MULTIPLE TRAVELLING WAVES FOR AN *SI*-EPIDEMIC MODEL

Arnaud Ducrot, Michel Langlais and Pierre Magal

Univ. Bordeaux, IMB, UMR 5251, F-33400 Talence, France
and
CNRS, IMB, UMR 5251, F-33400 Talence, France

Abstract. In this note we analyze a spatially structured SI epidemic model with vertical transmission, a logistic effect on vital dynamics and a density dependent incidence. The dynamics of the underlying system of ordinary differential equations are first shown to exhibit an infinite number of heteroclinic orbits connecting the trivial equilibrium with an interior equilibrium. Our mathematical study of the corresponding reaction-diffusion system is concerned with travelling wave solutions. Based on a detailed study of the center-unstable manifold around the interior equilibrium, we are able to prove the existence of an infinite number of travelling wave solutions connecting the trivial equilibrium and the interior equilibrium.

1. **Introduction.** The aim of this work is to consider a two component reaction-diffusion system,

$$\begin{cases} \partial_t S - \partial_{xx} S = bS + b(1-\theta)I - S\left[\mu + \frac{(b-\mu)}{\kappa}N\right] - \beta SI \\ \partial_t I - \partial_{xx} I = b\theta I - I\left[\mu + \frac{(b-\mu)}{\kappa}N\right] + \beta SI, \end{cases} \quad (1)$$

posed for $t > 0$ and $x \in \mathbb{R}$, wherein $N = S + I$. This system of equations models the spatio-temporal spread of a disease within a spatially structured population. Here $S(t,x)$ denotes the density of susceptibles at time $t > 0$ and location $x \in \mathbb{R}$ while $I(t,x)$ denotes the density of infective individuals. In a disease-free environment, the total population density, N, satisfies the following scalar equation

$$\partial_t N - \partial_{xx} N = N\left[b - \mu - \frac{(b-\mu)}{\kappa}N\right]. \quad (2)$$

When $\lambda = b - \mu > 0$, the above equation corresponds a logistic or Fisher-KPP equation [14, 18]. Classically, $b > 0$ denotes the birth rate, $\mu > 0$ corresponds to the death rate while $\kappa > 0$ denotes the carrying capacity of the environment. Going back to (1), parameter $\theta \in [0,1]$ describes the vertical transmission of the disease, that is a fraction $1 - \theta$ of offspring born from infective individuals are susceptible at birth while a proportion θ remains infective at birth. Finally one assumes a density dependent incidence, the usual law of mass action, with a parameter $\beta > 0$ denoting the efficient contamination rate. We refer for instance to the monograph

of Busenberg and Cooke [8] for more details on such models. We also refer to [1, 7, 9, 11, 17, 25] for more results on epidemic models; [22, 23] for recent survey papers; and [12, 13] for related works on travelling waves for SI models.

Adding the two equations in (1) yields an equivalent system of equations,

$$
\begin{cases}
\partial_t N - \partial_{xx} N = \lambda N \left[1 - \dfrac{N}{\kappa} \right] \\
\partial_t I - \partial_{xx} I = \left\{ \lambda - \theta b + \left(\beta - \dfrac{\lambda}{\kappa} \right) N \right\} I - \beta I^2.
\end{cases}
\tag{3}
$$

Note that the diffusive logistic equation for the N state variable in the above system is uncoupled from the second equation. The dynamics of such a logistic equation is well known and, in many cases, strongly related to travelling wave solutions. The literature about this topic is very wide. We only quote some of them, see for instance [2, 6, 16, 19, 20, 21, 27, 26] as well as references therein. Recall that for each $c \geq c^* = 2\sqrt{\lambda}$ this logistic equation has a unique (up to translation) travelling wave solution connecting $N = 0$ to $N = \kappa$. This means that for each $c \geq c^*$, there exists a non-increasing function $U \equiv U_c(x)$ such that

$$
\begin{cases}
U''(x) + cU'(x) + \lambda U(x) \left[1 - \dfrac{U(x)}{\kappa} \right] = 0, \ \ x \in \mathbb{R}, \\
U(-\infty) = \kappa, \ \ U(\infty) = 0.
\end{cases}
\tag{4}
$$

Going back to (3), we shall assume that the total population N is invading, that is it follows such a travelling wave solution dynamics, U, for some given speed $c \geq c^*$. This allows to reduce system (3) to a forced speed equation,

$$
\partial_t I - \partial_{xx} I = \left\{ \lambda - \theta b + \left(\beta - \dfrac{\lambda}{\kappa} \right) U(x - ct) \right\} I - \beta I^2,
\tag{5}
$$

where $c \geq c^*$ is a given wave speed while U is Fisher-KPP travelling front, namely a solution of (4).

Such a forced speed equation looks like equation (1.1) in Berestycki et al. [3] (see also Berestycki and Rossi [4, 5] for multi-dimensional frameworks, and Volpert and Suhov [28]). However in our study the "forcing" term, $\lambda - \theta b + \left(\beta - \frac{\lambda}{\kappa} \right) U(x - ct)$ in (5), remains positive at infinity and does not fit into the assumptions of [3] that would require it to be negative (see Assumption 4.4 in [3]).

In this work we are interested in special entire solutions of (5) of the form,

$$
I(t, x) = V(x - ct), \ \ \forall (t, x) \in \mathbb{R}^2,
$$

where $c \geq c^*$ is given while U a Fisher-KPP front solution of (4). This gives a new system of equations,

$$
\begin{cases}
U''(x) + cU'(x) + \lambda U(x) \left[1 - \dfrac{U(x)}{\kappa} \right] = 0, \ \ x \in \mathbb{R}, \\
V''(x) + cV'(x) + \left[\lambda - \theta b + \left(\beta - \dfrac{\lambda}{\kappa} \right) U(x) \right] V(x) - \beta V^2(x) = 0,
\end{cases}
\tag{6}
$$

supplemented with the following limiting behaviour

$$
\begin{cases}
U(-\infty) = \kappa, \ \ U(\infty) = 0 \\
V(-\infty) = v^*, \ \ V(\infty) = 0,
\end{cases}
\tag{7}
$$

wherein we have set

$$
v^* = \frac{\beta \kappa - \theta b}{\beta},
$$

that is well defined when $\beta\kappa - b\theta > 0$.

Let us first notice that when $v^* > 0$, $\beta - \frac{\lambda}{\kappa} = 0$ and $\lambda - \theta b > 0$, system (6) is fully uncoupled,

$$\begin{cases} U''(x) + cU'(x) + \lambda U(x)\left[1 - \dfrac{U(x)}{\kappa}\right] = 0, \quad x \in \mathbb{R}, \\ V''(x) + cV'(x) + V(x)\left[\lambda - \theta b - \beta V(x)\right] = 0. \end{cases} \tag{8}$$

For each $c \geq 2\sqrt{\lambda}$, due to the translation invariance, each equation of the above system has a one dimensional manifold of heteroclinic orbits. It follows that for each speed $c \geq 2\sqrt{\lambda}$, system (8) has a two dimensional manifold of heteroclinic orbits. The aim of this work is to prove that such a property will persist with coupling. To be more precise, the main result of our work reads as follows:

Theorem 1.1. *Assume that*

$$\lambda > \beta\kappa > \theta b.$$

Then for each $c \geq 2\sqrt{\lambda}$, system (8) has infinitely many heteroclinic solutions (U, V) connecting $(0,0)$ and $\left(\kappa, \frac{\beta\kappa - \theta b}{\beta}\right)$ and such that

$$0 \leq U \leq \kappa \text{ and } 0 \leq V.$$

The proof of this result is based on arguments from monotone semiflows as well as on a precise study of the center-unstable manifold for the fourth order ordinary differential equation corresponding to (6).

The local center-unstable manifold theorem for ordinary differential equations is recalled in Section 2. Section 3 deals with dynamical properties of the underlying ODE described by (1). In this case, the existence of infinitely many heteroclinic orbits is proved using invariant manifold arguments. Finally Section 4 is concerned with the proof of Theorem 1.1. In this last section, existence and uniqueness results for Fisher-KPP travelling fronts are added to present arguments based on the center-unstable manifold.

