NETWORKS AND HETEROGENEOUS MEDIA doi:10.3934/nhm.2012.7.941
©American Institute of Mathematical Sciences
Volume **7**, Number **4**, December **2012** pp. **941–966**

ENTROPY SOLUTIONS OF FORWARD-BACKWARD PARABOLIC EQUATIONS WITH DEVONSHIRE FREE ENERGY

Flavia Smarrazzo and Alberto Tesei

Department of Mathematics "G. Castelnuovo"
University of Rome "La Sapienza"
P.le A. Moro 5, I-00185 Rome, Italy

Abstract. A class of quasilinear parabolic equations of forward-backward type $u_t = [\phi(u)]_{xx}$ in one space dimension is addressed, under assumptions on the nonlinear term ϕ which hold for a number of mathematical models in the theory of phase transitions. The notion of a *three-phase* solution to the Cauchy problem associated with the aforementioned equation is introduced. Then the time evolution of three-phase solutions is investigated, relying on a suitable *entropy inequality* satisfied by such a solution. In particular, it is proven that transitions between stable phases must satisfy certain *admissibility conditions*.

1. Introduction.

This paper deals with the *forward-backward* parabolic equation

$$u_t = [\phi(u)]_{xx} \ , \tag{1}$$

where $\phi \in C^2(\mathbb{R})$ is a non-monotonic function. Therefore, initial-value problems for equation (1) are *ill-posed* whenever the solution u takes values where $\phi' < 0$.

In spite of this difficulty, motivations to study such forward-backward parabolic equations come from various contexts. The most well-known case is probably that of a cubic ϕ, which arises in the Landau theory of phase transitions. Different choices of ϕ are suggested by mathematical models of population dynamics, oceanography and image reconstruction (see [1, 9, 10]).

However, it has been pointed out that the Landau theory cannot account for phase transition phenomena exhibited by certain materials (*e.g.*, ferroelectrics; see [2]). Hence it has been proposed to modify the Landau free energy by including higher order terms – namely, to replace it by the *Devonshire free energy*

$$F(u) = F_0(T) + \alpha_1(T - T_c)u^2 - \alpha_2 u^4 + \alpha_3 u^6 \qquad (\alpha_1,\, \alpha_2,\, \alpha_3 > 0)$$

(here the absolute temperature T, regarded as a parameter, is assumed to be less than some critical value T_c). Then the *response function* ϕ becomes

$$\phi(u) = 2\alpha_1(T - T_c)u - 4\alpha_2 u^3 + 6\alpha_3 u^5. \tag{2}$$

It is the purpose of this paper to address equation (1) with a nonlinearity of the form (2), extending to this case some relevant results which hold when ϕ is cubic.

2000 *Mathematics Subject Classification.* Primary: 35K55, 35R25, 35Bxx; Secondary: 34C55.
Key words and phrases. Forward-backward parabolic equations, entropy inequality, Devonshire free energy, phase transitions, interfaces.

(i) Let us outline what is known for the case of a cubic-like $\phi \in C^2(\mathbb{R})$, satisfying the following assumption:

$$(H_0) \qquad \begin{cases} (i) \;\; \phi'(u) > 0 \;\text{ if }\; u \in (-\infty, a) \cup (b, \infty), \\ \qquad \phi'(u) < 0 \;\text{ if }\; u \in (a, b); \\ (ii) \;\; \phi(u) \to \pm\infty \text{ as } u \to \pm\infty\,. \end{cases}$$

Set $A := \phi(b)$, $B := \phi(a)$. Denote by

$$S_1 := \{(u, \phi(u)) \mid u \in (-\infty, a]\} = \{(s_1(v), v) \mid v \in (-\infty, B]\}\,,$$
$$S_2 := \{(u, \phi(u)) \mid u \in [b, \infty)\} = \{(s_2(v), v) \mid v \in [A, \infty)\}$$

the *stable branches*, and by

$$S_0 := \{(u, \phi(u)) \mid u \in (a, b)\} = \{(s_0(v), v) \mid v \in (A, B)\}$$

the *unstable branch* of the graph of ϕ (see Figure 1).

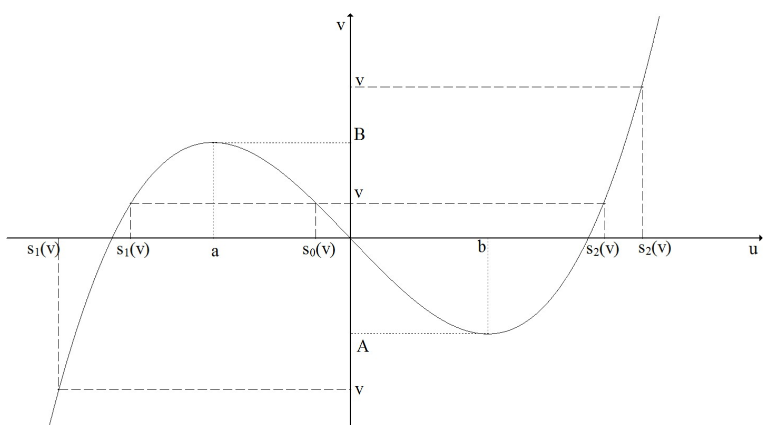

FIGURE 1. Assumption (H_0).

