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LIOUVILLE-TYPE THEOREMS FOR ELLIPTIC SCHRÖDINGER SYSTEMS ASSOCIATED WITH COPOSITIVE MATRICES

Philippe Souplet

Université Paris 13, Sorbonne Paris Cité
Laboratoire Analyse, Géométrie et Applications, CNRS, UMR 7539
93430 Villetaneuse, France

Abstract. We study non-cooperative, multi-component elliptic Schrödinger systems arising in nonlinear optics and Bose-Einstein condensation phenomena. We here reconsider the more delicate case of systems of $m \geq 3$ components. We prove a Liouville-type nonexistence theorem in space dimensions $n \geq 3$ for homogeneous nonlinearities of degree $p < n/(n-2)$, under the optimal assumption that the associated matrix is strictly copositive. This extends recent work of Tavares, Terracini, Verzini and Weth [22] and of Quittner and the author [17], where the results were limited to $n \leq 2$ dimensions or to $m = 2$ components. The proof of the Liouville theorem is done by combining and improving different arguments from [22] and [17], namely a feedback procedure based on Rellich-Pohozaev type identities and functional analytic inequalities on S^{n-1}, and suitable test-function arguments. We also consider a more general class of systems with gradient structure, for which our arguments show the triviality of solutions satisfying a suitable integral bound, a result which may be of independent interest.

1. Introduction and main results.

1.1. Position of the problem.

In this paper, we study elliptic Schrödinger systems of the form

$$-\Delta u_i = \sum_{j=1}^{m} \beta_{ij} u_i^q u_j^{q+1}, \quad x \in \mathbb{R}^n, \quad i = 1, \dots, m. \tag{1.1}$$

Here n, m are fixed positive integers, $q > 0$ and $B = (\beta_{ij})$ is a real $m \times m$ symmetric matrix. We denote $U = (u_1, \dots, u_m)$ and only consider *nonnegative classical solutions*, i.e. such that $u_i \in C^2(\mathbb{R}^n)$ and $u_i \geq 0$ in $\mathbb{R}^n$ for all $i = 1, \dots, m$. Also, we say that U is *positive* if $u_i > 0$ in $\mathbb{R}^n$ for all $i = 1, \dots, m$.

The nonlinearities on the RHS of (1.1) will be denoted by $F_i = F_i(U)$. The Euclidean norm in $\mathbb{R}^m$ or in $\mathbb{R}^n$ will be denoted by $|\cdot|$. Throughout this paper, $p_S = p_S(n) := (n+2)/(n-2)_+$ will denote the Sobolev exponent.

Such systems arise in mathematical models for various phenomena in physics, such as nonlinear optics and Bose-Einstein condensation; see e.g. [10, 8] and the references therein. System (1.1) has been investigated in numerous mathematical papers (see e.g. [11, 13, 1, 20, 12, 5, 2, 6, 22, 17, 6]). Central issues are the

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applicability of variational methods and the derivation of Liouville-type theorems, i.e. nonexistence results in the whole space.

In the recent work [22], interesting connections have been discovered between the existence/nonexistence of solutions of (1.1) and properties of the matrix B. Recall that the matrix B is said to be *strictly copositive* if the restriction of the associated quadratic form to the positive cone of $\mathbb{R}^n$ is positive definite, i.e.

$$\sum_{1\le i,j\le m} \beta_{ij} z_i z_j > 0, \quad \text{for all } z\in[0,\infty)^m,\ z\ne 0. \tag{1.2}$$

Property (1.2) plays a significant role in quadratic programming (see [9]). It was proved in [22] that the strict copositivity of B is a *necessary* condition for the nonexistence of positive solutions of (1.1) whenever $2q+1<p_S$. In fact, it was shown in [22] by variational methods that, if B is not strictly copositive, then there exists a positive solution which is periodic with respect to the unit cube. They also showed that it is a *sufficient* condition when $n\le 2$ and $q\le 1$. Then in [17], in the special case $m=2$, $n\le 4$, under the optimal growth condition $2q+1<p_S$, we showed that strict copositivity implies the nonexistence of nontrivial solutions. However, for $m\ge 3$, the results in [17] require stronger conditions on the matrix B (whereas for $n\ge 5$ they require stronger conditions on q).

In fact, as noted in [22] (see after Corollary 1.1 there), the cases $m=2$ and $m\ge 3$ exhibit an essential difference. Namely, when $m=2$, for any nonlinearity F arising from a strictly copositive matrix, there exists a convex combination of the F_i which is positive, and this is no longer true when $m\ge 3$.

In the present paper, we reconsider the more difficult case $m\ge 3$ and our main goal is to prove a Liouville theorem for strictly copositive matrices B in higher dimensions $n\ge 3$ for suitable q. This will be achieved by combining some refinements of the approach in [17] with ideas from [22] (see Section 1.3 for more details).

1.2. **Main results.** Our main Liouville-type theorem is the following.

Theorem 1. *Let $m,n\ge 3$ and $q>0$ satisfy $2q+1<n/(n-2)$. Let B be a real symmetric, strictly copositive matrix. Then system (1.1) has no positive, bounded solution.*

In view of the existence result from [22], we thus have the following characterization.

Corollary 1. *Let $m,n\ge 3$ and $q>0$ satisfy $2q+1<n/(n-2)$. Let B be a real symmetric matrix. Then system (1.1) has no positive, bounded solution if and only if B is strictly copositive.*

Remark 1.1. (a) Similarly as in [22], it is unknown whether Theorem 1 remains true for nonnegative solutions $U\not\equiv 0$, instead of positive solutions. Indeed, since $q<1$ (due to $2q+1<n/(n-2)\le 3$), the nonlinearities are not Lipschitz near zero and positivity of solutions is not a priori guaranteed. Note that the positivity restriction did not appear in the case $m=2$ because, thanks to the very simple structure of $(2,2)$ copositive matrices, some basic a priori estimate can be easily obtained without using negative powers of u_i as test-functions (see Section 1.3).

(b) The boundedness hypothesis in Theorem 1 can actually be replaced with the following, much weaker, exponential, upper growth assumption:

$$|U(x)|\le C\exp(|x|^q), \quad x\in\mathbb{R}^n, \quad \text{for some } C,q>0, \tag{1.3}$$

as a consequence of Theorem 2 below. Moreover, the proof of Theorem 1 shows that the result remains true for nonnegative solutions which are positive outside a set of finite measure (and which satisfy (1.3)).

(c) It is an interesting open problem whether nonexistence of positive solutions holds in the physically important, cubic case $q = 1$, $n = 3$ with $m \geq 3$. Note that Theorem 1 fails just on the borderline. More generally, for $m \geq 3$, if $n \geq 3$ and $n/(n-2) \leq 2q+1 < p_S$ or if $n \leq 2$ and $1 < q < \infty$, it is not known if strict copositivity implies (and is hence equivalent to) the Liouville property.

We next turn to the main technical result of the paper, upon which Theorem 1 depends. It asserts the triviality of nonnegative solutions satisfying a suitable integral bound. It is conveniently formulated for a more general class of systems with gradient structure. Namely we consider elliptic systems of the form

$$-D\Delta U = F(U), \quad x \in \mathbb{R}^n, \quad i = 1, \dots, m, \tag{1.4}$$

where D is an $m \times m$ diagonal matrix with positive (constant) entries $d_i > 0$, $U : \mathbb{R}^n \to \mathbb{R}^m$, $F \in C(\mathbb{R}^m; \mathbb{R}^m)$ and where we denote $\Delta U = (\Delta u_1, \dots, \Delta u_m)^T$. The components of F will be denoted by F_i. We assume that the system has gradient structure, i.e.

$$F = \nabla V, \text{ for some function } V \in C^1(\mathbb{R}^m, \mathbb{R}). \tag{1.5}$$

Fixing $p > 1$, we next assume that, for some constants $c_0, C_0 > 0$, the functions F and V satisfy the following conditions:

$$|F(U)| \leq C_0 |U|^p, \quad \text{for all } U \in [0, \infty)^m, \tag{1.6}$$

$$2nV(U) - (n-2) U \cdot \nabla V(U) \geq c_0 |U|^{p+1}, \quad \text{for all } U \in [0, \infty)^m. \tag{1.7}$$

Note that since $F(0) = 0$ by (1.6), $U \equiv 0$ is a solution of (1.4).

Theorem 2. *Let $n \geq 1$, $p > 1$, $s > 0$. Assume (1.5)-(1.7) and*

$$p < \min\Big(p_S, 1 + \frac{2s}{n-1}\Big). \tag{1.8}$$

Let $U \geq 0$ be a classical solution of (1.4) which satisfies the exponential growth condition (1.3). If

$$\int_{B_R} |U|^s \leq C R^{n-2s/(p-1)}, \qquad R \geq 1, \tag{1.9}$$

then $U \equiv 0$.

Remark 1.2. (on the conditions on s)

(a) Condition (1.9) can be rewritten as a decay condition on the L^s-averages, namely

$$\overline{\|U\|}_{s,B_R} := \left(\frac{1}{|B_R|} \int_{B_R} |U|^s \right)^{1/s} \leq C R^{-2/(p-1)}. \tag{1.10}$$

Note that the power on the RHS of (1.10) is consistent with the scaling properties of the system in case of homogeneous nonlinearities like in (1.1). To check the validity of (1.9) for suitable s, so as to derive Liouville-type results such as Theorem 1 from Theorem 2, one can use different test-function arguments, depending on the hypothesis made on the matrix B (see Section 1.3 below). The larger s may be found, the larger will be the allowable range for p (which in any case is limited from above by the Sobolev exponent).

