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FORMAL ASYMPTOTIC EXPANSIONS FOR SYMMETRIC ANCIENT OVALS IN MEAN CURVATURE FLOW

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In honor of Hiroshi Matano's sixtieth birthday and a belated thank you for his absolutely wonderful lectures given in Amsterdam, 1983.

Abstract. We provide formal matched asymptotic expansions for ancient convex solutions to MCF. The formal analysis leading to the solutions is analogous to that for the generic MCF neck pinch in [1].

For any p, q with $p + q = n$, $p \geq 1$, $q \geq 2$ we find a formal ancient solution which is a small perturbation of an ellipsoid. For $t \to -\infty$ the solution becomes increasingly astigmatic: q of its major axes have length $\approx \sqrt{2(q-1)(-t)}$, while the other p axes have length $\approx \sqrt{-2t\log(-t)}$.

We conjecture that an analysis similar to that in [2] will lead to a rigorous construction of ancient solutions to MCF with the asymptotics described in this paper.

1. **Introduction.** A family of immersed hypersurfaces $X^t : M \to \mathbb{R}^n$, $t_0 < t < t_1$, moves by Mean Curvature flow if X^t satisfies the PDE

$$\left(\frac{\partial X^t}{\partial t}\right)^{\perp} = H\boldsymbol{\nu} \tag{MCF}$$

where $\boldsymbol{\nu}$ is a unit normal to X^t, $(\cdots)^{\perp}$ is the normal component of a vector and H is the mean curvature of X^t in the direction of $\boldsymbol{\nu}$.

Ancient solutions to (MCF) are solutions which are defined for all $t \in (-\infty, -T)$, where we may assume without loss of generality that $T = 0$. Common examples of ancient solutions are given by self similar shrinking solutions, i.e. solutions defined for all $t < 0$ which have the form

$$X^t = \sqrt{-t}\, X^*$$

where $X^* : M \to \mathbb{R}^n$ is some fixed immersion. E.g. the cylinder with radius $r(t) = \sqrt{2(n-1)(-t)}$ for all $t < 0$ is an ancient solution. Other ancient solutions are provided by self-translating solitons, the simplest of which is the "Grim Reaper," i.e. the graph of $y = -\log\cos x + t$ in the xy plane.

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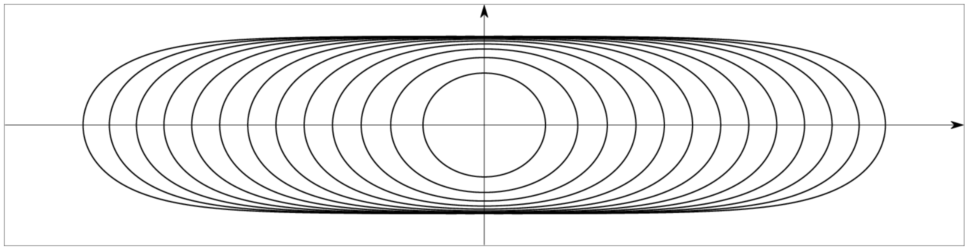

FIGURE 1. The paper clip solution: an ancient convex and compact solution of curve shortening.

Another ancient solution for Curve Shortening, the "shrinking paper clip" was found in [3]. In the (k, θ) representation (as used in [5]) it is given by

$$k(\theta, t) = \sqrt{B(\cos 2\theta + \coth Bt)} \tag{1}$$

where $B > 0$ is an arbitrary constant. Daskalopoulos, Hamilton and Sesum [4] showed that the circle and the paper clip are the only ancient solutions for CS which are convex and embedded.

Here we look for higher dimensional analogs of the paper clip solution. For simplicity we confine our search to solutions with rotational symmetry.