Consider for any $\epsilon > 0$ the Neumann initial-boundary value problem

$$\begin{cases} u^\epsilon_t = v^\epsilon_{xx} & \text{in } \Omega \times (0, \infty) := Q \\ v^\epsilon_x = 0 & \text{in } \partial\Omega \times (0, \infty) \\ u = u_0 & \text{in } \Omega \times \{0\}\,, \end{cases} \tag{3}$$

where $\Omega \subseteq \mathbb{R}$ is a bounded interval, and

$$v^\epsilon := \phi(u^\epsilon) + \epsilon u^\epsilon_t\,.$$

Problem (3) is the *pseudoparabolic* regularization of the Neumann problem

$$\begin{cases} u_t = [\phi(u)]_{xx} & \text{in } Q \\ [\phi(u)]_x = 0 & \text{in } \partial\Omega \times (0, \infty) \\ u = u_0 & \text{in } \Omega \times \{0\} \end{cases} \tag{4}$$

for equation (1), which is replaced by the Sobolev equation

$$u_t = [\phi(u)]_{xx} + \epsilon[u_t]_{xx} \qquad (\epsilon > 0)\,.$$

Global existence and uniqueness of a solution (u^ϵ, v^ϵ) to problem (3) – namely, of a couple $u^\epsilon \in C^1([0,T]; L^\infty(\Omega))$, $v^\epsilon \in C([0,T]; W^{2,\infty}(\Omega))$ which satisfies problem

(3) in the strong sense - was proven (see [8, 6]). Moreover, the following a priori estimates of the families $\{u^\epsilon\}$, $\{v^\epsilon\}$, uniform with respect to ϵ, are known to hold:

$$\| u^\epsilon \|_{L^\infty(Q)} \leq C\,, \tag{5}$$

$$\| v^\epsilon \|_{L^\infty(Q)} \leq C\,, \tag{6}$$

$$\| v^\epsilon_x \|_{L^2(Q)} + \sqrt{\epsilon}\, \| u^\epsilon_t \|_{L^2(Q)} \leq C\,, \tag{7}$$

for some number $C > 0$ which only depends on the initial data function u_0.

Estimates (5)-(6) suggest to study the limit as $\epsilon \to 0$ of the families $\{u^\epsilon\}$, $\{v^\epsilon\}$ *in the sense of Young measures*. This was made in [11] showing that, for some sequence $\{\epsilon_k\}$ with vanishing limit, the sequence $\{\tau^{\epsilon_k}\}$ of Young measures associated with the sequence $\{u^{\epsilon_k}\}$ converges in the *narrow topology* over $Q \times \mathbb{R}$ to a Young measure τ (*e.g.*, see [15]), whose disintegration $\nu_{(x,t)}$ is a superposition of three Dirac masses concentrated on the branches $S_1,\ S_2,\ S_0$ of the graph of ϕ. More precisely, there exist $\lambda_i \in L^\infty(Q)\ \ (i = 0, 1, 2),\ 0 \leq \lambda_i \leq 1,\ \sum_{i=0}^{2} \lambda_i = 1$, such that

$$\nu_{(x,t)} = \sum_{i=0}^{2} \lambda_i(x,t)\delta\big(\,\cdot - s_i(v(x,t))\big) \tag{8}$$

for almost every $(x,t) \in Q$, where $\lambda_1(x,t) = 1$ if $v(x,t) < A$, $\lambda_2(x,t) = 1$ if $v(x,t) > B$, and $v \in L^\infty(Q)$ is the weak* limit of the sequence $\{v^{\epsilon_k}\}$ in $L^\infty(Q)$ (see (6)):

$$v^{\epsilon_k} \overset{*}{\rightharpoonup} v \quad \text{ in } L^\infty(Q)\,.$$

By equality (8), the coefficients λ_i can be interpreted as *phase fractions*, and the solution u as a superposition of different phases (see (9) below). Moreover, by (8) and general properties of the narrow convergence of Young measures we have

$$f(u^{\epsilon_k}) \overset{*}{\rightharpoonup} f^* \qquad \text{in } L^\infty(Q)\,,$$

where

$$f^*(x,t) := \int_{\mathbb{R}} f(\xi)\, d\nu_{(x,t)}(\xi) = \sum_{i=0}^{2} \lambda_i(x,t) f(s_i(v(x,t)))$$

for almost every $(x,t) \in Q$. In particular, choosing $f(u) = u$ gives

$$u^{\epsilon_k} \overset{*}{\rightharpoonup} u := \sum_{i=0}^{2} \lambda_i s_i(v) \quad \text{ in } L^\infty(Q)\,. \tag{9}$$

Besides, by estimate (7) there holds $v \in L^2((0,T); H^1(\Omega))$ and

$$v^{\epsilon_k} \rightharpoonup v \quad \text{ in } L^2((0,T); H^1(\Omega))$$

for any $T > 0$. Finally, passing to the limit as $\epsilon_k \to 0$ in the weak formulation of problem (3) proves that the couple (u, v) satisfies the equality

$$\iint_Q \{u\psi_t - v_x\psi_x\}\, dxdt + \int_\Omega u_0(x)\psi(x,0)dx = 0$$

for any $\psi \in C^1(\bar{Q}),\ \psi(\cdot, T) = 0$ in Ω.

The above remarks suggest to regard the above quintuple $u,\ \lambda_0,\ \lambda_1,\ \lambda_2 \in L^\infty(Q),\ v \in L^\infty(Q) \cap L^2((0,T); H^1(\Omega))$ as a *weak measure-valued solution* of problem (4) in Q, in the sense of Young measures. A major feature of such solutions, called *entropy solutions* of problem (4) hereafter, is that they satisfy an *entropy inequality*, namely:

$$\iint_Q \left\{G^*\psi_t - g(v)v_x\psi_x - g'(v)v_x^2\psi\right\} dxdt + \int_\Omega G(u_0)\psi(x,0)dx \geq 0\,. \tag{10}$$

Here g is any nondecreasing function in $C^1(\mathbb{R})$,

$$G(u) := \int_0^u g(\phi(z))dz + k \qquad (k \in \mathbb{R})\,, \tag{11}$$

the function $G^* \in L^\infty(Q)$ is defined by

$$G^* := \sum_{i=0}^{2} \lambda_i G(s_i(v))\,,$$

and $\psi \in C^1(\bar{Q})$, $\psi \geq 0$, $\psi(\cdot, T) = 0$ in Ω. We refer the reader to [8, 11, 6] for the proof of (10), which plays a central role in the subsequent analysis.

