1(1/\varepsilon - R) > u/\varepsilon + (n+1)uR/2 \tag{66}$$

and

$$|\Psi(1/\varepsilon)| \leq c_1 e^{-u/\varepsilon - \alpha_1 R - nuR/2},$$
$$|\Psi(y)| \leq (c_1 + c_2) e^{\alpha_1(-2y+2/\varepsilon-R)-u/\varepsilon-(n-1)uR/2}.$$

By (66), we have $-nuR/2 \leq -2\alpha_1(1/\varepsilon - R) + u/\varepsilon + uR$ so that

$$|\Psi(1/\varepsilon)| \leq c_1 e^{\alpha_1(-2/\varepsilon+R)+uR}, \quad |\Psi(y)| \leq (c_1+c_2)e^{\alpha_1(-2y+R)+3uR/2}.$$

Using these inequalities and carrying out the same calculation as above, we obtain

$$|\Psi(y)| \leq ce^{\alpha_1(-2y+R)}, \quad |\Psi(y) - e^{B(y-1/\varepsilon)}\Psi(1/\varepsilon)| \leq ce^{\alpha_1(-2y+R)-uR/2}.$$

Hence

$$|\phi_-(y)| \leq ce^{-\alpha_1(y-R)}, \quad |\phi_-(1/\varepsilon) - 2e^{-\alpha(1/\varepsilon-R)}| \leq ce^{-\alpha_1(1/\varepsilon-R)-uR/2}.$$

Since it is easy to verify that (50) holds $k = 1, 2$, the proof is completed. □

Since (13), (14) are of Sturm-Liouville type, the solution ϕ of (13), (14), can be represented by two linearly independent solutions of (46). We obtained one solution ϕ_- in Lemma 4.2. We take another solution of (46) satisfying

$$\phi'(R) = 0, \quad \phi(1/\varepsilon) = 1,$$

denoted by ϕ_+. Then ϕ_+ and ϕ_- are linearly independent owing to their boundary conditions. We can show in the same way as in the proof of Lemma 4.2 that for sufficiently large $R > 0$, there exists $\varepsilon_0 > 0$ such that

$$\left|\frac{d^k}{dy^k}\phi_+\right| \leq c|\alpha|^k e^{-\alpha_1(1/\varepsilon-y)} \tag{67}$$

for any $y \in (R, 1/\varepsilon)$, $\varepsilon < \varepsilon_0$ and $k = 0, 1, 2$, and

$$|\phi_+'(1/\varepsilon) - \alpha| \leq c|\alpha|e^{-uR}, \tag{68}$$

$$|\phi_+(R) - 2e^{-\alpha(1/\varepsilon-R)}| \leq ce^{-\alpha_1(1/\varepsilon-R)-uR/2}. \tag{69}$$

By ϕ_+ and ϕ_-, ϕ can be expressed as

$$\begin{aligned}\phi(y) &= c_+\phi_+(y) + c_-\phi_-(y) \\ &\quad + \frac{1}{W}\left(\phi_-(y)\int_R^y \phi_+(\zeta)V(\zeta)d\zeta + \phi_+(y)\int_y^{1/\varepsilon}\phi_-(\zeta)V(\zeta)d\zeta\right),\end{aligned} \tag{70}$$

where c_+, c_- are constants and W is defined by

$$W = \phi_-'(R)\phi_+(R) - \phi_-(R)\phi_+'(R) = \phi_-'(R)\phi_+(R),$$

by using $\phi_+'(R) = 0$. Here we differentiate the both sides of (70) with respect to y and substitute $y = 1/\varepsilon$ into the resulting equation. Then, using $\phi_-'(1/\varepsilon) = 0$, we have

$$\begin{aligned}0 &= c_+\phi_+'(1/\varepsilon) + c_-\phi_-'(1/\varepsilon) \\ &\quad + \frac{1}{W}\left(\phi_-'(1/\varepsilon)\int_R^{1/\varepsilon}\phi_+(\zeta)V(\zeta)d\zeta + \phi_+'(1/\varepsilon)\int_{1/\varepsilon}^{1/\varepsilon}\phi_-(\zeta)V(\zeta)d\zeta\right) \\ &= c_+\phi_+'(1/\varepsilon).\end{aligned}$$

Then c_+ must be 0, because $\phi_+'(1/\varepsilon) \neq 0$ due to (68). Finally ϕ can be expressed as

$$\phi(y) = c_-\phi_-(y) + \frac{1}{W}\left(\phi_-(y)\int_R^y \phi_+(\zeta)V(\zeta)d\zeta + \phi_+(y)\int_y^{1/\varepsilon} \phi_-(\zeta)V(\zeta)d\zeta\right). \quad (71)$$

Here we estimate W from below as

$$|W| \geq \frac{1}{2}|\alpha|e^{-\alpha_1(1/\varepsilon - R)},$$

by using (51) and (69). The second and third terms are estimated as

$$\left|\frac{1}{W}\phi_-(y)\int_R^y \phi_+(\zeta)V(\zeta)d\zeta\right| \leq \frac{c}{|\alpha|}e^{-\alpha_1 y}\int_R^y e^{(\alpha_1 - v)\zeta}d\zeta \leq \frac{c}{|\alpha|}e^{-\rho y}, \quad (72)$$

where $\rho = \min\{\gamma, v\}$ and

$$\left|\frac{1}{W}\phi_+(y)\int_y^{1/\varepsilon} \phi_-(\zeta)V(\zeta)d\zeta\right| \leq \frac{c}{|\alpha|}e^{\alpha_1 y}\int_y^{1/\varepsilon} e^{-(\alpha_1 + v)\zeta}d\zeta \leq \frac{c}{|\alpha|}e^{-vy}. \quad (73)$$

If $\alpha_1 < v$ and α_1 is independent of ε, it is easy to prove that we can take $\rho = \alpha_1$ in (72). Hence if c_- is uniformly bounded in $\varepsilon > 0$, the proof of Lemma 2.2 is completed by using (50), (72) and (73). In order to prove that c_- is uniformly bounded in ε, we recall $\|\phi\|_{L^\infty(-1/\varepsilon,1/\varepsilon)} \leq \tilde{c}_0$ for a constant $\tilde{c}_0$ independent of ε, α, R because of $\|\phi\|_{H^1(-1/\varepsilon,1/\varepsilon)} \leq c_0$. Substituting $y = R$ into (71), we have $\tilde{c}_0 \geq |c_-| - ce^{-\rho R}$ thanks to $\phi_-(R) = 1$, which implies that $|c_-| \leq 2\tilde{c}_0$ if R is sufficiently large. Therefore (15) holds. This completes the proof of Lemma 2.2.

