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EFFECT OF BOUNDARY CONDITIONS ON THE DYNAMICS OF A PULSE SOLUTION FOR REACTION-DIFFUSION SYSTEMS

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Dedicated to Professor Hiroshi Matano on the occasion of his 60th birthday

Abstract. We consider pulse-like localized solutions for reaction-diffusion systems on a half line and impose various boundary conditions at one end of it. It is shown that the movement of a pulse solution with the homogeneous Neumann boundary condition is completely opposite from that with the Dirichlet boundary condition. As general cases, Robin type boundary conditions are also considered. Introducing one parameter connecting the Neumann and the Dirichlet boundary conditions, we clarify the transition of motions of solutions with respect to boundary conditions.

1. **Introduction.** Reaction diffusion systems have been widely treated to describe and study spatio-temporal patterns in dissipative systems. Among them, many reaction-diffusion systems which possess various types of localized solutions such as pulse-like localized solutions and front-like ones have been proposed while we omit the detail and merely refer to books ([8], [7]). To understand the dynamics of such solutions, reaction-diffusion systems have been studied under various situations such as one or higher dimensional spaces, bounded or unbounded domains, and the Neumann boundary conditions or the Dirichlet ones according to considered problems. In fact, the dynamics of solutions drastically change depending on the situations even if considered equations are same. As one example, we consider in this paper, the effect of boundary conditions on the dynamics of solutions, which is also important from a practical point of view. In fact, a boundary condition is one of the most important factors collateral on e.g. temperature, by which we can control inside states from outsides. Then we are interested in how inside patterns are controllable by adjusting boundary conditions.

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Here we show how the dynamics changes drastically depending on the boundary conditions. Let us consider the following one dimensional problems of the Allen-Cahn equation on the half line $\boldsymbol{R}_+ := (0, +\infty)$:

$$u_t = u_{xx} + \frac{1}{2}u(1-u^2),\ t > 0,\ x \in \boldsymbol{R}_+ \tag{1}$$

with the Dirichlet boundary condition $u = -1$ or the Neumann boundary condition $u_x = 0$ at $x = 0$, respectively.

Let $\Phi(z) := \tanh(z/2)$, which is a stable stationary front solution of (1) on the whole line $\boldsymbol{R}$. If the initial data of (1) is sufficiently close to $\Phi(x - l_0)$ for $l_0 \gg 1$, then the solution of (1) with the Dirichlet boundary condition moves toward the right hand side while the solution with the Neumann boundary condition moves toward the left as in Fig 1.1.

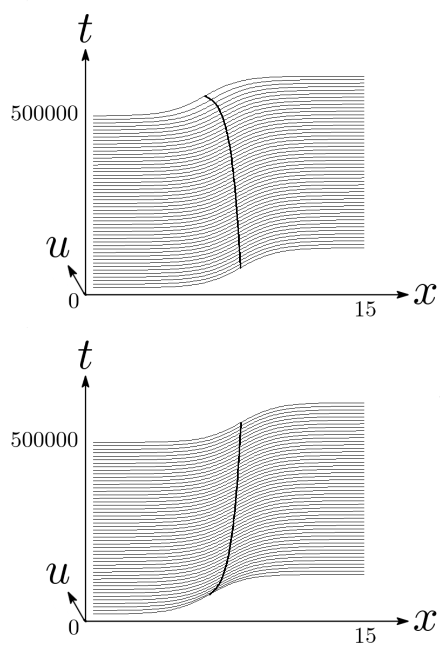


FIGURE 1.1. Movements of front solutions of (1) on $\boldsymbol{R}_+$. Left : Neumann boundary condition at $x = 0$, Right : Dirichlet boundary condition at $x = 0$. Solid lines denote the 0-level lines $\{x \in \boldsymbol{R}_+;\ u(t,x) = 0\}$.

These phenomena are understood as follows by reducing to the interaction between two front solutions: The problem with the Neumann boundary condition is interpreted as the interaction between a kink solution $u^+(t,x) := \Phi(x - l(t))$ and anti-kink solution $u^-(t,x) := \Phi(-x - l(t))$, which has been extensively investigated and attractive interaction was shown ([1], [5] and [6]). That is, the front solution close to $u^+(t,x)$ is attracted to the reflected solution $u^-(t,x)$, which means the front solution approaches the boundary $x = 0$.

On the other hand, the problem with the Dirichlet boundary condition is essentially interpreted as the interaction between $u^+(t,x) = \Phi(x - l(t))$ and $-u^-(t,x) =$

$-\Phi(-x-l(t))$. The repulsive interaction in this case can be shown in quite a similar way to the Neumann boundary case. Thus completely opposite motions appear only by changing boundary conditions.

Motivated by the fact, we introduce one parameter, say β, into boundary conditions as

$$\partial_x u = \beta(u+1),\ x = 0 \tag{2}$$

and investigate the dependency of motion of a front solution on β because the parameter β connects the Neumann boundary condition ($\beta = 0$) to the Dirichlet boundary condition ($\beta = \infty$). Applying results obtained in Section 2 to the problem (1) with (2), we can show that the motion of a front solution $u(t,x) \sim \Phi(x - l(t))$ is essentially given by

$$\frac{dl}{dt} = -\frac{12(1-\beta)}{1+\beta}e^{-2l},$$

which gives precise dependency of the motion on β. Specially, $\dfrac{dl}{dt} = -12e^{-2l} < 0$ when $\beta = 0$ and $\dfrac{dl}{dt} = 12e^{-2l} > 0$ when $\beta = \infty$, which correspond to the Neumann and the Dirichlet boundary conditions, respectively.

In this paper, we deal with fairly general types of reaction-diffusion systems

$$\boldsymbol{u}_t = D\boldsymbol{u}_{xx} + F(\boldsymbol{u}),\ t > 0,\ x \in \boldsymbol{R}_+,\ \boldsymbol{u} \in \boldsymbol{R}^N \tag{3}$$

with boundary conditions at $x = 0$ and mainly consider the dynamics of a pulse-like localized solution while a front-like localized solution is also treated by the similar way, where $D := diag\{d_1, \cdots, d_N\}$ and $F : \boldsymbol{R}^N \to \boldsymbol{R}^N$ is a sufficiently smooth function.

First we consider the problem (3) on $\boldsymbol{R}$

$$\boldsymbol{u}_t = D\boldsymbol{u}_{xx} + F(\boldsymbol{u}),\ t > 0,\ x \in \boldsymbol{R}, \tag{4}$$

and assume several conditions for (4) as follows:

A1) There exists a symmetric stationary pulse solution, say $S(x)$, satisfying $S(x) \to e^{-\alpha|x|}\boldsymbol{a}$ as $|x| \to \infty$ for $\alpha > 0$ and $\boldsymbol{a} \in \boldsymbol{R}^N$.

Here we note that $F(\boldsymbol{0}) = \boldsymbol{0}$ holds, where $\boldsymbol{0} := {}^t(0, \cdots, 0) \in \boldsymbol{R}^N$. We also assume the stability of $S(x)$ in $L^2(\boldsymbol{R})$ space as follows: Let $L := D\partial_{xx} + F'(S(x))$, the linearized operator of (4) with respect to $S(x)$ and $L_0 := D\partial_{xx} + F'(\boldsymbol{0})$.

A2) The spectrum $\sigma(L)$ of L is given by $\sigma(L) = \sigma_0 \cup \sigma_1$, where $\sigma_0 := \{0\}$ and $\sigma_1 \subset \{Re\lambda < -\gamma_0\}$ for $\gamma_0 > 0$. Moreover, 0 is a simple eigenvalue of L.

A2)' The spectrum $\sigma(L_0)$ satisfies $\sigma(L_0) \subset \{\lambda \in \boldsymbol{C};\ Re(\lambda) < -\gamma_0\}$.

By A1) and A2), there exists an eigenfunction $\phi^*(x)$ of the adjoint operator L^* of L satisfying $L^*\phi^* = 0$ and $\phi^*(x) \to e^{-\alpha x}\boldsymbol{a}^*$ as $x \to +\infty$ for $\boldsymbol{a}^* \in \boldsymbol{R}^N$. Note that we can take $\phi^*(x)$ as an odd function and $\phi^*(x)$ is uniquely determined by the normalization $\langle S_x, \phi^* \rangle_{L^2} = 1$.

Under A1), A2) and A2)', similar phenomena to (1) are observed in (3) according to the weak interaction analysis by [3]. If we impose the Neumann boundary condition at $x = 0$, the solution $\boldsymbol{u}(t,x)$ of (3) is approximated by $\boldsymbol{u}(t,x) \sim$

$S(x-l(t))+S(x+l(t))$ and the dynamics is essentially given by

$$\frac{dl}{dt}=2\alpha M_0 e^{-2\alpha l}, \tag{5}$$

where $M_0 := \langle\, D\boldsymbol{a}, \boldsymbol{a}^* \,\rangle$.

In order to consider the Dirichlet boundary condition $\boldsymbol{u}(t,0)=\boldsymbol{0}$, we assume $F(-\boldsymbol{u})=-F(\boldsymbol{u})$ holds for simplicity. Then $-S(x)$ is also a stable stationary pulse solution of (4) on $\boldsymbol{R}$ and the solution of (3) satisfying the Dirichlet boundary condition is approximated by $\boldsymbol{u}(t,x)\sim S(x-l(t))-S(x+l(t))$ and the dynamics is essentially given by

$$\frac{dl}{dt}=-2\alpha M_0 e^{-2\alpha l} \tag{6}$$

with the same constant M_0 as (5) ([3]). Thus, just opposite motions appear by changing only boundary conditions.

