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HOMOGENIZED DESCRIPTION OF MULTIPLE GINZBURG-LANDAU VORTICES PINNED BY SMALL HOLES

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Abstract. We consider a homogenization problem for the magnetic Ginzburg-Landau functional in domains with a large number of small holes. We establish a scaling relation between sizes of holes and the magnitude of the external magnetic field when the multiple vortices pinned by holes appear in nested subdomains and their homogenized density is described by a hierarchy of variational problems. This stands in sharp contrast with homogeneous superconductors, where all vortices are known to be simple. The proof is based on the Γ-convergence approach applied to a coupled continuum/discrete variational problem: continuum in the induced magnetic field and discrete in the unknown finite (quantized) values of multiplicity of vortices pinned by holes.

1. **Introduction.** Vortices determine electromagnetic properties of superconductors that are important for practical applications (e.g., resistance). A key practical issue is to decrease the energy dissipation in superconductors, which occurs due to the motion of vortices. This dissipation can be suppressed by the pinning of vortices. Moreover, problems of pinning in superconducting thin films lead to the analysis of a two-dimensional Ginzburg-Landau (GL) energy functional in a domain with periodic or random arrays of holes also called antidots in physical literature (see e.g., [6] and references therein).

In this work we consider a two-dimensional mathematical model of pinning of vortices by many holes in relatively small superconducting samples (comparable to the London depth). The sample is subjected to a uniform magnetic field, which is weak so that the vortices do not appear in the bulk of the superconducting sample and they may exist only in the holes.

Since modern experimental techniques allow for the creation of very small holes, the question arises if any specific scaling relations between the external magnetic field and the size of the holes can lead to novel physical effects. In particular, typical experimental results lead to uniform arrays of vortices in the entire domain.

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By contrast, in this work we derive a special "critical" scaling relation when a family of nested subdomains with pinned vortices of increasing multiplicity appears.

We next present a brief review of relevant mathematical work. The study of pinning by a *finite number of pinning sites* was pioneered in [12], where a simplified GL model with no magnetic field and discontinuous pinning term for a single inclusion was considered. The existence of d vortices of degree 1 inside the inclusion was established for the Dirichlet boundary data with degree d. The results of [12] were subsequently generalized for the magnetic GL functional [11], [4] and pinning by a single inclusion. A comprehensive study of pinning by finitely many normal inclusions and holes for the magnetic GL functional was performed in [2, 3]. More recently pinning by finitely many holes whose sizes goes to zero as the GL parameter goes to infinity was established in [7] for the simplified GL model.

Homogenization in the framework of magnetic GL model with continuous oscillating pinning term was considered in the pioneering work [1], where large number of vortices are described by the homogenized vorticity density. Since some composite superconductors are described by a discontinuous pinning term, in subsequent works [8, 9] homogenization problems for such a term were address in the context of simplified GL model and special Dirichlet boundary conditions, which result in either no vortices [8] or d vortices [9].

In this work we study a homogenization problem for a *large number of vortices* and large number of pinning holes that are described by a perforated domain Ω_ε (which corresponds to a discontinuous pinning term). This problem is described by the minimizers of the GL functional

$$GL(u,A) = \frac{1}{2}\int_{\Omega_\varepsilon} (|\nabla u - iAu|^2 + \frac{\kappa^2}{2}(1-|u|^2)^2)\,dx + \frac{1}{2}\int_\Omega (\mathrm{curl}A - h^\varepsilon_{ext})\,dx. \quad (1)$$

The unknowns here are the complex order parameter u and the vector potential of the magnetic field A, while h^ε_{ext} is given constant external magnetic field (a positive scalar quantity). The domain Ω_ε in (1) is obtained by perforating a given simply connected bounded domain $\Omega \subset \mathbb{R}^2$ by a large number N_ε of small holes. Holes are all identical disks with radius ρ_ε and have periodically distributed centers a^ε_j, with $\varepsilon > 0$ being a small spatial period. We assume that

$$|\log\kappa| \gg |\log\rho_\varepsilon| \text{ and } \rho_\varepsilon \ll \varepsilon \quad (2)$$

(hole radius much greater than vortex core and much less than the spatial period ε). Moreover, we consider the following scaling relations of the magnetic field and diameters of holes,

$$h^\varepsilon_{ext} = \sigma/\varepsilon^2,\ \mathrm{diam}(\omega^\varepsilon_j) = 2e^{-\gamma/\varepsilon^2}, \quad (3)$$

where σ and γ are fixed positive numbers. This scaling corresponds to the finite flux of the magnetic field over each periodicity cell $\int_{cell} h^\varepsilon_{ext}dx = O(1)$. Note that the effective core of a vortex pinned by a hole is of the order of the hole size ρ, and therefore its energy is of order $\log(1/\rho)$. Then if vortices inside holes have a finite degree, we get $\rho = \exp(-\gamma/\varepsilon^2)$, which is a much stronger separation than just $\rho \ll \varepsilon$ in (2). Our results show that the scaling (3) leads to a nonuniform spatial distribution of multivortices, whereas the other scalings lead to a simple homogenization limit (no vortices for "very small holes" and constant uniform vorticity for larger holes). Our analysis shows that the scaling (3) is special because the energy of the vortices and the bulk energy of the superconductor outside the vortex cores are of the same order.

Recall that in homogeneous superconductors there are no vortices if $0 < h^\varepsilon_{ext} < H_{c1} := C_1 \log \kappa$ (below the first critical field [14]). A further increase in the magnetic field $C_1 \log \kappa < h^\varepsilon_{ext} < C_2 \kappa^2$ results in a lattice of simple (multiplicity one) vortices. Our results show that in a superconductor with holes the interval $0 < h^\varepsilon_{ext} < H_{c1} := C_1 \log \kappa$ is divided into two subintervals: no vortices at all, and vortices inside holes but not in the bulk of the superconductor.

Formal calculations show that under the scale separation condition (2) there are no vortices in the bulk of the domain Ω_ε and the term with the integrand $\frac{\kappa^2}{4}(1-|u|^2)^2$ can be effectively replaced by the constraint $|u| = 1$. Mathematical justification of this replacement based on the energy decomposition will be presented elsewhere. For a domain with finitely many holes of finite size, analogous replacement was rigorously justified in [3]. However, even in the case of one small hole the analysis from [3] can not be carried out since the smaller the hole the more energy is needed for pinning by this hole (that is the energy of the vortex in the hole blows up as the size of the hole goes to zero).

This leads us to study the homogenization limit for the following minimization problem

$$M_\varepsilon = \inf\{F_\varepsilon(u, A);\ u \in H^1(\Omega_\varepsilon; S^1),\ A \in H^1(\Omega;\ \mathbb{R}^2)\} \tag{4}$$

for the functional

$$F_\varepsilon(u, A) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u - iAu|^2 dx + \frac{1}{2}\int_\Omega (\text{curl} A - h^\varepsilon_{ext})^2 dx,\ \varepsilon > 0. \tag{5}$$

Let d^ε_j be (integer) degrees of minimizer u^ε on $\partial\omega^\varepsilon_j$, then the main objective of this work is to obtain a *homogenized vorticity density* $D(x)$ defined as the weak limit of measures $D(x) = w - \lim_{\varepsilon\to 0} \varepsilon^2 \sum d^\varepsilon_j \delta_{a^\varepsilon_j}(x)$, that provides the limiting description of d^ε_j as $\varepsilon \to 0$.