2. **Preliminaries on local center-unstable manifold theorem.** In this section, we recall the local center-unstable manifold theorem for ordinary differential equations. This theorem is well known and we refer to the book of Chow, Li and Wang [10] for more results about this topic. Consider an ordinary differential equation in $\mathbb{R}^n$

$$\frac{dX(t)}{dt} = AX(t) + F(X(t)), \text{ for } t \geq 0, \text{ and } X(0) = X_0, \tag{9}$$

where $A \in M_n(\mathbb{R})$ is an $n \times n$ real matrix, and $F : \mathbb{R}^n \to \mathbb{R}^n$ is k-time continuously differentiable for some integer $k \geq 1$.

We will make the following assumption.

Assumption 2.1. *Assume that*

(i) $F(0) = 0_{\mathbb{R}^n}$ and $DF(0) = 0_{M_n(\mathbb{R})}$.
(ii) The center-unstable spectrum $\sigma_{cu}(A) = \{\lambda \in \sigma(A) : \text{Re}(\lambda) \geq 0\}$ is non empty.

Define

$$\sigma_s(A) := \sigma(A) \setminus \sigma_{cu}(A) = \{\lambda \in \sigma(A) : \text{Re}(\lambda) < 0\}.$$

By the Jordan's theorem, we can find two linear subspaces $X_{cu} \subset \mathbb{R}^n$ and $X_s \subset \mathbb{R}^n$ such that

$$AX_s \subset X_s, \ AX_{cu} \subset X_{cu},$$

$$\mathbb{R}^n = X_s \oplus X_{cu}.$$

For each $k = s, cu$ we define a linear operator $A_k : X_k \to X_k$ such that $\sigma(A_k) = \sigma_k(A)$ upon setting

$$A_k x = Ax, \forall x \in X_k.$$

Let $\Pi_{cu} : \mathbb{R}^n \to \mathbb{R}^n$ be the linear projector such that

$$\Pi_{cu} A = A\Pi_{cu}$$

and

$$\Pi_{cu}(\mathbb{R}^n) = X_{cu} \text{ and } (I - \Pi_{cu})(\mathbb{R}^n) = X_s.$$

The following theorem is obtained by a truncation procedure from the global center-unstable manifold theorem.

Theorem 2.2. *(Local center-unstable manifold) There exists* $\Psi_{cu} : X_{cu} \to X_s$ *a C^k-map satisfying*

$$\Psi_{cu}(0_{X_{cu}}) = 0_{X_s} \text{ and } D\Psi_{cu}(0) = 0_{\mathcal{L}(X_{cu}, X_s)}$$

and such that

$$M_{cu} = \{x_{cu} + \Psi_{cu}(x_{cu}) : x_{cu} \in X_{cu}\}$$

is locally invariant by the maximal semiflow generated by (9). More precisely, there exists a bounded neighbourhood Ω of 0 such that the following properties are satisfied:

   (i) If $I \subset \mathbb{R}$ is an interval and $u_{cu} : I \to X_{cu}$ is a solution of the system

$$u'_{cu}(t) = A_{cu} u_{cu}(t) + \Pi_{cu} F(u_{cu}(t) + \Psi_{cu}(u_{cu}(t))) \text{ \textbf{(Reduced Equation)}} \quad (10)$$

     and

$$u_{cu}(t) + \Psi_{cu}(u_{cu}(t)) \in \Omega, \forall t \in I$$

   then $u(t) = u_{cu}(t) + \Psi_{cu}(u_{cu}(t))$ is a solution of (9) on I.
  (ii) If $u : (-\infty, 0] \to \mathbb{R}^n$ is a solution of (9) such that

$$u(t) \in \Omega, \forall t \leq 0,$$

   then

$$u(t) \in M_{cu}, \forall t \leq 0,$$

   therefore $u_{cu}(t) = \Pi_{cu} u(t)$ is a solution of (10).

Remark 1. The fact that $D\Psi_{cu}(0) = 0$ implies that the manifold M_{cu} is tangent to X_{uc} at 0.

3. **The underlying ODE problem.** The aim of this section is to provide information about the existence and non-existence of heteroclinic connections for the underlying ODE system corresponding to system (1). This problem reads as the $SI-$epidemic model:

$$\begin{cases} S'(t) = bS(t) + b\theta I(t) - \mu S(t) - (b - \mu)\kappa^{-1}S(t)(I(t) + S(t)) - \beta S(t)I(t), \\ I'(t) = b(1 - \theta)I(t) - \mu I(t) - (b - \mu)\kappa^{-1}I(t)(I(t) + S(t)) + \beta S(t)I(t), \end{cases}$$
$$(11)$$

posed for time $t > 0$ and supplemented together with some initial values

$$S(0) = S_0 \geq 0 \text{ and } I(0) = I_0 \geq 0.$$

As explained in the introduction, the total number of individuals N defined by

$$N(t) := S(t) + I(t), \ t \geq 0,$$

satisfies the scalar logistic equation

$$N'(t) = (b - \mu) N(t) \left(1 - \kappa^{-1} N(t)\right), \forall t \geq 0, \ \text{and } N(0) = N_0 \geq 0. \qquad (12)$$

From here on we set and assume that

$$\lambda := b - \mu > 0.$$

Using these notations, the I-equation in model (11) reads

$$I'(t) = b(1 - \theta)I(t) - \mu I(t) - (b - \mu) \kappa^{-1} I(t) N(t) + \beta \left(N(t) - I(t)\right) I(t),$$

or equivalently as a non-autonomous logistic equation

$$I'(t) = \left[\lambda - \theta b + \left[\beta - \lambda \kappa^{-1}\right] N(t)\right] I(t) - \beta I(t)^2.$$

In the following we use the usual notion of global attractors for semiflows on a metric space. We refer for example to the book of Hale [15] for more precisions and results on this topic.

Lemma 3.1. *The system (11) generates a unique semiflow* $\{U(t)\}_{t \geq 0}$ *on* $[0, +\infty)^2$ *such that for each* $(S_0, I_0) \in [0, +\infty)^2$,

$$t \rightarrow (S(t), I(t)) := U(t)(S_0, I_0)$$

is the unique solution of (11). Moreover U has a global attractor $A \subset [0, +\infty)^2$ which is a connected set.

Remark 2. The compact attractor contains in particular all the heteroclinic orbits of the system.

Proof. The existence and uniqueness and the positivity of the semiflow follows from classical arguments. Basically, we have

$$(S'(t), I'(t)) = F(S(t), I(t))$$

where $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is Lipschitz continuous on bounded sets, and for each $M > 0$, there exists $\lambda = \lambda(M) > 0$

$$F(x) + \lambda x \in [0, +\infty)^2,$$

for each $x \geq 0$, such that $\|x\|_1 \leq M$. Here recall that the notation $x \geq 0$ for some $x \in \mathbb{R}^2$ means that both components of x are positive. The existence and the uniqueness of a positive maximal semiflow follows combined with the fact that $N(t)$ satisfies the logistic equation (for the global existence of solutions). Moreover since $N(t)$ satisfies a logistic equation, for each $\varepsilon > 0$ the bounded subset

$$B_\varepsilon = \left\{(S, I) \in [0, +\infty)^2 : S + I \leq \kappa + \varepsilon\right\}$$

is an absorbing set for U, and therefore U has a (unique) global attractor A in $[0, +\infty)^2$. Finally since $[0, +\infty)^2$ is convex, it follows that A is connected (see Hale [15] for more precisions). $\square$

Equilibria: The equilibrium $\overline{N} = 0$ of equation (12) corresponds to the equilibrium

$$\left(\overline{S}_0, \overline{I}_0\right) = (0, 0),$$

of system (11).

The equilibrium $\overline{N} = \kappa$ of equation (12) corresponds first to the disease free equilibrium

$$\left(\overline{S}_f, \overline{I}_f\right) = (\kappa, 0).$$

Since the I-equation can be rewritten as

$$I'(t) = \left[(b - \mu) - \theta b + \left[\beta - (b - \mu)\kappa^{-1}\right] N(t)\right] I(t) - \beta I(t)^2,$$

the endemic equilibrium (or interior equilibrium) is

$$\left(\overline{S}_e, \overline{I}_e\right) = \left(\kappa - \overline{I}_e, \overline{I}_e\right)$$

where

$$\overline{I}_e = \beta^{-1}\left[(b - \mu) - \theta b + \left[\beta\kappa - (b - \mu)\right]\right] = \beta^{-1}\left[\beta\kappa - b\theta\right] = \kappa - \frac{b\theta}{\beta}.$$

which is strictly positive if and only if

$$\beta\kappa > b\theta.$$

From now on, we assume that there exists an endemic equilibrium. Therefore we make the following assumption.