(ii) The above construction is reminiscent in several respects of that made for hyperbolic conservation laws (*e.g.*, see [12]). However, at variance with this case, uniqueness within the class of entropy solutions of problem (4) has not been proven. This motivated investigating well-posedness of the problem within a more restricted class of solutions of physical interest, called *two-phase solutions*, where only transitions between stable phases are allowed (see [4, 6, 7, 14]). More specifically, in [7] the Cauchy problem for equation (1):

$$\begin{cases} u_t = [\phi(u)]_{xx} & \text{in } \mathbb{R}\times(0,\infty) =: V \\ u = u_0 & \text{in } \mathbb{R}\times\{0\} \end{cases} \tag{12}$$

was addressed, assuming that

$$\bar{V} = V_1 \cup \gamma \cup V_2\,,$$

where

$$V_1 := \left\{(x,t)\in\bar{V} \mid x < \xi(t)\right\},$$
$$V_2 := \left\{(x,t)\in\bar{V} \mid x > \xi(t)\right\},$$
$$\gamma := \{(\xi(t),t)\} \qquad (t\in[0,\infty))$$

and

$$u = s_i(v) \text{ in } V_i \qquad (i = 1,2)\,,$$

the alternative choice being dealt with similarly. Observe that this amounts to take

$$\lambda_0 = 0 \quad \text{almost everywhere in } Q\,, \tag{13}$$
$$\lambda_1 = 1 \text{ and } \lambda_2 = 0 \text{ almost everywhere in } V_1\,,$$
$$\lambda_2 = 1 \text{ and } \lambda_1 = 0 \text{ almost everywhere in } V_2\,.$$

It should be mentioned that, to our knowledge, it is in general unknown whether two-phase solutions are limits of some sequence of approximating solutions $\{u^{\epsilon_k}\}$, $\{v^{\epsilon_k}\}$ as $\epsilon_k \to 0$ (see [13] for the proof of this fact in a significant case).

In some respects, two-phase solutions of problem (12) can be regarded as the counterpart of piecewise smooth solutions in the theory of hyperbolic conservation laws. Like these, they exhibit an interface which evolves according to a suitable

Rankine-Hugoniot condition. Moreover, they satisfy the entropy inequality (10), which now by equalities (13) takes the simpler form:

$$\iint_Q \left\{G(u)\psi_t - g(v)v_x\psi_x - g'(v)v_x^2\psi\right\} dx\,dt + \int_{\mathbb{R}} G(u_0)\psi(x,0)\,dx \geq 0\,. \tag{14}$$

A major point is that the entropy inequality (14) determines *admissibility conditions* for the evolution of the interface γ. Precisely, it can be proven that (see [4, 6]):

$$\begin{cases} \xi'(t) \geq 0 & \text{if } v(\xi(t),t) = A\,, \\ \xi'(t) \leq 0 & \text{if } v(\xi(t),t) = B\,, \\ \xi'(t) = 0 & \text{if } v(\xi(t),t) \neq A \text{ and } v(\xi(t),t) \neq B\,. \end{cases} \tag{15}$$

In particular, this shows that transitions between the stable phases S_1 and S_2 can only occur at the points $(x,t) \in \gamma$ where $v(x,t)$ takes the value A (jumps from S_2 to S_1) or B (jumps from S_1 to S_2). Therefore, the entropy conditions also select *admissible jumps* between the stable branches of ϕ. As first pointed out in [11], this gives rise to a *hysteresis loop* typical of first-order phase transitions (see [2]).

The a priori knowledge of the admissibility conditions (15) was instrumental to show that *two-phase solutions are a class of well-posedness for problem* (12). In fact, it was a basic tool in the proof of existence and uniqueness of such solutions, given in [7] for a piecewise linear ϕ (see [13] for a general cubic-like ϕ). Moreover, it provides information of heuristic interest to describe the qualitative behaviour of these physically relevant solutions.

(iii) Extending the above results to the case of the Devonshire potential needs a number of nontrivial steps. To be specific, let the response function ϕ satisfy the following assumption (see Figure 2):

$$(H_1) \qquad \begin{cases} (i) & \phi(s) \to \pm\infty \text{ as } s \to \pm\infty\,; \\ (ii) & \phi'(s) > 0 \text{ if } s \in (-\infty,a) \cup (b,c) \cup (d,\infty)\,; \\ (iii) & \phi'(s) < 0 \text{ if } s \in (a,b) \cup (c,d); \\ (iv) & \phi'(s) = 0\,,\ \phi''(s) \neq 0 \text{ if } s = a,b,c,d \qquad (a<b<c<d)\,. \end{cases}$$

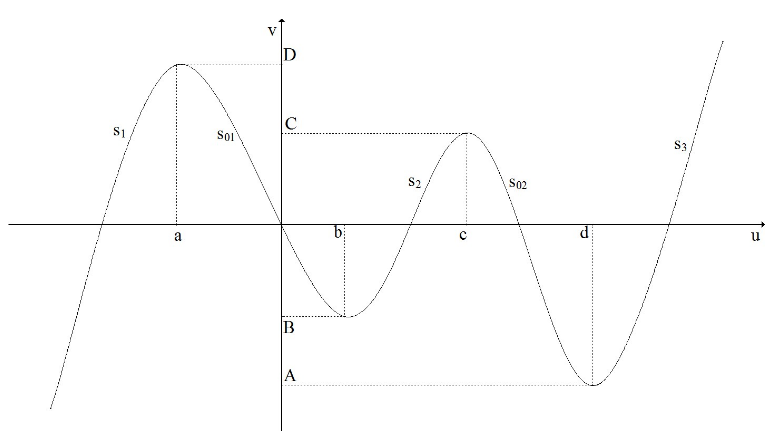

FIGURE 2. Assumptions (H_1) and (A_1).