For every fixed $\varepsilon > 0$ minimizers of $(u^\varepsilon, A^\varepsilon)$ of problem (4)-(5) can be expressed in terms of the degrees d^ε_j via the following problem for the induced magnetic field $h^\varepsilon = \text{curl} A^\varepsilon$ [3]

$$\begin{cases} -\Delta h^\varepsilon + h^\varepsilon = 0 \text{ in } \Omega_\varepsilon, \\ h^\varepsilon(x) = h^\varepsilon_{ext} \text{ on } \partial\Omega, \\ h^\varepsilon(x) = H^\varepsilon_j \ \text{ on } \ \omega^j_\varepsilon,\ j = 1, 2, \ldots, N_\varepsilon, \\ -\int_{\partial\omega^\varepsilon_j} \frac{\partial h^\varepsilon}{\partial\nu}\, ds = 2\pi d^\varepsilon_j - \int_{\omega^\varepsilon_j} h^\varepsilon(x) dx, \end{cases} \tag{6}$$

where H^ε_j are unknown constants that are part of the problem. Note that if we know the degrees d^ε_j, then the minimum (4) is given by

$$E_\varepsilon(h^\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla h^\varepsilon|^2 dx + \frac{1}{2}\int_\Omega (h^\varepsilon - h^\varepsilon_{ext})^2 dx, \tag{7}$$

and conversely, the minimum of (5) is obtained by minimizing (7) in integer d^ε_j, where $h^\varepsilon(x; \{d^\varepsilon_j\})$ is the unique solution of (6). Thus, the tuple $\{d^\varepsilon_j\}$ is obtained by minimizing (7) over tuples of integers and the infinite-dimensional problem (4) is reduced the following finite-dimensional (of dimension N_ε) minimization problem

$$M_\varepsilon = \inf\{E_\varepsilon(h^\varepsilon);\ h^\varepsilon = h^\varepsilon(x; \{d^\varepsilon_j\}) \text{ solves (6) for integer } d^\varepsilon_j\} \tag{8}$$

Remark 1. In the minimization problem (4) $\{d^\varepsilon_j\}$ are unknown integers defined as degrees of the S^1-valued function u. By contrast, problem (6) can be solved for any unknown real-valued tuple $d^\varepsilon_j,\ j = 1, ..., N_\varepsilon$. However, in order to restore the minimizer u_ε from the minimizer h^ε, it is necessary for the constants d^ε_j to be integers. Thus, the condition of d_j^ε being integers (a quantization) is a *constraint* in the minimization problem for h^ε.

In broad terms our result can be described as follows. The homogenized vorticity density $D(x)$ is a piecewise constant function that takes increasing integer values on a family of $j = j(\Omega, \gamma, \sigma)$ nested subdomains, and it is a real constant (not necessarily an integer) in the smallest (inner) domain. This picture describes a rise of vortices of increasing multiplicity. The precise formulation of these results is presented below in Section 4.

Heuristic explanation of the formation of multivortices in a superconductor with holes (unlike simple vortices in a homogenous SC) can be described as follows. If the number of vortices is much less than the number of holes, then all vortices are simple because of repulsion and there are enough available holes for all of them. If the number of vortices is comparable with the number of holes (same order), then the collective effect takes over, namely, vortices are accumulated in some holes resulting in the emergence of multivortices.

2. **Homogenization (corrector) and compactness results.**

We introduce rescaled quantities

$$\tilde{h}^\varepsilon = \varepsilon^2 h^\varepsilon,\ \tilde{h}^\varepsilon_{ext} = \varepsilon^2 h^\varepsilon_{ext},\ \tilde{E}_\varepsilon(\tilde{h}_\varepsilon) = \varepsilon^4 E_\varepsilon(\tilde{h}_\varepsilon), \tilde{M}_\varepsilon = \varepsilon^4 M_\varepsilon. \tag{9}$$

Note that h^ε (and therefore $\tilde{h}_\varepsilon$) is determined uniquely by the tuple of integers d_j^ε. Thus, abusing notation a little, we may write $\tilde{E}_\varepsilon(\tilde{h}^\varepsilon) = \tilde{E}_\varepsilon(\{d_j^\varepsilon\})$. Consider a minimizing tuple of degrees $\{d_j^\varepsilon\}$, so that the corresponding solution of (6), rescaled according to (9), satisfies $\tilde{E}_\varepsilon(\tilde{h}_\varepsilon) = \tilde{M}_\varepsilon$.

First we obtain a priori bounds for these degrees d_j^ε in the following

Lemma 2.1. *Let d_j^ε be degrees of the minimizer of (4), then*

$$\sum (d_j^\varepsilon)^2 \le C/\varepsilon^2, \tag{10}$$

where C is independent of ε.

Proof. The weak formulation of the problem for $\tilde{h}^\varepsilon$ reads, find $\tilde{h}^\varepsilon \in H^1(\Omega)$ such that $\nabla \tilde{h}^\varepsilon = 0$ in all ω_j^ε and $\tilde{h}^\varepsilon = \sigma$ on $\partial\Omega$, and

$$\int_\Omega (\nabla \tilde{h}^\varepsilon \cdot \nabla v + \tilde{h}^\varepsilon v)dx - 2\pi\varepsilon^2 \sum d_j^\varepsilon v|_{\omega_j^\varepsilon} = 0 \tag{11}$$

holds for every test function $v \in H_0^1(\Omega)$ such that $\nabla v = 0$ in all ω_j^ε. In particular, if we choose all $d_j^\varepsilon = 0$, and set $v = \tilde{h}^\varepsilon - \sigma$, we get an a priori bound (for $\tilde{h}$ that corresponds to this choice of degrees d_j^ε)

$$\|\tilde{h}^\varepsilon\|_{H^1(\Omega)} := \int_{\Omega_\varepsilon} (|\nabla \tilde{h}^\varepsilon|^2 + (\tilde{h}^\varepsilon)^2)dx \le C$$

therefore $\tilde{M}_\varepsilon \le C$, where C is independent of ε. Hence for the minimizing tuple $\{d_j^\varepsilon\}$ we have $\tilde{E}_\varepsilon(\tilde{h}^\varepsilon) \le C$ and thus $\|\tilde{h}^\varepsilon\|_{H^1(\Omega)} \le C$, with another independent of ε constant C. Now choose the test function $v = \sum d_j^\varepsilon L_j^\varepsilon(x)/\log(2\rho_\varepsilon/\varepsilon)$ in (11), where

$$L_j^\varepsilon(x) = \begin{cases} \log(2|x - a_j^\varepsilon|/\varepsilon) \text{ if } x \in B_{\varepsilon/2}(a_j^\varepsilon) \setminus B_{\rho_\varepsilon}(a_j^\varepsilon) \\ \log(2\rho_\varepsilon/\varepsilon) \text{ if } x \in B_{\rho_\varepsilon}(a_j^\varepsilon) \\ 0 \text{ if } x \notin B_{\varepsilon/2}(a_j^\varepsilon). \end{cases} \tag{12}$$

The heuristic idea behind the introduction of $L_j^\varepsilon(x)$ can be explained as follows. If the problem (6) is rescaled by ε^{-1}, it becomes a Laplacian to the leading term, whose solution in an annular domain (with a single hole) and constant values on the boundaries is $C_1 + C_2 \log|x|$. Note also that scaling in (12) is such that the supports of $L_j^\varepsilon(x)$ are disjoint.

Now simple computations lead to the required bound,

$$2\pi\varepsilon^2 \sum (d_j^\varepsilon)^2 = \int_\Omega (\nabla \tilde{h}^\varepsilon \cdot \nabla v + \tilde{h}^\varepsilon v) dx \le \sqrt{2\pi\varepsilon^2(1+o(\varepsilon))\sum (d_j^\varepsilon)^2/\gamma}\, \|\tilde{h}^\varepsilon\|_{H^1(\Omega)},$$

i.e. $2\pi\varepsilon^2 \sum (d_j^\varepsilon)^2 \le (1+o(\varepsilon))\|\tilde{h}^\varepsilon\|^2_{H^1(\Omega)}/\gamma \le C$, where γ is defined in (3). □

It follows from Lemma 2.1 that, up to extracting a subsequence,

$$\zeta^\varepsilon = \varepsilon^2 \sum d_j^\varepsilon \delta_{a_j^\varepsilon}(x) \rightharpoonup D(x) \text{ as distributions, and } D(x) \in L^2(\Omega). \tag{13}$$

From now on let $\{d_j^\varepsilon\}$ denote an arbitrary sequence of tuples of integers such that (10) and (13) hold, and $\tilde{h}^\varepsilon$ is the function associated to the tuple $\{d_j^\varepsilon\}$, i.e. $\tilde{h}^\varepsilon = \varepsilon^2 h^\varepsilon$, where h^ε is the solution of (6).