4.2. **Existence of a single-spike solution of (1).** In this subsection we prove that there exists a solution of (1) satisfying (2), (3). In fact, the existence of a solution was pointed out in [7] and [25]. However it seems unclear whether the solution given in [7] or [25] has these properties. Then we give the rigorous proof of the existence of a solution satisfying these properties.

Theorem 4.3. *Assume that $r > 1$. There exists a solution of (1), denoted by (a, h), such that it has the properties (2) and (3).*

Remark 2. The assumption $r > 1$ may be unnecessary and we may be able to prove the existence of a single-spike solution for $r > 0$ though we need it in our proof of Theorem 4.3.

In what follows, we assume that $\mu = 1$ in (1) without loss of generality by some scaling. Then we can take $c = 1$ in (2). We use the useful notations such as

$$L_e^2(I) = \{\phi \in L^2(I) \mid \phi(x) = \phi(-x)\}, \quad H_e^2(I) = H^2(I) \cap L_e^2(I),$$

and set $\|\cdot\|_{L^2} = \|\cdot\|_{L^2(-1/\varepsilon,1/\varepsilon)}$ and $\|\cdot\|_{H^2} = \|\cdot\|_{H^2(-1/\varepsilon,1/\varepsilon)}$ for simplicity. We need several lemmas to prove Theorem 4.3. We shall prove some of them after the proof of Theorem 4.3.

First of all, we need the existence of a unique solution of the problem

$$\begin{cases} dT'' - T + \dfrac{A}{T^s} = 0, & x \in (-1, 1), \\ T' = 0, & x = \pm 1 \end{cases} \quad (74)$$

for any nonnegative function $A \in H_e^2(-1, 1)$ (see Lemmas 4.9, 4.11). We denote the solution of the above equation by $T = T[A]$. Here we set $A = (w_\varepsilon(x/\varepsilon) + \phi(x/\varepsilon))^r/\varepsilon$

define $\tilde{T}[w_\varepsilon + \phi](y) = T[A](\varepsilon y)$ for $\phi \in B_{\sqrt{\varepsilon}}(0; H_e^2)$, where $B_{\sqrt{\varepsilon}}(0; H_e^2)$ is a closed ball in $H_e^2(-1/\varepsilon, 1/\varepsilon)$ with center 0, radius $\sqrt{\varepsilon}$, that is,

$$B_{\sqrt{\varepsilon}}(0; H_e^2) = \{\phi \in H_e^2(-1/\varepsilon, 1/\varepsilon) \mid \|\phi\|_{H^2} \le \sqrt{\varepsilon}\},$$

$w_\varepsilon(y) = \zeta^{q/(p-1)}\chi_\varepsilon(y)w(y)$, and χ_ε is a cut-off function, belonging to C^∞, which satisfies $0 \le \chi_\varepsilon \le 1$ and

$$\chi_\varepsilon(y) = \begin{cases} 1, & |y| \le \dfrac{1}{4\varepsilon}, \\ 0, & \dfrac{1}{2\varepsilon} \le |y| \le \dfrac{1}{\varepsilon}. \end{cases}$$

Next we consider a problem

$$\begin{cases} S[A] \equiv A'' - A + \dfrac{A^p}{\tilde{T}[A]^q} = 0, & y \in \ (-1/\varepsilon, 1/\varepsilon), \\ A' = 0, & y = \pm 1/\varepsilon. \end{cases}$$

To prove Theorem 4.3, we find a solution $\phi \in B_{\sqrt{\varepsilon}}(0; H_e^2)$ such as $S[w_\varepsilon + \phi] = 0$ with Neumann boundary condition. Now we set a linear operator $R_\varepsilon : L_e^2(-1/\varepsilon, 1/\varepsilon) \to (-\infty, \infty)$ such as

$$R_\varepsilon \phi = \frac{1}{1+s}\int_{-1/\varepsilon}^{1/\varepsilon} G(0,0)\frac{r w_\varepsilon^{r-1}}{\tilde{T}[w_\varepsilon]^s}\phi dz,$$

where $G = G(x,z)$ is the Green's function of a problem

$$\begin{cases} -d\phi'' + \phi = \delta_z, & x \in (-1,1), \\ \phi' = 0, & x = \pm 1 \end{cases}$$

and can be explicitly written by

$$G(x,z) = \begin{cases} \dfrac{1}{\sqrt{d}\sinh(2/\sqrt{d})}\cosh\left[\dfrac{1+x}{\sqrt{d}}\right]\cosh\left[\dfrac{1-z}{\sqrt{d}}\right], & -1 < x < z, \\ \dfrac{1}{\sqrt{d}\sinh(2/\sqrt{d})}\cosh\left[\dfrac{1-x}{\sqrt{d}}\right]\cosh\left[\dfrac{1+z}{\sqrt{d}}\right], & z < x < 1. \end{cases}$$

Here δ_z is the Dirac's δ-function at z. Note that G is a Lipschitz function, that is, there is a constant $c > 0$ such that

$$|G(x_1, z_1) - G(x_2, z_2)| \le c(|x_1 - x_2| + |z_1 - z_2|) \tag{75}$$

for $x_1, x_2, z_1, z_2 \in [-1, 1]$. In fact, it seems that R_ε is the linearized operator of $T[A]$ with respect to $A = w_\varepsilon$ in the following sense.

Lemma 4.4. *For any $\delta > 0$ and $0 < \alpha < 1$, there exists $\varepsilon_0 > 0$ such that*

$$\|e^{-\alpha|y|}(\tilde{T}[w_\varepsilon + \phi_1] - \tilde{T}[w_\varepsilon + \phi_2] - R_\varepsilon(\phi_1 - \phi_2))\|_{L^2} \le \delta\|\phi_1 - \phi_2\|_{L^2}, \tag{76}$$

for $\varepsilon < \varepsilon_0$ and $\phi_1, \phi_2 \in B_{\sqrt{\varepsilon}}(0; H_e^2)$ satisfying $|\phi_1|, |\phi_2| \le ce^{-\alpha|y|}$ for a constant $c > 0$ independent of ε, δ, and

$$\|e^{-\alpha|y|}(\tilde{T}[w_\varepsilon] - \zeta)\|_{L^2} \le c\varepsilon, \tag{77}$$

for $\varepsilon < \varepsilon_0$.