Set $A := \phi(d)$, $B := \phi(b)$, $C := \phi(c)$, and $D := \phi(a)$. The three *stable branches* of the graph of ϕ are

$$S_1 := \{(u, \phi(u)) \mid u \in (-\infty, a]\} = \{(s_1(v), v) \mid v \in (-\infty, D]\},$$
$$S_2 := \{(u, \phi(u)) \mid u \in [b, c]\} = \{(s_2(v), v) \mid v \in [B, C]\},$$
$$S_3 := \{(u, \phi(u)) \mid u \in [d, \infty)\} = \{(s_3(v), v) \mid v \in [A, \infty)\},$$

whereas the unstable branches are

$$S_{01} := \{(u, \phi(u)) \mid u \in (a, b)\} = \{(s_{01}(v), v)) \mid v \in (B, D)\},$$

and

$$S_{02} := \{(u, \phi(u)) \mid u \in (c, d)\} = \{(s_{02}(v), v)) \mid v \in (A, C)\}.$$

It is easily seen that even in this case there exists a unique solution of the regularized problem (4), which satisfies the same a priori estimates (5)-(7) of the case (H_0). Therefore, the limit (u, v) of the sequence $\{(u^{\epsilon_k}, v^{\epsilon_k})\}$ in the sense discussed above can be considered. The validity of the characterization of the limiting Young measure ν analogous to (8) will be addressed elsewhere (see [3]).

By analogy with two-phase solutions of the case (H_0), we can now define *three-phase solutions* of problem (12) (see Section 2). As in the case of a cubic-like ϕ, let the entropy inequality (14) hold. As we shall see below, again the entropy inequality determines admissibility conditions for the evolution of the interfaces, much as in the case of the unique interface of two-phase solutions (compare (15) with (18)-(20) below). In particular, jumps between stable branches can only occur to or from a local extremum of the graph of ϕ. The novel feature is that in the present case *only jumps from an extremum to a visible stable branch are allowed.*

To clarify this point, suppose that (see Figure 2):

$$(A_1) \qquad A < B < C < D.$$

In this case the branch S_1 is visible from the extremum (b, B), for the line $v = B$ intersects S_1 at some point of abscissa $\bar{u}$, and no other intersection of this line with the graph of ϕ has abscissa in the interval $(\bar{u}, b)$. Instead, the branch S_3 is not visible from (b, B), since the line $v = B$ intersects the unstable branch S_{02} before than S_3. Similarly, the branch S_3 is visible, whereas S_1 is not visible from the extremum (c, C). On the other hand, from neither of the two extrema (a, D), (d, A) the branch S_2 is visible, since it does not intersect the lines $v = A$, $v = D$.

Clearly, the visibility of each stable branch depends on the value of the quantities A, B, C and D. In turn, this determines the existence of hysteresis loops relative to a couple of stable phases, thus the local behaviour of the relative interfaces. As a consequence, a much richer structure of the interfaces arises with respect to the case of a cubic-like ϕ.

In the following we describe this structure under different assumptions on the value of A, B, C, D. In particular, we show that *splitting* or *merging of interfaces* – which corresponds to the *appearance* or *disappearance of a phase* – is possible. This can be regarded as a *bifurcation of interfaces*, which does not have a counterpart for a cubic-like ϕ. A complete description of the local structure of interfaces will also be given (*e.g.*, see Theorems 3.4 and 3.5 for the case (A_1)), which expectedly will be instrumental to prove the actual existence and uniqueness of three-phase solutions of problem (12).

It should be clear from the above remarks that the outlined phenomena depend on the assumptions on A, B, C, D, which determine the shape of the graph of ϕ. By the general assumption (H_1), the extrema (a, D) and (d, A) are a local maximum, respectively a local minimum. Therefore, beside (A_1) only the following possibilities arise (see Figures 8, 13, 18 and 22):

(A_2) $$B < A < C < D\,,$$

(A_3) $$B < A < D < C\,,$$

(A_4) $$A < B < D < C\,,$$

(A_5) $$B < D < A < C\,.$$

The plan of the paper is as follows. In Section 2, after introducing the definition of a three-phase solution to problem (12) and some related concepts, we state our main results (Theorems 2.4-2.6), which are proven by checking their validity in all cases $(A_1) - (A_5)$. In Section 3 a detailed proof is given for the case (A_1); this is the content of Theorems 3.1-3.5. To avoid tedious repetitions, in Section 4 we give only a qualitative discussion of the analogous results for cases (A_2)-(A_5), leaving statements and proofs to the reader.

2. **Mathematical framework and results.**

Let us give the definition of a three-phase solution to problem (12).

Definition 2.1. Let $u_0 \in L^\infty(\mathbb{R})$, $\phi(u_0) \in H^1_{loc}(\mathbb{R})$. By a *three-phase solution* of problem (12) we mean any quartet (u, v, ξ_1, ξ_2) such that:

(i) $u \in L^\infty(V)$, $v \in C(\bar{V}) \cap L^2((0,T); H^1_{loc}(\mathbb{R}))$ for every $T > 0$, and $v(x, 0) = \phi(u_0)(x)$ for all $x \in \mathbb{R}$;

(ii) $\xi_1, \xi_2 \in C^1([0,\infty))$, and $\xi_1 \le \xi_2$ on $[0,\infty)$. Moreover, for every $T > 0$ there exists a finite number of intervals $(t', t'') \subseteq (0, T]$ such that $\xi_2 - \xi_1 > 0$, $\xi_1' \ne 0$ and $\xi_2' \ne 0$ on (t', t'');

(iii) set

$$\begin{aligned} V_1 &:= \left\{(x,t) \in \bar{V} \mid x < \xi_1(t)\right\}, \\ V_2 &:= \left\{(x,t) \in \bar{V} \mid \xi_1(t) < x < \xi_2(t)\right\}, \\ V_3 &:= \left\{(x,t) \in \bar{V} \mid x > \xi_2(t)\right\} \qquad (t \in [0,\infty))\,. \end{aligned}$$

Then there is a one-to-one correspondence between the domains V_i and the stable branches S_j of the graph of ϕ, in the sense that for every i there exists a unique j such that $u(x,t) = s_j(v)(x,t)$ for almost every $(x,t) \in V_i$ $(i, j = 1, 2, 3)$;