(b) The admissibility assumption on s in (1.8) can be rewritten as a "supercriticality" condition:

$$s > s^*(n-1) := \frac{(n-1)(p-1)}{2}.$$

Observe that if s satisfies the stronger restriction $s > n(p-1)/2 = s^*(n)$, then the conclusion becomes trivial, letting $R \to \infty$ in (1.9). But under the condition $s > s^*(n-1)$, Theorem 2 is far for immediate. It is not known whether condition (1.8) is optimal for the validity of Theorem 2. However this condition is somehow natural from a heuristic point of view. Indeed, on the one hand, the number $s^*(n)$ is a well-known critical exponent for regularity criteria in semilinear elliptic and parabolic problems (see e.g. [23, 16]) and, on the other hand, our method can be seen as a combination of some regularity theory with the "dimensional reduction" (from n to $n-1$) achived via Rellich-Pohozaev identities (see Section 1.3 for a more detailed description of the method).

Remark 1.3. (on the growth restrictions for U)

(a) In previous work based on the approach described in section 1.3 below, a polynomial upper growth bound was required on U. This is here weakened to the exponential growth restriction (1.3). This improvement is made possible by a better choice of the auxiliary spaces used in the proof.

(b) If $n = 1$ or if s satisfies the additional conditions

$$s > p \qquad \text{and} \qquad s \geq \frac{2(n-1)p}{n+1}, \tag{1.11}$$

(the latter being effective only if $n \geq 4$), then Theorem 2 remains true without any growth restriction on U; see Appendix for $n = 1$ and Remark 3.1 below.

(c) In some cases, growth restrictions can be removed a posteriori. This is the case when Theorem 2 implies a Liouville-type theorem for bounded nonnegative solutions. Then, using a doubling-rescaling argument from [14], one deduces from such Liouville theorem a universal bound for all local solutions, and the latter implies a Liouville result without any growth restriction. See [17] for situations where this can be done. This does not seem to apply here in Theorems 1 and 2, because of the (component-wise) positivity restriction entailed by the non locally Lipschitz nonlinearities. Indeed, the rescaling method is based on a contradiction argument that produces a non trivial nonnegative solution in $\mathbb{R}^n$, but one cannot guarantee that this solution should be positive component-wise.

1.3. **Strategy of proof.** We note that the system (1.1) is of gradient type, but is **not** cooperative (which would require $\beta_{ij} \geq 0$ for all $i \neq j$). Therefore, maximum principle techniques, such as moving planes or moving spheres, which are the most usual methods for obtaining Liouville-type theorems for systems (see e.g. [4, 7, 18, 3]) are not applicable here. Also we would like to recall that, in spite of certain similarities, methods available for scalar equations need not straightforwardly apply to gradient systems, which are generally more difficult to handle (see [20, p. 205] and [5, p. 956] for more details).

The proof of Theorem 2 is long and technical. It relies on an extension of a feedback procedure from [21] and [17]. More precisely, by means of a Rellich-Pohozaev type identity, for any $r > 0$, the volume integral

$$H(r) := \int_{|x|<r} |U|^{p+1}\, dx$$

can be controled by surface integrals involving $|U|^{p+1}$, $|\nabla U|^2$ and $|U|^2$ at $|x| = r$. Fix $R, \delta > 0$. If, for *some* $r \in \big(R, (1+\delta)R\big)$, one can estimate these surface terms by $CR^{-a}H^b\big((1+2\delta)R\big)$ with $C, a > 0$, $b \in [0,1)$ independent of R, then one easily infers that $H(R)$ has very fast growth as $R \to \infty$, hence contradicting the assumption on U. It turns out that such estimation can be achieved by a careful analysis using Sobolev imbeddings and interpolation inequalities on S^{n-1}, elliptic estimates and a measure argument. In this scheme, the primary integral bound (1.9) is used in the interpolation step, where the surface terms are interpolated between various auxiliary norms. Heuristically, the efficiency of the method comes from the fact that one space dimension is "gained" via the Pohozaev type identity because, by applying functional analytic arguments on the $n-1$ dimensional unit sphere rather than directly on B_R, one can bootstrap from (1.9) under less stringent growth restrictions on the nonlinearity. Related arguments, without interpolation and feedback, and restricted to $n = 3$, first appeared in [19]. Both [19] and [21] were exclusively concerned with the Lane-Emden system $-\Delta u = v^p$, $-\Delta v = u^q$.

The Liouville-type theorems in [17] were obtained by applying this strategy in conjunction with an L^1 estimate of the RHS, namely the bound (1.9) for $s = p = 2q+1$. Such bound follows from a standard rescaled test-function argument, under the condition that some convex combination of the nonlinearities F_i is positive. But, as recalled above, this condition is not met for system (1.1) when the matrix B is merely assumed to be strictly copositive and $m \geq 3$. However, in this case, it turns out that a more involved test-function argument from [22] implies the validity of (1.9) for the smaller value $s = q+1 = (p+1)/2$ (see Lemma 3.1 and Remark 3.1 below). This motivated us to extend the approach of [17] under the assumption that (1.9) – or (1.10) – holds for some $s > 0$, thus leading to Theorem 2.

This extension leads to many additional difficulties as compared with those in [21] and [17] and requires some new ideas. In particular, in order to cover the full range (1.8) for s, we need to consider several separate cases. For instance, one has to use different ways of interpolation depending on certain relationships involving the parameters s, p, n. Also, the case $s < p$ presents some special difficulties due to the fact that ΔU is not initially controled in any Lebesgue space. Note that the full strength of Theorem 2 is not exploited by Theorem 1 (nor by the results in [17]). Namely, up to now, we have either $s = p$ or $s = (p+1)/2$ in our applications. However Theorem 2 might turn out to be useful again if, for such problems, primary bounds of the form (1.9) should be established for other values of s in the future.

Some useful preliminaries are given Section 2. The proofs of Theorem 2 and 1 are respectively given in Sections 3 and 4.

2. **Notation and preliminaries.** Denote

$$B_R = \{x \in \mathbb{R}^n;\ |x| < R\}, \quad R > 0, \quad \text{and} \quad S^{n-1} = \{x \in \mathbb{R}^n;\ |x| = 1\}.$$

We shall use the spherical coordinates (r, θ) with $r = |x|$, $\theta = x/|x| \in S^{n-1}$ (for $x \neq 0$). For a given function w of $x \in \mathbb{R}^n$, we write $w(x) = w(r, \theta)$ (using the same symbol w, without risk of confusion). For brevity we will use the following notation for volume and surface integrals

$$\int_{B_R} w = \int_{B_R} w(x)\,dx, \qquad \int_{S^{n-1}} w(R) = \int_{S^{n-1}} w(R,\theta)\,d\theta.$$

In the following Lemmas 2.1–2.4, the letter C will denote positive constants which are independent of the functions U, v, w and the number $R > 0$. We first give

Sobolev and interpolation inequalities on S^{n-1} which will be used repeatedly in the proof of Theorem 2.

Lemma 2.1. *(i) Let $n \geq 2$, $j \geq 1$ be integers, let $1 < z < \infty$ and let λ satisfy*

$$\begin{cases} \dfrac{1}{z} - \dfrac{1}{\lambda} \leq \dfrac{j}{n-1}, & \text{if } z < (n-1)/j, \\ 1 < \lambda < \infty, & \text{if } z = (n-1)/j, \\ \lambda = \infty, & \text{if } z > (n-1)/j. \end{cases}$$

Then, for any $w = w(\theta) \in W^{j,z}(S^{n-1})$, we have

$$\|w\|_\lambda \leq C(\|D_\theta^j w\|_z + \|w\|_1).$$

(ii) Let $1 \leq z < \infty$. For any $w = w(\theta) \in W^{2,z}(S^1)$, we have

$$\|w\|_\infty \leq C\big(\|w\|_1 + \|D_\theta^2 w\|_1\big)^{1/(z+1)} \|w\|_z^{z/(z+1)}.$$

(iii) For any $w = w(\theta) \in W^{1,1}(S^1)$, we have

$$\|w\|_\infty \leq C \inf_{S^1} |w| + C\|D_\theta w\|_1.$$

Proof. For assertion (i), see e.g. [19]. To prove assertion (ii), first note that

$$\begin{aligned} \operatorname{osc}(|w|^{\frac{z-1}{2}} w) &\leq \int_0^{2\pi} \big|D_\theta(|w|^{\frac{z-1}{2}} w)\big| \leq C \int_0^{2\pi} |w|^{\frac{z-1}{2}} |D_\theta w| \\ &\leq C\|w\|_z^{\frac{z-1}{2}} \|D_\theta w\|_{\frac{2z}{z+1}}, \end{aligned} \tag{2.1}$$

for any $w \in W^{1,2z/(z+1)}(S^1)$. If now $w \in W^{2,z}(S^1)$, inequality (2.1) combined with interpolation yields

$$\begin{aligned} \|w\|_\infty &\leq C\|w\|_1 + C\|w\|_z^{(z-1)/(z+1)} \|D_\theta w\|_{2z/(z+1)}^{2/(z+1)} \\ &\leq C\|w\|_1 + C\|w\|_z^{(z-1)/(z+1)} \Big(\big(\|D_\theta^2 w\|_1 + \|w\|_1\big)^{1/2} \|w\|_z^{1/2}\Big)^{2/(z+1)} \\ &\leq C\|w\|_1 + C\|w\|_z^{z/(z+1)} \Big(\|D_\theta^2 w\|_1 + \|w\|_1\Big)^{1/(z+1)}, \end{aligned}$$

hence (ii). As for assertion (iii) it follows immediately from (2.1) with $z = 1$. □

The next lemma follows from standard elliptic estimates for $R = 1$ and an obvious dilation argument.

Lemma 2.2. *Let $1 < z < \infty$ and $\delta > 0$. There exists $C > 0$ such that for all $R > 0$ and $v = v(x) \in W^{2,z}(B_R)$, we have*

$$\int_{B_R} |D_x^2 v|^z + R^{-z} \int_{B_R} |D_x v|^z \leq C\Big(\int_{B_{(1+\delta)R}} |\Delta v|^z + R^{-2z} \int_{B_{(1+\delta)R}} |v|^z\Big).$$

The following Rellich-Pohozaev type identity, cf. [15] and see also [17], plays a key role in the proof of Theorem 2.