It is rather easy to see that under the above mentioned conditions we can pass to the limit in (11) to get that $\tilde{h}^\varepsilon$ converges weakly-H^1 to the solution $\bar{h}$ of the homogenized problem

$$\begin{cases} -\Delta \bar{h} + \bar{h} = 2\pi D(x) \text{ in } \Omega \\ \bar{h} = \sigma \text{ on } \partial\Omega. \end{cases} \tag{14}$$

(for details see Lemma 2.3 below). However this weak-H^1 convergence is not sufficient for our principal goal of describing the limiting vorticity $D(x)$. Actually, this will be done by calculating the Γ-limit of the functionals $\tilde{E}^\varepsilon$, which requires convergence of energies for the optimal lower bound (that matches the upper bound). In order to obtain the strong-H^1 convergence, we next introduce a corrector.

Consider the ansatz,

$$\tilde{h}^\varepsilon(x) = \bar{h}^\varepsilon(x) - \varepsilon^2 \sum d_j^\varepsilon L_j^\varepsilon(x) = \bar{h}^\varepsilon(x) + R^\varepsilon, \tag{15}$$

where functions $L_j^\varepsilon(x)$ are given by (12). The problem for $\bar{h}^\varepsilon$ (in its weak form) is, find $\bar{h}^\varepsilon \in H^1(\Omega)$ such that $\nabla \bar{h}^\varepsilon = 0$ in all ω_j^ε and $\bar{h}^\varepsilon = \sigma$ on $\partial\Omega$, and

$$\int_\Omega (\nabla \bar{h}^\varepsilon \cdot \nabla v + \bar{h}_\varepsilon v) dx + \sum \int_{B_{\varepsilon/2}(a_j^\varepsilon)} v R^\varepsilon(x) dx = 2\varepsilon \sum d_j^\varepsilon \int_{\partial B_{\varepsilon/2}(a_j^\varepsilon)} v\, ds \tag{16}$$

holds for every test function $v \in H_0^1(\Omega)$ such that $\nabla v = 0$ in all ω_j^ε.

Lemma 2.2. *Under conditions (10), (13) functions $\bar{h}^\varepsilon$ converge strongly-H^1 to $\bar{h}$, the unique solution of (14).*

Remark 2. Lemma 2.2 shows that the function $R^\varepsilon(x) = -\varepsilon^2 \sum d_j^\varepsilon L_j^\varepsilon(x)$ is a corrector, so that $\bar{h}^\varepsilon = \tilde{h}^\varepsilon - R^\varepsilon$ converges strongly in $H^1(\Omega)$.

Proof. Using (10) one shows that R^ε converges weakly-H^1 to zero, and therefore, up to extracting a subsequence, $\bar{h}^\varepsilon \rightharpoonup \bar{h}$. To prove that $\bar{h}$ solves (14) we consider test functions $v^\varepsilon = v(x) + \sum \phi(x/\rho_\varepsilon)(v(a_j^\varepsilon) - v(x))$, where $v \in C_0^\infty(\Omega)$ is an arbitrary function, and $\phi(x)$ is a smooth cut-off function such that $\phi = 1$ if $|x| \le 1$ and $\phi = 0$ if $|x| > 2$. Set $v = v^\varepsilon$ in (16) and pass to the limit as $\varepsilon \to 0$ to get

$$\int_\Omega (\nabla \bar{h} \cdot \nabla v + \bar{h} v)\, dx = 2\pi \int_\Omega D(x) v\, dx.$$

Thus $\bar{h}$ solves (14).

Next we show that $\bar{h}^\varepsilon$ converges strongly-H^1 to $\bar{h}$. We set $v = \bar{h}^\varepsilon - \sigma$ to obtain in the limit $\varepsilon \to 0$,

$$\limsup \int_\Omega \nabla \bar{h}^\varepsilon \cdot \nabla \bar{h}^\varepsilon dx = 2 \limsup \varepsilon \sum d_j^\varepsilon \int_{\partial B_{\varepsilon/2}(a_j^\varepsilon)} (\bar{h}^\varepsilon - \sigma)\, ds + \int_\Omega (\sigma - \bar{h})\bar{h} dx.$$

By the Poincaré inequality, we have

$$\int_{\Pi_j^\varepsilon} \Big| \bar{h}^\varepsilon - \frac{1}{\pi\varepsilon} \int_{\partial B_{\varepsilon/2}(a_j^\varepsilon)} \bar{h}^\varepsilon\, ds \Big|^2 dx \leq C\varepsilon^2 \int_{\Pi_j^\varepsilon} |\nabla \bar{h}^\varepsilon|^2 dx$$

and

$$\int_{\Pi_j^\varepsilon} \Big| \bar{h}^\varepsilon - \frac{1}{\varepsilon^2} \int_{\Pi_j^\varepsilon} \bar{h}^\varepsilon dx \Big|^2 dx \leq C\varepsilon^2 \int_{\Pi_j^\varepsilon} |\nabla \bar{h}^\varepsilon|^2 dx,$$

hence

$$\frac{1}{\pi\varepsilon} \int_{\partial B_{\varepsilon/2}(a_j^\varepsilon)} \bar{h}^\varepsilon\, ds = \frac{1}{\varepsilon^2} \int_{\Pi_j^\varepsilon} \bar{h}^\varepsilon dx + O(1)\big(\int_{\Pi_j^\varepsilon} |\nabla \bar{h}^\varepsilon|^2 dx\big)^{1/2},$$

where Π_j^ε is the cell with the center at a_j^ε and the side length ε. Therefore

$$2 \lim \varepsilon \sum d_j^\varepsilon \int_{\partial B_{\varepsilon/2}(a_j^\varepsilon)} \bar{h}^\varepsilon\, ds = 2\pi \int_\Omega D(x)\bar{h}\, dx,$$

where we have used (10), (13) and the fact that $\bar{h}^\varepsilon \to \bar{h}$ strongly in $L^2(\Omega)$. Thus, taking into account (14), we finally get

$$\limsup \int_\Omega \nabla \bar{h}^\varepsilon \cdot \nabla \bar{h}^\varepsilon dx = 2\pi \int_\Omega D(x)(\bar{h} - \sigma)\, dx + \int_\Omega (\sigma - \bar{h})\bar{h} dx = \int_\Omega \nabla \bar{h} \cdot \nabla \bar{h} dx.$$

This implies that $\bar{h}^\varepsilon \to \bar{h}$ strongly in $H^1(\Omega)$. □

As a corollary of Lemma 2.2 we obtain the following

Lemma 2.3. *Under conditions (10), (13) the following energy expansion holds,*

$$\tilde{E}_\varepsilon(\tilde{h}^\varepsilon) = \bar{E}_1(\bar{h}) + \pi\gamma\varepsilon^2 \sum (d_j^\varepsilon)^2 + o(1), \tag{17}$$

where

$$\bar{E}_1(\bar{h}) = \frac{1}{2} \int_\Omega |\nabla \bar{h}|^2 dx + \frac{1}{2} \int_\Omega (\bar{h} - \sigma)^2 dx \tag{18}$$

The energy expansion (17) will be used in the proof of the $\Gamma-$ convergence result in Section 3. Note that the second term in (17) can also become vanishingly small, this is the case when the external field is weak, $h_{ext}^\varepsilon \ll \sigma_{\text{cr}1}/\varepsilon^2$, where $\sigma_{\text{cr}1}$ is the first critical value given by (43). On the other hand, if $\lim \varepsilon^2 h_{ext}^\varepsilon > \sigma_{\text{cr}1}$, then the second term in expansion (17) for minimizers $\tilde{h}^\varepsilon$ of (7) is bounded below by a positive constant.