(iv) for any $t_1, t_2 \in [0,\infty)$, $t_1 \le t_2$ and $\psi \in C^1([0,\infty); C^1_c(\mathbb{R}))$ there holds

$$\int_{t_1}^{t_2}\int_{\mathbb{R}} \{u\psi_t - v_x\psi_x\}\, dxdt = \int_{\mathbb{R}} u(x,t_2)\psi(x,t_2)\, dx - \int_{\Omega} u(x,t_1)\psi(x,t_1) dx\,; \quad (16)$$

(v) for any $t_1, t_2 \in [0,\infty)$, $t_1 \le t_2$ and any $g \in C^1(\mathbb{R})$, $g' \ge 0$, $\psi \in C^1([0,\infty); C^1_c(\mathbb{R}))$, $\psi \ge 0$ in S there holds

$$\begin{aligned} &\int_{t_1}^{t_2}\int_{\mathbb{R}} \left\{G(u)\psi_t - g(v)v_x\psi_x - g'(v)v_x^2\psi\right\}\, dxdt \ge \\ &\ge \int_{\mathbb{R}} G(u(x,t_2))\psi(x,t_2)\, dx - \int_{\mathbb{R}} G(u(x,t_1))\psi(x,t_1)\, dx\,. \end{aligned} \quad (17)$$

The graphs

$$\gamma_1 := \{(\xi_1(t), t)\}\,, \quad \gamma_2 := \{(\xi_2(t), t)\} \quad (t \in [0, \infty))$$

are called *interfaces* of the three-phase solution.

Observe that by the above definition

$$\bar{V} = V_1 \cup \gamma_1 \cup V_2 \cup \gamma_2 \cup V_3\,.$$

Moreover, we can have $\xi_1(t) = \xi_2(t)$ for any $t \in [0, \infty)$, thus $\gamma_1 \equiv \gamma_2$, $V_2 = \emptyset$; in this case we obtain two-phase solutions of problem (12) as a particular case of Definition 2.1. Also observe that $u \in C(\bar{V}_i)$ $(i = 1, 2, 3)$, since by assumption $v \in C(\bar{V})$. Moreover, u is a classical solution of the equation $u_t = [\phi(u)]_{xx}$ in subdomains of $V_i \cap S$ where $\phi'(u) > 0$ (*e.g.*, see [5]; $i = 1, 2, 3$).

A specific three-phase solution to problem (12) is defined by fixing the correspondence between domains V_i and stable branches S_j (see Definition 2.1-(*iii*) above). For instance, we can consider *(1,2,3)-solutions*, namely

$$u = s_i(v) \text{ in } V_i \qquad (i = 1, 2, 3)\,,$$

or *(1,3,2)-solutions*, namely

$$u = \begin{cases} s_1(v) & \text{in } V_1\,, \\ s_3(v) & \text{in } V_2\,, \\ s_2(v) & \text{in } V_3\,. \end{cases}$$

Other types of (k, l, m)-solutions $(k, l, m = 1, 2, 3)$ can be similarly addressed. On the other hand, observe that the actual existence of such solutions depends on the possibility of jumping between stable branches, which depends on the entropy conditions and the shape of the graph of ϕ. For instance, if the function ϕ satisfies assumption (A_5) below, only (1,2,3)- and (3,2,1)-solutions exist (see Subsection 4.4).

By the above regularity remarks, assigning a (k, l, m)-solution amounts to specify on which stable branches the initial data u_0 takes values. For instance, (1,2,3)-solutions are singled out by the assignment

$$u_0 = \begin{cases} s_1(\phi(u_0)) & \text{in } (-\infty, \xi_1(0))\,, \\ s_2(\phi(u_0)) & \text{in } (\xi_1(0), \xi_2(0))\,, \\ s_3(\phi(u_0)) & \text{in } (\xi_2(0), \infty)\,, \end{cases}$$

and similarly for (1,3,2)-solutions.

Once the initial data function u_0 is assigned, a three-phase solution is determined by the time evolution of its interfaces. This depends on the local behaviour of these curves, which, as outlined in the Introduction, is determined by admissibility conditions deriving from the entropy inequality. On the other hand, these admissibility conditions depend on the possibility of jumping between stable branches of the graph of ϕ (hence on their visibility, which depends on assumptions like (A_1) above and (A_2)-(A_5) below). Therefore, the local behaviour of an interface is expectedly related to which stable phases are contiguous across the interface itself. This motivates the following definitions.

Definition 2.2. Let $\gamma = \{(\xi(t), t)\}$ be either interface of a three-phase solution of problem (12). A point $\bar{P} \equiv (\bar{x}, \bar{t}) \in \gamma$ is called a *(k,l)-point* $(k, l = 1, 2, 3)$ if there exists a rectangle $\mathcal{R} \equiv (\bar{x} - \delta, \bar{x} + \delta) \times (\bar{t} - \tau, \bar{t} + \tau)$ $(\delta, \tau > 0)$ such that

$$\bar{x} - \delta < \xi(t) < \bar{x} + \delta \quad \text{for any } t \in (\bar{t} - \tau, \bar{t} + \tau) \cap [0, \infty)\,,$$

$$u(x,t) = s_k(v(x,t)) \quad \text{for any } x \in (\xi(t)-\delta, \xi(t))\,,\ t \in (\bar{t}-\tau, \bar{t}+\tau) \cap [0,\infty)\,,$$

$$u(x,t) = s_l(v(x,t)) \quad \text{for any } x \in (\xi(t), \xi(t)+\delta)\,,\ t \in (\bar{t}-\tau, \bar{t}+\tau) \cap [0,\infty)\,.$$

The (k,l)-points are called *regular points* of the solution interfaces. We say that *an arc* $\{(\xi(t),t) \,|\, t \in [t_1,t_2] \subseteq [0,\infty)\} \subseteq \gamma$ *separates the phases* k *and* l, if all its points are (k,l)-points $(k,l = 1,2,3)$.