Lemma 2.3. *Assume (1.5). Then, for any nonnegative solution U of (1.4) and any $R > 0$, there holds*

$$\begin{aligned} \int_{B_R} \big(2nV(U) - (n-2)U \cdot \nabla V(U)\big) &= 2R^n \int_{S^{n-1}} V(U(R)) \\ &+ R^n \sum_i d_i \int_{S^{n-1}} \Big(|\partial_r u_i|^2 - |\partial_\tau u_i|^2 + (n-2)R^{-1} u_i \partial_r u_i\Big)(R), \end{aligned}$$

where ∂_r and ∂_τ respectively denote the components of the gradient along $x/|x|$ and along its orthogonal complement. (For $n=1$, the above formula reads as

$$\int_{-R}^{R}\big(2V(U)+U\cdot\nabla V(U)\big)\,dx = R\sum_{x=\pm R}\Big[2V(U(x))+\sum_i d_i\big(|u_i'|^2-R^{-1}u_iu_i'\big)\Big](x).)$$

We will also need the following lemma.

Lemma 2.4. *Assume (1.6) and let $\delta>0$, $\beta\in(0,1)$. Then there exists $C>0$ such that, for any nonnegative solution U of (1.4) and any $R>0$, there holds*

$$\begin{aligned}\int_{B_R}|\nabla u_i|^2u_i^{-\beta} \quad &\le C\int_{B_{(1+\delta)R}}|U|^{p+1-\beta}\\ &\quad +CR^{-2}\int_{B_{(1+\delta)R}}|U|^{2-\beta},\quad R>0,\quad i=1,\cdots,m,\end{aligned}\tag{2.2}$$

where the LHS is understood to be 0 at points where $\nabla u_i=0$ and $u_i=0$.

Proof. Fix a cut-off function $0\le\chi\in C_0^\infty(\mathbb{R}^n)$, such that $\chi=1$ for $|x|\le 1$ and $\chi=0$ for $|x|\ge 1+\delta$. For $R>0$ we set $\varphi(x)=\varphi_R(x)=\chi(x/R)$. Let $\varepsilon>0$. Multiplying the i-th equation in (1.4) with $d_i^{-1}(u+\varepsilon)^{1-\beta}\varphi^2$ and integrating by parts, we obtain (setting $\int=\int_{\mathbb{R}^n}$ and $u=u_i$):

$$\begin{aligned}d_i^{-1}&\int F_i(U)(u+\varepsilon)^{1-\beta}\varphi^2\\ &=\int\nabla u\cdot\nabla((u+\varepsilon)^{1-\beta}\varphi^2)\\ &=(1-\beta)\int|\nabla u|^2(u+\varepsilon)^{-\beta}\varphi^2+2\int(u+\varepsilon)^{1-\beta}\varphi(\nabla u\cdot\nabla\varphi).\end{aligned}$$

Estimating the last term via

$$-\int(u+\varepsilon)^{1-\beta}\varphi(\nabla u\cdot\nabla\varphi)\le\frac{1-\beta}{4}\int|\nabla u|^2(u+\varepsilon)^{-\beta}\varphi^2+C\int|\nabla\varphi|^2(u+\varepsilon)^{2-\beta},$$

we get

$$\int|\nabla u|^2(u+\varepsilon)^{-\beta}\varphi^2\le C\int F_i(U)(u+\varepsilon)^{1-\beta}\varphi^2+C\int|\nabla\varphi|^2(u+\varepsilon)^{2-\beta}.$$

Letting $\varepsilon\to 0$ (applying monotone convergence on the LHS), and then using (1.6), we obtain (2.2). □

3. **Proof of Theorem 2.** Since the case $n=1$ can be treated by rather simple ODE arguments, in order not to distract ourselves from our main line of proof, we have relegated this case to an appendix and we assume $n\ge 2$ throughout this section.

For sake of clarity, since the proof is quite long and technical, we split it into several steps and lemmas.

Step 1. *Preparations.*

We fix two numbers $0<\varepsilon,\delta<1$, which will be chosen suitably small in subsequent steps of the proof. We set

$$\alpha=2/(p-1),\qquad k=(p+1)/p$$

and

$$\ell=\begin{cases} s/p, & \text{if } s>p,\\ 1+\varepsilon, & \text{otherwise.}\end{cases}\tag{3.1}$$

We may assume

$$s \le n(p-1)/2 < p+1, \tag{3.2}$$

(where the second inequality follows from $p < p_S$), since otherwise (1.9) immediately imply $U \equiv 0$.

Suppose for contradiction that (1.4) admits a nontrivial classical solution $U \ge 0$ which satisfies the exponential growth condition (1.3). Define

$$H(R) := \int_{B_R} |U|^{p+1}, \quad R > 0.$$

Picking $R_0 \ge 1$ such that $U \not\equiv 0$ on B_{R_0}, we have

$$H(R) \ge H(R_0) > 0, \quad R \ge R_0. \tag{3.3}$$

In this proof, the letter u will stand for any component of U and C will denote generic positive constants which are **independent of** R (but may possibly depend on the solution U and on all the other parameters, including ε, δ). Assuming $V(0) = 0$ without loss of generality, (1.6) implies that $V(U) \le C|U|^{p+1}$ for all $U \in [0,\infty)^m$. It then follows from (1.7) and the Rellich-Pohozaev identity in Lemma 2.3 that

$$H(R) \le CG_1(R) + CG_2(R), \quad R > 0, \tag{3.4}$$

where

$$\begin{cases} G_1(R) &= R^n \displaystyle\int_{S^{n-1}} |U(R)|^{p+1}, \\ G_2(R) &= R^n \displaystyle\int_{S^{n-1}} \big(|D_x U(R)|^2 + R^{-2}|U(R)|^2\big). \end{cases} \tag{3.5}$$

In order to reach a contradiction, our goal is to find constants $a, C > 0$ and $0 \le b < 1$ such that, for all $R \ge R_0$, the feedback estimate

$$G_1(\tilde R),\, G_2(\tilde R) \le CR^{-a} H^b\big((1+2\delta)R\big)$$

holds for some $\tilde R \in \big(R, (1+\delta)R\big)$. In the rest of the proof, for any given $R > 0$, we shall denote for brevity

$$R' = (1+\delta)R, \qquad R'' = (1+2\delta)R. \tag{3.6}$$

For given function $w = w(r,\theta)$, $0 < z \le \infty$ and $R > 0$, we denote

$$\|w\|_z = \|w(R,\cdot)\|_{L^z(S^{n-1})}, \tag{3.7}$$

when no risk of confusion arises. We note that $\|\cdot\|_z$ is not a norm when $0 < z < 1$, but we keep the same notation for convenience. Finally, we put

$$I_z(R) = \|D_x^2 U(R,\cdot)\|_z + R^{-1}\|D_x U(R,\cdot)\|_z + R^{-2}\|U(R,\cdot)\|_z.$$

For convenience of the readers, let us outline the rest of the proof, which is as follows:

Step 2: Estimation of $G_1(R)$ in terms of auxiliary norms for $n \ge 3$
Step 3: Estimation of $G_2(R)$ in terms of auxiliary norms for $n \ge 3$
Step 4: Estimation of $G_1(R)$ and $G_2(R)$ in terms of auxiliary norms for $n = 2$
Step 5: Control of averages of auxiliary norms in terms of R and $H(R'')$
Step 6: Measure argument
Step 7: Feedback estimate for G_1
Step 8: Feedback estimate for G_2
Step 9: Conclusion.

Step 2. *Estimation of $G_1(R)$ in terms of auxiliary norms for $n \ge 3$.* Let λ be given by

$$\lambda = \begin{cases} \dfrac{n-1}{n-3} \ \ (\infty \text{ for } n=3), & \text{if } s \le p, \\ \Big(\dfrac{1}{\ell} - \dfrac{2}{n-1}\Big)^{-1} \in (1, p+1), & \text{if } s > p \text{ and } \dfrac{1}{\ell} - \dfrac{2}{n-1} > \dfrac{1}{p+1}, \\ p+1, & \text{otherwise.} \end{cases} \tag{3.8}$$

Next, if $p+1 > \lambda$ then $\frac{p}{p+1} > \frac{2}{n-1}$ and we may define

$$\mu := \Big(\frac{p}{p+1} - \frac{2}{n-1}\Big)^{-1} \in (p+1, \infty) \tag{3.9}$$

(where $\mu > p+1$ follows from $p < p_S$).