Proof. Since $\bar{h}^\varepsilon \to \bar{h}$ strongly in $H^1(\Omega)$ while $R^\varepsilon \to 0$ weakly in $H^1(\Omega)$, we have

$$\tilde{E}_\varepsilon(\tilde{h}^\varepsilon) = \bar{E}_1(\bar{h}^\varepsilon) + \frac{1}{2} \int_\Omega |\nabla R^\varepsilon|^2\, dx + o(1) = \bar{E}_1(\bar{h}) + \frac{1}{2} \int_\Omega |\nabla R^\varepsilon|^2\, dx + o(1)$$

A straightforward calculation of the second term in this expansion yields (17). □

3. **Limiting vorticity via Γ-convergence.** The main result of this work describing the limiting vorticity is obtained by proving the Γ-convergence of functionals $\tilde{E}_\varepsilon$ with respect to the weak convergence (13) of vorticity measures,

$$\tilde{E}_\varepsilon(\{d_j^\varepsilon\}) \ \Gamma\text{-converge to } \bar{E}_0(D) = \bar{E}_1(\bar{h}) + \pi\gamma \int_\Omega \Phi(D(x))dx \text{ as } \varepsilon \to 0, \tag{19}$$

where $\bar{h}$ is the unique solution of (14). More precisely we demonstrate that

(i) ($\Gamma - liminf$ inequality) if conditions (10) and (13) are satisfied, then

$$\liminf \tilde{E}_\varepsilon(\{d_j^\varepsilon\}) \geq \bar{E}_0(D); \tag{20}$$

(ii) ($\Gamma - limsup$ inequality) $\forall D(x) \in L^2(\Omega)$ there is a (recovery) sequence of tuples $\{d_j^\varepsilon\}$ satisfying conditions (10) and (13) and such that

$$\limsup \tilde{E}_\varepsilon(\{d_j^\varepsilon\}) \leq \bar{E}_0(D). \tag{21}$$

Remark 3. Note that condition (13) implies that the limit $D(x)$ exists as a distribution. The condition (10) implies that $D(x)$ is, in fact, a function from $L^2(\Omega)$.

The function Φ in the limit functional (19) is a continuous piecewise linear function such that $\Phi(d) = d^2$ at integer points d. It describes the homogenized density of energies of individual vortices, whereas $\bar{E}_1$ corresponds to the interaction of a vortex with magnetic field due to other vortices and external field.

Thanks to the energy expansion (17), for "lim inf" inequality we need only to prove the lower bound

$$\liminf \varepsilon^2 \sum (d_j^\varepsilon)^2 \geq \int_\Omega \Phi(D(x))\, dx. \tag{22}$$

Since the left hand side of (22) is a nonlinear (quadratic) function of d_j^ε, we use an analog of Young measures (see Appendix).

3.1. **Lower bound.** Spread the measure ζ^ε (defined in (13)), which is the sum of point masses, over periodicity cells Π_j^ε by setting $D^\varepsilon = d_j^\varepsilon$ in Π_j^ε (Π_j^ε is the cell centered at a_j^ε). Then represent D^ε as

$$D^\varepsilon(x) = \sum_{k\in\mathbb{Z}} k\mu_k^\varepsilon(x), \text{ where } \mu_k^\varepsilon(x) = \begin{cases} 1 \text{ in } \Pi_j^\varepsilon, \text{ if } k = d_j^\varepsilon \\ 0, \text{ otherwise.} \end{cases} \tag{23}$$

We extend D^ε and μ_k^ε, $k \neq 0$, on Ω by setting $D^\varepsilon = \mu_k^\varepsilon = 0$ in $\Omega \setminus \cup\Pi_j^\varepsilon$ and also set $\mu_0 = 1$ in $\Omega \setminus \cup\Pi_j^\varepsilon$. The functions $\mu_k^\varepsilon(x)$ satisfy $\mu_k^\varepsilon \geq 0$ and $\sum \mu_k^\varepsilon = 1$ and therefore form a partition of unity. Clearly, we can extract a subsequence such that

$$\mu_k^\varepsilon \rightharpoonup \mu_k \text{ weakly in } L^2(\Omega)\ \forall k \in \mathbb{Z}. \tag{24}$$

Thanks to the bound (10) we have

$$\sum_{k\in\mathbb{Z}} k^2 \int_\Omega \mu_k^\varepsilon dx = \varepsilon^2 \sum (d_j^\varepsilon)^2 \leq C. \tag{25}$$

Hence the limit functions μ_k also form a partition of unity,

$$\mu_k \geq 0 \text{ and } \sum \mu_k = 1. \tag{26}$$

Indeed, the weak limit of non-negative functions is non-negative. Analogously $\sum \mu_k \leq 1$, while by (25) we have

$$\sum_{|k|\leq K} \int_\Omega \mu_k dx \geq |\Omega| - \limsup_{\varepsilon\to 0} \sum_{|k|>K} \int_\Omega \mu_k^\varepsilon dx \geq |\Omega| - C/K^2, \ \forall K > 0,$$

that yields $\sum \mu_k = 1$. Moreover, the function $D(x)$ defined in (13) admits the representation

$$D(x) = \sum k\mu_k(x) \tag{27}$$

and the equality in (25) implies that

$$\liminf \varepsilon^2 \sum (d_j^\varepsilon)^2 \geq \sum_{k\in\mathbb{Z}} k^2 \int_\Omega \mu_k(x). \tag{28}$$

In order to obtain a lower bound in terms of $D(x)$, introduce

$$\Phi(D) := \min_{\{\mu_k\}_{k\in\mathbb{Z}}} \left\{ \sum_{k\in\mathbb{Z}} k^2\mu_k;\ \mu_k \geq 0,\ \sum \mu_k = 1,\ \sum k\mu_k = D \right\}. \tag{29}$$

for every real number D. Then (28) implies the lower bound

$$\liminf \varepsilon^2 \sum (d_j^\varepsilon)^2 \geq \int_\Omega \Phi(D(x))dx \tag{30}$$

The function $\Phi(D)$ can be computed as follows

Lemma 3.1. *$\Phi(D) = (2k+1)|D| - k - k^2$ if $k \leq |D| < k+1$, $k = 0, 1, 2, \dots$. Moreover, if $D = d$ is an integer, then the unique minimizing sequence is $\mu_d = 1$, $\mu_k = 0 \ \ \forall k \neq d$. In the case D is non integer, represent D as the convex hull of the two nearest integers d and $d+1$, $D = \alpha d + (1-\alpha)(d+1)$, then the unique minimizing sequence is $\mu_d = \alpha$, $\mu_{d+1} = 1-\alpha$, $\mu_k = 0 \ \ \forall k \notin \{d, d+1\}$.*

Proof. If $D = d$ is an integer then, clearly, $\Phi(D) \leq d^2$, and by the Cauchy-Schwarz inequality we have

$$d^2 = (\sum k\mu_k)^2 = (\sum k\sqrt{\mu_k}\sqrt{\mu_k})^2 \leq \sum k^2\mu_k = \Phi(D). \tag{31}$$

Thus $\Phi(D) = d^2$ and the minimizing sequence is $m_d = 1$, $m_k = 0 \ \forall k \neq d$.

Consider now the case when D is non integer and $D = \alpha d + (1-\alpha)(d+1)$. Let $\{\mu_k\}$ be a minimizing sequence. If $\mu_k > 0$ for some $k < d$ then there is $\mu_l > 0$ for some $l \geq d+1$. Decrease μ_k and μ_l by a sufficiently small $\delta > 0$ and increase μ_{k+1} and μ_{l-1} by δ. This modification changes neither (26) nor (27) but decreases the value of the functional $\sum k^2\mu_k$. Therefore $\mu_k = 0 \ \forall k \notin \{d,\, d+1\}$. The case when $\mu_k > 0$ for some $k > d+1$ is similar. It follows from (26) and (27) that $\mu_d = \alpha$, $\mu_{d+1} = 1-\alpha$ and straightforward calculations yield the result. □

3.2. **Upper bound.** In order to complete the proof of Γ-convergence (19) we have to show the lim sup-inequality, i.e., given $D \in L^2(\Omega)$, we need to construct a (recovery) sequence of tuples $\{d_j^\varepsilon\}$ satisfying the boundedness condition (10), that converge to $D(x)$ in the sense of (13) and satisfy inequality (21).