Assumption 3.2. *Assume that $b > \mu$ and $\beta\kappa > b\theta$.*

Heteroclinic orbits for system (11): Since the region

$$\partial M := \{(S, 0) : S \geq 0\}$$

is invariant by the flow generated by (11), by using the properties of the logistic equation

$$S'(t) = (b - \mu) S \left(1 - \kappa^{-1} S\right)$$

we deduce that there exists a unique heteroclinic orbit $O_0 = \{(S_0(t), 0)\}_{t \in \mathbb{R}} \subset \partial M$ of system (11) going from $(0, 0)$ to $(\kappa, 0)$.

The segment

$$\Delta = \left\{(S, I) \in [0, +\infty)^2 : S + I = \kappa\right\}$$

is also invariant by the semiflow. The I-equation for $N = \kappa$ reads as

$$I'(t) = \left[\lambda - \theta b + \left[\beta - \lambda\kappa^{-1}\right]\kappa\right] I(t) - \beta I(t)^2$$

or

$$I'(t) = \left[\beta\kappa - \theta b\right] I(t) - \beta I(t)^2.$$

From this, there exists a unique heteroclinic orbit $O_1 = \{(S_1(t), I_1(t))\}_{t \in \mathbb{R}} \subset \Delta$ going from the disease free equilibrium $\left(\overline{S}_f, \overline{I}_f\right)$ to the endemic equilibrium $\left(\overline{S}_e, \overline{I}_e\right)$.

In term of heteroclinic orbits, it becomes less clear to understand if there exists a heteroclinic orbit $O_2 = \{(S_2(t), I_2(t))\}_{t \in \mathbb{R}} \subset (0, +\infty)^2$ going from the trivial equilibrium $(0, 0)$ to the endemic equilibrium $\left(\overline{S}_e, \overline{I}_e\right)$ in $(0, +\infty)^2$. The rest of this section is devoted to this question.

Linearized equation at $(0, 0)$: The linearized equation at $(0, 0)$ is the following

$$\left(\begin{array}{c} S' \\ I' \end{array}\right) = L_0 \left(\begin{array}{c} S \\ I \end{array}\right)$$

wherein we have set

$$L_0 = \left[\begin{array}{cc} b - \mu & b\theta \\ 0 & b(1 - \theta) - \mu \end{array}\right].$$

Note that the eigenvalues of the matrix L_0 are

$$\lambda^+ := b - \mu \geq \lambda^- := b(1 - \theta) - \mu,$$

while the corresponding eigenspaces are

$$E_{\lambda^+} = \left\{ (S, I) \in \mathbb{R}^2 : I = 0 \right\}$$

and

$$E_{\lambda^-} = \left\{ \begin{array}{l} \left\{ (S, I) \in \mathbb{R}^2 : S = -I \right\} \text{ if } \theta > 0, \\ \left\{ (S, I) \in \mathbb{R}^2 : S = 0 \right\}, \text{ if } \theta = 0. \end{array} \right.$$

Next consider the linear projector Π_{λ^+} on $\mathbb{R}^2$ defined by

$$\Pi_{\lambda^+} \left(\begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left\langle \left(\begin{array}{c} 1 \\ 1 \end{array} \right), \left(\begin{array}{c} x_1 \\ x_2 \end{array} \right) \right\rangle \left(\begin{array}{c} 1 \\ 0 \end{array} \right), \forall \left(\begin{array}{c} x_1 \\ x_2 \end{array} \right) \in \mathbb{R}^2.$$

Note that it satisfies

$$\Pi_{\lambda^+} L_0 = L_0 \Pi_{\lambda^+} = \lambda^+ \Pi_{\lambda^+}.$$

Next the following non-existence of heteroclinic connection holds true:

Lemma 3.3. *Let Assumption 3.2 be satisfied and assume furthermore that*

$$\lambda^- := b(1 - \theta) - \mu < 0.$$

Then there is no heteroclinic orbit going from $(0,0)$ to the endemic equilibrium $\left(\overline{S}_e, \overline{I}_e \right)$ in $(0, +\infty)^2$.

Proof. Since $\lambda^- < 0$ and $\lambda^+ > 0$, the center-unstable manifold at $(0,0)$ is one-dimensional. Let $\Psi_{cu} : E_{\lambda^+} \to E_{\lambda^-}$ be a C^1-map parametrizing the center-unstable manifold and so that the one dimensional manifold defined by

$$M_{cu} := \left\{ x_{cu} + \Psi_{cu}(x_{cu}) : x_{cu} \in E_{\lambda^+} \right\}$$

is locally invariant under the semiflow U around $(0,0)$. Then it furthermore satisfies

$$D_{x_{cu}} \Psi_{cu}(0) = 0,$$

meaning that the manifold that M_{cu} is tangent to E_{λ^+}. Moreover we know that there exists $\varepsilon > 0$, such that M_{cu} contains all negative orbits of U staying in the ball $B_{\mathbb{R}^2}(0, \varepsilon)$ for all negative times.

In order to prove the lemma, let us argue by contradiction, and assume that there exists an heteroclinic orbit $O_2 = \{(S_2(t), I_2(t))\}_{t \in \mathbb{R}} \subset (0, +\infty)^2$ connecting $(0,0)$ to the endemic equilibrium $\left(\overline{S}_e, \overline{I}_e \right)$ in $(0, +\infty)^2$. Since

$$\lim_{t \to -\infty} (S_2(t), I_2(t)) = (0,0) \text{ and } \lim_{t \to -\infty} (S_0(t), 0) = (0,0),$$

without loss of generality (i.e. using a translation in time) one may assume that

$$(S_2(t), I_2(t)) \in B_{\mathbb{R}^2}(0, \varepsilon) \text{ and } (S_0(t), 0) \in B_{\mathbb{R}^2}(0, \varepsilon), \forall t \leq 0.$$

This implies that

$$(S_2(t), I_2(t)) \in M_{cu} \text{ and } (S_0(t), 0) \in M_{cu}, \forall t \leq 0.$$

But since M_{cu} is the graph of a map from E_{λ^+} into E_{λ^-}, this leads us to a contradiction. Indeed, one has

$$\Pi_{\lambda^+} \left(\begin{array}{c} S_2(t) \\ I_2(t) \end{array} \right) = \left(\begin{array}{c} (S_2(t) + I_2(t)) \\ 0 \end{array} \right) \text{ and } (I - \Pi_{\lambda^+}) \left(\begin{array}{c} S_2(t) \\ I_2(t) \end{array} \right) = \left(\begin{array}{c} -I_2(t) \\ I_2(t) \end{array} \right)$$

$$\Pi_{\lambda^+} \left(\begin{array}{c} S_0(t) \\ 0 \end{array} \right) = \left(\begin{array}{c} S_0(t) \\ 0 \end{array} \right) \text{ and } (I - \Pi_{\lambda^+}) \left(\begin{array}{c} S_2(t) \\ 0 \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \end{array} \right).$$

If we fix $t_0 < 0$ and $t_2 < 0$ such that

$$S_0(t_0) = (S_2(t_2) + I_2(t_2)),$$

we obtain that

$$\Psi_{cu}\left(\left(\begin{array}{c} S_0(t_0) \\ 0 \end{array}\right)\right) = \left(\begin{array}{c} 0 \\ 0 \end{array}\right) = \left(\begin{array}{c} -I_2(t_2) \\ I_2(t_2) \end{array}\right),$$

that leads us to a contradiction since $I_2(t_2) > 0$. This completes the proof of the result. $\qquad\square$

Existence of non-trivial heteroclinic orbits: In order to deal with the existence of non-trivial heteroclinic orbits, we assume

$$\lambda^- = b(1-\theta) - \mu = \lambda - \theta b > 0,$$

where

$$\lambda := (b - \mu) > 0.$$

Then system (N, I) can be rewritten as

$$\left\{ \begin{array}{l} N'(t) = \lambda N \left(1 - \kappa^{-1} N\right) \\ I'(t) = \left[\lambda - \theta b + \left(\beta - \lambda \kappa^{-1}\right) N\right] I - \beta I^2 \\ N(0) = N_0 \geq 0, \text{ and } I(0) = I_0 \geq 0. \end{array} \right.$$