2. $\mathrm{SO}(p,\mathbb{R})\times\mathrm{SO}(q,\mathbb{R})$ **invariant hypersurfaces.** For any choice of integers p and q with $p+q=n$ we consider hypersurfaces in $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ which are invariant under the action of the subgroup $G = \mathrm{SO}(p,\mathbb{R})\times\mathrm{SO}(q,\mathbb{R})$ of $\mathrm{SO}(n,\mathbb{R})$. The quotient of $\mathbb{R}^n$ under the action of G is the first quadrant, i.e. $\mathbb{R}^n/G = [0,\infty)\times[0,\infty)$. Any G-invariant hypersurface is completely determined by its image under the map

$$\Pi : \mathbb{R}^n \to \mathbb{R}^n/G, \qquad \Pi : (X, Y) \mapsto \big(|X|, |Y|\big).$$

The image of any G-invariant hypersurface under Π is a curve in $\mathbb{R}^n/G$. In the particular case that this curve is the graph of a function $|Y| = u(|X|, t)$, evolution by (MCF) for the hypersurface is equivalent with the following PDE for u:

$$\frac{\partial u}{\partial t} = \frac{u_{xx}}{1+u_x^2} + \frac{p-1}{x}u_x - \frac{q-1}{u}. \tag{2}$$

This is the equation we will study. Our main result is a computation of a formal matched expansion of an ancient solution of (2). The asymptotic description of this formal solution as $t \to -\infty$ consists of three pieces.

Parabolic region. In the region $|x| = \mathcal{O}(\sqrt{-t})$ is close to the cylindrical solution. One has

$$u(x,t) \approx \sqrt{2(q-1)(-t)}\,\Big\{1 + \frac{x^2/(-t) - 2p}{4\log(-t)}\Big\} \qquad (t \to -\infty). \tag{3}$$

Intermediate region. When $\sqrt{-t} \ll |x| = \mathcal{O}(\sqrt{-t\log(-t)})$ is approximately ellipsoidal, with

$$\frac{u^2}{-t} + \frac{x^2}{-t\log(-t)} \approx 2(q-1), \qquad (t \to -\infty). \tag{4}$$

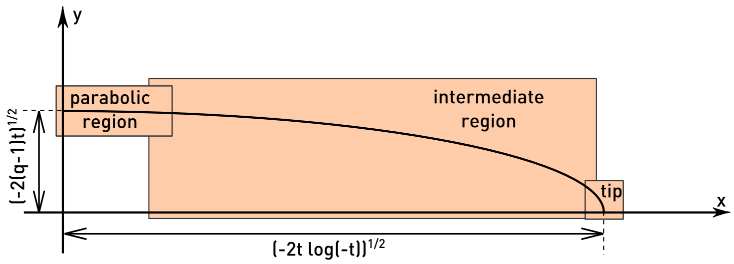

FIGURE 2. the three regions in the asymptotic expansion for the ancient ovals.

The tip. The end of the hypersurface is located at $x \approx \sqrt{-2t\log(-t)}$. Near this end point the surface is approximated by a translating soliton,

$$x(u,t) \approx \sqrt{-t\log(-t)}\,\left\{\sqrt{2} + \frac{1}{\log(-t)} P\Big(u\sqrt{\frac{\log(-t)}{-t}}\Big)\right\} \qquad (t \to -\infty). \tag{5}$$

Here $P(\xi)$ is the unique solution of

$$P''(\xi) + \frac{q-1}{\xi} P'(\xi) = -\tfrac{1}{2}\sqrt{2}, \qquad P(0) = P'(0) = 0$$

that is defined for all $\xi \geq 0$.

See Figure 2.

We now turn to the derivation of the asymptotic expansions, beginning with the inner "parabolic region."

3. **Solutions near the cylinder.** The cylinder, represented by

$$u(x,t) = \sqrt{-2(q-1)t}, \qquad (x \in \mathbb{R}, t < 0) \tag{6}$$

is a noncompact ancient solution. To find other ancient solutions we consider the rescaled variables

$$u(x,t) = \sqrt{-t}\, U\big(\frac{x}{\sqrt{-t}}, -\log(-t)\big), \quad y = \frac{x}{\sqrt{-t}}, \quad s = -\log(-t).$$

In these new variables we have

$$\frac{\partial U}{\partial s} = \frac{U_{yy}}{1+U_y^2} + \Big(\frac{p-1}{y} - \frac{y}{2}\Big)U_y + \frac{U}{2} - \frac{q-1}{U}. \tag{7}$$

To obtain ancient solutions for MCF we must find ancient solutions for (7), i.e. solutions which are defined for all $s \in (-\infty, s_0)$ for some $s_0 \in \mathbb{R}$.