Lemma 3.1. *Let $n \ge 3$. We have the estimate*

$$G_1(R) \le \begin{cases} CR^n \big(R^2 I_\ell^{1-\nu} I_k^{\nu}\big)^{p+1}, & \textit{if } p+1 \ge \lambda, \\ CR^n \big(R^{2(1-\nu)} I_\ell^{1-\nu} \|U\|_s^{\nu}\big)^{p+1}, & \textit{if } p+1 < \lambda, \end{cases} \tag{3.10}$$

where $\nu \in [0,1)$ is given by

$$\nu = \begin{cases} \Big(\dfrac{1}{\lambda} - \dfrac{1}{p+1}\Big)\Big(\dfrac{1}{\lambda} - \dfrac{1}{\mu}\Big)^{-1}, & \textit{if } p+1 \ge \lambda, \\ \Big(\dfrac{1}{\lambda} - \dfrac{1}{p+1}\Big)\Big(\dfrac{1}{\lambda} - \dfrac{1}{s}\Big)^{-1}, & \textit{if } p+1 < \lambda. \end{cases} \tag{3.11}$$

Proof. Lemma 2.1(i) and (3.8) imply that

$$\|u\|_\lambda \le C(\|D_\theta^2 u\|_\ell + \|u\|_1) \le C(R^2\|D_x^2 u\|_\ell + \|u\|_\ell) \le CR^2 I_\ell. \tag{3.12}$$

On the other hand, we deduce from Lemma 2.1(i) and (3.9) that

$$\begin{aligned} \|u\|_\mu &\le C(\|D_\theta^2 u\|_k + \|u\|_1) \\ &\le C(R^2\|D_x^2 u\|_k + \|u\|_k) \le CR^2 I_k, \qquad \text{if } p+1 > \lambda. \end{aligned} \tag{3.13}$$

Now, by (3.5), we have

$$G_1(R) = CR^n \|U\|_{p+1}^{p+1}. \tag{3.14}$$

If $p+1 = \lambda$, then (3.10) follows directly from (3.12). If $p+1 > \lambda$, then (3.14), Hölder's inequality and $p+1 < \mu$ imply

$$G_1(R) \le CR^n \big(\|U\|_\lambda^{1-\nu} \|U\|_\mu^{\nu}\big)^{p+1}$$

and (3.10) follows from (3.12), (3.13). Finally, if $p+1 < \lambda$, then (3.14), Hölder's inequality and $p+1 > s$ imply

$$G_1(R) \le CR^n \big(\|U\|_\lambda^{1-\nu} \|U\|_s^{\nu}\big)^{p+1}$$

and (3.10) follows from (3.12). □

Step 3. *Estimation of $G_2(R)$ in terms of auxiliary norms for $n \ge 3$.* Noting that $\ell < (p+1)/p < n-1$ due to $n \ge 3$, $p > 1$ and (3.2), we may define

$$\rho := \begin{cases} \dfrac{n-1}{n-2}, & \text{if } s \le p, \\ \Big(\dfrac{1}{\ell} - \dfrac{1}{n-1}\Big)^{-1}, & \text{if } s > p, \end{cases} \tag{3.15}$$

$$\gamma := \Big(\frac{p}{p+1} - \frac{1}{n-1}\Big)^{-1} \in \big(\max(\rho,2),\infty\big) \tag{3.16}$$

(where $\gamma > 2$ follows from $p < p_S$).

Lemma 3.2. *Let $n \geq 3$. We have the estimate*

$$G_2(R) \leq CR^{n+2} I_\ell^{2(1-\tau)} I_k^{2\tau}, \tag{3.17}$$

where

$$\tau := \Big(\frac{1}{\rho} - \frac{1}{\gamma}\Big)^{-1}\Big(\frac{1}{\rho} - \frac{1}{2}\Big)_+ \in [0,1). \tag{3.18}$$

Proof. Lemma 2.1(i), (3.15) and (3.16) imply that

$$\|D_x u\|_\rho \leq C(\|D_\theta D_x u\|_\ell + \|D_x u\|_1) \leq C(R\|D_x^2 u\|_\ell + \|D_x u\|_\ell) \leq CRI_\ell \tag{3.19}$$

and

$$\|D_x u\|_\gamma \leq C(\|D_\theta D_x u\|_k + \|D_x u\|_1) \leq C(R\|D_x^2 u\|_k + \|D_x u\|_k) \leq CRI_k. \tag{3.20}$$

If $\rho < 2$, then combining Hölder's inequality, $\gamma > 2$, (3.19) and (3.20), we obtain

$$\|D_x u\|_2 \leq \|D_x u\|_\rho^{1-\tau}\|D_x u\|_\gamma^\tau \leq C(RI_\ell)^{1-\tau}(RI_k)^\tau \leq CRI_\ell^{1-\tau}I_k^\tau. \tag{3.21}$$

If $\rho \geq 2$, then $\tau = 0$, so that (3.21) remains true. On the other hand, by Lemma 2.1(i), we have

$$R^{-1}\|u\|_2 \leq CR^{-1}\big(\|D_\theta u\|_2 + \|u\|_1\big) \leq C\big(\|D_x u\|_2 + R^{-1}\|u\|_1\big). \tag{3.22}$$

Since obviously $\|u\|_1 = \|u\|_1^{1-\tau}\|u\|_1^\tau \leq C\|u\|_\ell^{1-\tau}\|u\|_k^\tau \leq CR^2 I_\ell^{1-\tau}I_k^\tau$, estimate (3.17) follows from (3.5), (3.21) and (3.22). □

Step 4. *Estimation of $G_1(R)$ and $G_2(R)$ in terms of auxiliary norms for $n = 2$.* A similar procedure as for $n \geq 3$ could still be used, but this would eventually require a condition on s stronger than (1.8), because the Sobolev injection $W^{1,1}(S^1) \subset L^\infty(S^1)$ "loses" too much information. Instead, we shall estimate the $L^\infty(S^1)$ norm through the Gagliardo-Nirenberg type inequalities from Lemma 2.1(i)(ii) and then control $G_1(R)$ by interpolating between L^∞ and L^s. As for the estimate of $G_2(R)$, relying on a modification of an idea in [19], it is based on the inequality

$$\|D_x U\|_2^2 \leq \|U\|_\infty^\beta \sum_{i=1}^m \|u_i^{-\beta/2} D_x u_i\|_2^2, \qquad \beta > 0, \tag{3.23}$$

which will make it possible to appeal to Lemma 2.4 (in the subsequent averaging step). Note that in (3.23), the quantity $u_i^{-\beta/2}D_x u_i$ is understood to be 0 at points where $\nabla u_i = 0$ and $u_i = 0$.

Lemma 3.3. *Let $n = 2$. We have*

$$\|U\|_\infty \leq \begin{cases} C\Big(R^2\, I_\ell \|U\|_s^s\Big)^{1/(s+1)}, & \text{if } s > 1,\\ CR\, I_\ell^{1/2}\|U\|_s^{\frac{sp}{2(p+1-s)}}\big(\|U\|_{p+1}^{p+1}\big)^{\frac{1-s}{2(p+1-s)}}, & \text{if } s \leq 1. \end{cases} \tag{3.24}$$

Moreover, we have

$$G_1(R) \leq CR^2\|U\|_s^s\,\|U\|_\infty^{p+1-s} \tag{3.25}$$

and, for any $\beta > 0$,

$$G_2(R) \leq CR^2\|U\|_\infty^\beta \sum_{i=1}^m \|u_i^{-\beta/2}D_x u_i\|_2^2 + C\|U\|_s^2. \tag{3.26}$$

Proof. If $s>1$, then Lemma 2.1(ii) with $z=s$ implies

$$\|u\|_\infty \le C\big(\|u\|_1+\|D^2_\theta u\|_1\big)^{\frac{1}{s+1}}\|u\|_s^{\frac{s}{s+1}} \le C\big(\|u\|_\ell+R^2\|D^2_x u\|_\ell\big)^{\frac{1}{s+1}}\|u\|_s^{\frac{s}{s+1}}.$$

This yields estimate (3.24) for $s>1$. When $s\le 1$, applying Lemma 2.1(ii) with $z=1$, we obtain

$$\begin{aligned}\|u\|_\infty &\le C\big(\|u\|_1+\|D^2_\theta u\|_1\big)^{1/2}\|u\|_1^{1/2}\\ &\le C(\|u\|_\ell+R^2\|D^2_x u\|_\ell)^{1/2}\|u\|_1^{1/2}\le CRI_\ell^{1/2}\|U\|_1^{1/2}.\end{aligned}$$

Since $\|U\|_1 \le \|U\|_{p+1}^{(p+1)(1-s)/(p+1-s)}\|U\|_s^{sp/(p+1-s)}$ by Hölder's inequality, estimate (3.24) for $s\le 1$ follows.

Next, (3.25) is obvious from (3.5). To check (3.26), we first use Lemma 2.1(iii) to estimate

$$\|u\|_2\le C\|u\|_\infty \le C\inf_{S^1}|u|+C\|D_\theta u\|_1\le C\|u\|_s+C\|D_\theta u\|_2\le C\|u\|_s+CR\|D_x u\|_2.$$

From (3.5) and $n=2$, we deduce that

$$G_2(R)\le CR^2\|D_xU\|_2^2+C\|U\|_s^2,$$

hence (3.26), in view of (3.23). □

Step 5. *Control of averages of auxiliary norms in terms of R and $H(R'')$.* Recalling the notation (3.6), for any $z\in(1,\infty)$, we set $J_z(R)=\int_R^{R'} I_z^z(r)r^{n-1}\,dr$.

Lemma 3.4. *We have*

$$J_k+\int_R^{R'}\|U(r)\|_{p+1}^{p+1}r^{n-1}\,dr\le CH(R''),\qquad R\ge R_0, \tag{3.27}$$

$$J_\ell\le\begin{cases} CR^{n-s\alpha}, & R\ge R_0,\quad \text{if } s>p,\\ CR^\mu H^\eta(R''), & R\ge R_0,\quad \text{if } s\le p,\end{cases} \tag{3.28}$$

where

$$\mu=\mu_\varepsilon=\frac{(n-\alpha s)(1-p\varepsilon)}{p+1-s}\quad\text{and}\quad \eta=\eta_\varepsilon=\frac{p(1+\varepsilon)-s}{p+1-s}. \tag{3.29}$$

Moreover, when $n=2$, for any $\beta\in(0,1)$ satisfying $\beta\le p+1-s$, we have

$$\int_R^{R'}\sum_{i=1}^m\|u_i^{-\beta/2}D_xu_i(r)\|_2^2\,r\,dr\le CR^dH^e(R''),\qquad R\ge R_0, \tag{3.30}$$

(where C also depends on β), with

$$d=\frac{\beta(2-s\alpha)}{p+1-s},\quad e=\frac{p+1-\beta-s}{p+1-s}. \tag{3.31}$$

Proof. We first note that, for any $z\in(1,\infty)$, Lemma 2.2, (1.4) and (1.6) imply

$$\begin{aligned}J_z &\le C\int_{B_{R''}}\big(|D^2_xu|^z+R^{-z}|D_xu|^z+R^{-2z}u^z\big)\\ &\le C\Big(\int_{B_{R''}}|\Delta u|^z+R^{-2z}\int_{B_{R''}}|u|^z\Big)\\ &\le C\Big(\int_{B_{R''}}|U|^{pz}+R^{-2z}\int_{B_{R''}}|U|^z\Big)\\ &\le C\int_{B_{R''}}|U|^{pz}+CR^{-2z+\frac{n(p-1)}{p}}\Big(\int_{B_{R''}}|U|^{pz}\Big)^{\frac1p}\end{aligned}$$

hence,

$$J_z \le C \int_{B_{R''}} |U|^{pz} + CR^{n-\frac{2pz}{p-1}}. \tag{3.32}$$

First take $z = k$ in (3.32). Noting that $n - 2pk/(p-1) = n - 2(p+1)/(p-1) < 0$ due to $p < p_S$, and using (3.3), we deduce (3.27).