Set

$$n(t) = \kappa^{-1} N(t) \text{ and } i(t) = \kappa^{-1} I(t)$$

$$\left\{ \begin{array}{l} n'(t) = \lambda n \left(1 - n\right) \\ i' = (\lambda - \theta b + (\kappa\beta - \lambda) n) i - \kappa\beta i^2 \\ n(0) = n_0 \geq 0 \text{ and } i(0) = i_0 \geq 0. \end{array} \right. \qquad (13)$$

Let n_0 and i_0 such that

$$0 < i_0 \leq n_0 < 1.$$

Since by assumption $\beta\kappa > b\theta$ and $\theta \leq 1$, we have

$$\kappa\beta \geq \lambda = b - \mu,$$

therefore system (13) is monotone on $[0, +\infty)^2$ (see Smith [24] for more precisions and more results on this topic). Since $0 < i_0 \leq n_0 < 1$, we observe that

$$(n(t), i(t)) \rightarrow (1, \frac{\beta\kappa - b\theta}{\kappa\beta}) \text{ as } t \rightarrow +\infty.$$

Therefore it remains to investigate the behaviour of such a solution for negative times. To do so, let us first notice that we have

$$n(t) = \frac{e^{\lambda t} n_0}{1 + \lambda \int_0^t e^{\lambda l} n_0 dl} = \frac{e^{\lambda t} n_0}{1 + [e^{\lambda t} - 1] n_0}, \forall t \in \mathbb{R}.$$

Similarly the i-equation can be rewritten as

$$i'(t) = \Lambda(t) i(t) - \kappa\beta i(t)^2$$

wherein we have set

$$\Lambda(t) := (\lambda - \theta b + (\kappa\beta - \lambda) n(t)) = \frac{n'(t)}{n(t)} + \kappa\beta n(t) - \theta b.$$

Therefore for each $t > 0$ we have

$$i(t) = \frac{e^{\int_0^t \Lambda(l) dl} i_0}{1 + \kappa\beta \int_0^t e^{\int_0^\sigma \Lambda(l) dl} i_0 d\sigma}.$$

For $t < 0$, we obtain

$$i(t) = \frac{e^{-\int_t^0 \Lambda(t)dl} i_0}{\left(1 - \kappa\beta \int_t^0 e^{-\int_\sigma^0 \Lambda(t)dl} d\sigma i_0\right)}$$

$$= \frac{i_0}{e^{\int_t^0 \Lambda(t)dl} \left(1 - \kappa\beta \int_t^0 e^{-\int_\sigma^0 \Lambda(t)dl} d\sigma i_0\right)}$$

(14)

Since

$$\lim_{t \to -\infty} \Lambda(t) = \gamma = \lambda - \theta b > 0,$$

one deduces that

$$\lim_{t \to -\infty} e^{\int_t^0 \Lambda(l)dl} = +\infty,$$

and therefore

$$\chi := \lim_{t \to -\infty} \int_t^0 e^{-\int_\sigma^0 \Lambda(l)dl} d\sigma < +\infty.$$

Next we infer from (14) that for each $i_0 \in \left(0, (\chi\kappa\beta)^{-1}\right)$:

$$i(t) \to 0 \text{ as } t \to -\infty.$$

As a consequence, system (13) has a heteroclinic orbit from $(0,0)$ to $(1, \frac{\beta\kappa - b\theta}{\kappa\beta})$ passing through (n_0, i_0).

The above arguments can be summarized as follows:

Lemma 3.4. *Let Assumption 3.2 be satisfied. Assume furthermore that*

$$\lambda^- := b(1 - \theta) - \mu > 0.$$

Then there is an infinite number of heteroclinic orbits going from $(0,0)$ to the endemic equilibrium $\left(\overline{S}_e, \overline{I}_e\right)$ in $(0, +\infty)^2$. More precisely, the global attractor A is composed by the equilibria and all the heteroclinic orbits connected the equilibria.

Remark 3. When $\theta = 0$ (which implies $\lambda^- > 0$) the global attractor is

$$A = \left\{(S, I) \in [0, +\infty)^2 : S + I \leq \kappa\right\}$$

and every point of this domain is either an equilibrium or belongs to a heteroclinic orbit.

4. **Travelling wave problem.** The aim of this section is to prove Theorem 1.1. Before doing so, we will first come back to the Fisher-KPP travelling front problem described in (4). We will derive for this problem an existence and uniqueness result. This topic is widely developed (we refer for instance to [2, 27, 26]). Here we use attractor arguments for the existence proof while center-unstable manifold arguments are used to prove the uniqueness. The second part deals with system (6) and completes the proof of Theorem 1.1.

To simplify the notations, by using appropriate changes of variable (in time and space), we will assume that

$$\lambda := (b - \mu) = 1, \text{ and } \kappa = 1.$$

In order to assure the existence of the (interior) positive equilibrium we further assume

$$\beta > b\theta.$$

4.1. **ODE methods for the KPP equation.** In this section, we discuss the existence and uniqueness of solutions for (4) by using invariant manifold techniques. Let us notice that (4) can be re-written as

$$\left(\frac{d}{dx} + \frac{c}{2}\right)^2 U - \frac{c^2}{4}U + U(1 - U) = 0.$$

Next setting

$$\begin{cases} U_1 := U \\ U_2 := \left(\frac{d}{dx} + \frac{c}{2}\right)U. \end{cases}$$

we obtain the following first order system of ordinary differential equations

$$\begin{cases} \left(\frac{d}{dx} + \frac{c}{2}\right)U_1 = U_2 \\ \left(\frac{d}{dx} + \frac{c}{2}\right)U_2 = \frac{c^2}{4}U_1 - U_1(1 - U_1). \end{cases}$$

Set

$$\alpha := \frac{c}{2}$$

we obtain the system

$$\begin{cases} \frac{dU_1}{dx} = -\alpha U_1 + U_2 \\ \frac{dU_2}{dx} = -\alpha U_2 + \left(\alpha^2 - 1\right)U_1 + U_1^2 \end{cases} \tag{15}$$

Note that this system is monotone increasing on $[0, +\infty)^2$ whenever

$$\alpha \geq 1. \tag{16}$$

Moreover one has

$$\frac{d(\alpha U_1 + U_2)}{dt} = -U_1(x) + U_1(x)^2 = -U_1(x)(1 - U_1(x)) \tag{17}$$

and the points

$$\overline{U}^0 := (0, 0) \text{ and } \overline{U}^1 := (1, \alpha)$$

are the only equilibria of the system in $[0, +\infty)^2$.

4.1.1. *Existence of travelling waves.* Since $[0, 1] \times [0, \alpha]$ is invariant by the semiflow $\{T(t)\}_{t \geq 0}$ generated by the system (15). There exists a connected subset $A \subset [0, 1] \times [0, \alpha]$, which is the global attractor of the semiflow T on $[0, 1] \times [0, \alpha]$. Recalling that the global attractor is connected, since it contains both equilibria $(0, 0)$ and $(1, \alpha)$, by considering the linear functional $P : \mathbb{R}^2 \to \mathbb{R}$

$$P(U_1, U_2) = U_1$$

we deduce that $P(A)$ is compact and connected and contains $P(0, 0) = 0$ and $P(1, \alpha) = 1$. Hence one concludes that

$$P(A) = [0, 1].$$

Moreover

$$T(t)A = A, \forall t \geq 0.$$

Therefore $\{T(t)\}_{t \in \mathbb{R}}$ is a flow on A, and it follows that there exists a complete orbit $(U_1, U_2) \in C^1\left(\mathbb{R}, \mathbb{R}^2\right)$ of system (15) such that

$$(U_1(t), U_2(t)) \in A, \forall t \geq 0,$$

and passing $t = 0$ through

$$(U_1, U_2) \text{ with } U_1 = 1/2 \text{ and } U_2 \in [0, \alpha].$$

By using (17) we deduce that

$$\lim_{t \to +\infty} (U_1(t), U_2(t)) = (0,0) \text{ and } \lim_{t \to -\infty} (U_1(t), U_2(t)) = (1, \alpha).$$

Therefore A contains the equilibria, and all the travelling waves going from $(1, \alpha)$ to $(0, 0)$.