The cylinder, which is an ancient solution to MCF, corresponds to the constant solution of (7)

$$U_* = \sqrt{2(q-1)}. \tag{8}$$

Our formal construction of compact ancient solutions begins by looking for a solution which within the parabolic region $|x| = \mathcal{O}(\sqrt{-t})$ is a small perturbation of the cylinder. To study such perturbations we linearize (7) around the constant solution $U(y,s) = U_*$ by setting

$$U = U_* + v.$$

This substitution leads to an equation for v,

$$\frac{\partial v}{\partial s} = \frac{v_{yy}}{1+v_y^2} + \Big(\frac{p-1}{y} - \frac{y}{2}\Big)v_y + \frac{v}{2} + \frac{U_*}{2} - \frac{q-1}{U_* + v}.$$

Using $U_*^2 = 2(q-1)$ we rewrite this as

$$\frac{\partial v}{\partial s} = v_{yy} + \Big(\frac{p-1}{y} - \frac{y}{2}\Big)v_y + v - \frac{v^2}{2(U_*+v)} - \frac{v_y^2 v_{yy}}{1+v_y^2}.$$

or, expanded even further,

$$\frac{\partial v}{\partial s} = v_{yy} + \Big(\frac{p-1}{y} - \frac{y}{2}\Big)v_y + v - \frac{v^2}{2U_*} + \frac{v^3}{2U_*(U_*+v)} - \frac{v_y^2 v_{yy}}{1+v_y^2}.$$

The last two terms here are cubic, so that we can summarize this expansion as

$$\frac{\partial v}{\partial s} = v_{yy} + \Big(\frac{p-1}{y} - \frac{y}{2}\Big)v_y + v - \frac{v^2}{2U_*} + \mathcal{O}(v, v_y, v_{yy})^3. \tag{9}$$

The linear part of this equation is of the form

$$v_s = \mathcal{L}v$$

where

$$\mathcal{L}\phi = \phi_{yy} + \Big(\frac{p-1}{y} - \frac{y}{2}\Big)\phi_y + \phi \tag{10}$$

We are looking for solutions of this equation which in backward time ($s \to -\infty$) tend to zero. It is natural to separate variables and to look for solutions of the form $\phi \approx e^{\lambda s}\phi_\lambda(y)$, where ϕ_λ is an eigenfunction of $\mathcal{L}$ with eigenvalue λ. If we want $e^{\lambda s}\phi_\lambda(y) \to 0$ for $s \to -\infty$, then the eigenvalue λ should be positive, or at least nonnegative. If we find an eigenvalue $\lambda = 0$ (as we will), then the corresponding solution of the linearized equation (10) will be stationary. To see if a solution of the actual equation (7) exists which decays for $s \to -\infty$, we will have to take the second order terms in (9) into account. But first we compute the eigenfunctions and eigenvalues of $\mathcal{L}$.

4. **The spectrum of $\mathcal{L}$.** This operator with the appropriate domain is self adjoint in the Hilbert space

$$\mathfrak{H} = L^2\big(\mathbb{R}_+; y^{p-1}e^{-y^2/4}\big).$$

It also maps polynomials in y^2 to polynomials in y^2 with the same or lower degree. Therefore the matrix of $\mathcal{L}$ in the basis $\{1, y^2, y^4, y^6, \ldots\}$ of $\mathbb{R}[y^2]$ is upper triangular. In particular,

$$\mathcal{L}[y^m] = m(m+p-2)y^{m-2} + \big(1 - \frac{m}{2}\big)y^m.$$

Using this one finds that $\mathfrak{H}$ has a basis of orthogonal polynomials which are eigenfunctions of $\mathcal{L}$. More precisely,

$$\phi_{2m}(y) = y^{2m} + c_{m-1}y^{2(m-1)} + \cdots + c_2 y^2 + c_0$$

satisfies

$$\mathcal{L}[\phi_{2m}] = (1-m)\phi_{2m}$$

provided

$$c_{k-1} = -\frac{2k(p+2k-2)}{m-k+1}\,c_k.$$

The first few eigenfunctions are:

$$\begin{aligned} \phi_0(y) &= 1, & \lambda_0 &= +1,\\ \phi_2(y) &= y^2 - 2p, & \lambda_2 &= 0,\\ \phi_4(y) &= y^4 - 4(p+2)y^2 + 4p(p+2), & \lambda_4 &= -1. \end{aligned} \tag{11}$$

We note for future reference that

$$\phi_2(y)^2 = \phi_4(y) + 8\phi_2(y) + 8p\phi_0(y),$$

and hence

$$\langle \phi_2, (\phi_2)^2 \rangle = 8\|\phi_2\|^2,$$

where $\langle \ldots, \ldots \rangle$ and $\| \ldots \|$ are the inner product and norm on $\mathfrak{H}$.