Proof. Throughout this proof, we abbreviate $\int_{-1/\varepsilon}^{1/\varepsilon}$ as $\int$ for simplicity. Since the solution $T[A]$ of (74) satisfies

$$T[A](x) = \int_{-1}^{1} G(x,z)\frac{A}{T[A]^s}dz,$$

we have

$$\begin{aligned}&\tilde{T}[w_\varepsilon+\phi_1](y)-\tilde{T}[w_\varepsilon+\phi_2](y)\\&=\int G(\varepsilon y,\varepsilon z)\frac{(w_\varepsilon+\phi_1)^r}{\tilde{T}[w_\varepsilon+\phi_1]^s}dz-\int G(\varepsilon y,\varepsilon z)\frac{(w_\varepsilon+\phi_2)^r}{\tilde{T}[w_\varepsilon+\phi_2]^s}dz\\&=\int(f_1(\varepsilon,y,z)+g_1(\varepsilon,z))(\phi_1-\phi_2)dz\\&\quad-s\int(f_2(\varepsilon,y,z)+g_2(\varepsilon,z))(\tilde{T}[w_\varepsilon+\phi_1]-\tilde{T}[w_\varepsilon+\phi_2])dz,\end{aligned}$$

where $f_1(\varepsilon,y,z), f_2(\varepsilon,y,z), g_1(\varepsilon,z)$ and $g_2(\varepsilon,z)$ are defined by

$$f_1(\varepsilon,y,z)=(G(\varepsilon y,\varepsilon z)-G(0,0))\frac{r(w_\varepsilon+(1-\kappa)\phi_1+\kappa\phi_2)^{r-1}}{\tilde{T}[w_\varepsilon+\phi_1]^s},$$

$$f_2(\varepsilon,y,z)=(G(\varepsilon y,\varepsilon z)-G(0,0))\frac{(w_\varepsilon+\phi_2)^r}{(\tilde{T}[w_\varepsilon+\phi_1]+\theta(\tilde{T}[w_\varepsilon+\phi_2]-\tilde{T}[w_\varepsilon+\phi_1]))^{s+1}},$$

$$g_1(\varepsilon,z)=G(0,0)\frac{r(w_\varepsilon+(1-\kappa)\phi_1+\kappa\phi_2)^{r-1}}{\tilde{T}[w_\varepsilon+\phi_1]^s},$$

and

$$g_2(\varepsilon,z)=G(0,0)\frac{(w_\varepsilon+\phi_2)^r}{(\tilde{T}[w_\varepsilon+\phi_1]+\theta(\tilde{T}[w_\varepsilon+\phi_2]-\tilde{T}[w_\varepsilon+\phi_1]))^{s+1}}$$

for some $0\le\kappa\le 1$ and $0\le\theta\le 1$. Note that f_1, f_2, g_1 and g_2 satisfy

$$\begin{aligned}|f_1(\varepsilon,y,z)|&\le c\varepsilon(|y|+|z|)e^{-\alpha(r-1)|z|}, & g_1(\varepsilon,z)&\le ce^{-\alpha(r-1)|z|},\\|f_2(\varepsilon,y,z)|&\le c\varepsilon(|y|+|z|)e^{-\alpha r|z|}, & g_2(\varepsilon,z)&\le ce^{-\alpha r|z|}.\end{aligned}\tag{78}$$

Here we used (89), which will be proved later. Then we have

$$\begin{aligned}&\int g_2(\varepsilon,z)(\tilde{T}[w_\varepsilon+\phi_1]-\tilde{T}[w_\varepsilon+\phi_2])dz\\&=\int g_2(\varepsilon,z)\Big\{\int(f_1(\varepsilon,z,z_1)+g_1(\varepsilon,z_1))(\phi_1-\phi_2)dz_1\\&\quad-s\int(f_2(\varepsilon,z,z_1)+g_2(\varepsilon,z_1))(\tilde{T}[w_\varepsilon+\phi_1]-\tilde{T}[w_\varepsilon+\phi_2])dz_1\Big\}dz\\&=\int g_2(\varepsilon,z)\Big(\int f_1(\varepsilon,z,z_1)(\phi_1-\phi_2)dz_1\Big)dz+\Big(\int g_2dz\Big)\Big(\int g_1(\phi_1-\phi_2)dz\Big)\\&\quad-s\int g_2(\varepsilon,z)\Big(\int f_2(\varepsilon,z,z_1)(\tilde{T}[w_\varepsilon+\phi_1]-\tilde{T}[w_\varepsilon+\phi_2])dz_1\Big)dz\\&\quad-s\Big(\int g_2dz\Big)\Big(\int g_2(\tilde{T}[w_\varepsilon+\phi_1]-\tilde{T}[w_\varepsilon+\phi_2])dz\Big),\end{aligned}$$

from which we obtain

$$\begin{aligned}\int g_2(\varepsilon,z)(\tilde{T}[w_\varepsilon+\phi_1]-\tilde{T}[w_\varepsilon+\phi_2])dz&=\frac{\int g_2dz}{1+s\int g_2dz}\Big(\int g_1(\phi_1-\phi_2)dz\Big)\\&\quad+\frac{1}{1+s\int g_2dz}\int g_2(\varepsilon,z)\Big(\int f_1(\varepsilon,z,z_1)(\phi_1-\phi_2)dz_1\Big)dz\\&\quad-\frac{s}{1+s\int g_2dz}\int g_2(\varepsilon,z)\Big(\int f_2(\varepsilon,z,z_1)(\tilde{T}[w_\varepsilon+\phi_1]-\tilde{T}[w_\varepsilon+\phi_2])dz_1\Big)dz.\end{aligned}$$

So we have

$$\begin{aligned}
&\tilde{T}[w_\varepsilon+\phi_1](y)-\tilde{T}[w_\varepsilon+\phi_2](y)\\
&=\int f_1(\varepsilon,y,z)(\phi_1-\phi_2)dz+\frac{1}{1+s\int g_2dz}\int g_1(\phi_1-\phi_2)dz\\
&\quad-\frac{s}{1+s\int g_2dz}\int g_2(\varepsilon,z)\bigg(\int f_1(\varepsilon,z,z_1)(\phi_1-\phi_2)dz_1\bigg)dz\\
&\quad+\frac{s^2}{1+s\int g_2dz}\int g_2(\varepsilon,z)\bigg(\int f_2(\varepsilon,z,z_1)(\tilde{T}[w_\varepsilon+\phi_1]-\tilde{T}[w_\varepsilon+\phi_2])dz_1\bigg)dz\\
&\quad-s\int f_2(\varepsilon,y,z)(\tilde{T}[w_\varepsilon+\phi_1]-\tilde{T}[w_\varepsilon+\phi_2])dz.
\end{aligned}$$