In order to consider the relation between the Neumann and the Dirichlet boundary conditions, we impose the boundary condition

$$\boldsymbol{u}_x=\beta\boldsymbol{u}+\delta\boldsymbol{a},\ x=0 \tag{7}$$

on the problem (3) by introducing parameters β and a sufficiently small δ while we do not assume $F(-\boldsymbol{u})=-F(\boldsymbol{u})$. In fact, β connects the Neumann and the Dirichlet boundary conditions by taking $\beta=0$ and $\beta=\infty$, respectively. δ gives the perturbation of the boundary conditions from the ground state $\boldsymbol{0}$. Then Theorem 2.1 in Section 2 says that the solution $\boldsymbol{u}(t,x)$ of (3) with (7) is close to $S(x-l(t))$ and the motion of a pulse solution is essentially governed by the ODE

$$\frac{dl}{dt}=\frac{2\alpha}{\alpha+\beta}M_0\{(\alpha-\beta)e^{-2\alpha l}-\delta e^{-\alpha l}\} \tag{8}$$

for the same constant M_0. For simplicity, let us consider the case of $\delta=0$. Then (8) is

$$\frac{dl}{dt}=\frac{2\alpha(\alpha-\beta)}{\alpha+\beta}M_0 e^{-2\alpha l},$$

which implies that the direction of the motion of a pulse solution changes in the neighborhood of $\beta=\alpha$ and that the case of $\beta=0$ (the Neumann boundary condition) and the case of $\beta=\infty$ (the Dirichlet boundary condition) are with just opposite signs in the motion. Thus, the connection between two boundary conditions is clearly drawn.

2. **Main results.**

Throughout this paper, we define $X_+ := L^2(\boldsymbol{R}_+)$ with the L^2-norm denoted by $\|\cdot\|_+$ and the inner product in X_+ by $\langle\, \boldsymbol{u},\boldsymbol{v}\,\rangle_+ := \int_0^\infty \langle\, \boldsymbol{u}(x),\boldsymbol{v}(x)\,\rangle\, dx$ for $\boldsymbol{u},\ \boldsymbol{v}\in X_+$.

2.1. **Motion of pulse solution.**

In this subsection, we will give some preliminaries and the results on the dynamics of pulse-like localized solutions.

A2)' means the followings: Let $\boldsymbol{k}_j(x;\lambda)$ $(j=1,2,\cdots,2N)$ be the fundamental functions of the ODE

$$(L_0-\lambda)\boldsymbol{u}=\boldsymbol{0},\ x\in\boldsymbol{R}. \tag{9}$$

Then $\boldsymbol{k}_j(x;\lambda)\in\boldsymbol{R}^N$ for $\lambda\in\boldsymbol{C}$ with $Re(\lambda)>-\gamma_0$ are the forms of

$$\boldsymbol{k}_j(x;\lambda)=(x+1)^{k_j(\lambda)}e^{\pm\mu_j(\lambda)x}\boldsymbol{b}_j(\lambda),\ k_j(\lambda)=0,1,\cdots,\ \boldsymbol{b}_j(\lambda)\in\boldsymbol{R}^N,\ \mu_j(\lambda)>0$$

by the assumption A2)'. We assume $0 < \mu_1(\lambda) \le \cdots \le \mu_N(\lambda)$, $\boldsymbol{k}_j(x;\lambda) = (x+1)^{k_j(\lambda)} e^{-\mu_j(\lambda)x} \boldsymbol{b}_j(\lambda)$ for $j = 1, \cdots, N$ and $\boldsymbol{k}_j(x;\lambda) = (x+1)^{k_j(\lambda)} e^{\mu_j(\lambda)x} \boldsymbol{b}_j(\lambda)$ for $j = N+1, \cdots, 2N$. Defining $\alpha_j := \mu_j(0)$, $m_j := k_j(0)$, $\boldsymbol{a}_j := \boldsymbol{b}_j(0)$ and $\boldsymbol{m}_j(x) := \boldsymbol{k}_j(x;0) = (x+1)^{k_j} e^{\pm\alpha_j x} \boldsymbol{a}_j$, we add one more assumption for the operator L_0.

A2)" $\boldsymbol{m}_1(x) = e^{-\alpha x}\boldsymbol{a}$ and $0 < \alpha_1 = \alpha < \alpha_2 \le \cdots \le \alpha_N$.

Remark 1. Assumption A2)" means α in the assumption A1) is a simple root of $\det(\mu^2 D + F'(\mathbf{0})) = 0$ and $\boldsymbol{a}$ is an eigenvector satifying $(\alpha^2 D + F'(\mathbf{0}))\boldsymbol{a} = \mathbf{0}$.

Next coming back to the original problem (3) on the half line $\boldsymbol{R}_+$, we impose the boundary condition (7) on it. Then we have

Theorem 2.1. *Assume A1), A2), A2)' and A2)". If the initial data $\boldsymbol{u}(0,x)$ is sufficiently close to $S(x - l_0)$ for $l_0 \gg 1$, then the solution $\boldsymbol{u}(t,x)$ of (3) satisfies*

$$\|\boldsymbol{u}(t,\cdot) - S(\cdot - l(t))\|_\infty \le O(e^{-\alpha l(t)} + \delta)$$

as long as $l(t) > \bar{l}$ for $\bar{l} \gg 1$. $l(t)$ satisfies

$$\frac{dl}{dt} = \frac{2\alpha M_0}{\alpha+\beta}\{(\alpha-\beta)e^{-\alpha l} - \delta\}e^{-\alpha l}(1 + O(e^{-\gamma' l} + \delta^2))$$

for a positive constant γ'.

Let $H_0(l) := \dfrac{2\alpha M_0}{\alpha+\beta}\{(\alpha-\beta)e^{-\alpha l} - \delta\}e^{-\alpha l}$. Then

Corollary 1. *If there exists $l^* > \bar{l}$ such that $H_0(l^*) = 0$ and $\dfrac{dH_0}{dl}(l^*) < 0$ (> 0 resp.). Then there exists a stable (unstable resp.) stationary solution $\boldsymbol{u}^*(x)$ of (3) satisfying*

$$\|\boldsymbol{u}^* - S(\cdot - l^*)\|_\infty \le O(e^{-\alpha l^*} + \delta).$$

2.2. **Motion of front solution.** Next, we consider the case of front solutions for (3). We assume the followings for (4) on $\boldsymbol{R}$.

A3) There exists a stationary front solution, say $S(x)$ satisfying $S(x) \to S_\pm + e^{-\alpha_\pm |x|}\boldsymbol{a}_\pm$ as $x \to \pm\infty$ for $\alpha_\pm > 0$, $\boldsymbol{a}_\pm \in \boldsymbol{R}^N$ and $S_\pm \in \boldsymbol{R}^N$.

Here we note that $F(S_\pm) = \mathbf{0}$ hold. Let $L := D\partial_{xx} + F'(S(x))$, the linearized operator of (4) with respect to $S(x)$ and $L_\pm := D\partial_{xx} + F'(S_\pm)$.

A4) The spectrum $\sigma(L)$ of L is given by $\sigma(L) = \sigma_0 \cup \sigma_1$, where $\sigma_0 := \{0\}$ and $\sigma_1 \subset \{Re\lambda < -\gamma_0\}$ for $\gamma_0 > 0$. Moreover, 0 is a simple eigenvalue of L.

A4)' The spectrum $\sigma(L_\pm)$ satisfies $\sigma(L_\pm) \subset \{Re\lambda < -\gamma_0\}$.

In the following, we only give an assumption for L_- for simplicity while we assume a similar assumption on L_+. Let $\boldsymbol{k}_j(x;\lambda)$ $(j = 1, 2, \cdots, 2N)$ be the fundamental functions of the ODE

$$(L_- - \lambda)\boldsymbol{u} = \mathbf{0}, \ x \in \boldsymbol{R}. \tag{10}$$

Then $\boldsymbol{k}_j(x;\lambda) \in \boldsymbol{R}^N$ for $\lambda \in \boldsymbol{C}$ with $Re(\lambda) > -\gamma_0$ are the forms of $\boldsymbol{k}_j(x;\lambda) = (x+1)^{k_j(\lambda)} e^{\pm\mu_j(\lambda)x} \boldsymbol{b}_j(\lambda)$ for $k_j(\lambda) = 0, 1, \cdots$, $\boldsymbol{b}_j(\lambda) \in \boldsymbol{R}^N$ and $\mu_j(\lambda) > 0$ by the assumption A4)'. We assume $0 < \mu_1(\lambda) \le \cdots \le \mu_N(\lambda)$, $\boldsymbol{k}_j(x;\lambda) = (x+1)^{k_j(\lambda)} e^{-\mu_j(\lambda)x} \boldsymbol{b}_j(\lambda)$ for $j = 1, \cdots, N$ and $\boldsymbol{k}_j(x;\lambda) = (x+1)^{k_j(\lambda)} e^{\mu_j(\lambda)x} \boldsymbol{b}_j(\lambda)$ for

$j = N+1, \cdots, 2N$. Defining $\alpha_j := \mu_j(0)$, $m_j := k_j(0)$, $\boldsymbol{a}_j := \boldsymbol{b}_j(0)$ and $\boldsymbol{m}_j(x) := \boldsymbol{k}_j(x;0) = (x+1)^{k_j} e^{\pm\alpha_j x}\boldsymbol{a}_j$, we assume:

A4)" $\boldsymbol{m}_1(x) = e^{-\alpha_- x}\boldsymbol{a}$ and $0 < \alpha_1 = \alpha_- < \alpha_2 \le \cdots \le \alpha_N$. A similar assumption is assumed for L_+.

By A3) and A4), there exists an eigenfunction $\phi^*(x)$ of the adjoint operator L^* of L satisfying $L^*\phi^* = 0$ and $\phi^*(x) \to e^{-\alpha_\pm|x|}\boldsymbol{a}^*_\pm$ as $x \to \pm\infty$ for $\boldsymbol{a}^*_\pm \in \boldsymbol{R}^N$. By the normalization $\langle\, S_x, \phi^* \,\rangle_{L^2} = 1$, $\phi^*(x)$ is uniquely determined.

Next coming back to the original problem (3) on the half line $\boldsymbol{R}_+$, we impose the following boundary condition on it:

$$\boldsymbol{u}_x = \beta(\boldsymbol{u} - S_-) + \delta\boldsymbol{a}_-,\ x = 0 \tag{11}$$

for sufficiently small δ. Then we have

Theorem 2.2. *Assume A3), A4), A4)' and A4)". If the initial data $\boldsymbol{u}(0,x)$ is sufficiently close to $S(x - l_0)$ for $l_0 \gg 1$, then the solution $\boldsymbol{u}(t,x)$ of (3) satisfies*

$$\|\boldsymbol{u}(t,\cdot) - S(\cdot - l(t))\|_\infty \le O(e^{-\alpha l(t)})$$

as long as $l(t) > \bar{l}$ for $\bar{l} \gg 1$. $l(t)$ satisfies

$$\frac{dl}{dt} = -\frac{2\alpha_- M_-}{\alpha_- + \beta}\{(\alpha_- - \beta)e^{-2\alpha_- l} - \delta e^{-\alpha_- l}\}(1 + O(e^{-\gamma' l} + \delta^2))$$

for a positive constant γ', where $M_- := \langle\, D\boldsymbol{a}_-, \boldsymbol{a}^*_- \,\rangle$.