Definition 2.3. Let (u,v,ξ_1,ξ_2) be any three-phase solution of problem (12) such that $\xi_1(t^*) = \xi_2(t^*) =: x^*$ for some $t^* \in [0,\infty)$.
(i) The point $P^* \equiv (x^*,t^*)$ $(t^* > 0)$ is called a *backward (k,l,m)-point* $(k,l,m = 1,2,3)$ if there exists a rectangle $\mathcal{R} \equiv (x^*-\delta, x^*+\delta) \times (t^*-\tau, t^*+\tau)$ $(\delta, \tau \in (0,t^*))$ such that

$$\bar{x}-\delta < \xi_1(t) < \xi_2(t) < \bar{x}+\delta \quad \text{for any } t \in (t^*-\tau, t^*)\,,$$

$$\bar{x}-\delta < \xi_1(t) = \xi_2(t) < \bar{x}+\delta \quad \text{for any } t \in (t^*, t^*+\tau)\,,$$

$$u(x,t) = s_k(v(x,t)) \quad \text{for any } x \in (\xi_1(t)-\delta, \xi_1(t))\,,\ t \in (t^*-\tau, t^*+\tau)\,,$$

$$u(x,t) = s_l(v(x,t)) \quad \text{for any } x \in (\xi_1(t), \xi_2(t))\,,\ t \in (t^*-\tau, t^*)\,,$$

$$u(x,t) = s_m(v(x,t)) \quad \text{for any } x \in (\xi_2(t), \xi_2(t)+\delta)\,,\ t \in (t^*-\tau, t^*+\tau)\,.$$

(ii) The point $P^* \equiv (x^*,t^*)$ is called a *forward (k,l,m)-point* $(k,l,m = 1,2,3)$ if there exists a rectangle $\mathcal{R} \equiv (x^*-\delta, x^*+\delta) \times (t^*-\tau, t^*+\tau)$ $(\delta, \tau > 0)$ such that

$$\bar{x}-\delta < \xi_1(t) = \xi_2(t) < \bar{x}+\delta \quad \text{for any } t \in (t^*-\tau, t^*) \cap [0,\infty)\,,$$

$$\bar{x}-\delta < \xi_1(t) < \xi_2(t) < \bar{x}+\delta \quad \text{for any } t \in (t^*, t^*+\tau)\,,$$

$$u(x,t) = s_k(v(x,t)) \quad \text{for any } x \in (\xi_1(t)-\delta, \xi_1(t))\,,\ t \in (t^*-\tau, t^*+\tau) \cap [0,\infty)\,,$$

$$u(x,t) = s_l(v(x,t)) \quad \text{for any } x \in (\xi_1(t), \xi_2(t))\,,\ t \in (t^*, t^*+\tau)\,,$$

$$u(x,t) = s_m(v(x,t)) \quad \text{for any } x \in (\xi_2(t), \xi_2(t)+\delta)\,,\ t \in (t^*-\tau, t^*+\tau) \cap [0,\infty)\,.$$

Backward and forward (k,l,m)-points are called *bifurcation points* of the solution interfaces.

Now our main results can be concisely described as follows.[1] Define

$$m_i := \inf\{z \in \mathbb{R} \,|\, (z,\phi(z)) \in S_i\}\,,\ M_i := \sup\{z \in \mathbb{R} \,|\, (z,\phi(z)) \in S_i\} \qquad (i = 1,2,3)\,;$$

observe that by assumption (H_1) m_1, m_2, m_3, M_1, M_2 and M_3 are all different. By abuse of notation, write $\phi(m_1) = \phi(-\infty) = -\infty$ and $\phi(M_3) = \phi(\infty) = \infty$. Then the following statements hold.

Theorem 2.4. *Let assumption* (H_1) *hold and* $(\xi(t),t)$ *be a (k,l)-point* $(k,l = 1,2,3)$.
(i) *If* $\xi'(t) > 0$*, then either*
$m_k > M_l$*,* $\phi(M_k) > \phi(M_l) = v(\xi(t),t)$*, and* $\phi(z) < \phi(M_l)$ *for all* $z \in (M_l, m_k)$*,*
or,
$M_k < m_l$*,* $\phi(m_k) < \phi(m_l) = v(\xi(t),t)$*, and* $\phi(z) > \phi(m_l)$ *for all* $z \in (M_k, m_l)$*.*
(ii) *If* $\xi'(t) < 0$*, then either*
$M_k < m_l$*,* $\phi(M_k) = v(\xi(t),t) < \phi(M_l)$*, and* $\phi(z) < \phi(M_k)$ *for all* $z \in (M_k, m_l)$*,*
or,
$m_k > M_l$*,* $\phi(m_k) = v(\xi(t),t) > \phi(m_l)$*, and* $\phi(z) > \phi(m_k)$ *for all* $z \in (M_l, m_k)$*.*
(iii) *If* $\xi'(t) = 0$*, then* $v(\xi(t),t) \in \overline{(\phi(m_k), \phi(M_k)) \cap (\phi(m_l), \phi(M_l))}$*.*

---

[1] We are indebted to one of the referees for the formulation of Theorems 2.4-2.6.

Theorem 2.5. *Let assumption (H_1) hold and $(\xi_1(t),t) = (\xi_2(t),t)$ be a backward (k,l,m)-point $(k,l,m = 1,2,3)$. Then there exists $\epsilon \in (0,t)$ such that either*

$$\xi_1' > 0 \text{ on } (t-\epsilon,t), \text{ and } \xi_2' = 0 \text{ on } (t-\epsilon,t+\epsilon),$$

or,

$$\xi_1' = 0 \text{ on } (t-\epsilon,t+\epsilon), \text{ and } \xi_2' < 0 \text{ on } (t-\epsilon,t).$$

Theorem 2.6. *Let assumption (H_1) hold and $(\xi_1(t),t) = (\xi_2(t),t)$ be a forward (k,l,m)-point $(k,l,m = 1,2,3)$. Then there exists $\epsilon \in (0,t)$ such that either*

$$\xi_1' < 0 \text{ on } (t,t+\epsilon), \text{ and } \xi_2' = 0 \text{ on } (t-\epsilon,t+\epsilon),$$

or,

$$\xi_1' = 0 \text{ on } (t-\epsilon,t+\epsilon), \text{ and } \xi_2' > 0 \text{ on } (t,t+\epsilon).$$

3. The case (A_1).

3.1. **Results.** It can be expected from the above remarks that the local behaviour of an interface at a (k,l)-point is the opposite of that at an (l,k)-point – namely, if the contiguous phases are exchanged (in this respect, see Remark 1). Therefore, it is sufficient to describe the local behaviour at $(1,2)$, (2,3) and (1,3)-points, under assumptions (A_1)-(A_5) which determine the visibility of the stable branches. A first result of this kind is the content of the following theorem.