4.1.2. *Uniqueness of the travelling waves.* In order to prove the uniqueness of the heteroclinic orbit going from $(1, \alpha)$ to $(0,0)$ we will study the center-unstable manifold around the equilibrium $(1, \alpha)$.

Linearized equation at $\overline{U}^1 = (1, \alpha)$: The matrix of the linearized equation of system (15) at $(1, \alpha)$ is

$$L_U = \left[\begin{array}{cc} -\alpha & 1 \\ \alpha^2 + 1 & -\alpha \end{array} \right],$$

the characteristic equation is given by

$$(\alpha + \lambda)^2 - \alpha^2 - 1 = 0 \Leftrightarrow \lambda^2 + 2\alpha\lambda + \alpha^2 - \alpha^2 - 1 = 0$$
$$\Leftrightarrow \lambda^2 + 2\alpha\lambda - 1 = 0,$$

hence the spectrum of L_U is given by

$$\sigma(L_U) = \left\{ \lambda_U^-, \lambda_U^+ \right\},$$

with

$$\lambda_U^- := -\alpha - \sqrt{\alpha^2 + 1} < 0 < \lambda_U^+ := -\alpha + \sqrt{\alpha^2 + 1}.$$

It follows that the center-unstable manifold at $(1, \alpha)$ is a one dimensional locally invariant manifold. Therefore by using the same arguments as in Lemma 3.3, it follows that the travelling wave going from $(1, \alpha)$ to $(0, 0)$ is unique (we refer for instance to [26] for an other proof). The precise result proven is the following:

Lemma 4.1. *Assume that $\alpha \geq 1$ (that reads $c \geq 2$). Then there exists at most one travelling wave going from $(1, \alpha)$ to $(0, 0)$ for (4). More precisely, there exists a unique solution $U^*(x) = (U_1^*(x), U_2^*(x))$ of system (15) satisfying*

$$\lim_{x \to -\infty} U^*(x) = (1, \alpha) \text{ and } \lim_{x \to +\infty} U^*(x) = (0, 0).$$

Since we will use it in the following, we can detail a little bit more the system reduced to the center-unstable manifold around the interior equilibrium. First the projector in the center-unstable space reads as

$$\Pi_{cu} \left(\begin{array}{c} U_1 \\ U_2 \end{array} \right) = \left(\sqrt{\alpha^2 + 1}\, U_1 + U_2 \right) \left(\begin{array}{c} \left(\sqrt{\alpha^2 + 1}\right)^{-1} \\ 1 \end{array} \right).$$

Moreover the reduced system of (15) takes the following form

$$U'_{cu} = \Pi_{cu} F \left(U_{cu} + \widetilde{\Psi}_{cu}(U_{cu}) \right) \tag{18}$$

where F is the second member of system (15), and the unstable manifold is the graph of $\widetilde{\Psi}_{cu} : \Pi_{cu}(\mathbb{R}^2) \to (I - \Pi_{cu})(\mathbb{R}^2)$. Since Π_{cu} has a one-dimensional rank, equation (18) around the equilibrium $\Pi_{cu}(1, \alpha)$ can be identified to a scalar ordinary differential equation of the form

$$u'_{cu} = f(u_{cu}) \tag{19}$$

with

$$f(0) = 0 \text{ and } f'(0) = \lambda_U^+. \tag{20}$$

4.2. **Travelling waves for the full model.** The aim of this section is to prove Theorem 1.1. To do so we consider system (6). Recalling that $\lambda = \kappa = 1$ so that the system under consideration reads as

$$\begin{cases} U'' + cU' = -\lambda U\left[1 - U\right], \\ V'' + cV' = -\left(1 - b\theta + (\beta - 1)U\right)V + \beta V^2. \end{cases}$$

As before we can reformulate the U-equation as system (15). Similarily by setting

$$\begin{cases} V_1 := V \\ V_2 := \left(\dfrac{d}{dx} + \dfrac{c}{2}\right)V_1, \end{cases}$$

and by using the fact that

$$\left(\frac{d}{dx} + \frac{c}{2}\right)^2 V - \left(\frac{c}{2}\right)^2 V = -\left(1 - b\theta + (\beta - 1)U\right)V + \beta V^2$$

the V-equation becomes

$$\begin{cases} V_1' = -\alpha V_1 + V_2 \\ V_2' = -\alpha V_2 + \left[\alpha^2 - (1 - b\theta) + (1 - \beta)U_1\right]V_1 + \beta V_1^2 \end{cases} \tag{21}$$

wherein we have set $\alpha = \frac{c}{2}$. Note that as above one has

$$\left(\alpha V_1 + V_2\right)' = -V_1\left[(1 - b\theta + (\beta - 1)U_1) - \beta V_1\right]. \tag{22}$$

Then the fourth-dimensional system (15) and (21) is monotone increasing on $[0, +\infty)^4$ whenever

$$\alpha^2 \geq 1 - b\theta \text{ and } 1 \geq \beta. \tag{23}$$

Moreover, in order to obtain a positive equilibrium, we impose that

$$\beta > b\theta. \tag{24}$$

By combining the conditions (16) (23) and (24), we obtain the following set of conditions that will be assumed in the rest of the paper:

Assumption 4.2. *We assume that*

$$\alpha \geq 1 \text{ and } 1 \geq \beta > b\theta.$$

Equilibria for the V-system: When $U_1 = 1$, the non-negative equilibria for the V-system are

$$\overline{V}^0 = (0, 0) \text{ and } \overline{V}^1 = \left(\left(1 - \frac{b\theta}{\beta}\right), \alpha\left(1 - \frac{b\theta}{\beta}\right)\right).$$

For $U_1 = 0$, the non-negative equilibria for the V-equation are

$$\overline{V}^0 = (0, 0) \text{ and } \overline{V}^2 = \left(\frac{(1 - b\theta)}{\beta}, \alpha\frac{(1 - b\theta)}{\beta}\right).$$

Remark 4. One may observe that we need to impose

$$0 \leq i \leq n,$$

which implies

$$0 \leq V_1 \leq U_1.$$

Therefore the biological constraint permits to exclude the equilibrium $U_1 = 0$ and

$$V \equiv \left(\frac{(1 - b\theta)}{\beta}, \alpha \frac{(1 - b\theta)}{\beta} \right).$$

When $\beta = 1$ the U-equation and the V-equation are uncoupled. In this case, we also observe that the positive equilibria

$$\overline{V}^1 = \overline{V}^2.$$

By apply the results for the Fisher-KPP equation to the V-equation, we obtain the following lemma.

Lemma 4.3. *(Decoupled case) Let Assumption 4.2 be satisfied. Assume in addition that*

$$\beta = 1.$$

Then there exists a unique solution $V^*(x) = (V_1^*(x), V_2^*(x))$ of system

$$\begin{cases} V_1' = -\alpha V_1 + V_2 \\ V_2' = -\alpha V_2 + \left[\alpha^2 - (1 - b\theta) \right] V_1 + \beta V_1^2 \end{cases} \tag{25}$$

satisfying

$$\lim_{x \to -\infty} V^*(x) = ((1 - b\theta), \alpha (1 - b\theta)) \ \text{ and } \ \lim_{x \to +\infty} V^*(x) = (0, 0).$$

Moreover for each $\delta \in \mathbb{R}$

$$U_\delta(x) := U^*(x + \delta) \ \text{ and } \ V_\delta(x) = V^*(x)$$

is a heteroclinic orbit of system (15) and (21). Therefore system (15) and (21) has an infinite number of heteroclinic orbits going from $\left(\overline{U}^1, \overline{V}^1 \right)$ to $(0, 0)$.