Now we define an operator $K: L^2_e(-1/\varepsilon,1/\varepsilon)\to L^2_e(-1/\varepsilon,1/\varepsilon)$ by

$$\begin{aligned}
K\psi=&\frac{s^2e^{-\alpha|y|}}{1+s\int g_2dz}\int g_2(\varepsilon,z)\bigg(\int f_2(\varepsilon,z,z_1)e^{\alpha|z_1|}\psi dz_1\bigg)dz\\
&-se^{-\alpha|y|}\int f_2(\varepsilon,y,z)e^{\alpha|z|}\psi dz.
\end{aligned}$$

From (78), we have $\|K\psi\|_{L^2}\le c\varepsilon\|\psi\|_{L^2}$, which implies that $I-K$ has the invertible operator $(I-K)^{-1}$ with $\|(I-K)^{-1}\|\le c$, where $\|\cdot\|$ is the usual norm for operators. So we have

$$\begin{aligned}
&e^{-\alpha|y|}(\tilde{T}[w_\varepsilon+\phi_1](y)-\tilde{T}[w_\varepsilon+\phi_2](y))\\
&=\frac{e^{-\alpha|y|}}{1+s\int g_2dz}\int g_1(\phi_1-\phi_2)dz\\
&+(I-K)^{-1}\bigg\{K\frac{e^{-\alpha|y|}}{1+s\int g_2dz}\int g_1(\phi_1-\phi_2)dz+e^{-\alpha|y|}\int f_1(\varepsilon,y,z)(\phi_1-\phi_2)dz\\
&-\frac{se^{-\alpha|y|}}{1+s\int g_2dz}\int g_2(\varepsilon,z)\bigg(\int f_1(\varepsilon,z,z_1)(\phi_1-\phi_2)dz_1\bigg)dz\bigg\}.
\end{aligned}$$

Using (78) again and $w_\varepsilon\le ce^{-|y|}$, we have

$$\begin{aligned}
&\|e^{-\alpha|y|}(\tilde{T}[w_\varepsilon+\phi_1]-\tilde{T}[w_\varepsilon+\phi_2]-R_\varepsilon(\phi_1-\phi_2))\|_{L^2}\\
&\le c\|\phi_1-\phi_2\|_{L^2}\\
&\cdot\bigg(\varepsilon+e^{-\alpha(r-1)R}+\|e^{-\alpha|y|}(\tilde{T}[w_\varepsilon+\phi_1]-\tilde{T}[w_\varepsilon])\|_{L^2}+\bigg|\int g_2dz-1\bigg|\bigg),
\end{aligned}\tag{79}$$

for sufficiently large $R>0$. Because of (78), (79) and $|R_\varepsilon\phi|\le c\|\phi\|_{L^2}$, we have

$$\|e^{-\alpha|y|}(\tilde{T}[w_\varepsilon+\phi_1]-\tilde{T}[w_\varepsilon+\phi_2])\|_{L^2}\le c\|\phi_1-\phi_2\|_{L^2}.\tag{80}$$

Similarly, we can also prove (77) because of $\zeta=\int_{-\infty}^{\infty}G(0,0)\zeta^{\frac{qr}{p-1}}w^r/\zeta^s dz$. Then, because of $\int_{-\infty}^{\infty}G(0,0)\ (\zeta^{\frac{q}{p-1}}w)^r/\zeta^{s+1}dz=1$, $|\phi_2|\le ce^{-\alpha|y|}$ and (80), we have

$$\begin{aligned}
\bigg|\int g_2dz-1\bigg|&\le c(\|e^{-\alpha|y|}(\tilde{T}[w_\varepsilon+\phi_1]-\tilde{T}[w_\varepsilon])\|_{L^2}\\
&\quad+\|e^{-\alpha|y|}(\tilde{T}[w_\varepsilon+\phi_2]-\tilde{T}[w_\varepsilon])\|_{L^2}+\|e^{-\alpha|y|}(\tilde{T}[w_\varepsilon]-\zeta)\|_{L^2}+\varepsilon)\\
&\le c\sqrt{\varepsilon}.
\end{aligned}$$

Hence it follows from (79) that

$$\|e^{-\alpha|y|}(\tilde{T}[w_\varepsilon+\phi_1]-\tilde{T}[w_\varepsilon+\phi_2]-R_\varepsilon(\phi_1-\phi_2))\|_{L^2}\le\delta\|\phi_1-\phi_2\|_{L^2}.$$

Thus we complete the proof. □

We define another linear operator $L_\varepsilon : D(L_\varepsilon) \to L^2_e(-1/\varepsilon, 1/\varepsilon)$ by

$$L_\varepsilon \phi = \phi'' - \phi + p\frac{w_\varepsilon^{p-1}}{\tilde{T}[w_\varepsilon]^q}\phi - q\frac{w_\varepsilon^p}{\tilde{T}[w_\varepsilon]^{q+1}}R_\varepsilon \phi,$$

where the domain of L_ε, denoted by $D(L_\varepsilon)$, is defined by

$$D(L_\varepsilon) = \{\phi \in H^2_e(-1/\varepsilon, 1/\varepsilon) \mid \phi'(\pm 1/\varepsilon) = 0\}.$$

As well as R_ε, we expect that L_ε is the linearized operator of $S[A]$ with respect to $A = w_\varepsilon$. In addition we set a nonlinear map $M(\phi) : D(L_\varepsilon) \to L^2_e(-1/\varepsilon, 1/\varepsilon)$ such as $M(\phi) = S[w_\varepsilon + \phi] - S[w_\varepsilon] - L_\varepsilon\phi$. Note that

$$M(\phi) = \frac{(w_\varepsilon + \phi)^p}{\tilde{T}[w_\varepsilon + \phi]^q} - \frac{w_\varepsilon^p}{\tilde{T}[w_\varepsilon]^q} - p\frac{w_\varepsilon^{p-1}}{\tilde{T}[w_\varepsilon]^q}\phi + q\frac{w_\varepsilon^p}{\tilde{T}[w_\varepsilon]^{q+1}}R_\varepsilon\phi.$$

Since we would like to seek a solution such as $S[w_\varepsilon + \phi] = 0$, it suffices to show that there is a solution $\phi \in D(L_\varepsilon)$ such as $M(\phi) = -S[w_\varepsilon] - L_\varepsilon\phi$, namely,

$$\phi = N(\phi) \equiv -L_\varepsilon^{-1}(M(\phi) + S[w_\varepsilon]), \tag{81}$$

because L_ε has an invertible operator L_ε^{-1}.