Let $H_0(l) := -\dfrac{2\alpha_- M_-}{\alpha_- + \beta}\{(\alpha_- - \beta)e^{-\alpha_- l} - \delta\}e^{-\alpha_- l}$. Then

Corollary 2. *If there exists $l^* > \bar{l}$ such that $H_0(l^*) = 0$ and $\dfrac{dH_0}{dl}(l^*) < 0$ (> 0 resp.). Then there exists a stable (unstable resp.) stationary solution $\boldsymbol{u}^*(x)$ of (3) satisfying*

$$\|\boldsymbol{u}^* - S(\cdot - l^*)\|_\infty \le O(e^{-\alpha_- l^*} + \delta).$$

Proofs of Theorem 2.2 and Corollary 2 are quite similar to those of Theorem 2.1 and Corollary 1 and we will give proofs only of Theorem 2.1 and Corollary 1 in Section 4.

3. **Applications.** In this section, we will apply results in Section 2 to examples of a front solution for the Allen-Cahn equation and a pulse solution for the Gray-Scott model.

3.1. **Dynamics of a front solution for the Allen-Cahn equation.** In this subsection, we consider the dynamics of a front solution for the Allen-Cahn equation

$$u_t = u_{xx} + f(u), \quad t > 0, \quad x \in \boldsymbol{R}_+, \tag{12}$$

where $u \in \boldsymbol{R}^1$ and $f(u) = \frac{1}{2}u(1-u^2)$. Then, the problem (12) on $\boldsymbol{R}$ has a stable standing front solution $S(x)$ given by

$$S(x) = \tanh\frac{x}{2},$$

which is linearly stable and connecting $S_\pm = \pm 1$ ([4]).

Let us consider the dynamics of a front solution $u(t,x) \sim S(x-l)$ for $l \gg 1$ with the boundary condition

$$u_x = \beta(u+1) + \delta a_-, \quad x = 0, \tag{13}$$

where $\beta > 0$ and $|\delta| \ll 1$. The asymptotic form of $S(x)$ as $x \to \pm\infty$ is

$$S(x) \to \mp 2e^{-|x|} \pm 1.$$

On the other hand, the eigenfunction $\phi^*(x)$ is easily obtained as

$$\phi^*(x) = \frac{3}{2} S_x(x)$$

together with its asymptotic form $\phi^*(x) = 3e^{-|x|}$ as $x \to \pm\infty$. Thus, all of necessary quantities in Theorem 2.2 are given by

$$D = 1, \quad \alpha_- = 1, \quad a_- = 2, \quad a_-^* = 3.$$

Hence, M_- can be calculated as

$$M_- = D \cdot a_- \cdot a_-^* = 1 \cdot 2 \cdot 3 = 6$$

and we get the equation of l as

$$\dot{l} := \frac{dl}{dt} = \frac{12}{1+\beta}((\beta-1)e^{-2l} + \delta e^{-l})(1 + O(e^{-l} + \delta^2)). \tag{14}$$

The effect of boundary conditions on the dynamics of a front solution for the Allen-Chan equation is analyzed by considering the function

$$q_{AC}(l;\beta,\delta) := (\beta-1)e^{-l} + \delta.$$

We consider four cases with respect to the values of β and δ as follows:

(I) If $\beta > 1$ and $\delta > 0$, then $q_{AC}(l) > 0$ for any l. From (14), this implies that the velocity $\dot{l}$ is positive for any position $l(t) > 0$, which means that the front solution goes away from the boundary $x = 0$. The direction of this motion is same as the Dirichlet boundary case($\beta = +\infty$).

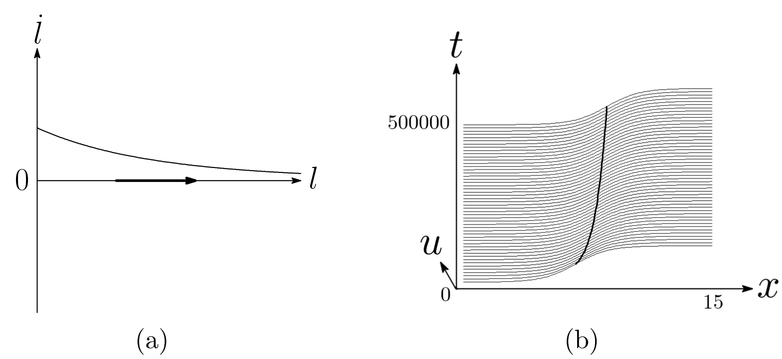
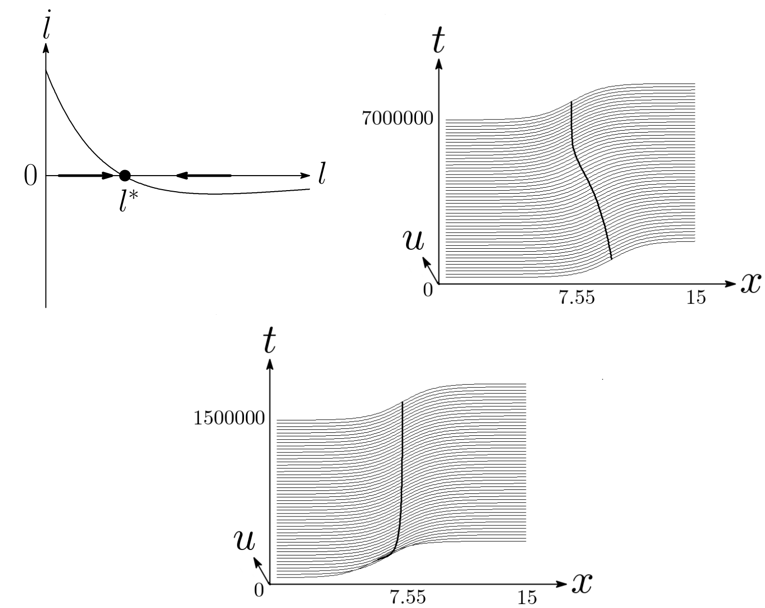


FIGURE 3.1. (a) The flow of the ODE (14). ((b) The movement of a front solution for (12) with (13) in the case of $\beta > 1$ and $\delta > 0$ ($\beta = 20.0$ and $\delta = 0.01$). Solid line denotes the 0-level line $\{x \in \boldsymbol{R}_+;\ u(t,x) = 0\}$.

(II) If $\beta > 1$ and $\delta < 0$, the sign of $q_{AC}(l)$ depends on the value of l. Now, we define

$$l^* := \log \frac{1-\beta}{\delta}.$$

If $l > l^*$, then $q_{AC}(l) < 0$, which means $\dot{l} < 0$. If $l < l^*$, then $q_{AC}(l) > 0$, which means $\dot{l} > 0$. That is, $l = l^*$ is a stable equilibrium of (14), which corresponds to the stable stationary front solution $\bar{u}(x) = S(x - l^*) + O(e^{-l^*} + \delta)$.



(a) (b)

FIGURE 3.2. (a) Flows of the ODE (14). (b) Movements of front solutions for (12) with (13) in the case of $\beta > 1$ and $\delta < 0$ ($\beta = 20.0$ and $\delta = -0.01$). Solid lines denote the 0-level lines $\{x \in \boldsymbol{R}_+;\ u(t,x) = 0\}$.

(III) If $0 < \beta < 1$ and $\delta > 0$, the sign of $q_{AC}(l)$ depends on the value of l, Now, we define

$$l^{**} := \log \frac{1-\beta}{\delta}.$$

If $l > l^{**}$, then $q_{AC}(l) > 0$, which means $\dot{l} > 0$. If $l < l^{**}$, then $q_{AC}(l) < 0$, which means $\dot{l} < 0$. That is, $l = l^{**}$ is an unstable equilibrium of (14), which corresponds to an unstable stationary front solution $\bar{u}(x) = S(x - l^{**}) + O(e^{-l^{**}} + \delta)$.

(IV) If $0 < \beta < 1$ and $\delta < 0$, then $q_{AC}(l) < 0$ for any l. From (14), this implies that the velocity $\dot{l}$ is negative for any position $l(t) > 0$, which means that the front solution approaches the boundary $x = 0$. The direction of this motion is same as the Neumann boundary case($\beta = 0$).

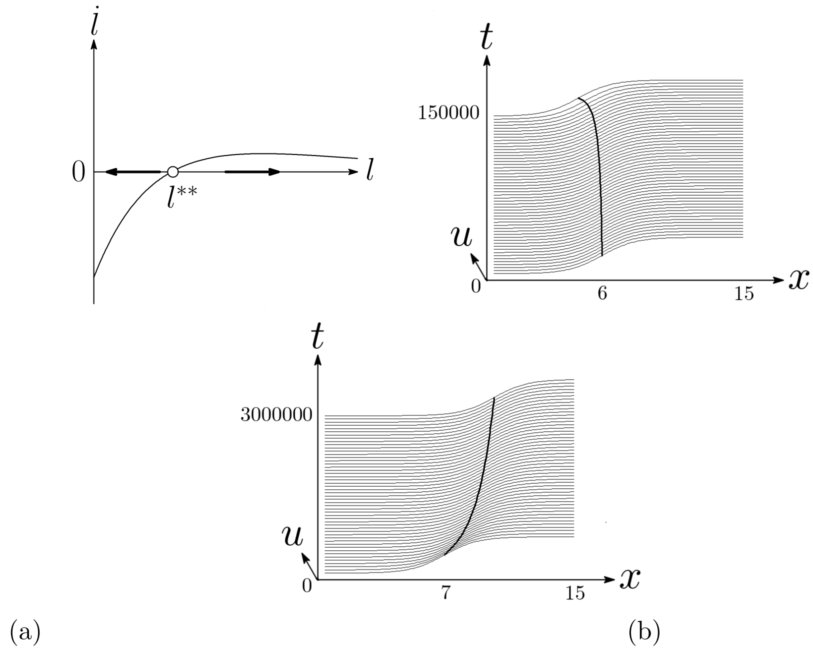


FIGURE 3.3. (a) Flow of the ODE (14). (b) Movements of front solutions for (12) with (13) in the case of $\beta < 1$ and $\delta > 0$ ($\beta = 0.3$ and $\delta = 0.001$).