The limiting functional $\bar{E}_0(D)$ is continuous with respect to the strong convergence in $L^2(\Omega)$ therefore it is sufficient to establish the lim sup-inequality for $D \in C_0^\infty(\Omega)$ and then use density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$. Moreover, due to Lemma 2.3

in order to establish (21), it is sufficient to construct a recovery sequence of tuples $\{d_j^\varepsilon\}$ that satisfies

$$\varepsilon^2 \sum (d_j^\varepsilon)^2 \leq \int_\Omega \Phi(D(x))\,dx. \tag{32}$$

Note that we not only need the converge of tuples in the sense of (13) but more importantly we need the convergence of energies which does not follow from (13).

The key issue in the construction of the upper bound is that different configurations of vortices may lead to the same homogenized vorticity $D(x) = \sum k\mu_k(x)$, however, these configurations can be distinguished by $\sum k^2\mu_k(x)$ (which is equal to $\Phi(D)$ for the optimal μ_k given by Lemma 3.1).[1] Thus we need to choose d_j^ε that define μ_k^ε via (23) so that the limiting values μ_k are optimal in the sense of (29).

According to Lemma 3.1 if we represent $D(x)$ for fixed $x \in \Omega$ as a convex hull of the nearest integers d and $d+1$, $D(x) = \alpha d + (1-\alpha)(d+1)$, then we must have only holes with degree d and $d+1$ in a small neighborhood of x and in this neighborhood $\#\{\text{holes with degree } d\}/\#\{\text{holes with degree } d+1\}$ should be approximately equal to $\alpha/(1-\alpha)$.

Recall that the centers of holes a_j^ε form an ε-lattice and therefore partition the domain Ω into fine squares Π_j^ε of side length ε. In order to construct the distribution of vortex degrees that yields the optimal probabilities $\mu_k(x)$ in the homogenization limit (averaging over the ε scale), we need a partition of Ω on a coarser scale. To this end we fix a sufficiently large integer M and consider squares K_k with side length $(2M+1)\varepsilon$ and centers at points a_k^ε, which form an $(2M+1)\varepsilon$-periodic lattice. Thus, we introduce a mesoscale $(2M+1)\varepsilon$. This would allow to the construction of $\tilde{D}(x)$ that approximates $D(x)$ in $L^2(\Omega)$ for small ε and large M.

Consider a square K_k that lies strictly inside Ω (it contains exactly $(2M+1)^2$ holes). Let D_k denote the mean value of $D(x)$ on K_k,

$$D_k := \frac{1}{(2M+1)^2\varepsilon^2} \int_{K_k} D(x)dx,$$

and let d_k be the integer such that $d_k \leq D_k < d_k + 1$. We next use Lemma 3.1 to recover the degrees of vortices. To this end we represent D_k as the convex combination of d_k and d_k+1, $D_k = \alpha_k d_k + (1-\alpha_k)(d_k+1)$ $(0 \leq \alpha_k < 1)$. Thus we need to find a distribution of degrees d_k and d_{k+1} over cells that lie in K_k. Since K_k has finitely many cells, this cannot be done exactly and we choose the largest integer $R > 0$ such that $R/(2M+1)^2 \leq \alpha_k$ and set $d_j^\varepsilon := d_k$ for R holes ω_j^ε in K_k and $d_j^\varepsilon := d_k + 1$ for the remaining holes in K_k. We repeat this procedure for all squares K_k lying strictly inside Ω and set degrees of the remaining holes to be zero. Thus the constructed distribution of degrees defines the functions μ_k^ε by (23),

$$\varepsilon^2 \sum (d_j^\varepsilon)^2 = \sum_{l \in Z} l^2 \int_\Omega \mu_l^\varepsilon(x)\,dx.$$

[1] This important point can be illustrated heuristically as follows. In domain Ω consider ε-periodic checkerboard microstructure with black cells having degree -1 and white cells having degree 1. Then in the limit $\varepsilon \to 0$ we get $\mu_{-1} = \mu_1 = 1/2$. Therefore the first moment is zero, $D = (-1)\frac{1}{2} + (1)\frac{1}{2} = 0$, whereas the second moment is $\sum k^2\mu_k = 1$. Next consider checkerboard with both black and white cells having degrees 0. This yields $\mu_0 = 1$ and $\mu_k = 0,\ k \neq 0$ in the limit $\varepsilon \to 0$, and therefore $D = 0$ as in the previous case. However now the second moment is $\sum k^2\mu_k = \Phi = 0$ and it is optimal from the energy standpoint.

We claim that for sufficiently small $\varepsilon > 0$

$$\tilde{E}_\varepsilon(\tilde{h}^\varepsilon)) \leq \bar{E}_0(D) + \delta_M, \tag{33}$$

where $\delta_M \to 0$ as $M \to \infty$. Indeed, due to boundedness, we have that, up to extracting a subsequence, $\mu_l^\varepsilon \to \mu_l(x)$ weakly in $L^2(\Omega)$ for all $l \in Z$. The corresponding limiting vorticity $\tilde{D}(x)$ (which depends on M) is given by $\tilde{D}(x) = \sum l\mu_l(x)$. Due to the above construction of tuples $\{d_j^\varepsilon\}$ on squares, we have two cases.

First, when $D(x)$ is not close to integers, $\operatorname{dist}(D(x), \mathbb{Z}) \geq 1/(2M)^2$, we have $|\mu_d(x) - \alpha| \leq 1/(2M)^2$ and $|\mu_{d+1}(x) - (1-\alpha)| \leq 1/(2M)^2$, where $\alpha d + (1-\alpha)(d+1) = D(x)$ is the representation of $D(x)$ as the convex combination of nearest integers. Second, when $|D(x) - d| \leq 1/(2M)^2$ for some integer d, we have $|\mu_d(x) - 1| \leq 2/(2M)^2$. It follows that

$$\lim_{\varepsilon \to 0} \varepsilon^2 \sum (d_j^\varepsilon)^2 = \sum l^2 \int_\Omega \mu_l(x) \leq \int_\Omega \Phi(\tilde{D}(x))\, dx + C/M^2. \tag{34}$$

Analogously by the construction of tuples $\{d_j^\varepsilon\}$ we have $|\tilde{D}(x) - D(x)| \leq C/M^2$ in Ω. Thus, by virtue of Lemma 2.3 we obtain (33) by taking the limit $\varepsilon \to 0$ for a given M. Now we consider the limit $M \to \infty$ so that the last term in (34) vanishes, which proves the desired upper bound.

3.3. **Γ-convergence theorem.** We summarize the results of this Section in the following

Theorem 3.2. *The functionals $\tilde{E}_\varepsilon$ Γ-converge to $\bar{E}_0(D)$ as $\varepsilon \to 0$. The limit functional $E_0(D)$ is given by*

$$E_0(D) = \frac{1}{2} \int_\Omega \left(|\nabla \bar{h}|^2 + (\bar{h} - \sigma)^2\right) dx + \pi\gamma \int_\Omega \Phi(D(x))dx, \tag{35}$$

where $\bar{h} = \bar{h}(D)$ is the unique solution of (14) and $\Phi(D) = (2k+1)|D| - k - k^2$ if $k \leq |D| < k+1$, $k = 0, 1, 2, \dots$, see Fig 1.

This yields the main homogenization result of this work

Theorem 3.3 (homogenized vorticity)**.** *Let $\{d_j^\varepsilon\}$ be a tuple of integer degrees solving the minimization problem (4), (5) (i.e. the solution h^ε of problem (6) minimizes the functional (7)), then*

$$\varepsilon^2 \sum d_j^\varepsilon \delta_{a_j^\varepsilon}(x) \rightharpoonup D(x), \text{ in the sense of distributions}, \tag{36}$$

where D is the unique minimizer of the functional $E_0(D)$ in $L^2(\Omega)$, $E_0(D)$ being given by (35).