Lemma 4.5. *There is a constant $\kappa > 0$ independent of ε such that $\|L_\varepsilon\phi\|_{L^2} \geq \kappa\|\phi\|_{H^2}$ for any $\varepsilon < \varepsilon_0$ and $\phi \in D(L_\varepsilon)$. In addition, L_ε is surjective.*

This lemma is similar to Lemma 4.1 in [4]. However, the authors of [4] considered the case of two-dimensional whole space, which is different from our case. So we need to prove Lemma 4.5, which shall be done later.

We construct a Cauchy sequence in $H^2(-1/\varepsilon, 1/\varepsilon)$ in order to seek a solution of (81). We take a sequence $\phi_n \in D(L_\varepsilon)$ such as

$$\begin{cases} \phi_0 = 0, \\ \phi_{n+1} = -L_\varepsilon^{-1}(M(\phi_n) + S[w_\varepsilon]). \end{cases} \tag{82}$$

It can be shown from the following three lemmas that $\{\phi_n\}$ is a Cauchy sequence in $H^2(-1/\varepsilon, 1/\varepsilon)$.

Lemma 4.6. *There is a constant $c > 0$ such as $\|S[w_\varepsilon]\|_{L^2} \leq c\varepsilon$ and $|S[w_\varepsilon]| \leq ce^{-|y|}$ in $y \in (-1/\varepsilon, 1/\varepsilon)$.*

Proof. We obtain

$$\begin{aligned} S[w_\varepsilon] &= w_\varepsilon'' - w_\varepsilon + \frac{w_\varepsilon^p}{\tilde{T}[w_\varepsilon]^q} = \frac{w_\varepsilon^p}{\tilde{T}[w_\varepsilon]^q} - \frac{(\zeta^{\frac{q}{p-1}}w)^p}{\zeta^q}\chi_\varepsilon + \zeta^{\frac{q}{p-1}}(2w'\chi_\varepsilon' + w\chi_\varepsilon'') \\ &= w_\varepsilon^p\left(\frac{1}{\tilde{T}[w_\varepsilon]^q} - \frac{1}{\zeta^q}\right) + \zeta^{\frac{q}{p-1}}w^p(\chi_\varepsilon^p - \chi_\varepsilon) + \zeta^{\frac{q}{p-1}}(2w'\chi_\varepsilon' + w\chi_\varepsilon''). \end{aligned}$$

Because of Lemma 4.4, (89) and $w \leq ce^{-|y|}$, we have

$$\|S[w_\varepsilon]\|_{L^2} \leq c\|e^{-p|y|}(\tilde{T}[w_\varepsilon] - \zeta)\|_{L^2} + ce^{-1/4\varepsilon} \leq c\varepsilon.$$

Also, it is easy to verify the second inequality in the statement of the lemma. So we omit the details of the proof. □

Lemma 4.7. *For each $\delta > 0$ and $0 < \alpha < 1$, there exists $\varepsilon_0 > 0$ such that*

$$\|M(\phi)\|_{L^2} \le \delta\|\phi\|_{L^2}, \quad \|M(\phi_1) - M(\phi_2)\|_{L^2} \le \delta\|\phi_1 - \phi_2\|_{L^2} \tag{83}$$

for $\varepsilon < \varepsilon_0$ and $\phi, \phi_1, \phi_2 \in D(L_\varepsilon) \cap B_{\sqrt{\varepsilon}}(0; H_e^2)$ satisfying $|\phi(y)|, |\phi_1(y)|, |\phi_2(y)| \le ce^{-\alpha|y|}$.

Proof. We first have

$$\begin{aligned} M(\phi) &= \frac{(w_\varepsilon + \phi)^p}{\tilde{T}[w_\varepsilon + \phi]^q} - \frac{w_\varepsilon^p}{\tilde{T}[w_\varepsilon]^q} - p\frac{w_\varepsilon^{p-1}}{\tilde{T}[w_\varepsilon]^q}\phi + q\frac{w_\varepsilon^p}{\tilde{T}[w_\varepsilon]^{q+1}}R_\varepsilon\phi \\ &= p\left(\frac{(w_\varepsilon + \theta\phi)^{p-1}}{\tilde{T}[w_\varepsilon + \phi]^q} - \frac{w_\varepsilon^{p-1}}{\tilde{T}[w_\varepsilon]^q}\right)\phi + w_\varepsilon^p\left(\frac{1}{\tilde{T}[w_\varepsilon + \phi]^q} - \frac{1}{\tilde{T}[w_\varepsilon]^q} + \frac{qR_\varepsilon\phi}{\tilde{T}[w_\varepsilon]^{q+1}}\right) \end{aligned}$$

for some $0 \le \theta \le 1$. Hence we obtain

$$\begin{aligned} &\|M(\phi)\|_{L^2} \\ &\le c\|\phi\|_{L^2}(\|(w_\varepsilon + \theta\phi)^{p-1} - w_\varepsilon^{p-1}\|_{L^2} + \|w_\varepsilon^{p-1}(\tilde{T}[w_\varepsilon + \phi] - \tilde{T}[w_\varepsilon])\|_{L^2}) \\ &+ c(\|w_\varepsilon^p(\tilde{T}[w_\varepsilon + \phi] - \tilde{T}[w_\varepsilon] - R_\varepsilon\phi)\|_{L^2} + \|w_\varepsilon^p(\tilde{T}[w_\varepsilon + \phi] - \tilde{T}[w_\varepsilon])R_\varepsilon\phi\|_{L^2}). \end{aligned}$$

From $|\phi| \le ce^{-\alpha|y|}$, it holds that

$$\|(w_\varepsilon + \theta\phi)^{p-1} - w_\varepsilon^{p-1}\|_{L^2} \le c(e^{-\alpha R} + \|(w_\varepsilon + \theta\phi)^{p-1} - w_\varepsilon^{p-1}\|_{L^2(-R,R)}).$$

From $\|\phi\|_{H^2} \le \sqrt{\varepsilon}$ and Lemma 4.4, we can readily show $\|M(\phi)\|_{L^2} \le \delta\|\phi\|_{L^2}$. Similarly, we also have $\|M(\phi_1) - M(\phi_2)\|_{L^2} \le \delta\|\phi_1 - \phi_2\|_{L^2}$. We omit the details of the proof. □