Theorem 3.1. *Let assumptions (H_1) and (A_1) hold, and $\bar{P} \equiv (\bar{x},\bar{t}) \in \gamma$, $\gamma = \{(\xi(t),t)\}$ being either interface of any three-phase solution (u,v,ξ_1,ξ_2) of problem (12). Then:*

(i) if $\bar{P}$ is a $(1,2)$-point of γ, there holds

$$\xi'(\bar{t}) \begin{cases} = 0 & \text{if } v(\bar{P}) \in (B,C]\,, \\ \geq 0 & \text{if } v(\bar{P}) = B\,; \end{cases} \tag{18}$$

(ii) if $\bar{P}$ is a (2,3)-point of γ, there holds

$$\xi'(\bar{t}) \begin{cases} = 0 & \text{if } v(\bar{P}) \in [B,C)\,, \\ \leq 0 & \text{if } v(\bar{P}) = C\,; \end{cases} \tag{19}$$

(iii) if $\bar{P}$ is a (1,3)-point of γ, there holds

$$\xi'(\bar{t}) \begin{cases} \geq 0 & \text{if } v(\bar{P}) = A\,, \\ = 0 & \text{if } v(\bar{P}) \in (A,D)\,, \\ \leq 0 & \text{if } v(\bar{P}) = D\,. \end{cases} \tag{20}$$

Remark 1. The same methods used in the proof of Theorem 3.1 show that under assumption (A_1):

(i) if $\bar{P}$ is a (2,1)-point of γ, then

$$\xi'(\bar{t}) \begin{cases} = 0 & \text{if } v(\bar{P}) \in (B,C]\,, \\ \leq 0 & \text{if } v(\bar{P}) = B\,; \end{cases}$$

(ii) if $\bar{P}$ is a $(3,2)$-point of γ, then

$$\xi'(\bar{t}) \begin{cases} = 0 & \text{if } v(\bar{P}) \in [B,C)\,, \\ \geq 0 & \text{if } v(\bar{P}) = C\,; \end{cases} \tag{21}$$

(iii) if $\bar{P}$ is a $(3,1)$-point of γ, then

$$\xi'(\bar{t}) \begin{cases} \leq 0 & \text{if } v(\bar{P}) = A\,, \\ = 0 & \text{if } v(\bar{P}) \in (A,D)\,, \\ \geq 0 & \text{if } v(\bar{P}) = D\,. \end{cases}$$

It follows from Theorem 3.1 and Remark 1 that the transitions

$$\text{phase 2} \rightsquigarrow \text{phase 1}, \qquad \text{phase 2} \rightsquigarrow \text{phase 3}$$

are allowed by the entropy conditions, whereas the opposite transitions are forbidden. The former only occur at $(1,2)$-points $\bar{P}$ of the interface where $v(\bar{P}) = B$, the latter at (2,3)-points where $v(\bar{P}) = C$. On the other hand, the transition

$$\text{phase 1} - \text{phase 3}$$

is allowed in both directions, and occurs at (1,3)-points where either $v(\bar{P}) = A$ or $v(\bar{P}) = D$.

Clearly, the situation depicted above depends on the shape of the graph of ϕ consistent with assumption (A_1), which suggests the existence of a unique hysteresis loop between the stable branches S_1 and S_3 (see Figure 2). This also suggests that:

- if the phase 2 disappears at some positive time, it will not reappear;
- if the phase 3 appears at some positive time, it will not disappear.

In fact, this is a consequence of the following theorem.

Theorem 3.2. *Let assumptions (H_1) and (A_1) hold.*
(i) Let (u, v, ξ_1, ξ_2) be a (1,2,3)-solution of problem (12). If $\xi_1(t^*) = \xi_2(t^*)$ for some $t^* \in [0, \infty)$, then $\xi_1(t) = \xi_2(t)$ for any $t \in (t^*, \infty)$.
(ii) Let (u, v, ξ_1, ξ_2) be any (1,3,2)-solution of problem (12). If $\xi_1(t^*) = \xi_2(t^*)$ for some $t^* \in (0, \infty)$, then $\xi_1(t) = \xi_2(t)$ for any $t \in [0, t^*)$.

By Theorem 3.2 we have the following result, which describes the structure of (1,2,3)- and (1,3,2)-solutions of problem (12) when (A_1) holds.

Theorem 3.3. *Let assumptions (H_1) and (A_1) hold.*
(i) Let (u, v, ξ_1, ξ_2) be a (1,2,3)-solution of problem (12). Then exactly one of the following possibilities occurs:

(a) there holds $\xi_1(t) < \xi_2(t)$ for any $t \in [0, \infty)$;
(b) there holds $\xi_1(t) = \xi_2(t)$ for any $t \in [0, \infty)$;
(c) there exists a unique backward (1,2,3)-point of the interface.

(ii) Let (u, v, ξ_1, ξ_2) be any (1,3,2)-solution of problem (12). Then exactly one of the following possibilities occurs:

(a) there holds $\xi_1(t) < \xi_2(t)$ for any $t \in [0, \infty)$;
(b) there holds $\xi_1(t) = \xi_2(t)$ for any $t \in [0, \infty)$;
(c) there exists a unique forward (1,3,2)-point of the interface.

Let us now describe the local structure of the interface of (1,2,3)- and (1,3,2)-solutions (similar results for other solutions are easily derived from Remark 1). In sufficiently small neighbourhoods of regular points the following holds (see Figure 3).