Proof. Note that the Γ-limit functional $E_0(D)$ is strictly convex, continuous and coercive, therefore it has a unique minimizer. On the other hand, by Lemma 2.1 the weak limit $\varepsilon^2 \sum d_j^\varepsilon \delta_{a_j^\varepsilon}(x) \rightharpoonup D(x)$ exists (up to extracting a subsequence) and $D \in L^2(\Omega)$. Therefore, due to the classical properties of Γ-convergence, D is the unique minimizer of $E_0(D)$. □

Remark 4. Note that this theorem provides the homogenized vorticity $D(x)$. However, our goal is to describe the distribution of multiplicities of vortices in terms of probabilities $\mu_k(x)$ of having a vortex of degree k in a neighborhood of a point x . It can be seen from (27) that $\mu_k(x)$ are not uniquely defined by $D(x)$. However, the optimality condition (29) selects $\mu_k(x)$ uniquely so that (30) becomes an equality (otherwise (30) will be a strict inequality). The recipe of computing $\mu_k(x)$ follows

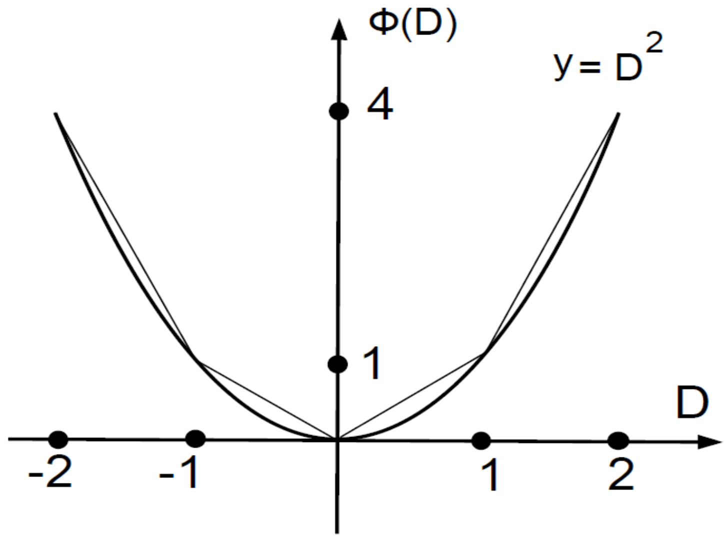

FIGURE 1. The function $\Phi(D)$

Lemma 3.1. If $D(x) = d$ is an integer , then the unique minimizing sequence is $\mu_d(x) = 1$, $\mu_k(x) = 0 \quad \forall k \neq d$. In the case $D(x)$ is non integer, represent $D(x)$ as the convex hull of two nearest integers d and $d+1$, $D(x) = \alpha d + (1-\alpha)(d+1)$, then the unique minimizing sequence is $\mu_d(x) = \alpha$, $\mu_{d+1}(x) = 1-\alpha$, $\mu_k(x) = 0$ $\forall k \notin \{d, d+1\}$.

4. **Analysis of the limit problem via convex duality. Hierarchy of multiplicities.** We use convex duality (see, e.g., [10]) to pass from the problem

$$M_\sigma = \min\Big\{\frac{1}{2}\int_\Omega \big(|\nabla(h-\sigma)|^2 + (h-\sigma)^2 + 2\pi\gamma\Phi((-\Delta h + h)/(2\pi))\big)\,dx;$$

$$(h-\sigma) \in H^1_0(\Omega),\ -\Delta h + h \in L^2(\Omega)\Big\} \quad (37)$$

to the dual one

$$-M_\sigma = \min\left\{\frac{1}{2}\int_\Omega (|\nabla f|^2 + f^2)\,dx + \mathcal{F}^*(-f);\ f \in H^1_0(\Omega)\right\}, \quad (38)$$

where $\mathcal{F}^*(f)$ is the Legendre transform of the functional

$$\mathcal{F}(\kappa) = \pi\gamma\int_\Omega \Phi((-\Delta\kappa + \kappa + \sigma)/(2\pi)),$$

i.e.

$$\mathcal{F}^*(f) = \sup\left\{\int_\Omega (\nabla f\cdot\nabla\kappa + f\kappa)\,dx - \mathcal{F}(\kappa);\ \kappa \in H^1_0(\Omega)\right\}.$$

Due to the fact that $\mathcal{F}(\kappa)$ is lower semicontinuous, the minimizer $(\bar{h}-\sigma)$ of (37) and minimizer $\bar{f}$ of (38) coincide (moreover M_σ in (37) and (38) is the same). Thus, we have

$$2\pi D(x) = -\Delta\bar{h} + \bar{h} = -\Delta\bar{f} + \bar{f} + \sigma \quad (39)$$

The calculation of the Legendre transform $\mathcal{F}^*(f)$ is reduced to the calculation of the Legendre transform $\Phi^*(f)$ of the function $\pi\gamma\Phi(z/(2\pi))$. Indeed, if we use integration by parts we derive

$$\int_\Omega (\nabla f\cdot\nabla\kappa + f\kappa)\,dx - \mathcal{F}(\kappa) =$$
$$\int_\Omega(-\Delta\kappa+\kappa+\sigma)f\,dx - \int_\Omega\big(\pi\gamma\Phi\big((-\Delta\kappa+\kappa+\sigma)/(2\pi)\big)-\sigma f\big)dx,$$

and therefore

$$\mathcal{F}^*(-f) = \int_\Omega (\Phi^*(-f)+\sigma f)dx. \tag{40}$$

The Legendre transform $\Phi^*(f)$ of $\pi\gamma\Phi(z/(2\pi))$ is given by $\Phi^*(f)=0$ for $|f|\le\gamma/2$ and $\Phi^*(f)=2\pi k|f|-\pi\gamma k^2$ for $k\gamma-\gamma/2\le|f|\le k\gamma+\gamma/2$.

Thus (37) is equivalent to the problem

$$\min\Big\{\frac{1}{2}\int_\Omega\big(|\nabla f|^2+f^2+2\Phi^*(f)+2\sigma f\big)\,dx;\, f\in H_0^1(\Omega)\Big\}, \tag{41}$$

and the limit vorticity is defined in terms of the minimizer $\bar f$ by the formula (39).

The following Lemma shows the monotone dependence of minimizers on σ. It is an important tool in the analysis of the dependence of the vorticity domains on the magnitude of the magnetic field.

Lemma 4.1. *Let $\bar f_\sigma$ be the minimizer of (41), then $\bar f_\sigma\le 0$ in Ω for every $\sigma>0$, and $\bar f_\alpha\le\bar f_\beta$ in Ω if $\alpha>\beta$.*

Proof. Set

$$\tilde\Phi_\delta^*(f)=\frac{1}{2\delta}\int_{f-\delta}^{f+\delta}\Phi^*(z)\,dz,$$

since it is an integral of a continuous function, $\tilde\Phi^*_\delta\in C^1(\mathbb{R})$. Moreover, $\tilde\Phi^*_\delta$ is convex thanks to the convexity of $\Phi^*(f)$.

Let $\tilde f_\sigma$ be the minimizer of the functional (41) with $\tilde\Phi^*_\delta$ in place of Φ^*. It is obvious that this minimizer is a continuous function. Assume that the function $\tilde f=\tilde f_\beta-\tilde f_\alpha$ has a negative minimum. Subtracting the Euler-Lagrange equation for $\tilde f_\alpha$ from that for $\tilde f_\beta$ we obtain

$$-\Delta\tilde f+\tilde f+(\beta-\alpha)+(\tilde\Phi^*_\delta)'(f_\beta)-(\tilde\Phi^*_\delta)'(f_\alpha)=0$$

At the minimum point of $\tilde f$ we have that $-\Delta\tilde f\le 0$, $\tilde f<0$, $\beta-\alpha<0$ and $(\tilde\Phi^*_\delta)'(\tilde f_\beta)-(\tilde\Phi^*_\delta)'(\tilde f_\alpha)\le 0$. Thus we have a contradiction and therefore $\tilde f_\beta\ge\tilde f_\alpha$. In particular, setting $\beta=0$ we get $\tilde f_\sigma\le 0$ for $\sigma>0$.