The rest of the paper is devoted to the coupled case

$$1 > \beta$$

which is of course more delicate. To prove the existence of an infinite number of heteroclinic orbits going from $\left(\overline{U}^1, \overline{V}^1 \right)$ to $(0, 0)$ for system (15) and (21) we will analyze the local unstable manifold around $\left(\overline{U}^1, \overline{V}^1 \right).$

Trivial heteroclinic orbit: We observe that when we fix

$$U \equiv \overline{U}^1 = \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$$

the V-equation becomes

$$\begin{cases} V_1' = -\alpha V_1 + V_2 \\ V_2' = -\alpha V_2 + \left[\alpha^2 - (\beta - b\theta) \right] V_1 + \beta V_1^2 \end{cases} \tag{26}$$

There exists a unique solution $\widehat{V}(x) = \left(\widehat{V}_1(x), \widehat{V}_2(x) \right)$ of system (26) satisfying

$$\lim_{x \to -\infty} \widehat{V}(x) = \left(\left(1 - \frac{b\theta}{\beta} \right), \alpha \left(1 - \frac{b\theta}{\beta} \right) \right) \ \text{ and } \ \lim_{x \to +\infty} \widehat{V}(x) = (0, 0).$$

Linearized equation at $\left(\overline{U}^1, \overline{V}^1\right)$**:** The matrix of the linearized equation is

$$L = \begin{bmatrix} L_U & 0 \\ M & L_V \end{bmatrix}$$

where

$$L_V := \begin{bmatrix} -\alpha & 1 \\ \alpha^2 + (\beta - b\theta) & -\alpha \end{bmatrix} \text{ and } M := \begin{bmatrix} 0 & 0 \\ \left(1 - \frac{b\theta}{\beta}\right)(1 - \beta) & 0 \end{bmatrix}.$$

The characteristic equation of L_V is

$$(\alpha + \lambda)^2 - \left[\alpha^2 + (\beta - b\theta)\right] = 0 \Leftrightarrow \lambda^2 + 2\alpha\lambda - (\beta - b\theta) = 0$$

so that the spectrum of L_V is given by

$$\sigma\left(L_V\right) = \left\{\lambda_V^-, \lambda_V^+\right\},$$

where

$$\lambda_V^- := -\alpha - \sqrt{\alpha^2 + (\beta - b\theta)} < 0 < \lambda_V^+ := -\alpha + \sqrt{\alpha^2 + (\beta - b\theta)}.$$

The spectrum of L is

$$\sigma\left(L\right) = \left\{\lambda_U^-, \lambda_U^+, \lambda_V^-, \lambda_V^+\right\}.$$

Since $\beta < 1$, one has

$$\lambda_U^- = -\alpha - \sqrt{\alpha^2 + 1} < \lambda_V^- < 0,$$

and

$$0 < \lambda_V^+ < \lambda_U^+ = -\alpha + \sqrt{\alpha^2 + 1}.$$

As a consequence the center-unstable manifold is two dimensional. We need to specify further the linear center-unstable space X_{cu} for L. The right eigenvector of L associated to λ_U^+ satisfies

$$L \begin{pmatrix} U_{\lambda_U^+} \\ V_{\lambda_U^+} \end{pmatrix} = \lambda_U^+ \begin{pmatrix} U_{\lambda_U^+} \\ V_{\lambda_U^+} \end{pmatrix} \Leftrightarrow \begin{cases} L_U U_{\lambda_U^+} = \lambda_U^+ U_{\lambda_U^+} \\ M U_{\lambda_U^+} + L_V V_{\lambda_U^+} = \lambda_U^+ V_{\lambda_U^+}. \end{cases}$$

So we can fix

$$U_{\lambda_U^+} := \begin{pmatrix} \left(\sqrt{\alpha^2 + 1}\right)^{-1} \\ 1 \end{pmatrix}. \tag{27}$$

In order to compute $V_{\lambda_U^+}$, observe that

$$M U_{\lambda_U^+} + L_V V_{\lambda_U^+} = \lambda_U^+ V_{\lambda_U^+} \Leftrightarrow \left(\lambda_U^+ I - L_V\right) V_{\lambda_U^+} = M U_{\lambda_U^+}.$$

Since $\lambda_U^+ > \lambda_V^+$, the matrix $\lambda_U^+ I - L_V$ is invertible, and one has

$$\left(\lambda_U^+ I - L_V\right)^{-1} = \int_0^{+\infty} e^{-\lambda_U^+ t} e^{L_V t} dt >> 0,$$

a componetwized non-negative matrix. Next due to Assumption 4.2 and since $\beta < 1$, one has

$$\left(1 - \frac{b\theta}{\beta}\right)(1 - \beta) > 0,$$

hence

$$V_{\lambda_U^+} := \left(\lambda_U^+ I - L_V\right)^{-1} M U_{\lambda_U^+} >> 0 \tag{28}$$

is a componentwized non-negative matrix. Therefore the eigenspace of L associated to λ_U^+ is given by

$$E_{\lambda_U^+} = \mathbb{R} W_{\lambda_U^+}$$

where

$$W_{\lambda_U^+} = \begin{pmatrix} U_{\lambda_U^+} \\ V_{\lambda_U^+} \end{pmatrix} \gg 0.$$

To summarize, we have the following lemma.

Lemma 4.4. *The center-unstable space is given by*

$$X_{cu} = \mathbb{R}W_{\lambda_U^+} \oplus \mathbb{R}W_{\lambda_V^+}.$$

where

$$W_{\lambda_U^+} = \begin{pmatrix} U_{\lambda_U^+} \\ V_{\lambda_U^+} \end{pmatrix} \gg 0 \ and \ W_{\lambda_V^+} := \begin{pmatrix} 0_{\mathbb{R}^2} \\ V_{\lambda_V^+} \end{pmatrix},$$

with

$$V_{\lambda_V^+} := \begin{pmatrix} \left(\sqrt{\alpha^2 + (\beta - b\theta)}\right)^{-1} \\ 1 \end{pmatrix}.$$

Remark 5. The projector of the eigenspace associated to λ_U^+ for L is given by

$$\Pi_{\lambda_U^+}^L \begin{pmatrix} U \\ V \end{pmatrix} = \left(\sqrt{\alpha^2 + 1}U_1 + U_2\right) W_{\lambda_U^+}.$$

The projector of the eigenspace associated to λ_V^+ for L is given by

$$\Pi_{\lambda_V^+}^L \begin{pmatrix} U \\ V \end{pmatrix} = \left(aU_1 + bU_2 + \sqrt{\alpha^2 + (\beta - b\theta)}V_1 + V_2\right) W_{\lambda_V^+}.$$

where $\left(U^T, V^T\right) = \left(a, b, \sqrt{\alpha^2 + (\beta - b\theta)}, 1\right)$ is a left eigenvector of L associated to λ_V^+, and

$$U^T = (a, b) = V^T M \left(\lambda_V^+ I - L_U\right)^{-1}$$

with

$$V^T = \left(\sqrt{\alpha^2 + (\beta - b\theta)}, 1\right).$$

Indeed, a left eigenvector of L associated to λ_V^+ should satisfy

$$\left(U^T, V^T\right) L = \lambda_V^+ \left(U^T, V^T\right) \Leftrightarrow \begin{cases} U^T L_U + V^T M = \lambda_V^+ U^T \\ V^T L_V = \lambda_V^+ V^T. \end{cases}$$

System reduced to the center-unstable manifold around $\left(\overline{U}^1, \overline{V}^1\right)$**:** The system formed by the U-equation and the V-equation can be rewritten as

$$\begin{cases} U' = F(U) \\ V' = G(U, V). \end{cases} \tag{29}$$

The center-unstable space of the linearized equation at $\left(\overline{U}^1, \overline{V}^1\right)$ is given

$$X_{cu} = \mathbb{R}W_{\lambda_U^+} \oplus \mathbb{R}W_{\lambda_V^+}.$$

The projector on the center-unstable space X_{cu} reads as

$$\Pi_{cu} \begin{pmatrix} U \\ V \end{pmatrix} = \left(\Pi_{\lambda_U^+}^L + \Pi_{\lambda_V^+}^L\right) \begin{pmatrix} U \\ V \end{pmatrix}.$$