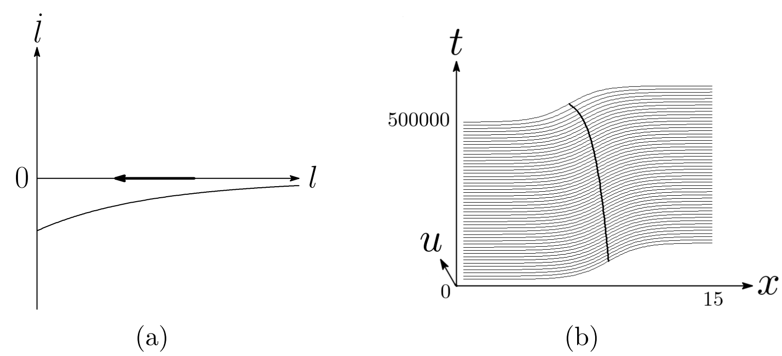


FIGURE 3.4. (a) The flow of the ODE (14). (b) The movement of a front solution for (12) with (13) in the case of $\beta < 1$ and $\delta < 0$ ($\beta = 0.3$ and $\delta = -0.001$).

3.2. **Dynamics of a pulse solution for the Gray-Scott model.** In this subsection, we consider the dynamics of a pulse solution for the Gray-Scott model

$$\begin{cases} u_t = u_{xx} - uv^2 + A(1-u), \\ v_t = D_v v_{xx} - Bv + uv^2, \end{cases} \quad t > 0, \quad x \in \boldsymbol{R}_+ \tag{15}$$

where $\boldsymbol{u} := {}^t(u, v) \in \boldsymbol{R}^2$, A and B are positive constants. For the problem (15) on $\boldsymbol{R}$, Doelmann *et al*. ([2]) showed the existence of a stable standing pulse solution, say $S(x)$ under the assumptions

$$D_v = A = \varepsilon^2, \quad B = \varepsilon^\nu$$

for a small parameter $\varepsilon > 0$ and $0 < \nu < 1$. The adjoint eigenfunction $\phi^*(x)$ was also calculated under the same assumptions as above in [3]. Since $S(x) \to {}^t(1,0) =: S^*$ as $|x| \to +\infty$, let us consider the dynamics of pulse solutions $u(t,x) \sim S(x-l)$ for $l \gg 1$ with the boundary condition

$$\boldsymbol{u}_x = \beta(\boldsymbol{u} - S^*) + \delta \boldsymbol{a}, \quad x = 0, \tag{16}$$

where $\beta > 0$ and $|\delta| \ll 1$. The asymptotic forms $S(x)$ and $\phi^*(x)$ are respectively given by [3]

$$S(x) \to e^{-\varepsilon|x|}\boldsymbol{a} + S^*, \quad \phi^*(x) \to e^{-\varepsilon|x|}\boldsymbol{a}^*$$

as $x \to \pm\infty$, and all of necessary quantities in Theorem 2.1 are given by

$$D = diag(1, \varepsilon^2), \quad \alpha = \varepsilon, \quad \boldsymbol{a} = {}^t(-a_+,\ 0), \quad \boldsymbol{a}^* = {}^t(-\varepsilon^{3/4}a_+^*,\ 0)$$

for positive constants $a_+,\ a_+^*$. Hence, the constant M_0 in Theorem 2.1 is

$$M_0 = \langle\, D\boldsymbol{a}, \boldsymbol{a}^*\, \rangle = \varepsilon(-a_+)(-\varepsilon^{3/4}a_+^*) = \varepsilon^{7/4}a_+a_+^* > 0.$$

Thus, we get the equation of l as

$$\dot{l} := \frac{dl}{dt} = \frac{2\varepsilon}{\varepsilon + \beta} M_0((\varepsilon - \beta)e^{-2\varepsilon l} - \delta e^{-\varepsilon l})(1 + O(e^{-\varepsilon l} + \delta^2)). \tag{17}$$

The effect of boundary conditions on the dynamics of a pulse solution for the Gray-Scott model (17) is considered to be captured in

$$q_{GS}(l; \beta, \delta) := (\varepsilon - \beta)e^{-\varepsilon l} - \delta.$$

We consider four cases about the value of β and δ as follows:
(I) If $0 < \beta < \varepsilon$ and $\delta < 0$, then $q_{GS}(l) > 0$ for any l. From (17), this implies that the velocity $\dot{l}$ is positive for any position $l(t)$, which means that the pulse solution goes away from the boundary $x = 0$.

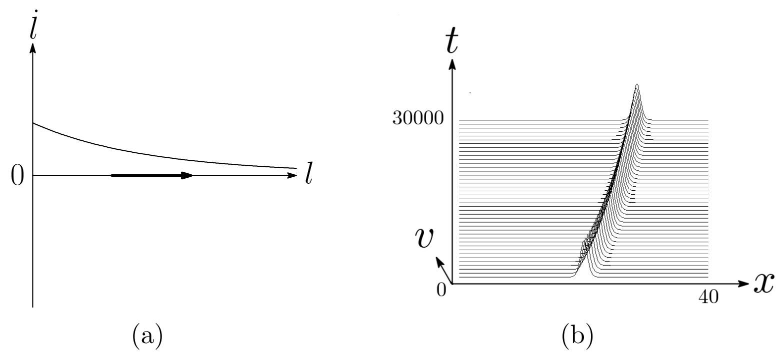
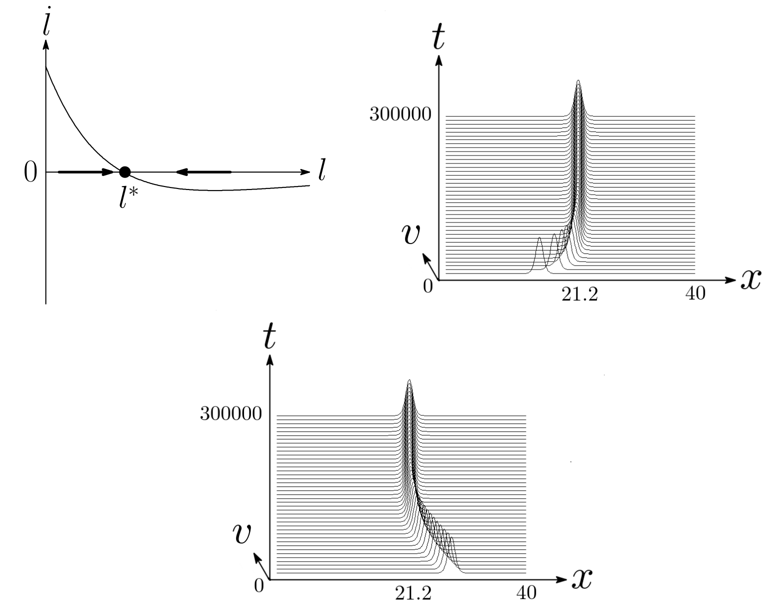


FIGURE 3.5. (a) The flow of the ODE (17). (b) The movement of a pulse solution for (15) with (16) in the case of $\beta < \varepsilon$ and $\delta < 0$ ($\beta = 0.01,\ \delta = -0.01$ and $\varepsilon = 0.1$).

(II) If $0 < \beta < \varepsilon$ and $\delta > 0$, the sign of $q_{GS}(l)$ depends on the value of l. Now, we define

$$l^* := \frac{1}{\varepsilon} \log \frac{\varepsilon - \beta}{\delta}.$$

If $l > l^*$, then $q_{GS}(l) < 0$, which means $\dot{l} < 0$. If $l < l^*$, then $q_{GS}(l) > 0$, which means $\dot{l} > 0$. That is, $l = l^*$ is a stable equilibrium of (17), which shows the existence of a stable stationary solution $\bar{\boldsymbol{u}}(x) = S(x - l^*) + O(e^{-\varepsilon l^*} + \delta)$.



(a) (b)

FIGURE 3.6. (a) Flows of the ODE (17). (b) Movements of pulse solutions for (15) with (16) in the case of $\beta < \varepsilon$ and $\delta > 0$ ($\beta = 0.01,\ \delta = 0.01$ and $\varepsilon = 0.1$).

(III) If $\beta > \varepsilon$ and $\delta < 0$, the sign of $q_{GS}(l)$ depends on the value of l. Now, we define

$$l^{**} := \frac{1}{\varepsilon} \log \frac{\varepsilon - \beta}{\delta}.$$

If $l > l^{**}$, then $q_{GS}(l) > 0$, which means $\dot{l} > 0$. If $l < l^{**}$, then $q_{GS}(l) < 0$, which means $\dot{l} < 0$. That is, $l = l^{**}$ is an unstable equilibrium of (17), which shows the existence of an unstable stationary solution $\bar{\boldsymbol{u}}(x) = S(x - l^{**}) + O(e^{-\varepsilon l^{**}} + \delta)$

(IV) If $\beta > \varepsilon$ and $\delta > 0$, then $q_{GS}(l) < 0$ for any l. From (17), this implies that the velocity $\dot{l}$ is negative for any position $l(t)$, which means that the pulse solution approaches the boundary $x = 0$.