The result follows by passing to the limit $\delta\to 0$. □

4.1. **Weak magnetic fields: Zero vorticity.** Let us consider weak magnetic fields $h^\varepsilon_{ext}=\sigma/\varepsilon^2$, such that $\sigma>0$ is small. It is natural to expect that for such magnetic fields the minimizer $\bar f_\sigma$ of the problem (41) satisfies $-\gamma/2<f_\sigma\le 0$ therefore $\Phi^*(f_\sigma+v)=0$ in Ω for every sufficiently small smooth test function $v\in C_0^\infty(\Omega)$, and this implies that $\bar f_\sigma$ must solve the problem

$$\begin{cases}\Delta f=f+\sigma \text{ in } \Omega\\ f=0 \text{ on } \partial\Omega.\end{cases} \tag{42}$$

More precisely, the case of zero vorticity is described by

Proposition 1. *Let f_1 be the solution of (42) for $\sigma = 1$ and γ is defined in (3). If*

$$\sigma \le \sigma_{\rm cr1} := \frac{\gamma}{2\max|f_1|} \tag{43}$$

then the minimizer $\bar{f}_\sigma$ of (41) is given by $\bar{f}_\sigma = \sigma f_1$, and, according to (39), $D(x) = 0$.

Proof. Since

$$\frac{1}{2}\int_\Omega \big(|\nabla f|^2 + f^2 + 2\Phi^*(f) + 2\sigma f\big)\,dx \ge \frac{1}{2}\int_\Omega \big(|\nabla f|^2 + f^2 + 2\sigma f\big)\,dx \tag{44}$$

for every f, and (44) becomes an equality if f is the minimizer of the right hand side, $f = \sigma f_1$, the result follows. □

4.2. **Moderate magnetic fields: Simple vortices.** If the parameter σ begin to exceed $\sigma_{\rm cr1}$, then σf_1 is no longer the minimizer of (41), and (41) is reduced to the following variational problem

$$\min\left\{\frac{1}{2}\int_\Omega \big(|\nabla f|^2 + f^2 + 4\pi(|f| - \gamma/2)_+ + 2\sigma f\big)\,dx;\ f \in H_0^1(\Omega)\right\}. \tag{45}$$

Proposition 2. *Let g_σ be the minimizer of (45) and let*

$$\sigma_{\rm cr2} := \max\{\sigma > 0; \max|g_\sigma| \le 3\gamma/2\}. \tag{46}$$

Consider $\sigma \in (\sigma_{\rm cr1}, \sigma_{\rm cr2})$, then the minimizer $\bar{f}_\sigma$ of (41) coincides with that of (45).

Proof. The proof is identical to that of Proposition 1. Note that we make use here of Lemma 4.1 to get that the minimizer g_σ of (45) satisfies the inequality $g_\sigma \ge -3\gamma/2$ in Ω when $\sigma \le \sigma_{\rm cr2}$. □

In order to describe the limiting vorticity function $D(x)$ we have to study problem (45) in more details.

Proposition 3. *If $\sigma_{\rm cr1} < \sigma \le 2\pi + \gamma/2$, then (45) is reduced to the obstacle problem*

$$\min\left\{\frac{1}{2}\int_\Omega \big(|\nabla f|^2 + f^2 + 2\sigma f\big)\,dx;\ f \in H_0^1(\Omega),\ |f| \le \gamma/2\right\}. \tag{47}$$

The minimizer $\bar{f}_\sigma$ takes the value $-\gamma/2$ on a set with nonzero Lebesgue measure on $\mathbb{R}^2$. Moreover, the vorticity $D(x)$ is zero in the domain where $\bar{f}(x) > -\gamma/2$ and $D(x) = (\sigma - \gamma/2)/(2\pi)$ otherwise.

If $\max\{\sigma_{\rm cr1}, 2\pi + \gamma/2\} < \sigma \le \sigma_{\rm cr2}$ then $D(x) = 1$ when $\bar{f}_\sigma(x) < -\gamma/2$ and $D(x) = 0$ if $\bar{f}_\sigma(x) \ge -\gamma/2$, where $\bar{f}_\sigma$ is the minimizer of (45).

Remark 5. It follows from Proposition 3 that the vorticity $D(x)$ is zero in a subdomain of Ω where $\bar{f}_\sigma > -\gamma/2$ and $0 < D(x) \le 1$ when $\bar{f}_\sigma \le -\gamma/2$. Moreover, there are two scenarios, if $\sigma_{\rm cr1} < \sigma \le 2\pi + \gamma/2$ then $0 < D(x) < 1$ in the set where $\bar{f}_\sigma(x) \le -\gamma/2$ (holes with degrees one and zero), and if $\max\{\sigma_{\rm cr1}, 2\pi + \gamma/2\} < \sigma \le \sigma_{\rm cr2}$ then $D(x) = 1$ when $\bar{f}_\sigma(x) \le -\gamma/2$ (all holes in this set have degree one).

Proof. Let $\sigma_{\rm cr1} < \sigma \le 2\pi + \gamma/2$ then $f^2 + 4\pi(|f| - \gamma/2)_+ + 2\sigma f > \gamma^2/4 - \sigma\gamma$ when $f < -\gamma/2$. It follows that the minimizer $\bar{f}_\sigma$ of (45) satisfies the pointwise inequality $\bar{f}_\sigma \ge -\gamma/2$. Clearly we also have $\bar{f}_\sigma \le 0$. Thus $\bar{f}_\sigma$ minimizes (47). If we assume that $\bar{f}_\sigma$ satisfies the strict inequality $\bar{f}_\sigma(x) > -\gamma/2$ for a.e. $x \in \Omega$, then $-\Delta\bar{f}_\sigma(x) + \bar{f}_\sigma(x) + \sigma = 0$ for a.e. $x \in \Omega$ and therefore $\bar{f}_\sigma$ is the solution of problem (42). Since $\sigma > \sigma_{\rm cr1}$ we have a contradiction with the pointwise bound $\bar{f}_\sigma \ge -\gamma/2$.

In the case when $\max\{\sigma_{\rm cr1}, 2\pi+\gamma/2\} < \sigma \le \sigma_{\rm cr2}$ we clearly have $-\Delta \bar f_\sigma(x) + \bar f_\sigma(x) + \sigma = 0$ when $-\gamma/2 < \bar f_\sigma \le 0$, and $-\Delta \bar f_\sigma(x) + \bar f_\sigma(x) + \sigma = 2\pi$ when $\bar f_\sigma < -\gamma/2$. Thus we only need to show that the level set $\bar f_\sigma = -\gamma/2$ has zero measure. To this end consider the set $W = \{x \in \Omega;\, -\gamma/2 \ge \bar f_\sigma > -\gamma/2 - \delta\}$, where $\delta > 0$. For sufficiently small δ the boundary of W can be divided into two nonempty parts $S_1 = \{x \in \partial W;\, \bar f_\sigma = -\gamma/2\}$ and $S_2 = \{x \in \partial W;\, \bar f_\sigma = -\gamma/2 - \delta\}$. Both sets S_1 and S_2 have zero measure. Consider the function U such that $\Delta U = 0$ in the interior of W, $U = -\gamma/2$ on S_1, and $U = -\gamma/2 - \delta$ on S_2. By the maximum principle $U < -\gamma/2$ in the interior of W. On the other hand $\bar f_\sigma \le U$ (otherwise $\min\{\bar f_\sigma, U\}$ is a minimizer). Thus the level set $\bar f_\sigma = -\gamma/2$ coincides with S_1 and has zero measure. □

4.3. **Stronger magnetic fields: Multiple vortices.** For $\sigma > \sigma_{\rm cr2}$ vortices with multiplicity two appear. Similarly to the case of simple vortices there are two scenarios depending on whether $\sigma_{\rm cr2} < \gamma/2 + 2\pi$ or $\sigma_{\rm cr2} \ge \gamma/2 + 2\pi$. Define

$$\sigma_{\rm cr3} := \max\{\sigma > 0; \max|g_\sigma| \le 5\gamma/2\}, \tag{48}$$

where g_σ is the minimizer of the problem

$$\min\left\{\frac{1}{2}\int_\Omega \big(|\nabla f|^2 + f^2 + 4\pi((|f|-\gamma/2)_+ + (|f|-3\gamma/2)_+) + 2\sigma f\big)\,dx;\right.$$
$$\left. f \in H_0^1(\Omega)\right\}. \tag{49}$$