The center-unstable manifold of system (29) around $\left(\overline{U}^1, \overline{V}^1\right)$ is therefore characterized as follows:

$$\begin{pmatrix} U \\ V \end{pmatrix} \in M_{cu} \Leftrightarrow$$

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} \overline{U}^1 + \Theta_1\left(U - \overline{U}^1\right) \\ \overline{V}^1 + \Theta_2\left(U - \overline{U}^1, V - \overline{V}^1\right) \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} \overline{U}^1 \\ \overline{V}^1 \end{pmatrix} + \Pi_{cu}\begin{pmatrix} U - \overline{U}^1 \\ V - \overline{V}^1 \end{pmatrix} + \Psi_{cu}\begin{pmatrix} U - \overline{U}^1 \\ V - \overline{V}^1 \end{pmatrix}$$

where

$$\Psi_{cu}\begin{pmatrix} U - \overline{U}^1 \\ V - \overline{V}^1 \end{pmatrix} = \begin{pmatrix} \Psi_{cu}^1\left(U - \overline{U}^1\right) \\ \Psi_{cu}^2\left(U - \overline{U}^1, V - \overline{V}^1\right) \end{pmatrix} \in (I - \Pi_{cu})\,\mathbb{R}^4$$

and

$$D\Psi_{cu}(0) = 0.$$

Lemma 4.5. *We have the following properties:*

(i) *The trivial heteroclinic orbit $\left(\overline{U}^1, \widehat{V}\right)$ belongs to the center-unstable manifold of system (29) locally around $\left(\overline{U}^1, \overline{V}^1\right)$.*

(ii) *The eigenspace $E_{\lambda_V^+} = \mathbb{R}W_{\lambda_V^+}$ is tangent to the trivial heteroclinic orbit $\left(\widehat{U}, \widehat{V}\right)$ at $\left(\overline{U}^1, \overline{V}^1\right)$.*

Proof. The prove (i), it is sufficient to apply the property (ii) of Theorem 2.2. To show (ii), it is sufficient to observe that $\widehat{U} = \overline{U}^1$, therefore $\widehat{V}$ is an heteroclinic orbit going from $\overline{V}^1$ to 0 solution of the system

$$\widehat{V}'(x) = G(\overline{U}^1, \widehat{V}(x)). \tag{30}$$

Therefore, $\widehat{V}$ belongs to the local center-unstable manifold of (30) locally around $\overline{V}^1$. Hence, by applying Remark 1 to this system the result follows. $\qquad\square$

Lemma 4.6. *Locally around $\overline{U}^1$, the heteroclinic orbit U^* for the U-equation provided by Lemma 4.1 belongs to PM_{cu}, where $P : \mathbb{R}^4 \to \mathbb{R}^2$ is defined by*

$$P\begin{pmatrix} U \\ V \end{pmatrix} := U.$$

Moreover U^* is tangent to $E_{\lambda_U^+} = \mathbb{R}W_{\lambda_U^+}$ at $\overline{U}^1$.

More precisely, we have for $\varepsilon > 0$ small enough that

$$O_\varepsilon = \{U^*(x) : x \in \mathbb{R}\} \cap B_{\mathbb{R}^2}\left(\overline{U}^1, \varepsilon\right) = PM_{cu} \cap \left[0, \overline{U}^1\right[\cap B_{\mathbb{R}^2}\left(\overline{U}^1, \varepsilon\right),$$

where

$$\left[0, \overline{U}^1\right[= \left\{U : 0 \le U < \overline{U}^1\right\}.$$

The main result of this section is the following theorem.

Theorem 4.7. (Coupled case) *From each point of the trivial heteroclinic orbit*
$\begin{pmatrix} \overline{U}^1 \\ \widehat{V}(x) \end{pmatrix}$ *(with $x \in \mathbb{R}$) close enough to* $\begin{pmatrix} \overline{U}^1 \\ \overline{V}^1 \end{pmatrix}$, *there exists a sequence of points*
$\left\{ \begin{pmatrix} U_n \\ V_n \end{pmatrix} \right\}_{n \geq 0} \in \left[0, \overline{U}^1 \right] \times \left[0, \overline{V}^1 \right]$ *satisfying the following properties:*

(i) $\begin{pmatrix} U_n \\ V_n \end{pmatrix} \to \begin{pmatrix} \overline{U}^1 \\ \widehat{V}(x) \end{pmatrix}$ *as $n \to +\infty$;*

(ii) *For each integer $n \geq 0$, $U_n = U^*(y_n)$ for some $y_n \in \mathbb{R}$ and $V_n \neq \widehat{V}(\widehat{x})$ for each $\widehat{x} \in \mathbb{R}$;*

(iii) *For each integer $n \geq 0$, there exists an heteroclinic orbit of (29) passing through $\begin{pmatrix} U_n \\ V_n \end{pmatrix}$ and going from $\begin{pmatrix} \overline{U}^1 \\ \overline{V}^1 \end{pmatrix}$ to $\begin{pmatrix} 0_{\mathbb{R}^2} \\ 0_{\mathbb{R}^2} \end{pmatrix}$.*

Consequently (since by construction $y_n \to -\infty$) there exists an infinite number of heteroclinic orbits of system (29) going from $\begin{pmatrix} \overline{U}^1 \\ \overline{V}^1 \end{pmatrix}$ to $\begin{pmatrix} 0_{\mathbb{R}^2} \\ 0_{\mathbb{R}^2} \end{pmatrix}$ and passing in any fixed neighbourhood of $\begin{pmatrix} \overline{U}^1 \\ \widehat{V}(x) \end{pmatrix}$.

Proof. **Existence of heteroclinic orbits:** The system reduced on the unstable manifold around $\begin{pmatrix} \overline{U}^1 \\ \overline{V}^1 \end{pmatrix}$ can be identify to a system of two scalar ordinary differential equations of the form

$$\begin{cases} u'_{cu} = f(u_{cu}) \\ v'_{cu} = g(u_{cu}, v_{cu}) \end{cases} \tag{31}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are two maps of the class C^1 that satisfy the following properties

$$f(0) = 0 \text{ and } f'(0) = \lambda_U^+ > 0$$

and

$$g(0,0) \text{ and } \frac{\partial g(0,0)}{\partial v_{cu}} = \lambda_V^+ > 0.$$

Therefore by apply the Hartman-Grobman theorem, or the exponential stability theorem (back in time) to system (31), we deduce that for each ball $B_{\mathbb{R}^2}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \varepsilon \right)$ (with $\varepsilon > 0$) there exists a ball $B_{\mathbb{R}^2}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \eta \right)$ (with $0 < \eta \leq \varepsilon$) such that for each point

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in B_{\mathbb{R}^2}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \eta \right)$$

there exists a negative orbit passing through $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ at time $t = 0$ and staying in $B_{\mathbb{R}^2}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \varepsilon \right)$ for all negative time and converging to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to -\infty$.

It follows that there exists $\varepsilon^* > 0$ such that for each point

$$\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} \in \left[0, \overline{U}^1 \right] \times \left[0, \overline{V}^1 \right] \cap M_{cu} \cap B_{\mathbb{R}^4} \left(\begin{pmatrix} \overline{U}^1 \\ \overline{V}^1 \end{pmatrix}, \varepsilon^* \right) \tag{32}$$

with

$$V_0 << \overline{V}^1, \tag{33}$$

there exists a negative orbit $\left\{ \begin{pmatrix} U(x) \\ V(x) \end{pmatrix} \right\}_{x \leq 0}$ of system (29) passing through

$\begin{pmatrix} U_0 \\ V_0 \end{pmatrix}$ as $x = 0$ and such that

$$\begin{pmatrix} U(x) \\ V(x) \end{pmatrix} \to \begin{pmatrix} \overline{U}^1 \\ \overline{V}^1 \end{pmatrix} \quad \text{as } x \to -\infty.$$

Moreover by (32) we have

$$\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} \leq \begin{pmatrix} \overline{U}^1 \\ \widehat{V}(x) \end{pmatrix} \quad \text{for some } x \in \mathbb{R}$$

therefore by using the monotony of system (29) we deduce that

$$\begin{pmatrix} U(x) \\ V(x) \end{pmatrix} \to 0 \quad \text{as } x \to +\infty.$$

Therefore each initial value satisfying (32) and (33) is a point of an heteroclinic

orbit of system (29) going from $\begin{pmatrix} \overline{U}^1 \\ \overline{V}^1 \end{pmatrix}$ to $\begin{pmatrix} 0_{\mathbb{R}^2} \\ 0_{\mathbb{R}^2} \end{pmatrix}$.