Lemma 4.8. *Suppose that $1/p < \beta < 1$. Then there are $\varepsilon_0 > 0$ independent of n and $c > 0$ independent of n, ε such that $\phi_n \in B_{\sqrt{\varepsilon}}(0; H_e^2)$ and $|\phi_n| \le c\varepsilon^{\frac{1}{4}}e^{-\beta|y|}$ for $\varepsilon < \varepsilon_0$ and $y \in (-1/\varepsilon, 1/\varepsilon)$.*

Proof. Throughout this proof, we suppose that a constant c is independent of ε and n. Since $L_\varepsilon\phi_{n+1} = -M(\phi_n) - S[w_\varepsilon]$, it follows from Lemmas 4.6, 4.7 that

$$\|\phi_{n+1}\|_{H^2} \le \frac{1}{\kappa}(\|M(\phi_n)\|_{L^2} + \|S[w_\varepsilon]\|_{L^2}) \le \frac{1}{\kappa}(\delta\|\phi_n\|_{L^2} + c\varepsilon) \le c\delta\sqrt{\varepsilon} \tag{84}$$

if $\phi_n \in B_{\sqrt{\varepsilon}}(0; H_e^2)$ satisfies $|\phi_n| \le c\varepsilon^{\frac{1}{4}}e^{-\beta|y|}$.

Next we consider an ordinary differential equation

$$\phi_{n+1}'' - (1 - p\frac{w_\varepsilon^{p-1}}{\tilde{T}[w_\varepsilon]^q})\phi_{n+1} = -M(\phi_n) - S[w_\varepsilon] + q\frac{w_\varepsilon^p}{\tilde{T}[w_\varepsilon]^{q+1}}R_\varepsilon\phi_{n+1} \equiv F_n(y),$$

recalling (82). Since the above equation with Neumann boundary condition is of Sturm-Liouville type, as well as (70), we have

$$\begin{aligned} \phi_{n+1}(y) &= c_{n+1}\phi_-(y) \\ &+ \frac{1}{W}\left\{\phi_-(y)\int_R^y \phi_+(z)F_n(z)dz + \phi_+(y)\int_y^{1/\varepsilon}\phi_-(z)F_n(z)dz\right\}, \end{aligned} \tag{85}$$

where both of ϕ_+, ϕ_- are solutions of

$$\phi'' - (1 - p\frac{w_\varepsilon^{p-1}}{\tilde{T}[w_\varepsilon]^q})\phi = 0,$$

satisfying the boundary conditions $\phi_+'(R) = 0$, $\phi_+(1/\varepsilon) = 1$ and $\phi_-(R) = 1$, $\phi_-'(1/\varepsilon) = 0$, respectively, and W is defined by

$$W = \phi_-'(R)\phi_+(R) - \phi_-(R)\phi_+'(R) = \phi_-'(R)\phi_+(R).$$

Here we can show by the similar argument to the proof of Lemma 2.2 that

$$\frac{1}{|W|} \le ce^{1/\varepsilon - R}, \quad |\phi_-(y)| \le ce^{-(y-R)},$$
$$|\phi_+(y)| \le ce^{-(1/\varepsilon - y)}, \quad \|\phi_-\|_{L^2(R,1/\varepsilon)} \ge \frac{1}{c}. \tag{86}$$

Using the principle of induction, we show that

$$|\phi_n| \le c\varepsilon^{\frac{1}{4}} e^{-\beta y} \tag{87}$$

for $R \le y \le 1/\varepsilon$. Because of $\phi_0 \equiv 0$, it is clear that (87) holds for $n = 0$. Next we assume that (87) holds for n. Then it follows from (84), (86), (87) and $|R_\varepsilon \phi| \le c\|\phi\|_{L^2}$ that

$$\left| \frac{1}{W}\phi_-(y) \int_R^y \phi_+(z) S[w_\varepsilon] dz \right| \le c\varepsilon e^{-\beta y},$$
$$\left| \frac{1}{W}\phi_+(y) \int_y^{1/\varepsilon} \phi_-(z) S[w_\varepsilon] dz \right| \le c\varepsilon e^{-y},$$
$$\left| \frac{1}{W}\phi_-(y) \int_R^y \phi_+(z) q\frac{w_\varepsilon^p}{\tilde{T}[w_\varepsilon]} R_\varepsilon \phi_{n+1} dz \right| \le c\sqrt{\varepsilon} e^{-y},$$
$$\left| \frac{1}{W}\phi_+(y) \int_y^{1/\varepsilon} \phi_-(z) q\frac{w_\varepsilon^p}{\tilde{T}[w_\varepsilon]} R_\varepsilon \phi_{n+1} dz \right| \le c\sqrt{\varepsilon} e^{-y}.$$

Furthermore we easily see that

$$|M(\phi_n)| \le c\{((w_\varepsilon + \theta\phi_n)^{p-1} - w_\varepsilon^{p-1})|\phi_n| + w_\varepsilon^{p-1}|\tilde{T}[w_\varepsilon + \phi_n] - \tilde{T}[w_\varepsilon]||\phi_n|$$
$$+ w_\varepsilon^p|\tilde{T}[w_\varepsilon + \phi_n] - \tilde{T}[w_\varepsilon]| + w_\varepsilon^p|R_\varepsilon\phi_n|\}$$

so that we have

$$\left| \frac{1}{W}\phi_-(y) \int_R^y \phi_+(z) M(\phi_n) dz \right| \le c(\varepsilon^{\frac{\beta p - 1}{8(1-\beta)}} + \varepsilon^{\frac{1}{4}})\varepsilon^{\frac{1}{4}} e^{-\beta y},$$
$$\left| \frac{1}{W}\phi_+(y) \int_y^{1/\varepsilon} \phi_-(z) M(\phi_n) dz \right| \le c(\varepsilon^{\frac{\beta p - 1}{8(1-\beta)}} + \varepsilon^{\frac{1}{4}})\varepsilon^{\frac{1}{4}} e^{-\beta y}.$$