5. **The parabolic region.** In the inner region $|y| \leq C$ we assume that $v(y,s)$ can be expanded in terms of the eigenfunctions $v(y,s) = \sum_j a_j(s)\phi_{2j}(y)$. We also assume that there is one dominating term, so that $v(y,s) \sim a_j(s)\phi_{2j}(y)$ for some j. The linear equation predicts that $a_j'(s) = (1-j)a_j(s)$, so that $a_j(s) = e^{(1-j)s}a_j(0)$. The nonlinear terms in the equation modify this to $a_j'(s) = (1-j+o(1))a_j(s)$, and thus

$$a_j(s) \approx a_j(0)e^{(1-j+o(1))s}. \tag{12}$$

For $j > 1$ this tells us that $v(y,s)$ grows exponentially as $s \to -\infty$, thereby violating our assumption that v is always close to the cylinder. For $j < 1$, and here there is only one possibility, namely $j = 0$, we get a solution v which does decay exponentially as $s \to -\infty$. Since $\phi_0(y) = 1$ this solution is spatially constant. It corresponds to another cylinder solution of MCF which focusses at a time other than $t = 0$.

Thus the only interesting case for us here is $j = 1$. In this case the linearized equation predicts $a_1'(s) = o(a_1(s))$, from which we cannot even tell if a_1 grows or decays. For more information we look at the second order terms. Assuming the expansion (12), and substituting it in the nonlinear equation (9) we find

$$\sum_k a_{2k}'(s)\phi_{2k}(y) = \mathcal{L}v - \frac{v^2}{2U_*} + \mathcal{O}(v, v_y, v_{yy})^3.$$

Take the $\mathfrak{H}$ inner product with ϕ_2, to get

$$\begin{aligned} a_2'(s)\|\phi_2\|^2 &= \Big\langle \phi_2, \mathcal{L}v - \frac{v^2}{2U_*} + \mathcal{O}(v, v_y, v_{yy})^3 \Big\rangle \\ &= \big\langle \phi_2, \mathcal{L}v \big\rangle - \frac{1}{2U_*}\big\langle \phi_2, v^2 \big\rangle + \big\langle \phi_2, \mathcal{O}(v, v_y, v_{yy})^3 \big\rangle. \end{aligned}$$

The first term vanishes since $\langle \phi_2, \mathcal{L}v \rangle = \langle \mathcal{L}\phi_2, v \rangle = 0$. Assume that the ϕ_2 term in the expansion of v dominates, and that v can therefore be approximated by $v \approx a_2(s)\phi_2$. Also assume that the $\mathcal{O}(\cdots)^3$ term is in fact $\mathcal{O}(a_2)^3$. Then we find that

$$a_2'(s) = -\frac{a_2(s)^2}{2U_*}\frac{\langle \phi_2, \phi_2^2 \rangle}{\|\phi_2\|^2} + \mathcal{O}(a_2^3) = -\frac{4}{U_*}a_2^2 + \mathcal{O}(a_2^3).$$

Ignoring the $\mathcal{O}(a_2^3)$ term we get a simple differential equation for a_2 whose solution is

$$a_2(s) = -\frac{U_*}{4}\frac{1}{S-s},$$

for some constant S. The arbitrary constant S appears here because the equations (7) and (9) which we are trying to solve are autonomous. Any solution we find can therefore be translated in the s-time variable to produce new solutions. This translation corresponds to the parabolic rescaling $(x,t) \mapsto (\lambda x, \lambda^2 t)$ of space time. Parabolic rescaling leaves (MCF) and our cylindrical self similar solution $u(y,s) = U_*$ invariant. However, the ancient solutions we will construct are not invariant under this rescaling, and therefore we will get a one parameter family of such solutions. Since these solutions are all related by a simple rescaling we may choose $S = 0$ without losing generality. In choosing $S = 0$ we are essentially fixing a length and time scale in the original (x, t) variables.