Infinite number of heteroclinic orbits: Let $r \in \mathbb{R}$ such that

$$\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} := \begin{pmatrix} \overline{U}^1 \\ \widehat{V}(r) \end{pmatrix} \in B_{\mathbb{R}^4} \left(\begin{pmatrix} \overline{U}^1 \\ \overline{V}^1 \end{pmatrix}, \varepsilon^* \right).$$

Then by construction $\begin{pmatrix} U_0 \\ V_0 \end{pmatrix}$ satisfies (32) and (33). By perturbing the U-compo-

nent, and by using Lemma 4.6, we can find a sequence

$$\left\{ \begin{pmatrix} U_n \\ V_n \end{pmatrix} \right\} \subset M_{cu} \cap B_{\mathbb{R}^4} \left(\begin{pmatrix} \overline{U}^1 \\ \overline{V}^1 \end{pmatrix}, \varepsilon^* \right) \to \begin{pmatrix} \overline{U}^1 \\ \widehat{V}(r) \end{pmatrix} = \begin{pmatrix} U_0 \\ V_0 \end{pmatrix}$$

with

$$U_n = U^*(y_n) << \overline{U}^1$$

and $y_n \to -\infty$ as $n \to +\infty$.

Since $\widehat{V}(x) << \overline{V}^1$, for each $n \geq 0$ positive large enough

$$V_n << \overline{V}^1.$$

So for each $n \geq 0$ large enough, the point

$$\begin{pmatrix} U_n \\ V_n \end{pmatrix}$$

satisfies (32) and (33). Therefore for each $n \geq 0$ large enough we can find an

heteroclinic orbit going from $\begin{pmatrix} \overline{U}^1 \\ \overline{V}^1 \end{pmatrix}$ to $\begin{pmatrix} 0_{\mathbb{R}^2} \\ 0_{\mathbb{R}^2} \end{pmatrix}$.

To conclude it remains to verify that we can fix

$$V_n \neq \widehat{V}(r).$$

If $V_n = \widehat{V}(x)$, we consider $\begin{pmatrix} U_n(x) \\ V_n(x) \end{pmatrix}$ the solution of system (29) passing through $\begin{pmatrix} U_n \\ V_n \end{pmatrix}$ as $x = 0$. By construction, we have

$$U_n(x) = U^*(x + y_n) << \overline{U}^1$$

and by using the monotonicity of system (29) we have

$$V_n' = G(U_n, V_n) < G(\overline{U}^1, V_n) \text{ and } V_n(0) = \widehat{V}(x).$$

Therefore

$$V_n(\varepsilon) < \widehat{V}(x + \varepsilon), \forall \varepsilon > 0.$$

So when $V_n = \widehat{V}(x)$, by replacing $\begin{pmatrix} U_n \\ V_n \end{pmatrix}$ by $\begin{pmatrix} U^*(\varepsilon + y_n) \\ V_n(\varepsilon) \end{pmatrix}$ for $\varepsilon > 0$ small enough, the problem is unchanged and assertion (ii) is verified. This complete the proof of the result. $\square$

REFERENCES

[1] R. M. Anderson and R. M. May, "Infectious Diseases of Humans: Dynamics and Control," Oxford Univ. Press, Oxford, U.K., 1991.

[2] D. G. Aronson and H. F. Weinberger, *Multidimensional nonlinear diffusions arising in population genetics*, Adv. Math., **30** (1978), 33–76.

[3] H. Berestycki, O. Diekmann, K. Nagelkerke and P. Zegeling, *Can a species keep pace with a shifting climate?*, Bull. Math. Biol., **71** (2008), 399–429.

[4] H. Berestycki and L. Rossi, *Reaction-diffusion equations for population dynamics with forced speed, I - The case of the whole space*, Discrete Continuous Dynam. Systems - B, **21** (2008), 41–67.

[5] H. Berestycki and L. Rossi, *Reaction-diffusion equations for population dynamics with forced speed, II - Cylindrical type domains*, Discrete Continuous Dynam. Systems - A, **25** (2009), 19–61.

[6] M. Bramson, *Convergence of solutions of the Kolmogorov equation to travelling waves*, Mem. Amer. Math. Soc., **285** (1983).

[7] F. Brauer and C. Castillo-Chavez, "Mathematical Models in Population Biology and Epidemiology," Springer, New York, 2000.

[8] S. Busenberg and K. C. Cooke, "Vertically Transmitted Diseases," Biomathematics, **23**, Springer-Verlag, New York, 1993.

[9] V. Capasso, "Mathematical Structures of Epidemic Systems," Lecture Notes in Biomathematics **97**, Springer-Verlag, Berlin, 1993.

[10] S.-N. Chow, C. Li and D. Wang, "Normal Forms and Bifurcation of Planar Vector Fields," Cambridge University Press, Cambridge, 1994.

[11] O. Diekmann and J. A. P. Heesterbeek, "Mathematical Epidemiology of Infectious Diseases: Model Building, Analysis and Interpretation," Wiley, Chichester, U.K., 2000.

[12] A. Ducrot and M. Langlais, *Travelling waves in invasion processes with pathogens*, Math. Models Methods Appl. Sci., **18** (2008), 325–349.

[13] A. Ducrot, M. Langlais and P. Magal, *Qualitative analysis and traveling wave solutions for the SI model with vertical transmission*, Communications in Pure and Applied Analysis, **11** (2012), 97–113.

[14] R. A. Fisher, *The wave of advance of advantageous genes*, Ann. of Eugenics, **7** (1937), 355–369.

[15] J. K. Hale, "Asymptotic Behavior of Dissipative Systems," Mathematical Surveys and Monographs 25, Amer. Math. Soc., Providence, RI, 1988.

[16] F. Hamel and L. Roques, *Fast propagation for KPP equations with slowly decaying initial conditions*, J. Diff. Equ., **249** (2010), 1726–1745.

[17] Y. Hosono and B. Ilyas, *Traveling waves for a simple diffusive epidemic model*, Math. Models Methods Appl. Sci., **5** (1995), 935–966.

[18] A. N. Kolmogorov, I. G. Petrovski and N. S. Piskunov, *Etude de l'équation de la chaleur avec croissance de la quantité de matière et son application à un problème biologique*, Bull. Moskov. Gos. Univ. Mat. Mekh., **1** (1937).

[19] D. A. Larson, *Transient bounds and time-asymptotic behavior of solutions to nonlinear equations of Fisher type*, SIAM J. Appl. Math., **34** (1978), 93–103.

[20] K.-S. Lau, *On the nonlinear diffusion equation of Kolmogorov, Petrovsky and Piscounov*, J. Diff. Equ., **59** (1985), 44–70.

[21] F. Rothe, *Convergence to travelling fronts in semilinear parabolic equations*, Proc. Roy. Soc. Edinburgh Sect. A, **80** (1978), 213–234.

[22] S. Ruan, *Spatial-temporal dynamics in nonlocal epidemiological models*, in "Mathematics for Life Science and Medicine" (eds. Y. Takeuchi, K. Sato and Y. Iwasa), Springer-Verlag, Berlin, (2007), 99–122.

[23] S. Ruan and J. Wu, *Modeling spatial spread of communicable diseases involving animal hosts*, in "Spatial Ecology", Chapman & Hall/CRC, Boca Raton, FL, (2009), 293–316.

[24] H. L. Smith, "Monotone Dynamical Systems, an Introduction to the Theory of Competitive and Cooperative Systems," Math. Surveys and Monographs, **41**, American Mathematical Society, Providence, Rhode Island 1995.

[25] H. R. Thieme, "Mathematics in Population Biology," Princeton Univ. Press, Princeton, NJ, 2003.

[26] K. Uchiyama, *The behavior of solutions of some nonlinear diffusion equations for large time*, J. Math. Kyoto Univ., **18** (1978), 453–508.

[27] A. Volpert, V. Volpert and V. Volpert, "Travelling Wave Solutions of Parabolic Systems," Translations of Mathematical Monographs, **140**, AMS Providence, RI, 1994.

[28] V. A. Volpert and Y. M. Suhov, *Stationary solutions of non-autonomous Kolmogorov-Petrovsky-Piskunov equations*, Ergodic Theory and Dynamical Systems, **19** (1999), 809–835.

E-mail address: arnaud.ducrot@u-bordeaux2.fr
E-mail address: michel.langlais@u-bordeaux2.fr
E-mail address: pierre.magal@u-bordeaux2.fr