4. **Proofs.**

4.1. **Proof of Theorem 2.1.**

Define the function $g(x) = g(x; l)$ on $\boldsymbol{R}_+$ by the solution of

$$\left\{ \begin{array}{lcl} \mathbf{0} & = & D g_{xx} + F'(\mathbf{0}) g,\ x > 0, \\ g_x & = & \beta g + \beta S - S_x + \delta \boldsymbol{a}\ at\ x = 0,\ g(\infty) = \mathbf{0}, \end{array} \right.$$

where $S = S(x-l)$, whose existence is due to A2)'. Since the fundamental functions of the ODE $\mathbf{0} = D\boldsymbol{u}_{xx} + F'(\mathbf{0})\boldsymbol{u}$ are $\boldsymbol{m}_j(x)$, $g(x; l) = \sum_{j=1}^{N} c_j(l) \boldsymbol{m}_j(x)$ holds by the

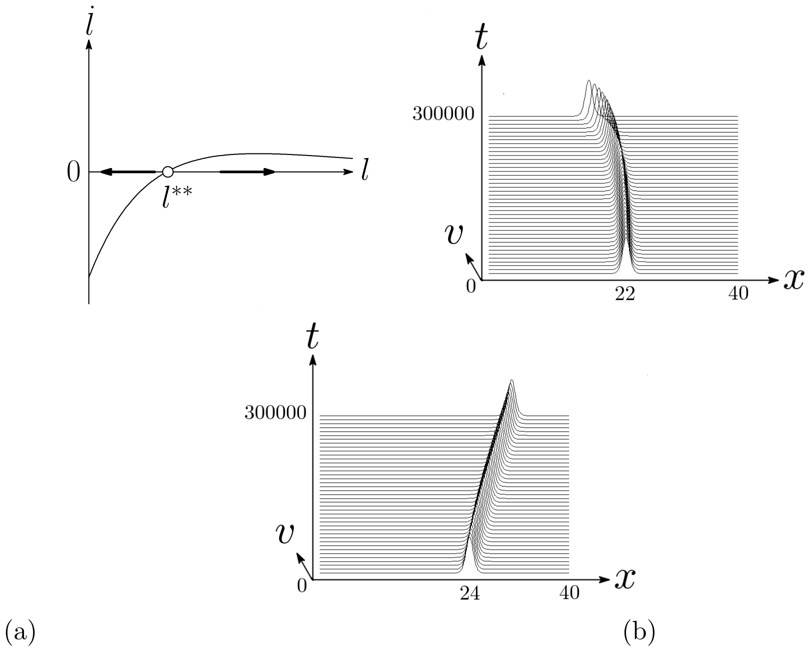


FIGURE 3.7. (a) Flows of the ODE (17). (b) Movements of front solutions for (15) with (16) in the case of $\beta > \varepsilon$ and $\delta < 0$ ($\beta = 0.2,\ \delta = -0.01$ and $\varepsilon = 0.1$).

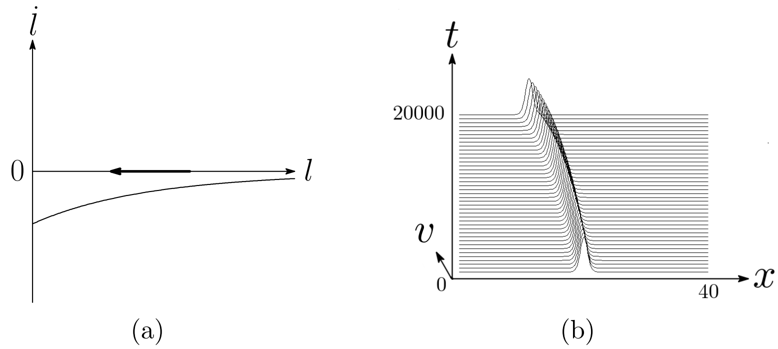


FIGURE 3.8. (a) The flow of the ODE (17). (b) The movement of a front solution for (12) with (13) in the case of $\beta > \varepsilon$ and $\delta > 0$ ($\beta = 0.2,\ \delta = 0.01$ and $\varepsilon = 0.1$).

condition $g(+\infty) = \mathbf{0}$. In a neighborhood of $x = 0$, $S(x - l)$ has the asymptotic profile as $l \to \infty$

$$S(x-l) \to e^{\alpha(x-l)}\boldsymbol{a},\ S_x(x-l) \to \alpha e^{\alpha(x-l)}\boldsymbol{a}.$$

Hence

$$c_1(l) = C(l) := \frac{1}{\alpha+\beta}\{(\alpha-\beta)e^{-\alpha l} - \delta\},\ |c_j(l)| \le O(e^{-\alpha_2 l})\ (j \ge 2)$$

as $l \to \infty$ in order that $S(x-l)+g(x;l)$ satisfies the boundary condition (7). Thus, we see $g(x;l) = C(l)e^{-\alpha x}\boldsymbol{a} + O(e^{-\alpha_2'(l+x)})$ with $\|g\|_+ \le O(e^{-\alpha l}+\delta)$ for $\alpha_2' > \alpha$. We put the solution $\boldsymbol{u}(t,x)$ of (3) with (7) as $\boldsymbol{u}(t,x) = S(x-l(t)) + g(x;l(t)) + \boldsymbol{v}(t,x)$. Substituting it into (3), we have

$$\dot{l}\{-S_x + g_l\} + \boldsymbol{v}_t = L(l)\boldsymbol{v} + L(l)g + G(g+\boldsymbol{v}), \tag{18}$$

where $L(l) := D\partial_x^2 + F'(S(x-l))$, $G(\boldsymbol{u}) = G(\boldsymbol{u};l) := F(S(x-l)+\boldsymbol{u}) - F(S(x-l)) - F'(S(x-l))\boldsymbol{u} = O(|\boldsymbol{u}|^2)$ and $\boldsymbol{v}$ satisfies the boundary condition at $x=0$

$$\boldsymbol{v}_x = \beta\boldsymbol{v}. \tag{19}$$

Lemma 4.1. *Let L_0 be the operator on $X_+ = L^2(\boldsymbol{R}_+)$ with the boundary condition (19). Then $\{\lambda \in \boldsymbol{C};\ Re(\lambda) > -\gamma_1\} \subset \rho(L_0)$, the resolvent set of L_0 and $\|(L_0-\lambda)^{-1}\|_+ \le \dfrac{C_1}{|\lambda|+\gamma_1}$ holds for $\lambda \in \boldsymbol{C}$ with $Re(\lambda) > -\gamma_1$, where $0 < \gamma_1 < \gamma_0$ and $C_1 > 0$.*

Proof. Give arbitrary $\boldsymbol{f} \in X_+$, $\lambda \in \boldsymbol{C}$ with $Re(\lambda) > -\gamma_1$ and consider the equation $(L_0-\lambda)\boldsymbol{u} = \boldsymbol{f}$. We define the operators K_N and K_D by

$$K_N[\boldsymbol{f}](x) := \begin{cases} \boldsymbol{f}(x), & x > 0, \\ \boldsymbol{f}(-x), & x < 0, \end{cases} \quad K_D[\boldsymbol{f}](x) := \begin{cases} \boldsymbol{f}(x), & x > 0. \\ -\boldsymbol{f}(-x), & x < 0, \end{cases}$$

Then $K_N[\boldsymbol{f}]$, $K_D[\boldsymbol{f}] \in L^2(\boldsymbol{R})$ and by the assumption A2)', there exist unique solutions W^N, $W^D \in H^2(\boldsymbol{R})$ satisfying $(L_0-\lambda)W^N = K_N[\boldsymbol{f}]$ and $(L_0-\lambda)W^D = K_D[\boldsymbol{f}]$, respectively. Since $(L_0-\lambda)(W^D - W^N) = \boldsymbol{0}$ holds for $x > 0$, $W^D(x)$ is represented as $W^D(x) = W^N(x) + \sum_{j=1}^N a_j \boldsymbol{k}_j(x;\lambda)$ for $a_j \in \boldsymbol{R}$. Since $W^D(0) = \boldsymbol{0}$ and $\partial_x W^N(0) = \boldsymbol{0}$ hold, we see

$$W^N(0) = -\sum_{j=1}^N a_j \boldsymbol{k}_j(0;\lambda) = -\sum_{j=1}^N a_j \boldsymbol{b}_j(\lambda),\ \partial_x W^D(0) = \sum_{j=1}^N a_j (k_j(\lambda) - \mu_j(\lambda))\boldsymbol{b}_j(\lambda) \tag{20}$$

and there exists a matrix U_0 such that $U_0 \partial_x W^D(0) = W^N(0)$. Here, we note that we may assume $k_j(\lambda) - \mu_j(\lambda) \ne 0$ and we can take U_0. In fact, if $k_j(\lambda) - \mu_j(\lambda) = 0$ for some j, then it suffices to replace the fundamental function by $\boldsymbol{k}_j(x;\lambda) = (x+a)^{k_j(\lambda)} e^{-\mu_j(\lambda)x} \boldsymbol{b}_j(\lambda)$ for an appropriate a such that $k_j(\lambda) - a\mu_j(\lambda) \ne 0$. Let $U(x)$ $(x \ge 0)$ be a smooth matrix function satisfying $U(0) = U_0$ and $U(x) = 0$ for $x \ge \eta$ for a sufficiently small $\eta > 0$ and define

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{u}[\boldsymbol{f}](x) \\ &:= W^N(x) + \beta U(x) W^D(x) \\ &= (L_0-\lambda)^{-1} K_N[\boldsymbol{f}](x) + \beta U(x)(L_0-\lambda)^{-1} K_D[\boldsymbol{f}](x) \end{aligned}$$

for $x > 0$. Then

$$\begin{aligned} \partial_x \boldsymbol{u}(0) &= \partial_x W^N(0) + \beta U_0 \partial_x W^D(0) + \beta \partial_x U(0) W^D(0) \\ &= \beta W^N(0) \\ &= \beta\{W^N(0) + \beta U(0) W^D(0)\} \\ &= \beta \boldsymbol{u}(0) \end{aligned}$$

holds. That is, the boundary condition (19) is satisfied by $\boldsymbol{u}$.