Next, in case $s > p$, we apply (3.32) with $z = \ell = s/p$. Using assumption (1.9), we obtain (3.28) for $s > p$.

In case $s \le p$, applying (3.32) with $z = \ell = 1 + \varepsilon$, and using Hölder's inequality and assumption (1.9), we obtain

$$\begin{aligned} J_\ell &\le C \int_{B_{R''}} |U|^{p(1+\varepsilon)} + CR^{n-\frac{2p(1+\varepsilon)}{p-1}} \\ &\le C\Big(\int_{B_{R''}} |U|^s\Big)^{1-\eta}\Big(\int_{B_{R''}} |U|^{p+1}\Big)^{\eta} + CR^{n-\frac{2p(1+\varepsilon)}{p-1}} \\ &\le CR^\mu H^\eta(R'') + CR^{n-\frac{2p(1+\varepsilon)}{p-1}}, \end{aligned}$$

where μ, η are given by (3.31). By a simple computation, using $p < p_S$, we see that $\mu > n - 2p(1+\varepsilon)/(p-1)$ for $\varepsilon > 0$ small. In view of (3.3), we deduce (3.28) for $s \le p$.

Let us turn to (3.30). Using Lemma 2.4, Hölder's inequality, $n = 2$ and (3.3), we obtain, for $R \ge R_0$,

$$\begin{aligned} \sum_{i=1}^m \int_{B_R} |\nabla u_i|^2 u_i^{-\beta} &\le C \int_{B_{R''}} |U|^{p+1-\beta} + CR^{-2+n\frac{p-1}{p+1-\beta}}\Big(\int_{B_{R''}} |U|^{p+1-\beta}\Big)^{\frac{2-\beta}{p+1-\beta}} \\ &\le C \int_{B_{R''}} |U|^{p+1-\beta} \\ &\le C\Big(\int_{B_{R''}} |U|^{p+1}\Big)^{\frac{p+1-\beta-s}{p+1-s}}\Big(\int_{B_{R''}} |U|^s\Big)^{\frac{\beta}{p+1-s}}. \end{aligned}$$

Estimate (3.30) then follows from assumption (1.9). □

Step 6. *Measure argument.*

For a given constant $K > 0$, let us define the sets

$$\Gamma_1(R) := \big\{r \in (R, R');\ \|U(r)\|_s > KR^{-\alpha}\big\}, \tag{3.33}$$

$$\Gamma_2(R) := \big\{r \in (R, R');\ I_k^k(r) + \|U(r)\|_{p+1}^{p+1} > KR^{-n}H(R'')\big\}, \tag{3.34}$$

$$\Gamma_3(R) := \begin{cases} \big\{r \in (R, R');\ I_\ell^\ell(r) > KR^{-s\alpha}\big\}, & \text{if } s > p, \\ \big\{r \in (R, R');\ I_\ell^\ell(r) > KR^{\mu-n}H^\eta(R'')\big\}, & \text{if } s \le p, \end{cases} \tag{3.35}$$

$$\Gamma_4(R) := \begin{cases} \Big\{r \in (R, R');\ \sum_{i=1}^m \|u_i^{-\beta/2} D_x u_i(r)\|_2^2 > KR^{d-2}H^e(R'')\Big\}, & \text{if } n = 2, \\ \emptyset, & \text{if } n \ge 3. \end{cases} \tag{3.36}$$

Lemma 3.5. *For each $R \ge R_0$, we can find*

$$\tilde{R} \in (R, R') \setminus \bigcup_{i=1}^4 \Gamma_i(R) \ne \emptyset. \tag{3.37}$$

Proof. For $R \geq R_0$, by assumption (1.8), we have

$$CR^{n-\alpha s} \geq \int_R^{R'} \|U(r)\|_s^s \, r^{n-1}\, dr \geq |\Gamma_1(R)| R^{n-1} K^s R^{-\alpha s} = |\Gamma_1(R)| K^s R^{n-\alpha s-1}$$

and, by estimate (3.27) in Lemma 3.4,

$$\begin{aligned} CH(R'') &\geq \int_R^{R'} \big(I_k^k(r) + \|U(r)\|_{p+1}^{p+1}\big) r^{n-1}\, dr \\ &\geq |\Gamma_2(R)| R^{n-1} K R^{-n} H(R'') = |\Gamma_2(R)| K R^{-1} H(R''). \end{aligned}$$

Consequently, $|\Gamma_1(R)| \leq \delta R/5$ for $K \geq (5\,C/\delta)^{1/s}$ and $|\Gamma_2(R)| \leq \delta R/5$ for $K \geq 5\,C/\delta$. In a similar way, it follows from estimates (3.28) and (3.30) in Lemma 3.4 that $|\Gamma_3(R)|, |\Gamma_4(R)| \leq \delta R/5$, for $K > 0$ large enough (independent of $R \geq R_0$). The lemma follows. □

Step 7. *Feedback estimate for G_1.*

Building on the results of the previous steps, we shall prove the following feedback estimate.

Lemma 3.6. *There exist numbers $a > 0, b \in [0,1)$ and, in case $s \leq p$, a number $\varepsilon \in (0,1)$ in (3.1), such that*

$$G_1(\tilde{R}) \leq CR^{-a} H^b(R''), \quad R \geq R_0, \tag{3.38}$$

where $\tilde{R}$ is given by Lemma 3.5. Moreover, a, b, ε depend only on n, p and s.

Proof. The proof involves only elementary but long calculations, since we need to distinguish several cases, according to the values of n, s, p. Recall that $\eta = \eta_\varepsilon$ and $\mu = \mu_\varepsilon$ are defined in (3.29).

Case 1: $n \geq 3$**,** $s > p$ (hence $p + 1 \geq \lambda$). We deduce from Lemmas 3.1 and 3.5, (3.34), (3.35) and $\ell = s/p$ that

$$G_1(\tilde{R}) \leq CR^n \big[R^2 R^{-p\alpha(1-\nu)} \big(R^{-n} H(R'')\big)^{\nu/k}\big]^{p+1},$$

where, in view of (3.8), (3.9) and (3.11), ν is given by

$$\nu = \frac{p+1}{p+1-s}\Big[1 - \frac{s}{p}\Big(\frac{2}{n-1} + \frac{1}{p+1}\Big)\Big]_+. \tag{3.39}$$

This yields (3.38) with

$$a = (p+1)\Big[(1-\nu)p\alpha + \frac{n\nu}{k} - 2 - \frac{n}{p+1}\Big], \qquad b = \nu p. \tag{3.40}$$

Now, we have either $b = 0$, or else

$$1 - b = 1 + \frac{p+1}{p+1-s}\Big[s\Big(\frac{2}{n-1} + \frac{1}{p+1}\Big) - p\Big] = \frac{p+1}{p+1-s}\Big[\frac{2s}{n-1} + 1 - p\Big] > 0$$

by assumption (1.8). Moreover, we have

$$\frac{a}{p+1} = \Big(p\alpha - 2 - \frac{n}{p+1}\Big) - \Big(\alpha - \frac{n}{p+1}\Big)\nu p = \Big[\frac{2}{p-1} - \frac{n}{p+1}\Big](1-b) > 0,$$

where we used $p < p_S$.

Case 2: $n \geq 3$**,** $s \leq p$ **and** $p + 1 \geq \lambda$ (hence $n \geq 4$ by (3.8)). We deduce from Lemmas 3.1 and 3.5, (3.34), (3.35) and $\ell = 1 + \varepsilon$ that

$$G_1(\tilde{R}) \leq CR^n \big[R^2 \big(R^{\mu-n} H^\eta(R'')\big)^{\frac{1-\nu}{1+\varepsilon}} \big(R^{-n} H(R'')\big)^{\nu/k}\big]^{p+1},$$

where, in view of (3.8), (3.9) and (3.11), $\nu = \frac{(n-3)(p+1)}{n-1} - 1$. Therefore, we have (3.38) with

$$a = a_\varepsilon := (p+1)\Big[\frac{(1-\nu)(n-\mu)}{1+\varepsilon} + \frac{n\nu}{k} - 2 - \frac{n}{p+1}\Big], \qquad b = b_\varepsilon := \nu p + \frac{(1-\nu)(p+1)\eta}{1+\varepsilon}.$$

Taking $\varepsilon > 0$ small, it suffices to show that $a_0 > 0$ and $b_0 < 1$. We compute

$$\begin{aligned} 1 - b_0 &= 1 - \nu p - \frac{(1-\nu)(p+1)(p-s)}{p+1-s} \\ &= \frac{(p+1-s)(1-\nu p) + (1-\nu)(p+1)(s-p)}{p+1-s} \\ &= \frac{(p-\nu)s + 1 - p^2}{p+1-s} = \frac{\frac{2(p+1)s}{n-1} + 1 - p^2}{p+1-s} \\ &= \frac{p+1}{p+1-s}\Big[\frac{2s}{n-1} + 1 - p\Big] > 0, \end{aligned}$$

due to (1.8). On the other hand, after some computation, we observe that

$$n - \mu_0 = (n - \alpha(p+1))\eta_0 + \alpha p, \tag{3.41}$$

hence

$$\begin{aligned} \frac{a_0}{p+1} &= (1-\nu)\big((n-\alpha(p+1))\eta_0 + \alpha p\big) + \frac{n\nu p}{p+1} - 2 - \frac{n}{p+1} \\ &= \Big(\alpha - \frac{n}{p+1}\Big)\Big(1 - \nu p - (1-\nu)(p+1)\eta_0\Big) \\ &= \Big[\frac{2}{p-1} - \frac{n}{p+1}\Big](1-b_0) > 0. \end{aligned}$$