Thus we arrive at the following approximation for $a_2(s)$

$$a_2(s) = \frac{U_*}{4s}. \tag{13}$$

In the parabolic region our solution is given by

$$U(y, s) = U_* + \frac{U_*}{4s}\phi_2(y) = U_*\Big\{1 + \frac{y^2 - 2p}{4s}\Big\}. \tag{14}$$

6. **The intermediate region.** We introduce the coordinate

$$z = \frac{y}{\sqrt{-s}}$$

and consider U as a function of z instead of y. In fact, to simplify notation we will not quite do this. Instead, we consider the quantities x, t, s, u, U, v, y, z as functions on the solution of (MCF) viewed as a submanifold of space-time. When we then specify a time derivative $\frac{\partial U}{\partial s}$ we must, for completeness, also specify if in this derivative y is kept constant or z is kept constant. With this notational convention we avoid having to introduce a new variable to denote the quantity U regarded as a function of z.

If we write

$$\frac{\partial U}{\partial s}\bigg|_z$$

for the rate at which U changes with respect to s when z is kept constant, and if $\frac{\partial U}{\partial s}\big|_y$ denotes the analogous derivative with constant y, then by the chain rule we have

$$\frac{\partial U}{\partial s}\bigg|_z = \frac{\partial U}{\partial s}\bigg|_y + \frac{\partial y}{\partial s}\bigg|_z \frac{\partial U}{\partial y},$$

and thus

$$\frac{\partial U}{\partial s}\bigg|_z = \frac{U_{yy}}{1 + U_y^2} + \Big(\frac{p-1}{y} - \frac{y}{2}\Big)U_y + \frac{U}{2} - \frac{q-1}{U} - \frac{1}{2}(-s)^{-1/2} z U_y \tag{15a}$$

$$= \frac{U_{zz}}{-s + U_z^2} - \Big(\frac{z}{2} - \frac{z}{2s} + \frac{p-1}{zs}\Big)U_z + \frac{U}{2} - \frac{q-1}{U} \tag{15b}$$

Assuming that for z fixed U converges as $s \to -\infty$, we find that the limit

$$U^{\mathrm{m}}(z) = \lim_{s\to-\infty} U|_{z\ \mathrm{const}}$$

must satisfy the ODE

$$\frac{z}{2}\frac{dU^{\mathrm{m}}}{dz} = \frac{U^{\mathrm{m}}}{2} - \frac{q-1}{U^{\mathrm{m}}} \tag{16}$$

This equation is easily integrated. Keeping in mind that $U_* = \sqrt{2(q-1)}$ one can write the solution as

$$U^{\mathrm{m}}(z) = \sqrt{U_*^2 + Kz^2} \tag{17}$$

where K is the integration constant. To determine K we compare with the inner solution (14) which tells us that

$$U = U_* - \frac{U_*}{4}z^2 + o(z^2) \qquad (z \searrow 0),$$

while (17) claims

$$U^{\mathrm{m}}(z) = U_* + \frac{K}{2U_*}z^2 + o(z^2).$$

For these two expressions to be compatible at small z we must choose $K = -U_*^2/2 = -(q-1)$. The solution in the intermediate region is thus

$$U^{\mathrm{m}}(z) = \sqrt{q-1}\,\sqrt{2-z^2}. \tag{18}$$

7. **The tip.** Since U_z^{m} and U_{zz}^{m} both diverge as $z \nearrow \sqrt{2}$, the formal solution $U^{\mathrm{m}}(z)$ breaks down near the tip $z = \sqrt{2}$. To analyze the solution near this tip we follow the formal computations in [1, p.23].