On the other hand, for $x > 0$,

$$(L_0 - \lambda)\boldsymbol{u} = \boldsymbol{f} + \beta(L_0 - \lambda)U(x)(L_0 - \lambda)^{-1}K_D[\boldsymbol{f}] =: (Id + B)\boldsymbol{f}$$

holds, where Id denotes identity and $B := \beta(L_0 - \lambda)U(x)(L_0 - \lambda)^{-1}K_D$. Here we can take $U(x)$ by taking appropriately small η such that $\|B\|_+$ becomes small, which means $Id+B$ is invertible. Then $\boldsymbol{u} = \boldsymbol{u}[(Id+B)^{-1}\boldsymbol{f}]$ satisfies $(L_0-\lambda)\boldsymbol{u} = \boldsymbol{f}$ for $x > 0$ and the boundary condition (19). Thus $\lambda \in \rho(L_0)$ and the estimate of the resolvent is easily shown. The proof is completed. □

Lemma 4.2. *Let $L_b(l)$ be the operator $L(l)$ on X_+ with the boundary condition (19). Then the spectrum $\sigma(l)$ of $L_b(l)$ consists of $\sigma_0(l) \subset \{\lambda \in \boldsymbol{C};\ |\lambda| < O(e^{-\frac{1}{2}\alpha l})\}$ and $\sigma_1(l) \subset \{\lambda \in \boldsymbol{C};\ Re(\lambda) < -\gamma_1\}$.*

Proof. Fix $\bar{l} > 0$ sufficiently large and take $l > \bar{l}$. Let $B(x;l) := F'(S(x-l)) - F'(\boldsymbol{0})$. Then $L_b(l) = L_0 + B(l)$, where $L_0 = D\partial_x^2 + F'(\boldsymbol{0})$ is the operator on X_+ with (19). Let $l_j := \frac{j}{5}l$ $(j = 1, 2, 3, 4)$, $I_1 := (l_2, l_3)$, $I_2 := (l_1, l_4)$ and $\chi_1(x)$, $\chi_2(x)$, $\chi_3(x)$ be smooth functions satsfying

$$\chi_1(x) = \begin{cases} 1, & x \le l_2, \\ 0, & x \ge l_3, \end{cases} \quad \chi_2(x) = \begin{cases} 0, & x \le l_2, \\ 1, & x \ge l_3, \end{cases} \quad 0 \le \chi_j(x) \le 1,\ \chi_1(x) + \chi_2(x) \equiv 1$$

and

$$\chi_3(x) = \begin{cases} 1, & x \in I_1, \\ 0, & x \notin I_2, \end{cases} \quad 0 \le \chi_3(x) \le 1.$$

We consider the equation $(L_b(l)-\lambda)\boldsymbol{u} = \boldsymbol{f}$ for $\boldsymbol{f} \in X_+$. Define the operator $D(\lambda)$ on X_+ by $D(\lambda) := \chi_1(L_0-\lambda)^{-1} + \chi_2(L(l)-\lambda)^{-1}K_N$ for $\lambda \in \rho(L(l)) \cap \rho(L_0)$, where $L(l)$ is regarded as the operator on $L^2(\boldsymbol{R})$ here. Then for $\boldsymbol{f} \in X_+$, $(L_b(l)-\lambda)D(\lambda)\boldsymbol{f} = \boldsymbol{f}$ holds for $x \ge l_3$. For $0 < x < l_2$, $D(\lambda)\boldsymbol{f} = (L_0 - \lambda)^{-1}\boldsymbol{f}$ and hence

$$\begin{aligned} (L_b(l) - \lambda)D(\lambda)\boldsymbol{f} &= (L_0 + B(l) - \lambda)(L_0 - \lambda)^{-1}\boldsymbol{f} \\ &= \boldsymbol{f} + B(l)(L_0 - \lambda)^{-1}\boldsymbol{f} \\ &= \boldsymbol{f} + O(e^{-\alpha l})\boldsymbol{f} \end{aligned}$$

holds because $|B(x;l)| \le O(e^{-\alpha l})$ for $0 < x < l_2$.

Let us consider the case for $l_2 \le x \le l_3$. Let $\boldsymbol{u}_1 := (L_0 - \lambda)^{-1}\boldsymbol{f}$ and $\boldsymbol{u}_2 := (L(l) - \lambda)^{-1}K_N[\boldsymbol{f}]$. Then for $x > 0$,

$$\begin{aligned} (L_0 - \lambda)\boldsymbol{u}_2 &= (L_b(l) - B(l) - \lambda)(L(l) - \lambda)^{-1}K_N[\boldsymbol{f}] \\ &= \boldsymbol{f} - B(l)(L(l) - \lambda)^{-1}K_N[\boldsymbol{f}] \end{aligned}$$

holds. Since $|B(x;l)| \le O(e^{-\alpha l})$ for $x \in I_2 = (l_1, l_4)$ and hence $|\chi_3(x)B(x;l)| \le O(e^{-\alpha l})$ holds for $x \in \boldsymbol{R}$, we have $\|\chi_3 B(l)(L(l) - \lambda)^{-1}\|_+ \le \dfrac{C_2 e^{-\alpha l}}{|\lambda|}$ and $\|\chi_3 B(l)(L(l)-\lambda)^{-1}\|_+ \le e^{-\frac{1}{2}\alpha l}$ holds for $|\lambda| \ge C_2 e^{-\frac{1}{2}\alpha l}$. Defining $\Lambda(l) := \{\lambda \in \boldsymbol{C};\ Re(\lambda) \ge -\gamma_1,\ |\lambda| \ge C_2 e^{-\frac{1}{2}\alpha l}\}$ and $\boldsymbol{g}_2 := -\chi_3 B(l)(L(l) - \lambda)^{-1}K_N[\boldsymbol{f}]$, we let $W_2 \in H^2(\boldsymbol{R}_+)$ be the solution of $(L_0 - \lambda)W_2 = \boldsymbol{g}_2$ for $\lambda \in \Lambda(l)$, which satisfies

$$\|W_2\|_{H^2(\boldsymbol{R}_+)} \le C_3\|\boldsymbol{g}_2\|_+ \le C_4 e^{-\frac{1}{2}\alpha l}\|\boldsymbol{f}\|_+$$

for C_3 and $C_4 > 0$. Since $(L_0 - \lambda)(\boldsymbol{u}_2 - \boldsymbol{u}_1) = \boldsymbol{g}_2$ holds, we see $(L_0 - \lambda)(\boldsymbol{u}_2 - \boldsymbol{u}_1 - W_2) = \boldsymbol{0}$. Then $\boldsymbol{u}_2 - \boldsymbol{u}_1 - W_2$ is given by the fundamental functions $\boldsymbol{k}_j(x;\lambda)$

$(j = 1, \cdots, 2N)$ of the ODE $(L_0 - \lambda)\boldsymbol{u} = \boldsymbol{0}$ as

$$\boldsymbol{u}_2(x) - \boldsymbol{u}_1(x) - W_2(x) = \sum_{j=1}^{2N} a_j \boldsymbol{k}_j(x;\lambda) \tag{21}$$

for $x > 0$. Here we note that $\mu_j = \mu_j(\lambda)$ are positive constants satisfying $\mu_j(\lambda) > \gamma_2$ for a constant $\gamma_2 > 0$ independent of $\lambda \in \Lambda(l)$ by $Re(\lambda) \geq -\gamma_1$. Since $\|\boldsymbol{u}_2(x) - \boldsymbol{u}_1(x) - W_2(x)\|_{H^2(\boldsymbol{R}_+)} \leq C_5\|\boldsymbol{f}\|_+$ holds, $\boldsymbol{u}_2(x) - \boldsymbol{u}_1(x) - W_2(x)$ is bounded on I_2, specially $|\boldsymbol{u}_2(x) - \boldsymbol{u}_1(x) - W_2(x)| \leq C_6\|\boldsymbol{f}\|_+$ holds at $x = l_1,\ l_4$ for $C_5,\ C_6 > 0$. This fact and (21) imply $\|\boldsymbol{u}_2 - \boldsymbol{u}_1 - W_2\|_{H^2(I_1)} \leq C_7 e^{-\frac{1}{5}\gamma_2 l}\|\boldsymbol{f}\|_+$ for $C_7 > 0$. Then

$$\begin{aligned}\chi_1(x)\boldsymbol{u}_1(x) + \chi_2(x)\boldsymbol{u}_2(x) &= \chi_1(x)\boldsymbol{u}_1(x) + \chi_2(x)\{\boldsymbol{u}_1(x) + \boldsymbol{u}_2(x) - \boldsymbol{u}_1(x)\}\\ &= \boldsymbol{u}_1(x) + \chi_2(x)\{\boldsymbol{u}_2(x) - \boldsymbol{u}_1(x)\}\\ &= \boldsymbol{u}_1(x) + W_3(x)\end{aligned}$$

holds on I_1, where $W_3(x) := \chi_2(x)\{\boldsymbol{u}_2(x) - \boldsymbol{u}_1(x)\} = \chi_2(x)\{\boldsymbol{u}_2(x) - \boldsymbol{u}_1(x) - W_2(x) + W_2(x)\}$ satisfying $\|W_3\|_{H^2(I_1)} \leq C_8 e^{-\gamma_3 l}\|\boldsymbol{f}\|_+$ for $0 < \gamma_3 \leq \min\{\frac{1}{2}\alpha, \frac{1}{5}\gamma_2\}$. Therefore, it follows that

$$\begin{aligned}(L_b(l) - \lambda)D(\lambda)\boldsymbol{f} &= (L_b(l) - \lambda)(\boldsymbol{u}_1 + W_3)\\ &= (L_0 - \lambda + B(l))\boldsymbol{u}_1 + (L_b(l) - \lambda)W_3\\ &= \boldsymbol{f} + B(l)\boldsymbol{u}_1 + (L_b(l) - \lambda)W_3\\ &= \boldsymbol{f} + O(e^{-\gamma_3 l})\boldsymbol{f}\end{aligned}$$

on I_1.

Thus, we have for any $x > 0$

$$(L_b(l) - \lambda)D(\lambda)\boldsymbol{f} = \boldsymbol{f} + B_1(\lambda; l)\boldsymbol{f}$$

with $\|B_1(\lambda; l)\|_+ \leq O(e^{-\gamma_3 l})$. By taking $\bar{l}$ so large that $\|B_1(\lambda; l)\|_+ \leq \frac{1}{2}$ for $l > \bar{l}$, we see $(Id + B_1(l))^{-1}$ exists and

$$(L_b(l) - \lambda)D(\lambda)(Id + B_1(\lambda; l))^{-1} = Id$$

holds, which means $\lambda \in \rho(L_b(l))$ and

$$(L_b(l) - \lambda)^{-1} = D(\lambda)(Id + B_1(\lambda; l))^{-1}.$$

Thus $\Lambda(l) \subset \rho(L_b(l))$ and the proof is completed □

Let $Q_b(l)$, $R_b(l)$ respectively be projections on X_+ corresponding to spectral sets $\sigma_0(l)$, $\sigma_1(l)$ and let Q, R respectively be projections on $L^2(\boldsymbol{R})$ corresponding to spectral sets σ_0, σ_1 of L. We also define $E_b(l) := Q_b(l)X_+$, $E_b^{\perp}(l) := R_b(l)X_+$, $E := QL^2(\boldsymbol{R})$ and $E^{\perp} := RL^2(\boldsymbol{R})$. Then the assumption A2) implies $Q\boldsymbol{u} = \langle\, \boldsymbol{u}, \phi^*\, \rangle_{L^2} S_x$ for $\boldsymbol{u} \in L^2(\boldsymbol{R})$.