Case 3: $n \geq 3$, $s \leq p$ and $p+1 < \lambda$. We deduce from Lemmas 3.1 and 3.5, (3.33) and (3.35) that

$$G_1(\tilde R) \leq CR^n\Big[R^{-\alpha\nu}\big(R^{\mu+2(1+\varepsilon)-n}H^\eta(R'')\big)^{\frac{1-\nu}{1+\varepsilon}}\Big]^{p+1},$$

where ν is given by (3.11). We deduce (3.38) with

$$a = a_\varepsilon := (p+1)\Big[\alpha\nu + \frac{(1-\nu)(n-2(1+\varepsilon)-\mu)}{1+\varepsilon} - \frac{n}{p+1}\Big],$$

$$b = b_\varepsilon := \frac{(1-\nu)(p+1)\eta}{1+\varepsilon}.$$

Again it suffices to show that $a_0 > 0$ and $b_0 < 1$. Recalling that $\lambda = (n-1)/(n-3)$ due to $s \leq p$, we have

$$1 - \nu = \begin{cases} \dfrac{(p+1-s)\lambda}{(p+1)(\lambda - s)}, & \text{if } n \geq 4, \\ \dfrac{p+1-s}{p+1}, & \text{if } n = 3. \end{cases}$$

Therefore, owing to (3.8) and (1.8), we have

$$1 - b_0 = 1 - \frac{(p-s)\lambda}{\lambda - s} = \frac{\lambda-1}{\lambda-s}\Big(s - \frac{\lambda(p-1)}{\lambda-1}\Big) = \frac{\lambda-1}{\lambda-s}\Big(s - \frac{(n-1)(p-1)}{2}\Big) > 0$$

if $n \geq 4$ and $1 - b_0 = s - p + 1 > 0$ if $n = 3$.

On the other hand, after some computation, we observe that $(p+1-s)(n-2-\mu_0-\alpha)=(n-\alpha(p+1))(p-s)$. Therefore, for $n\geq 4$, we get

$$\begin{aligned}\frac{a_0}{p+1}&=(1-\nu)(n-2-\mu_0-\alpha)+\alpha-\frac{n}{p+1}\\&=\Big(\frac{n}{p+1}-\alpha\Big)\frac{(p-s)\lambda}{\lambda-s}+\alpha-\frac{n}{p+1}\\&=\Big(\frac{2}{p-1}-\frac{n}{p+1}\Big)\Big(1-\frac{(p-s)\lambda}{\lambda-s}\Big)=\Big[\frac{2}{p-1}-\frac{n}{p+1}\Big](1-b_0)>0.\end{aligned}$$

The computation for $n=3$ is similar.

Case 4: $n=2$**.** First assume $s>1$. Since $s<p$ by (3.2), it follows from (3.24) in Lemma 3.3, Lemma 3.5, (3.33) and (3.35) that

$$\|U(\tilde{R})\|_\infty\leq C\Big(R^{2-\alpha s}\left(R^{\mu-2}H^\eta(R'')\right)^{1/(1+\varepsilon)}\Big)^{1/(s+1)},\tag{3.42}$$

hence

$$G_1(\tilde{R})\leq CR^{2-\alpha s}\Big(R^{2-\alpha s}\left(R^{\mu-2}H^\eta(R'')\right)^{1/(1+\varepsilon)}\Big)^{(p+1-s)/(s+1)}$$

by (3.25). We deduce (3.38) with

$$a=a_\varepsilon=\alpha s-2+\frac{p+1-s}{s+1}\Big(\alpha s-2+\frac{2-\mu}{1+\varepsilon}\Big),\qquad b=b_\varepsilon=\frac{p+1-s}{(s+1)(1+\varepsilon)}\,\eta.$$

To show that $a_0>0$ and $b_0<1$, we compute

$$\begin{aligned}a_0&=\frac{(\alpha s-2)(s+1)+\alpha s(p+1-s)+\alpha s-2}{s+1}\\&=\frac{\big(\alpha(p+3)-2\big)s-4}{s+1}=\frac{4(\alpha s-1)}{s+1}>0\end{aligned}$$

and $b_0=(p-s)/(s+1)<1$, due to (1.8).

Next assume $s\leq 1$. Then (3.24) in Lemma 3.3, Lemma 3.5 and (3.33)–(3.35) imply

$$\|U(\tilde{R})\|_\infty\leq CR\left(R^{\mu-2}H^\eta(R'')\right)^{\frac{1}{2(1+\varepsilon)}}R^{\frac{-\alpha sp}{2(p+1-s)}}\left(R^{-2}H(R'')\right)^{\frac{1-s}{2(p+1-s)}},\tag{3.43}$$

hence

$$G_1(\tilde{R})\leq CR^{2-\alpha s}\Big(R\left(R^{\mu-2}H^\eta(R'')\right)^{\frac{1}{2(1+\varepsilon)}}R^{\frac{-\alpha sp}{2(p+1-s)}}\left(R^{-2}H(R'')\right)^{\frac{1-s}{2(p+1-s)}}\Big)^{p+1-s}$$

by (3.25). We deduce (3.38) with

$$a=a_\varepsilon=\alpha s-2+\Big(\frac{2-\mu}{2(1+\varepsilon)}-1\Big)(p+1-s)+\frac{\alpha sp}{2}+1-s,$$

$$b=b_\varepsilon=\frac{\eta(p+1-s)}{2}+\frac{1-s}{2}.$$

Again, we have

$$a_0=\alpha s-2-\frac{2-\alpha s}{2}+\frac{\alpha sp}{2}+1-s=\Big(\frac{\alpha(p+3)}{2}-1\Big)s-2=2(\alpha s-1)>0$$

and $b_0=-s+(p+1)/2<1$, due to (1.8). □

Step 8. *Feedback estimate for G_2.*

Similarly as for G_1 in Step 7, we shall prove the following feedback estimate for G_2.

Lemma 3.7. *There exist numbers $\tilde{a} > 0, \tilde{b} \in [0, 1)$ and, in case $s \le p$, a number $\varepsilon \in (0, 1)$ in (3.1), such that*

$$G_2(\tilde{R}) \le C R^{-\tilde{a}} H^{\tilde{b}}(R''), \quad R \ge R_0, \tag{3.44}$$

where $\tilde{R}$ is given by Lemma 3.5. Moreover, $\tilde{a}, \tilde{b}, \varepsilon$ depend only on n, p and s.

Proof. We again have to split the proof into several cases (which are different from those in Step 7).

Case 1: $n \ge 3$ and $s > p$. We deduce from Lemmas 3.2 and 3.5, (3.34), (3.35) and $\ell = s/p$ that

$$G_2(\tilde{R}) \le C R^{n+2} (R^{-p\alpha})^{2(1-\tau)} (R^{-n} H(R''))^{2\tau/k},$$

where τ is given by (3.15), (3.16), (3.18). This yields (3.44) with

$$\tilde{a} = 2(1-\tau)p\alpha + 2\frac{n\tau}{k} - (n+2), \qquad \tilde{b} = 2\tau/k.$$

We compute

$$\tilde{b} = \frac{2p}{p+1}\Big(\frac{p}{s} - \frac{p}{p+1}\Big)^{-1}\Big(\frac{p}{s} - \frac{n+1}{2(n-1)}\Big)_+ = (p+1-s)^{-1}\Big(2p - \frac{(n+1)s}{n-1}\Big)_+. \tag{3.45}$$

Therefore, we have either $\tilde{b} = 0$, or else

$$1 - \tilde{b} = (p+1-s)^{-1}\Big(\frac{2s}{n-1} + 1 - p\Big) > 0$$

by (1.8). Moreover, we have

$$\tilde{a} = \frac{2\tau}{k}(n - p\alpha k) + 2p\alpha - 2 - n = \Big[\frac{2(p+1)}{p-1} - n\Big](1 - \tilde{b}) > 0,$$

where we used $p < p_S$.