First we choose U as the coordinate on the surface and compute the evolution of z. One finds

$$\left.\frac{\partial z}{\partial s}\right|_U = \frac{z_{UU}}{1 - sz_U^2} + \Big(\frac{q-1}{U} - \frac{U}{2}\Big)z_U + \frac{z}{2} - \frac{z}{2s} + \frac{p-1}{sz}. \tag{19}$$

Assuming, as we did above, that when U is kept fixed, z converges as $s \to -\infty$, one finds that the limit z^{m} should satisfy

$$\Big(\frac{q-1}{U} - \frac{U}{2}\Big)z_U + \frac{z}{2} = 0 \tag{20}$$

The solution of this ODE on the interval $0 < U < U_*$ which matches the inner solution (14) at $U = U_*$ is

$$z^{\mathrm{m}} = \sqrt{2 - \frac{U^2}{q-1}}. \tag{21}$$

In obtaining (20) from (19) we let $s \to -\infty$, and assumed $\partial z/\partial s \to 0$; we also assumed that $z_U \neq 0$ so that for large $-s$ the second derivative term becomes negligible. If $-sz_U^2 = \mathcal{O}(1)$, then the second order term in (19) becomes important and we must treat (19) as a heat equation. To find the scale at which this happens we apply the condition $(-s)z_U^2 \gg 1$ to the intermediate solution z^{m}. The result is that (20) fails to approximate (19) when $U = \mathcal{O}((-s)^{-1/2})$. We therefore introduce a new coordinate

$$V = \sqrt{-s}\,U.$$

We also rescale in the z direction, choosing

$$z = \sqrt{2} + \frac{w}{-s},$$

where w is the new coordinate. We compute the following PDE for the evolution of w:

$$\frac{1}{-s}\left.\frac{\partial w}{\partial s}\right|_V = \frac{w_{VV}}{1+w_V^2} + \frac{q-1}{V}w_V + \frac{\sqrt{2}}{2}$$
$$\frac{1}{-2s}\Big\{-\sqrt{2} + w - Vw_V - \frac{2(p-1)}{\sqrt{2}+w/(-s)}\Big\} + \frac{w - Vw_V}{2(-s)^2}. \tag{22}$$

Assuming that w converges as $s \to -\infty$ with V fixed, we get an ODE for the limit by setting $\partial w/\partial s = 0$ on the left, and taking the limit on the right. We get

$$\frac{w_{VV}}{1+w_V^2} + \frac{q-1}{V}w_V = -\frac{\sqrt{2}}{2}. \tag{23}$$

This equation also appeared in [1]. The graph of any solution of (23) is the profile of a rotationally symmetric translating soliton for (MCF) in $\mathbb{R}^{q+1}$ with velocity $\frac{1}{2}\sqrt{2}$.

In [1, prop.2.1] it was shown that there is, up to an additive constant, a unique solution of (23) which is defined for all $V > 0$. These solutions all have an asymptotic expansion for large V which begins with

$$w = -\frac{V^2}{2\sqrt{2}(q-1)} + \mathcal{O}(\log V) \qquad (V \to \infty) \tag{24}$$

On the other hand, our solution z^{m} in the intermediate region is given by (21), which in terms of (w, V) implies

$$\begin{aligned} w^{\mathrm{m}} &= (-s)\bigl(z^{\mathrm{m}} - \sqrt{2}\bigr) \\ &= (-s)\Bigl\{\sqrt{2} - \frac{U^2}{2\sqrt{2}(q-1)} + \mathcal{O}(U^4) - \sqrt{2}\Bigr\} \\ &= -\frac{V^2}{2\sqrt{2}(q-1)} + o(V^2). \end{aligned}$$

Therefore the inner approximation in which w is given by any solution of (23) matches to lowest order with the approximate solution z^{m} in the intermediate region.

REFERENCES

[1] S. B. Angenent and J. J. L. Velázquez, *Degenerate neckpinches in mean curvature flow*, J. Reine Angew. Math., **482** (1997), 15–66.

[2] Sigurd Angenent, Cristina Caputo and Dan Knopf, *Minimally invasive surgery for Ricci flow singularities*, J. Reine Angew. Math., **672** (2012), 39C87(to appear).

[3] Sigurd Angenent, *Shrinking doughnuts*, Nonlinear diffusion equations and their equilibrium states, 3 (Gregynog, 1989), Progr. Nonlinear Differential Equations Appl., **7** Birkhäuser Boston, Boston, MA, (1992), 21–38.

[4] Panagiota Daskalopoulos, Richard Hamilton and Natasa Sesum, *Classification of compact ancient solutions to the curve shortening flow*, J. Differential Geom., **84** (2010), 455–464.

[5] M. Gage and R. S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geom., **23** (1986), 69–96.

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