By Lemma 4.2, the resolvent $(L_b(l) - \lambda)^{-1}$ for $\lambda \in \Lambda(l)$ is expressed as

$$(L_b(l) - \lambda)^{-1} = D(\lambda)(Id + B_2(\lambda; l))$$

with $\|B_2(\lambda; l)\|_+ \leq O(e^{-\gamma_3 l})$. Then taking the Dunford integral around $\sigma_0(l)$, we easily know $Q_b(l) = \chi_2 Q(l)K_N + B_3(\lambda; l)$ with $\|B_3(\lambda; l)\|_+ \leq O(e^{-\gamma_3 l})$, where $\{Q(l)\boldsymbol{u}\}(x) := \langle\, \boldsymbol{u}, \phi^*(\cdot - l)\, \rangle_{L^2} S_x(x - l)$ for $\boldsymbol{u} \in L^2(\boldsymbol{R})$. This means $\|Q_b(l) - Q(l)K_N\|_+ \leq O(e^{-\gamma_3 l}) \ll 1$ and therefore $E_b(l)$ and $Q(l)K_N X_+$ are homeomorphic each other. Thus, there exist $\lambda_0(l) \in \boldsymbol{R}$, $\phi(l) \in X_+$ such that $\sigma_0(l) = \{\lambda_0(l)\}$ with $|\lambda_0(l)| \leq O(e^{-\frac{1}{2}\alpha l})$ and $E_b(l) = span\{\phi(l)\}$ with $\phi(l)(x) = \chi_2(x)S_x(x - l) + O(e^{-\gamma_3 l}) = S_x(x - l) + O(e^{-\gamma_3 l})$. The adjoint operator $L_b^*(l)$ of $L_b(l)$ also has the same properties, specially there exists $\phi^*(l) \in X_+$ satisfying $L_b^*(l)\phi^*(l) = \lambda_0(l)\phi^*(l)$, $\langle\ \phi(l), \phi^*(l)\ \rangle_+ = 1$ and $\phi^*(l)(x) = \phi^*(x-l) + O(e^{-\gamma_3 l})$. We note that $E_b^\perp(l)$ is given by $E_b^\perp(l) = \{\boldsymbol{v} \in X_+;\ \langle\ \boldsymbol{v}, \phi^*(l)\ \rangle_+ = 0\}$.

Now we consider the solution $\boldsymbol{u} = \boldsymbol{u}(t,x)$ of (3) with the boundary condition (7) and put $\boldsymbol{u}(t,x) = S(x - l(t)) + g(x; l(t)) + \boldsymbol{v}(t,x)$ with $\boldsymbol{v}(t) \in E_b^\perp(l)$. Then (3) becomes (18), that is,

$$\dot{l}\{-S_x + g_l\} + \boldsymbol{v}_t = L_b(l)\boldsymbol{v} + L(l)g + G(g + \boldsymbol{v}), \tag{22}$$

where $S_x = S_x(x - l(t))$. Here we note to distinguish between the operators $L_b(l)$ and $L(l)$ by regarding them as the operator on X_+ with the boundary condition (19) and the operator on $L^2(\boldsymbol{R})$ restricted within X_+, respectively. By operating $Q_b(l)$ and $R_b(l)$ on (22), we have

$$\begin{cases} \dot{l} & = & H, \\ R_b(l)\boldsymbol{v}_t & = & L_b(l)\boldsymbol{v} + J, \end{cases} \tag{23}$$

where

$$\begin{aligned} H &= H(l, \boldsymbol{v}) := -\frac{\langle\ L(l)g + G(g + \boldsymbol{v}), \phi^*(l)\ \rangle_+}{\langle\ S_x, \phi^*(l)\ \rangle_+ - \langle\ g_l, \phi^*(l)\ \rangle_+}, \\ J &= J(l, \boldsymbol{v}) := R_b(l)\{L(l)g + G(g + \boldsymbol{v})\} + H(l, \boldsymbol{v})R_b(l)\{S_x - g_l\}. \end{aligned}$$

Lemma 4.3.

$$H(l, \boldsymbol{v}) = 2\alpha M_0 C(l) e^{-\alpha l}(1 + O(e^{-\gamma_4 l})) + O((e^{-2\alpha l} + \delta^2)e^{-\alpha l} + \|\boldsymbol{v}\|_\infty^2)$$

holds for $\gamma_4 > 0$.

Proof. First, we will show $|\langle\ G(g + \boldsymbol{v}), \phi^*(l)\ \rangle_+| \le O((e^{-2\alpha l} + \delta^2)e^{-\alpha l} + \|\boldsymbol{v}\|_\infty^2)$. Since $|G(g + \boldsymbol{v})| \le C_8(|g|^2 + |\boldsymbol{v}|^2)$ holds, it follows that

$$\begin{aligned} |\langle\ G(g + \boldsymbol{v}), \phi^*(l)\ \rangle_+| &\le C_8 \int_0^\infty \{|g(x)|^2 + |\boldsymbol{v}(x)|^2\}|\phi^*(l)(x)|dx \\ &\le C_8 \int_0^\infty |g(x)|^2|\phi^*(l)(x)|dx + C_8\|\boldsymbol{v}\|_\infty^2 \int_0^\infty |\phi^*(l)(x)|dx \\ &\le C_9|C(l)|^2 \int_0^\infty e^{-2\alpha x}e^{-\alpha|x-l|}dx + C_9\|\boldsymbol{v}\|_\infty^2 \\ &\le O(|C(l)|^2 e^{-\alpha l} + \|\boldsymbol{v}\|_\infty^2) \\ &\le O((e^{-2\alpha l} + \delta^2)e^{-\alpha l} + \|\boldsymbol{v}\|_\infty^2) \end{aligned}$$

because $g(x; l) = C(l)e^{-\alpha x}\boldsymbol{a} + O(e^{-\alpha_2'(l+x)})$ with $\alpha_2' > \alpha$ and $|\phi^*(l)(x)| \le O(e^{-\alpha|x-l|})$ hold.

Next, we show $\langle\ L(l)g, \phi^*(l)\ \rangle_+ = -2\alpha M_0 C(l)e^{-\alpha l}(1 + O(e^{-\gamma_4 l}))$ for $\gamma_4 > 0$. As in (21), let $\boldsymbol{k}_j^*(x;\lambda) = (x+1)^{k_j}e^{\pm\mu_j x}\boldsymbol{b}_j^* \in \boldsymbol{R}^N$ be the fundamental solutions of the ODE $(L_0^* - \lambda)\boldsymbol{u} = 0$, where $\mu_j = \mu_j(\lambda)$, $k_j = k_j(\lambda)$ and $\boldsymbol{b}_j^* = \boldsymbol{b}_j^*(\lambda)$ with $0 < \mu_1(\lambda) \le \mu_2(\lambda) \le \cdots$. Note that $\mu_j(0) = \alpha_j$ and $\boldsymbol{b}_1^*(0) = \boldsymbol{a}^*$. Since $L_b^*(l)\phi^*(l) = \lambda_0(l)\phi^*(l)$ and $\phi^*(l)(x) = \phi^*(x-l) + O(e^{-\gamma_3 l})$ hold, we see $\phi^*(l)(x) = -e^{\mu_1(\lambda_0(l))(x-l)}\boldsymbol{b}_1^*(\lambda_0(l)) + O(e^{\mu_2(\lambda_0(l))(x-l)})$ as $l \to \infty$, specially

$$\phi^*(l)(0) = -e^{-\mu_1(\lambda_0(l))l}\boldsymbol{b}_1^*(\lambda_0(l)) + O(e^{-\mu_2(\lambda_0(l))l}) = -e^{-\alpha l}\boldsymbol{a}^*(1 + O(e^{-\gamma_4 l})) \tag{24}$$

and

$$\partial_x\phi^*(l)(0) = -\alpha e^{-\alpha l}\boldsymbol{a}^*(1 + O(e^{-\gamma_4 l})) \tag{25}$$

for $\gamma_4 > 0$. Here we used $\mu_1(\lambda_0(l)) = \alpha + O(e^{-\frac{1}{2}\alpha l})$, $\boldsymbol{b}_1^*(\lambda_0(l)) = \boldsymbol{a}^* + O(e^{-\frac{1}{2}\alpha l})$ and $\mu_2(\lambda_0(l)) > \alpha + \gamma_4$. Hence we have

$$
\begin{aligned}
&\langle\, L(l)g, \phi^*(l)\,\rangle_+ \\
&= [\langle\, D\partial_x g(x), \phi^*(l)(x)\,\rangle - \langle\, Dg(x), \partial_x\phi^*(l)(x)\,\rangle]_0^\infty + \langle\, g, L^*(l)\phi^*(l)\,\rangle_+ \\
&= -\langle\, D\partial_x g(0), \phi^*(l)(0)\,\rangle + \langle\, Dg(0), \partial_x\phi^*(l)(0)\,\rangle + \lambda_0(l)\langle\, g, \phi^*(l)\,\rangle_+ \\
&= -2\alpha C(l)e^{-\alpha l}\langle\, D\boldsymbol{a}, \boldsymbol{a}^*\,\rangle(1 + O(e^{-\gamma_4 l})) + O(|\lambda_0(l)C(l)|le^{-\alpha l}) \\
&= -2\alpha M_0 C(l)e^{-\alpha l}(1 + O(e^{-\gamma_4 l})).
\end{aligned}
$$

Finally, the denominator in the definition of $H(l, \boldsymbol{v})$ is $1 + O(e^{-\alpha l})$ because $\|g_l\|_+ \le O(e^{-\alpha l})$. Thus the proof is completed. □

Remark 2. Let $H(l, \boldsymbol{v}) = H_0(l) + H_1(l, \boldsymbol{v})$, where $H_0(l) := 2\alpha M_0 C(l)e^{-\alpha l}$ and $H_1(l, \boldsymbol{v}) := H(l, \boldsymbol{v}) - H_0(l)$. Then

$$|H_1(l, \boldsymbol{v})| \le O\left((e^{-\alpha l} + \delta)e^{-\alpha' l} + \|\boldsymbol{v}\|_\infty^2\right)$$

and

$$
\begin{aligned}
&|H_1(l, \boldsymbol{v}) - H_1(l', \boldsymbol{v}')| \\
\le& O\left(\{(e^{-\alpha' l} + \delta^2)e^{-\alpha l} + (e^{-\alpha' l'} + \delta^2)e^{-\alpha l'}\}|l - l'| + (\|\boldsymbol{v}\|_\infty + \|\boldsymbol{v}'\|_\infty)\|\boldsymbol{v} - \boldsymbol{v}\|_\infty\right)
\end{aligned}
$$

hold for $\alpha' > \alpha$.