Proposition 4. *If $\sigma_{\rm cr2} < \sigma \le 4\pi + 3\gamma/2$ then (49) is reduced to the obstacle problem*

$$\min\left\{\frac{1}{2}\int_\Omega \big(|\nabla f|^2 + f^2 + 4\pi(|f|-\gamma/2)_+ + 2\sigma f\big)\,dx;\ f \in H_0^1(\Omega),\ |f| \le 3\gamma/2\right\}. \tag{50}$$

The minimizer $\bar f_\sigma$ takes the value $-3\gamma/2$ on a set with nonzero Lebesgue measure on $\mathbb{R}^2$. Moreover, the vorticity $D(x)$ is zero in the domain where $\bar f_\sigma(x) > -\gamma/2$, $D(x) = 1$ when $-3\gamma/2 < \bar f_\sigma(x) < -\gamma/2$ and $D(x) = (\sigma - 3\gamma/2)/(2\pi)$ when $\bar f_\sigma(x) = -3\gamma/2$.

If $\max\{\sigma_{\rm cr2}, 4\pi + 3\gamma/2\} < \sigma \le \sigma_{\rm cr3}$ then $D(x) = 0$ when $\bar f_\sigma(x) > -\gamma/2$, $D(x) = 1$ if $-3\gamma/2 < \bar f_\sigma(x) < -\gamma/2$ and $D(x) = 2$ when $f_\sigma(x) < -3\gamma/2$, where $\bar f_\sigma$ is the minimizer of (49).

Remark 6. In the case when $\sigma_{\rm cr2} < \sigma \le 4\pi + 3\gamma/2$ we see that vortices with multiplicities one and two coexist in the set where $\bar f_\sigma(x) = -3\gamma/2$, while all holes have degree one in the domain where $-3\gamma/2 < \bar f_\sigma(x) < -\gamma/2$ and zero degree in the domain where $f_\sigma(x) > -\gamma/2$. If $\max\{\sigma_{\rm cr2}, 4\pi + 3\gamma/2\} < \sigma \le \sigma_{\rm cr3}$ there are three subdomains, where $f_\sigma(x) > -\gamma/2$, $-3\gamma/2 < \bar f_\sigma(x) < -\gamma/2$ and $\bar f_\sigma(x) < -\gamma/2$. All holes in these domains have vortices of degrees zero, one, and two correspondingly.

The proof of this result is similar to the previous ones. Further increase of the magnetic field leads to vortices with higher multiplicities in nested subdomains. Thus our results can be summarized in the following

Theorem 4.2. *There exists a strictly increasing sequence of critical values $\sigma_{{\rm cr}j} = \sigma_{{\rm cr}j}(\gamma, \Omega)$, $j = 1, 2, \ldots$ such that if $\sigma_{{\rm cr}j} < \sigma < \sigma_{{\rm cr}(j+1)}$ the limiting vorticity takes constant values in subsets $\Omega_k \setminus \Omega_{k+1}$, where $\Omega_k = \Omega_k(\sigma)$, $k = 0, 1, \ldots, j$ are strictly nested sets (vorticity sets) and $\Omega_0 = \Omega$. Namely, the vorticity $D(x) = 0$ in $\Omega_0 \setminus \Omega_1$ and $D(x) = k$ in $\Omega_k \setminus \Omega_{k+1}$, $k \le j - 1$. Finally, when $x \in \Omega_j$, then there are* *two scenarios: (i) if* $\sigma < 2\pi j + (j-1/2)\gamma$ *then* $(j-1) < D(x) < j$ *otherwise (ii)* $D(x) = j$.

Finally, the following Lemma describes the dependence of the vorticity sets on the magnitude of the external magnetic field σ.

Lemma 4.3. *If* $\sigma \to \infty$*, then for every* $k \geq 1$*, the domains* $\Omega_k(\sigma)$ *monotonically expand to* Ω*. On the other hand if* $\sigma \to \sigma_{cr1}$*, then* $\Omega_1(\sigma)$ *shrinks to the set of finitely many points of minima of the function* f_1 *defined in* (1)*. If the domain* Ω *is convex, then this set consists of exactly one point and the domain* $\Omega_1(\sigma)$ *shrinks to this point.*

Proof. Let us seek the minimizer f_σ of (41) in the form $f_\sigma = \sigma w_\sigma$. Then w_σ minimizes the functional

$$\min\Big\{\frac{1}{2}\int_\Omega \big(|\nabla w_\sigma|^2 + w_\sigma^2 + 2\Psi_\sigma(w_\sigma) + 2w_\sigma\big)\,dx; w_\sigma \in H_0^1(\Omega)\Big\}, \tag{51}$$

where

$$\Psi_\sigma(t) = \frac{1}{\sigma^2}\Phi^*(\sigma t).$$

Note that $\Psi_\sigma(t)$ is a piecewise linear interpolation of the function $\pi t^2/\gamma - \pi\gamma/(4\sigma^2)$ at points $k\gamma/\sigma + \gamma/(2\sigma)$, $k \in \mathbb{Z}$. Therefore $\Psi_\sigma(t)$ converges uniformly to the function $\pi t^2/\gamma$ as $\sigma \to \infty$. It follows that w_σ converges strongly in $H^1(\Omega)$ to the minimizer w of the problem

$$\min\Big\{\frac{1}{2}\int_\Omega \big(|\nabla w|^2 + (1 + 2\pi/\gamma)w^2 + 2w\big)\,dx; w \in H_0^1(\Omega)\Big\}, \tag{52}$$

which solves $-\Delta w + (1 + 2\pi/\gamma)w = -1$ in Ω. By the maximum principle $w < 0$ in Ω. Since $f_\sigma = \sigma w_\sigma$, we have that every sublevel set $f_\sigma \leq -t$ $(t > 0)$ expands to the domain Ω. This proves that every set $\Omega_k(\sigma)$ expands to Ω.

The behavior of $\Omega_1(\sigma)$ as $\sigma \to \sigma_{cr1}$ is a consequence of the structure of the solution of (42) described in [14] . □

5. **Appendix.** The derivation of the lower bound in Section 3 makes use of the concept of Young measures. Recall that a Young measure is a parametrized family of probability measures m_x associated with a family of functions $\phi_\varepsilon(x)$ $(\phi_\varepsilon : \Omega \to \mathbb{R})$ such that $F(\phi_\varepsilon(x))$ converges to $\int_\mathbb{R} F(\sigma)\,dm_x(\sigma)$ in $L^\infty(\Omega)$ weak star for every bounded continuous function $F(\sigma)$, i.e.,

$$\lim_{\varepsilon\to 0}\int_\Omega F(\phi_\varepsilon(x))\psi(x)dx = \int_\Omega\int_\mathbb{R} F(\sigma)\,dm_x(\sigma)\psi(x)dx, \qquad \forall\psi \in L^1(\Omega). \tag{53}$$

It is known (see, e. g., [5], [13], [15]) that under some a priori bounds on the sequence of function $\phi_\varepsilon(x)$ there exists a family m_x such that (53) holds.

In this work we consider a sequence of integer-valued functions $D^\varepsilon(x)$. We construct a partition of unity $\mu_k(x)$ associated this sequence via (23)-(24), so that the corresponding Young measure m_x on $\mathbb{R}^1$ is the sum of δ-functions centered at integer points, $m_x(\sigma) = \sum \mu_k(x)\delta_k(\sigma)$. For fixed k, the value $\mu_k(x)$ represents the probability to find a hole ω_j^ε with degree $d_j^\varepsilon = k$ in a small vicinity of the point x (i.e. $\mu_k(x)$ represents the ratio of holes with degree k in a small neighborhood of x to the total number of holes in this neighborhood).

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