From these inequalities, (84) and (85), we obtain

$$c\delta\sqrt{\varepsilon} \ge \|\phi_{n+1}\|_{L^2(R,1/\varepsilon)} \ge \frac{c_{n+1}}{c} - c\delta\varepsilon^{\frac{1}{4}}$$

if R is sufficiently large and ε is sufficiently small. Here we suppose $\|e^{-y}\|_{L^2} \le c$. Therefore we have $c_{n+1} \le c\delta\varepsilon^{\frac{1}{4}}$. Using the above inequalities and (85) again, we have

$$|\phi_{n+1}| \le c\delta\varepsilon^{\frac{1}{4}} e^{-y+R} + c\delta\varepsilon^{\frac{1}{4}} e^{-\beta y} \le c\varepsilon^{\frac{1}{4}} e^{-\beta y}$$

if δ is sufficiently small. On the other hand, using Sobolev's embedding, we have $|\phi_n| \le c\sqrt{\varepsilon}$ for $-R \le y \le R$ so that we finally obtain

$$|\phi_n| \le c\varepsilon^{\frac{1}{4}} e^{-\beta|y|}$$

for $-1/\varepsilon < y < 1/\varepsilon$. Thus the proof is completed. □

From Lemmas 4.6, 4.7, 4.8, we have

$$\|\phi_{n+1}\|_{H^2} \le \frac{1}{\kappa}(\|M(\phi_n)\|_{L^2} + \|S[w_\varepsilon]\|_{L^2}) \le \frac{1}{\kappa}(\delta\|\phi_n\|_{L^2} + c\varepsilon) \le \sqrt{\varepsilon},$$
$$\|\phi_{n+1} - \phi_n\|_{H^2} \le \frac{\delta}{\kappa}\|M(\phi_n) - M(\phi_{n-1})\|_{L^2} \le \frac{\delta}{\kappa}\|\phi_n - \phi_{n-1}\|_{L^2}.$$

Therefore it follows that $\{\phi_n\}$ is a Cauchy sequence in $H^2(-1/\varepsilon, 1/\varepsilon)$, which implies that there exists a function $\phi \in H^2(-1/\varepsilon, 1/\varepsilon)$ such as $\phi_n \to \phi$ in $H^2(-1/\varepsilon, 1/\varepsilon)$ strongly as $n \to \infty$ and $\phi = N(\phi)$. Hence we have $S[w_\varepsilon + \phi] = 0$. Furthermore, it is easy to show that the pair of $\tilde{a} = w_\varepsilon + \phi$ and $\tilde{h} = \tilde{T}[w_\varepsilon + \phi]$ is a solution of (1) and satisfies (2), (3). This completes the proof.

In what follows, we prove the lemmas which have not been done yet. We first prove the existence of $T[A]$.

Lemma 4.9. *For any $\delta_1 > 0$ and any nonnegative function $A \in H_e^2(-1,1)$, there exists a unique solution $T_{\delta_1} = T_{\delta_1}[A]$ of the problem*

$$\begin{cases} dT''_{\delta_1} - T_{\delta_1} + \dfrac{A}{(T_{\delta_1} + \delta_1)^s} = 0, & x \in (-1,1), \\ T'_{\delta_1} = 0, & x = \pm 1. \end{cases} \tag{88}$$

Furthermore $T_{\delta_1}[A]$ is even.

Proof. We take a sequence T_n defined by

$$T_0 = 0, \quad T_{n+1} = \int_{-1}^{1} G(x,z) \frac{A}{(T_n + \delta_1)^s} dz.$$

Then we can easily show that

$$|T_n(x)| \le \frac{G(0,0)}{\delta_1^s} \int_{-1}^{1} A dz,$$
$$|T_n(x_1) - T_n(x_2)| \le \frac{c}{\delta_1^s} |x_1 - x_2| \int_{-1}^{1} A dz.$$

Therefore, it follows from Ascoli-Arzela's Theorem that there exist a subsequence T_{n_k} and a function T such that T_{n_k} converges to T uniformly on $[-1,1]$ as $k \to \infty$ and T is a solution of (88). Next we show the uniqueness of a solution of (88). Suppose that T_1, T_2 are solutions of (88). Then we have

$$\begin{aligned} & d(T_1 - T_2)'' - (T_1 - T_2) + \frac{A}{(T_1 + \delta_1)^s} - \frac{A}{(T_2 + \delta_1)^s} \\ &= d(T_1 - T_2)'' - (T_1 - T_2) - s \frac{A}{(T_1 + \delta_1 + \kappa(T_2 - T_1))^{s+1}} (T_1 - T_2) = 0 \end{aligned}$$

for some $0 \le \kappa \le 1$. Multiplying $T_1 - T_2$ to the both sides and integrating it by parts, we have

$$\begin{aligned} & d\|T_1' - T_2'\|^2_{L^2(-1,1)} + \|T_1 - T_2\|^2_{L^2(-1,1)} \\ & + \int_{-1}^{1} s \frac{A}{(T_1 + \delta_1 + \kappa(T_2 - T_1))^{s+1}} (T_1 - T_2)^2 dx = 0, \end{aligned}$$

which implies $T_1 - T_2 = 0$. Hence we obtain the uniqueness of a solution of (88). Furthermore it can also be shown from the uniqueness that the solution of (88) is even. This completes the proof. $\square$

Lemma 4.10. *Let T_{δ_1} be a solution of (88) for a nonnegative function $A \in H_e^2(-1, 1)$. Then it must satisfy*

$$e^{-2/d} \left\{ \left(\frac{1}{2} \int_{-1}^{1} A dx \right)^{\frac{1}{s+1}} - \delta_1 \right\} \le T_{\delta_1} \le e^{2/d} \left(\frac{1}{2} \int_{-1}^{1} A dx \right)^{\frac{1}{s+1}}.$$

This lemma can be shown by the same argument as in [19] and so we omit the details.

Next we show that $T_{\delta_1}[A]$ approaches a solution of the problem (74) as $\delta_1 \to 0$.

Lemma 4.11. *Let $T_{\delta_1}[A]$ be a unique solution of (88) given in Lemma 4.9 for $\delta_1 > 0$. Then $T_{\delta_1}[A]$ converges to a unique solution of (74), denoted by $T[A]$, uniformly on $[-1,1]$ as $\delta_1 \to 0$. Furthermore $T[A]$ is even.*

Proof. Since $T_{\delta_1}[A]$ is a solution, we have

$$T_{\delta_1}[A](x) = \int_{-1}^{1} G(x,z) \frac{A}{(T_{\delta_1}[A] + \delta_1)^s} dz.$$

Then it follows from Lemma 4.10 and (75) that $T_{\delta_1}[A]$ is uniformly bounded in $\delta_1 > 0$ and

$$|T_{\delta_1}[A](x_1) - T_{\delta_1}[A](x_2)| \le \int_{-1}^{1} |G(x_1,z) - G(x_2,z)| \frac{A}{(T_{\delta_1}[A] + \delta_1)^s} dz \le c|x_1 - x_2|,$$

for a constant $c > 0$ independent of $\delta_1 > 0$. From Ascoli-Arzela's Theorem, we have a subsequence $\delta_{1,k}$ of δ_1 for $k = 1, 2, \ldots$, and a solution $T[A]$ of (74) such that $\delta_{1,k} \to 0$ and $T_{\delta_{1,k}}[A] \to T[A]$ uniformly on $[-1,1]$ as $k \to \infty$. Furthermore we can prove the uniqueness of a solution of (74) by using the similar argument for the uniqueness of $T_{\delta_1}[A]$. Hence $T_{\delta_1}[A]$ converges to $T[A]$ uniformly on $[-1,1]$ as $\delta_1 \to 0$. Since $T_{\delta_1}[A]$ is even, it is clear that $T[A]$ is also even. $\square$

From Lemma 4.11, it is easy to show that the following result holds. So we omit the details of the proof.