Case 2: $n \ge 3$ and $s \le p$. We deduce from Lemmas 3.2 and 3.5, (3.34), (3.35) and $\ell = 1 + \varepsilon$ that

$$G_2(\tilde{R}) \le C R^{n+2} (R^{\mu-n} H^{\eta}(R''))^{\frac{2(1-\tau)}{1+\varepsilon}} (R^{-n} H(R''))^{2\tau/k},$$

where τ is as in Case 1 and $\eta = \eta_\varepsilon, \mu = \mu_\varepsilon$ are defined in (3.29). This yields (3.44) with

$$\tilde{a} = \tilde{a}_\varepsilon = \frac{2(1-\tau)(n-\mu)}{1+\varepsilon} + 2\frac{n\tau}{k} - (n+2), \qquad \tilde{b} = \tilde{b}_\varepsilon = \frac{2\tau}{k} + \frac{2(1-\tau)\eta}{1+\varepsilon}.$$

Taking $\varepsilon > 0$ small, it suffices to show that $\tilde{a}_0 > 0$ and $\tilde{b}_0 < 1$. By (3.15), (3.16), (3.18), we have $\tau = (p+1)(n-3)/2(n-1)$. Consequently,

$$\begin{aligned} 1 - \tilde{b}_0 &= 1 - \frac{2\tau}{k} - \frac{2(1-\tau)(p-s)}{p+1-s} = \frac{(p+1-s)(1-(2\tau/k)) + 2(1-\tau)(s-p)}{p+1-s} \\ &= \frac{(1 - \frac{2\tau}{p+1})s + 1 - p}{p+1-s} = \frac{\frac{2s}{n-1} + 1 - p}{p+1-s} > 0, \end{aligned}$$

by assumption (1.8). On the other hand, using (3.41), we find that

$$\begin{aligned} a_0 &= 2(1-\tau)\big[(n - \alpha(p+1))\eta_0 + \alpha p\big] + 2\frac{n\tau}{k} - (n+2) \\ &= \Big[\frac{2(p+1)}{p-1} - n\Big]\Big(1 - 2(1-\tau)\eta_0 - 2\frac{\tau}{k}\Big) = \Big[\frac{2(p+1)}{p-1} - n\Big](1 - \tilde{b}_0) > 0. \end{aligned}$$

Case 3: $n = 2$. By (3.26) in Lemma ?.3, Lemma 3.5, (3.33) and (3.36), we have

$$\begin{aligned} G_2(\tilde{R}) &\le CR^2 \|U(\tilde{R})\|_\infty^\beta \sum_{i=1}^m \|u_i^{-\beta/2} D_x u_i(\tilde{R})\|_2^2 + C\|U(\tilde{R})\|_s^2 \\ &\le CR^d H^e(R'')\, \|U(\tilde{R})\|_\infty^\beta + CR^{-2\alpha} =: G_{2,1} + CR^{-2\alpha}, \end{aligned} \tag{3.46}$$

where d, e are defined in (3.31).

First consider the case $s > 1$. Then (3.42) implies

$$G_{2,1} \le CR^d H^e(R'') \Big(R^{2-\alpha s} \left(R^{\mu-2} H^\eta(R'') \right)^{1/(1+\varepsilon)} \Big)^{\beta/(s+1)}.$$

By (3.46) and (3.3), we deduce (3.44) with

$$\tilde{a} = \tilde{a}_\varepsilon = \min(\bar{a}_\varepsilon, 2\alpha), \qquad \bar{a}_\varepsilon = -d + \frac{\beta}{s+1} \Big(\alpha s - 2 + \frac{2-\mu}{1+\varepsilon} \Big)$$

and

$$\tilde{b} = \tilde{b}_\varepsilon = e + \frac{\eta\beta}{(s+1)(1+\varepsilon)}.$$

It suffices to show that $\bar{a}_0 > 0$ and $\tilde{b}_0 < 1$. Using (1.8), we obtain

$$\beta^{-1}(1 - \tilde{b}_0) = \frac{1}{p+1-s} - \frac{p-s}{(p+1-s)(s+1)} = \frac{2s-p+1}{(p+1-s)(s+1)} > 0$$

and

$$\begin{aligned} \beta^{-1}\bar{a}_0 &= \frac{\alpha s - 2}{p+1-s} + \frac{\alpha s - \mu}{s+1} = \frac{(\alpha s - 2)(s+1) + \alpha s(p+1-s) + \alpha s - 2}{(p+1-s)(s+1)} \\ &= \frac{2(\alpha s - 2) - 2s + \alpha s(p+1)}{(s+1)(p+1-s)} = \frac{4(\frac{2s}{p-1} - 1)}{(s+1)(p+1-s)} > 0. \end{aligned}$$

Next assume $s \le 1$. Then (3.43) implies

$$G_{2,1} \le CR^d H^e(R'') \Big[R \left(R^{\mu-2} H^\eta(R'') \right)^{\frac{1}{2(1+\varepsilon)}} R^{\frac{-\alpha s p}{2(p+1-s)}} \left(R^{-2} H(R'') \right)^{\frac{1-s}{2(p+1-s)}} \Big]^\beta.$$

By (3.46) and (3.3), we deduce (3.44) with

$$\tilde{a} = \tilde{a}_\varepsilon = \min(\bar{a}_\varepsilon, 2\alpha), \qquad \bar{a}_\varepsilon = -d + \beta \Big(-1 + \frac{2-\mu}{2(1+\varepsilon)} + \frac{2(1-s) + \alpha s p}{2(p+1-s)} \Big)$$

and

$$\tilde{b} = \tilde{b}_\varepsilon = e + \beta \Big(\frac{\eta}{2(1+\varepsilon)} + \frac{1-s}{2(p+1-s)} \Big).$$

Using (1.8), we obtain

$$\beta^{-1}(1 - \tilde{b}_0) = \frac{1}{p+1-s} - \frac{(p-s) + (1-s)}{2(p+1-s)} = \frac{2s-p+1}{2(p+1-s)} > 0$$

and

$$\begin{aligned} \beta^{-1}\bar{a}_0 &= \frac{\alpha s - 2}{p+1-s} - 1 + \frac{2-\mu}{2} + \frac{2(1-s) + \alpha s p}{2(p+1-s)} \\ &= \frac{2(\alpha s - 2) + \alpha s - 2 + 2(1-s) + \alpha s p}{2(p+1-s)} = \frac{2(\frac{2s}{p-1} - 1)}{p+1-s} > 0. \end{aligned}$$

This completes the proof of the Lemma. □

Step 9. *Conclusion.*

Assumption (1.3) guarantees

$$H(R) \le \tilde{C} R^n \exp[C(p+1)R^q], \quad R > 1, \tag{3.47}$$

for some $\tilde{C} > 0$. On the other hand, (3.3), (3.4), Lemmas 3.6–3.7 and the nondecreasing property of H imply that

$$H(R) \le C_0 R^{-\hat{a}} H^{\hat{b}}\big((1+2\delta)R\big), \quad R \ge R_0, \tag{3.48}$$

for some constant $C_0 > 0$, with $\hat{a} = \min(a, \tilde{a}) > 0$ and $\hat{b} = \max(b, \tilde{b}) \in [0, 1)$.

To reach a contradiction, we want to show that (3.48) entails an arbitrary fast exponential lower bound on $H(R)$. We first claim that $H(R) \to \infty$ as $R \to \infty$. Indeed, inequality (3.3) guarantees that $H^{1-\hat{b}}\big((1+2\delta)R\big) \ge 2C_0 R^{-\hat{a}}$ for $R \ge \bar{R}_0$ large enough. Consequently, (3.48) yields $H\big((1+2\delta)R\big) \ge 2H(R)$ for $R \ge \bar{R}_0$, which implies the claim.

We may thus chose $\tilde{R}_0 \ge R_0$ such that $\log H(\tilde{R}_0) \ge 1$ and $C_0 R^{-\hat{a}} \le 1$. Setting $\gamma = \hat{b}^{-1} > 1$, (3.48) then guarantees

$$\log H\big((1+2\delta)R\big) \ge \gamma \log H(R), \quad R \ge \tilde{R}_0,$$

hence

$$\log H\big((1+2\delta)^i \tilde{R}_0\big) \ge \gamma^i, \quad i = 1, 2, \dots$$

that is,

$$H(R_i) \ge \exp[C R_i^{\lambda}], \quad i = 1, 2, \dots$$

with $R_i = (1+2\delta)^i \tilde{R}_0$, $\lambda = \log\gamma / \log(1+2\delta)$, $C = \tilde{R}_0^{-\lambda}$. But λ can be made arbitrarily large by choosing $\delta > 0$ small enough (recalling that b and $\tilde{b}$, hence γ, depend only on n, p, s). Therefore, we reach a contradiction with (3.47). Theorem 2 is proved. □

Remark 3.1. Let us justify the claim made in Remark 1.3(b) that, under assumption (1.11), Theorem 2 remains true without making any growth restriction on U. Indeed, if (1.11) is true, then we have $b = \tilde{b} = 0$, due to (3.39), (3.40) and (3.45). Consequently $\hat{b} = 0$ in (3.48), and we obtain $U \equiv 0$ upon letting $R \to \infty$.

4. **Proof of Theorem 1.** We need the following Lemma, which is based on a small modification of a test-function argument from [22].

Lemma 4.1. *Let $n \ge 1$, $0 < q \le 1$ and B be strictly copositive. Let U be a nonnegative solution of (1.1) and assume that $u_i > 0$ in $\mathbb{R}^n \setminus \Sigma$ for all $i = 1, \cdots, m$, where $\Sigma \subset \mathbb{R}^n$ is a measurable set such that $|\Sigma| = 0$. Then U satisfies*

$$\int_{B_R} |U|^{q+1} \le C R^{n-(q+1)/q}, \qquad R \ge 1, \tag{4.1}$$

with $C = C(n, q, B) > 0$.

Remark 4.1. (a) In [22], for $n \le 2$, the authors obtained a similar estimate in the case $U > 0$ (with an additional decaying logarithmic factor in case $n = 2$), which allowed them to conclude $U \equiv 0$ directly by sending $R \to \infty$. By a choice of test-function slightly different from that in [22], one gets (4.1) for all $n \ge 1$ and this could still be used directly to yield a Liouville theorem, but would lead to the more stringent restriction $2q+1 < (n+1)/(n-1)$. Instead, so as to cover the larger range $2q+1 < n/(n-2)$, we will use Lemma 3.1 as a primary a priori bound, in order to satisfy assumption (1.9) from Theorem 2 (with $s = q+1$).

(b) In [17], a stronger assumption on the matrix B allowed one to get estimate (4.1) with $(q+1)/q$ replaced with the larger value $(2q+1)/q$ (i.e. $s=p=2q+1$ in (1.9)). The special case of the (proof of) Theorem 2 established in [17] for that particular value of s hence led to a Liouville theorem under the optimal assumption $p<p_S$ for $n\le 4$ (or $p<(n-1)/(n-3)$ for $n\ge 5$).