Lemma 4.4.

$$\|J(l, \boldsymbol{v})\|_+ \le O((e^{-\alpha l} + \delta)e^{-\frac{3}{4}\alpha l} + \|\boldsymbol{v}\|_\infty^2)$$

holds.

Proof. It suffices to show $\|L(l)g\|_+ \le O((e^{-\alpha l} + \delta)e^{-\frac{3}{4}\alpha l})$. Since $g(x; l) = C(l)e^{-\alpha x}\boldsymbol{a} + O(e^{-\alpha_2(l+x)})$ with $\alpha_2 > \alpha$ and $|S(x - l)| \le O(e^{-\alpha|x-l|})$ hold, we see

$$
\begin{aligned}
|L(l)g(x)| &\le |L_0 g(x)| + |(L(l) - L_0)g(x)| \\
&\le |F'(S(x - l)) - F'(\mathbf{0})| \cdot |g(x)| \\
&\le O(e^{-\alpha|x-l|} \cdot |C(l)|e^{-\alpha x}).
\end{aligned}
$$

Then it follows that $\|L(l)g\|_+ \le O(|C(l)|\sqrt{l}e^{-\alpha l}) \le O((e^{-\alpha l} + \delta)e^{-\frac{3}{4}\alpha l})$. □

Now we define the map $\Pi(l) : E_b^\perp(\bar{l}) \to E_b^\perp(l)$ by $\boldsymbol{v}(l) := \Pi(l)\boldsymbol{w}$ for $\boldsymbol{w} \in E_b^\perp(\bar{l})$, where $\boldsymbol{v}(l)$ is the solution of

$$
\left\{
\begin{aligned}
\frac{d\boldsymbol{v}}{dl} &= -\langle\, \boldsymbol{v}, \phi_l^*(l)\,\rangle_+ \phi(l), \\
\boldsymbol{v}(\bar{l}) &= \boldsymbol{w} \in E_b^\perp(\bar{l}).
\end{aligned}
\right.
$$

The existence of $\Pi^{-1}(l)$ is obvious because the solution $\boldsymbol{v}(l)$ in the above equation is solvable in the inverse direction with respect to l. Tranforming (23) by $\boldsymbol{v}(t) = \Pi(l(t))\boldsymbol{w}(t)$, we have

$$
\left\{
\begin{aligned}
\dot{l} &= H^*, \\
\boldsymbol{w}_t &= A(l)\boldsymbol{v} + J^*,
\end{aligned}
\right.
\tag{26}
$$

where $H^* = H^*(l, \boldsymbol{w}) := H(l, \Pi(l)\boldsymbol{w})$, $J^* = J^*(l, \boldsymbol{w}) := \Pi^{-1}(l)J(l, \Pi(l)\boldsymbol{w})$ and $A(l) := \Pi^{-1}(l)L_b(l)\Pi(l)$. Let X^ω $(1/4 < \omega < 1)$ be the fractional powered space of

X_+ by $A(l)$ embedded into $L^\infty(\boldsymbol{R}_+)$ whose norm is denoted by $\|\cdot\|_\omega$. Since

$$\begin{aligned}&\|J(l,\boldsymbol{v})-J(l',\boldsymbol{v}')\|_+\\ \le& O\left(\{(e^{-\alpha l}+\delta)e^{-\frac{3}{4}\alpha l}+(e^{-\alpha l'}+\delta)e^{-\frac{3}{4}\alpha l'}\}|l-l'|+(\|\boldsymbol{v}\|_\infty+\|\boldsymbol{v}'\|_\infty)\|\boldsymbol{v}-\boldsymbol{v}\|_\infty\right)\end{aligned}$$

hold, J^* also have the estimates

$$\begin{aligned}&\|J^*(l,\boldsymbol{v})-J^*(l',\boldsymbol{v}')\|_+ \\ \le\;& O\left(\{(e^{-\alpha l}+\delta)e^{-\frac{3}{4}\alpha l}+(e^{-\alpha l'}+\delta)e^{-\frac{3}{4}\alpha l'}\}|l-l'|+(\|\boldsymbol{v}\|_\omega+\|\boldsymbol{v}'\|_\omega)\|\boldsymbol{v}-\boldsymbol{v}\|_\omega\right)\end{aligned} \tag{27}$$

Here we may assume the estimates of Lemmas 4.3, 4.4 and (27) for any $l\in\boldsymbol{R}$ by using appropriate cut-off functions.

Let $W(D_1,D_2):=\{\boldsymbol{w}(\cdot)\in C(\boldsymbol{R};E_b^\perp(\bar{l})\cap X^\omega);\ \|\boldsymbol{w}(l)\|_\omega\le D_1((e^{-\alpha l}+\delta)e^{-\frac{3}{4}\alpha l}),$ $\|\boldsymbol{w}(l)-\boldsymbol{w}(l')\|_\omega\le D_2\{(e^{-\alpha l}+\delta)e^{-\frac{3}{4}\alpha l}+(e^{-\alpha l'}+\delta)e^{-\frac{3}{4}\alpha l'}\}|l-l'|\}$. Then by quite a similar way to the proof of Theorem 2.1 in [3], we can show the existence of an exponentially attractive invariant manifold $M:=\{(l,\Sigma(l));\ l\in\boldsymbol{R},\ \Sigma\in W(D_1,D_2)\}$ for (26) by taking appropriate positive constants D_1 and D_2. Thus $\boldsymbol{u}(t,x)=S(x-l(t))+g(x;l(t))+(\Pi(l(t))\Sigma(l(t)))(x)$ is the solution of (3) and the proof is completed. □

4.2. **Proof of Corollary 1.** The dynamics on the manifold M constructed in the proof of Theorem 2.1 is given by

$$\frac{dl}{dt}=\overline{H}(l):=H^*(l,\Pi(l)\Sigma(l)). \tag{28}$$

As in the Remark 2, we see $\overline{H}(l)=H_0(l)+\overline{H}_1(l)$ with $\overline{H}_1(l):=H_1(l,\Pi(l)\Sigma(l))$ and $\overline{H}_1(l)$ satisfies

$$\begin{aligned}&|\overline{H}_1(l)-\overline{H}_1(l')|\\ &\le O\Big(\{(e^{-\alpha' l}+\delta^2)e^{-\alpha l}+(e^{-\alpha' l'}+\delta^2)e^{-\alpha l'}\}|l-l'|\\ &\qquad\qquad +(\|\Sigma(l)\|_\infty+\|\Sigma(l')\|_\infty)\|\Sigma(l)-\Sigma(l')\|_\infty)\\ &\le O\Big(\{(e^{-\alpha' l}+\delta^2)e^{-\alpha l}+(e^{-\alpha' l'}+\delta^2)e^{-\alpha l'}\}|l-l'|\\ &\qquad +\{(e^{-\alpha l}+\delta)e^{-\frac{3}{4}\alpha l}+(e^{-\alpha l'}+\delta)e^{-\frac{3}{4}\alpha l'}\}\\ &\qquad\qquad \times\{(e^{-\alpha l}+\delta)e^{-\frac{3}{4}\alpha l}+(e^{-\alpha l'}+\delta)e^{-\frac{3}{4}\alpha l'}\}|l-l'|\Big)\\ &\le O\left(\{(e^{-\alpha'' l}+\delta^2)e^{-\alpha l}+(e^{-\alpha'' l'}+\delta^2)e^{-\alpha l'}\}|l-l'|\right)\end{aligned}$$

for $\alpha''>\alpha$. Let $l^*>\bar{l}$ be the value satisfying $H_0(l^*)=0$ and $\dfrac{dH_0}{dl}(l^*)=:-\gamma^*<0$.

We shall show the existence of an equilibrium $\widehat{l}$ of $\overline{H}$ satisfying $\widehat{l}=l^*(1+O(e^{-\gamma_5 l^*}))$ for $\gamma_5>0$ and $\overline{H}(l)$ is monotone decreasing in the neighborhood of $\widehat{l}$, which means $\widehat{l}$ is a stable equilibrium of the ODE (28).

Substituting $\widehat{l}=l^*+h$ into $\overline{H}(l^*+h)=0$ for $|h|\ll 1$, we have $0=-\gamma^*h+O(h^2)+\overline{H}_1(l^*+h)$ and

$$h=\frac{1}{\gamma^*}\overline{H}_1(l^*+h)+O(h^2). \tag{29}$$

Since $\gamma^*=O\left((e^{-\alpha l^*}+\delta)e^{-\alpha l^*}\right)$, $\overline{H}_1(l^*+h)=O\left((e^{-\alpha l^*}+\delta)e^{-\alpha' l^*}\right)$ and $\alpha'>\alpha$, $\frac{1}{\gamma^*}\overline{H}_1(l^*+h)=O(e^{-\gamma_6 l^*})$ holds for $\gamma_6>0$, which means to become sufficiently small by taking $l^* > \bar{l}$ sufficiently large. Then it is easy to show the right hand side of (29) is contraction in the set $W_1 := \{|h| \leq D_3 e^{-\gamma_6 l^*}\}$ for an appropriate $D_3 > 0$. Thus the fixed value, say $h^* = O(e^{-\gamma_6 l^*})$ gives the equilibrium $\widehat{l} = l^* + h^*$ of $\overline{H}(l)$.

In the neighborhood of $\widehat{l}$, $\overline{H}(l)$ is written as

$$\overline{H}(\widehat{l} + h) = -\gamma^* h + O(h^2 + (h^*)^2) + \overline{H}_1(l^* + h^* + h) - \gamma^* h^*.$$

Hence by the estimate of $|\overline{H}_1(l) - \overline{H}_1(l')|$, we have

$$\overline{H}(\widehat{l} + h) - \overline{H}(\widehat{l} + h') = \{-\gamma^* + O\left(|h| + |h'| + (e^{-\alpha'' l^*} + \delta^2)e^{-\alpha l^*}\right)\}(h - h').$$

$\alpha'' > \alpha$ means $\overline{H}(\widehat{l} + h)$ is monotone decreasing for sufficiently small h and h'.

Other cases are similarly shown. □

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