Corollary 1. *Let T be a solution of the problem (74) for $A \in H_e^2(-1,1)$. Then it must satisfy*

$$e^{-2/d} \left(\frac{1}{2} \int_{-1}^{1} A dx \right)^{\frac{1}{s+1}} \le T \le e^{2/d} \left(\frac{1}{2} \int_{-1}^{1} A dx \right)^{\frac{1}{s+1}}.$$

From this corollary, an estimate

$$e^{-2/d} \left(\frac{1}{2} \int_{-1/\varepsilon}^{1/\varepsilon} (w_\varepsilon + \phi)^r dy \right)^{\frac{1}{s+1}} \le \tilde{T}[w_\varepsilon + \phi] \le e^{2/d} \left(\frac{1}{2} \int_{-1/\varepsilon}^{1/\varepsilon} (w_\varepsilon + \phi)^r dx \right)^{\frac{1}{s+1}}$$

holds for $\phi \in B_{\sqrt{\varepsilon}}(0; H_e^2)$. Furthermore, if $|\phi| \le ce^{-\alpha|y|}$ with $0 < \alpha < \min\{p-1, r-1, 1/2\}$, we obtain

$$\frac{1}{2} e^{-2/d} \left(\frac{1}{2} \int_{-\infty}^{\infty} \zeta^{\frac{qr}{p-1}} w^r dy \right)^{\frac{1}{s+1}} \le \tilde{T}[w_\varepsilon + \phi] \le 2e^{2/d} \left(\frac{1}{2} \int_{-\infty}^{\infty} \zeta^{\frac{qr}{p-1}} w^r dy \right)^{\frac{1}{s+1}}. \quad (89)$$

Next we prove that L_ε is an invertible operator. In the proof, we consider an operator $L_0 : H_e^2(-\infty,\infty) \to L_e^2(-\infty,\infty)$ such as

$$L_0 \phi = \phi'' - \phi + p w^{p-1} \phi - \frac{qr}{s+1} \frac{\int_{-\infty}^{\infty} w^{r-1} \phi dz}{\int_{-\infty}^{\infty} w^r dz} w^p.$$

We consider that L_0 is the limiting operator of L_ε as $\varepsilon \to 0$. Then the following lemma is useful.

Lemma 4.12. *L_0 is invertible.*

This lemma seems to be similar to Lemma 4.1 in [4]. It is true that the authors of [4] consider the case of the two-dimensional whole space, which is different from our case. However, we can prove the above lemma by using the similar argument to Lemma 4.1 in [4] because the domain of L_0 is $H^2_e(-\infty,\infty)$. So we omit the details.

Finally we prove Lemma 4.5.

Proof. Suppose that there exists $\phi_\varepsilon \in D(L_\varepsilon)$ such that $\|\phi_\varepsilon\|_{H^2} = 1$ and $\|L_\varepsilon\phi_\varepsilon\|_{L^2} \to 0$ as $\varepsilon \to 0$. From $\|\phi_\varepsilon\|_{H^2} = 1$, there exist a subsequence ϕ_{ε_k} and a function $\phi_0 \in H^2_e(-\infty,\infty)$ such as $\phi_{\varepsilon_k} \to \phi_0$ as $\varepsilon \to 0$ weakly in $H^2(-\infty,\infty)$. From Lemma 4.4 and $\|L_\varepsilon\phi_\varepsilon\|_{L^2} \to 0$, it holds that

$$L_{\varepsilon_k}\phi_{\varepsilon_k} \to L_0\phi_0 \quad \text{weakly in } L^2(-\infty,\infty)$$

as $k \to \infty$. Therefore we obtain $\phi_0 = 0$ from Lemma 4.12.

Let χ_R be a cut-off function satisfying

$$\chi_R = \begin{cases} 1, & |y| \le R, \\ 0, & |y| \ge R+1. \end{cases}$$

Since $\phi_{\varepsilon_k} \to 0$ as $k \to \infty$ weakly in $H^2(-\infty,\infty)$, we have $\phi_{\varepsilon_k}\chi_R \to 0$ as $k \to \infty$ strongly in $H^1(-\infty,\infty)$ by using compact embedding, which implies $\phi_{\varepsilon_k}\chi_R \to 0$ as $k \to \infty$ strongly in $H^2(-\infty,\infty)$ because of $\|L_\varepsilon\phi_\varepsilon\|_{L^2} \to 0$ as $\varepsilon \to 0$. Here we may replace k with an appropriate subsequence if needed, but we use the same notation.

Next we have an estimate of $\psi_k = \phi_{\varepsilon_k}(1-\chi_R)$. Multiplying ψ_k to both sides of $L_\varepsilon\psi_k = L_\varepsilon\phi_{\varepsilon_k} - L_\varepsilon(\phi_{\varepsilon_k}\chi_R)$ and integrating it by parts, we have

$$\|\psi_k\|_{H^1} \le \|L_\varepsilon\phi_{\varepsilon_k}\|_{L^2} + \|L_\varepsilon(\phi_{\varepsilon_k}\chi_R)\|_{L^2} + ce^{-(p-1)R}$$

by using $w \le ce^{-|y|}$. Then we obtain

$$\limsup_{k\to\infty} \|\psi_k\|_{H^1} \le ce^{-(p-1)R}$$

because $\|L_\varepsilon\phi_{\varepsilon_k}\|_{L^2}$ and $\|L_\varepsilon(\phi_{\varepsilon_k}\chi_R)\|_{L^2}$ tends to 0 as $k \to \infty$. Furthermore, from the definition of L_ε, we can easily verify

$$\limsup_{k\to\infty} \|\psi_k\|_{H^2} \le ce^{-(p-1)R}.$$

Hence we have

$$1 \le \limsup_{k\to\infty}(\|\phi_{\varepsilon_k}\chi_R\|_{H^2} + \|\phi_{\varepsilon_k}(1-\chi_R)\|_{H^2}) \le ce^{-(p-1)R} \to 0$$

as $R \to \infty$, which is a contradiction. □

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