Proof of Lemma 4.1. Let $0\le\varphi\in C_0^\infty(\mathbb{R}^n)$ and let $a\ge 2$. Multiplying the i-th equation in (1.1) with $(u_i+\varepsilon)^{-q}\varphi^a$ and integrating by parts, it follows that

$$\begin{aligned}\sum_{j=1}^m\quad & \beta_{ij}\int_{\mathbb{R}^n}u_j^{q+1}\Big(\frac{u_i}{u_i+\varepsilon}\Big)^q\varphi^a\,dx\\ &=\int_{\mathbb{R}^n}(-d_i\Delta u_i)(u_i+\varepsilon)^{-q}\varphi^a\,dx=d_i\int_{\mathbb{R}^n}\nabla u_i\cdot\nabla\big((u_i+\varepsilon)^{-q}\varphi^a\big)\,dx\\ &=d_i\int_{\mathbb{R}^n}\nabla u_i\cdot\Big(a\varphi^{a-1}(u_i+\varepsilon)^{-q}\nabla\varphi-q\varphi^a(u_i+\varepsilon)^{-q-1}\nabla u_i\Big)\,dx\\ &=-d_i\int_{\mathbb{R}^n}\left|\sqrt{q}\varphi^{\frac{a}{2}}(u_i+\varepsilon)^{-\frac{q+1}{2}}\nabla u_i-\frac{a}{2\sqrt{q}}\varphi^{\frac{a}{2}-1}(u_i+\varepsilon)^{\frac{1-q}{2}}\nabla\varphi\right|^2\,dx\\ &\qquad\qquad+\frac{a^2d_i}{4q}\int_{\mathbb{R}^n}(u_i+\varepsilon)^{1-q}\varphi^{a-2}|\nabla\varphi|^2\,dx\\ &\le\frac{a^2d_i}{4q}\int_{\mathbb{R}^n}(u_i+\varepsilon)^{1-q}\varphi^{a-2}|\nabla\varphi|^2\,dx.\end{aligned}$$

Letting $\varepsilon\to 0$ and using dominated convergence, we obtain

$$\sum_{j=1}^m\beta_{ij}\int_{\mathbb{R}^n}u_j^{q+1}\chi_{\{u_i>0\}}\varphi^a\,dx\le\frac{a^2d_i}{4q}\int_{\mathbb{R}^n}u_i^{1-q}\varphi^{a-2}|\nabla\varphi|^2\,dx,$$

hence

$$\sum_{j=1}^m\beta_{ij}\int_{\mathbb{R}^n}u_j^{q+1}\varphi^a\,dx\le\frac{a^2d_i}{4q}\int_{\mathbb{R}^n}u_i^{1-q}\varphi^{a-2}|\nabla\varphi|^2\,dx,\tag{4.2}$$

due to our assumption that $\big|\{u_i=0\}\big|=0$. By homogeneity, the strict copositivity assumption (1.2) implies the existence of $\kappa>0$ such that

$$\sum_{1\le i,j\le m}\beta_{ij}z_iz_j\ge\kappa\|z\|^2,\quad\text{for all } z\in[0,\infty)^m.\tag{4.3}$$

Set $c_i=c_i(\varphi):=\int_{\mathbb{R}^n}u_i^{q+1}\varphi^a\,dx$ and $d=\sup_i d_i$. Multiplying inequality (4.2) with c_i, summing over i and using (4.3), we obtain, for some $\tilde\kappa>0$,

$$\tilde\kappa\Big(\sum_{i=1}^m c_i\Big)^2\le\sum_{i,j=1}^m\beta_{ij}c_ic_j\le\frac{a^2d}{4q}\sum_{i=1}^m c_i\int_{\mathbb{R}^n}u_i^{1-q}\varphi^{a-2}|\nabla\varphi|^2\,dx,$$

hence

$$\sum_{i=1}^m c_i\le\frac{a^2d}{4q\tilde\kappa}\sum_{i=1}^m\int_{\mathbb{R}^n}u_i^{1-q}\varphi^{a-2}|\nabla\varphi|^2\,dx.\tag{4.4}$$

Now, for each $R>0$, we choose $\varphi=\varphi_R$ given by $\varphi_R(x)=\chi(x/R)$, where $0\le\chi\in C^\infty(\mathbb{R}^n)$ is such that $\chi(x)=1$ for $|x|\le 1$ and $\chi(x)=0$ for $|x|\ge 2$. We set $a=(q+1)/q$. If $q<1$, applying Hölder's inequality and using $a=(a-2)(q+1)/(1-q)$, we get

$$\begin{aligned}\sum_{i=1}^m c_i(\varphi_R) &\le \frac{a^2 d}{4q\tilde\kappa}\Big(\int_{\mathbb{R}^n} |\nabla\varphi_R|^{\frac{q+1}{q}}\,dx\Big)^{\frac{2q}{q+1}} \Big(\sum_{i=1}^m \int_{\mathbb{R}^n} u_i^{q+1}\varphi_R^{\frac{(a-2)(q+1)}{1-q}}\Big)^{\frac{1-q}{q+1}}\\ &\le \frac{a^2 d}{4q\tilde\kappa}\Big(\int_{\mathbb{R}^n} |\nabla\varphi_R|^{\frac{q+1}{q}}\,dx\Big)^{\frac{2q}{q+1}} \Big(\sum_{i=1}^m c_i(\varphi_R)\Big)^{\frac{1-q}{q+1}}.\end{aligned}$$

It follows that

$$\int_{B_R} |U|^{q+1}\,dx \le \sum_{i=1}^m c_i(\varphi_R) \le C\int_{\mathbb{R}^n} |\nabla\varphi_R|^{\frac{q+1}{q}}\,dx \le CR^{n-(q+1)/q}. \tag{4.5}$$

Finally, if $q=1$, then (4.5) follows directly from (4.4). □

Proof of Theorem 1. First observe that system (1.1) can be rewritten in the form (1.4), where $D = \mathrm{Id}$ and (1.5) is satisfied for

$$V(U) := \frac{1}{2q+2}\sum_{1\le i,j\le m} \beta_{ij}|u_i|^{q+1}|u_j|^{q+1}.$$

Let us check the assumptions of Theorem 2. Setting $p = 2q+1$, inequality (1.6) is obvious. Next, by the strict copositivity property (4.3), and since $p < p_S$, it follows that

$$2nV(U) - (n-2)U\cdot\nabla V(U) = \big(2n-(n-2)(p+1)\big)V(U) \ge c\sum_{1\le i\le m} u_i^{2(q+1)} \ge \tilde c|U|^{p+1}$$

for some $c, \tilde c > 0$, hence (1.7).

Now, by Lemma 4.1, if (1.1) admits a bounded positive solution U, then U satisfies estimate (1.9) with $s = q+1 = (p+1)/2$, and condition (1.8) is satisfied due to $2q+1 < n/(n-2)$. It thus follows from Theorem 2 that $U \equiv 0$, a contradiction. □

Appendix. Proof of Theorem 2 in the case $n = 1$.

Let us point out that no assumption on the growth of U at infinity will be used in this proof.

First note that (1.3) guarantees the existence of a sequence $R_j \to \infty$ such that

$$|U(R_j)| + |U(-R_j)| \le CR_j^{-\alpha} \to 0, \quad j\to\infty. \tag{A.1}$$

Assume $V(0) = 0$ without loss of generality and denote $' = d/dx$. By taking the inner product of (1.4) with $2U'$, taking (1.5) into account and integrating, we see that

$$\sum_{i=1}^m d_i|U_i'(x)|^2 + 2V(U(x)) = C = Const., \quad x\in\mathbb{R}. \tag{A.2}$$

We claim that $C = 0$. Assume the contrary. Then, due to (A.1) and $V(0) = 0$, we have $C > 0$ and (A.2) implies the existence of $L > K > 0$ and $\eta\in(0,1)$ such that

$$K \le |U'(x)| \le L \ \text{ whenever } \ |U(x)| \le \eta. \tag{A.3}$$

One easily deduces from (A.3) that, if $|U(x)| \le \eta/2$, then $|U(y)| \le \eta$ whenever $|y-x| \le \eta/2L$. Since $-DU'' = F(U)$, $F(0) = 0$ and F is continuous, by taking η possibly smaller, it follows that

$$|U''(y)| \le \frac{KL}{m} \ \text{ whenever } |U(x)| \le \frac{\eta}{2} \text{ and } |y-x| \le \frac{\eta}{2L}. \tag{A.4}$$

Now pick x_0 such that $|U(x_0)| < (4mL)^{-1}K\eta < \eta/2$ (such x_0 does exist in view of (A.1)). By (A.3), there exists $i \in \{1,\dots,m\}$ such that $|U_i'(x_0)| \geq K/m$. Assume for instance $U_i'(x_0) \leq -K/m$ (the other case being similar). Then, by (A.4), we have

$$U_i'(y) \leq -\frac{K}{m} + \frac{KL}{m}\frac{\eta}{2L} \leq -\frac{K}{2m} \quad \text{for } |y - x_0| \leq \frac{\eta}{2L}.$$

Therefore,

$$U_i\Big(x_0 + \frac{\eta}{2L}\Big) \leq U_i(x_0) - \frac{K\eta}{4mL} \leq |U(x_0)| - \frac{K\eta}{4mL} < 0:$$

a contradiction. This proves our claim that $C = 0$.

Now, as a consequence of (A.2), (1.6) and $V(0) = 0$, we have

$$|V(U)| + |U'|^2 \leq C_1|U|^{p+1}, \quad x \in \mathbb{R}, \tag{A.5}$$

for some constant $C_1 > 0$. Combining Lemma 2.4, (1.7), (A.5) and (A.1) yields the estimate

$$\int_{-R_j}^{R_j} |U|^{p+1} \leq CR_j \sum_{\sigma=\pm1}\Big[|V(U)| + |U'|^2 + R_j^{-2}|U|^2\Big](\sigma R_j) \leq CR_j^{1-2(p+1)/(p-1)}.$$

Since the RHS goes to 0 as $j \to \infty$, we conclude that $U \equiv 0$. □

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E-mail address: souplet@math.univ-paris13.fr