Theorem 3.4. *Let assumptions (H_1) and (A_1) hold.*
(i) Let $\{(\xi(t), t) \,|\, t \in [t_1, t_2] \subset [0, \infty)\} \subseteq \gamma$ separate the phases 1 and 2. Then the interval $[t_1, t_2]$ is the union of a finite number of subintervals $[t', t'']$ such that either $\xi'(t) = 0$ and $v(\xi(t), t) \in [B, C]$, or $\xi'(t) > 0$ and $v(\xi(t), t) = B$ for any $t \in (t', t'')$. Moreover, if $t' > 0$ there holds $\xi'(t') = \xi'(t'') = 0$, $v(\xi(t'), t') = v(\xi(t''), t'') = B$.
(ii) Let $\{(\xi(t), t) \,|\, t \in [t_1, t_2] \subset [0, \infty)\} \subseteq \gamma$ separate the phases 2 and 3. Then the interval $[t_1, t_2]$ is the union of a finite number of subintervals $[t', t'']$ such that either $\xi'(t) = 0$ and $v(\xi(t), t) \in [B, C]$, or $\xi'(t) < 0$ and $v(\xi(t), t) = C$ for any $t \in (t', t'')$. Moreover, if $t' > 0$ there holds $\xi'(t') = \xi'(t'') = 0$, $v(\xi(t'), t') = v(\xi(t''), t'') = C$.

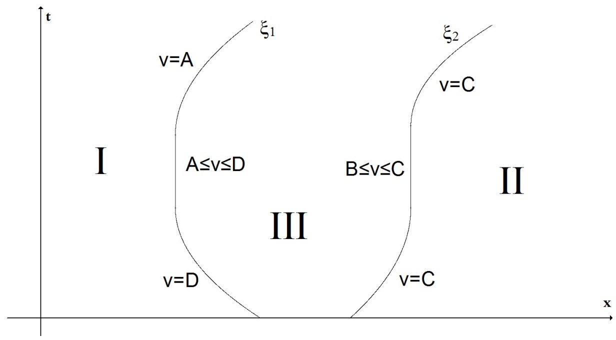

FIGURE 3. Assumption (A_1): (1,3,2)-solution without bifurcation.

(iii) Let $\{(\xi(t),t) \,|\, t \in [t_1,t_2]\} \subset [0,\infty)\} \subseteq \gamma$ *separate the phases 1 and 3. Then the interval* $[t_1,t_2]$ *is the union of a finite number of subintervals* $[t',t'']$ *such that either* $\xi'(t)=0$ *and* $v(\xi(t),t) \in [A,D]$*, or* $\xi'(t)<0$ *and* $v(\xi(t),t)=D$*, or* $\xi'(t)>0$ *and* $v(\xi(t),t)=A$ *for any* $t \in (t',t'')$*. Moreover, if* $t'>0$ *there holds* $\xi'(t')=\xi'(t'')=0$ *and either* $v(\xi(t'),t')=A$*,* $v(\xi(t''),t'')=D$*, or* $v(\xi(t'),t')=D$*,* $v(\xi(t''),t'')=A$*.*

Concerning small neighbourhoods of bifurcation points of the interface, we have the following result (see Figures 4 and 5).

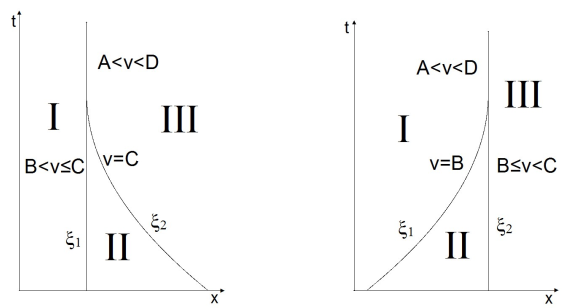

FIGURE 4. Assumption (A_1): Backward (1,2,3)-points.

Theorem 3.5. *Let assumptions* (H_1) *and* (A_1) *hold.*
(i) Let (u,v,ξ_1,ξ_2) *be a* (1,2,3)*-solution of problem* (12)*, and* $P^* \equiv (x^*,t^*)$ *a backward* (1,2,3)*-point of the interface. Then, either*

(α) $v(x^*,t^*)=B$ *and there exists* $\theta>0$ *such that* $\xi_1'(t)>0$*,* $\xi_2'(t)=0$*,* $v(\xi_1(t),t)=B$*,* $v(\xi_2(t),t) \in [B,C)$*, for any* $t \in (t^*-\theta,t^*)$*, or*

(β) $v(x^*,t^*)=C$ *and there exists* $\theta>0$ *such that* $\xi_1'(t)=0$*,* $\xi_2'(t)<0$*,* $v(\xi_1(t),t) \in (B,C]$*,* $v(\xi_2(t),t)=C$ *for any* $t \in (t^*-\theta,t^*)$*.*

Moreover, there holds $\xi_1(t)=\xi_2(t)=:\xi(t)$ *for any* $t \in [t^*,\infty)$ *and* $\xi'(t)=0$*,* $v(\xi(t),t) \in (A,D)$ *for any* $t \in (t^*,t^*+\theta)$*.*

(ii) Let (u,v,ξ_1,ξ_2) *be a* (1,3,2)*-solution of problem* (12)*, and* $P^* \equiv (x^*,t^*)$ *a forward* (1,3,2)*-point of the interface. Then* $v(x^*,t^*)=C$ *and there exists* $\theta>$

0 such that $\xi_1'(t) = 0$, $\xi_2'(t) > 0$, $v(\xi_1(t), t) \in (A, D)$, $v(\xi_2(t), t) = C$ *for any* $t \in (t^*, t^* + \theta)$.
Moreover, there holds $\xi_1(t) = \xi_2(t) =: \xi(t)$ *for any* $t \in [0, t^*]$ *and* $\xi'(t) = 0$, $v(\xi(t), t) \in (B, C]$ *for any* $t \in (t^* - \theta, t^*)$.

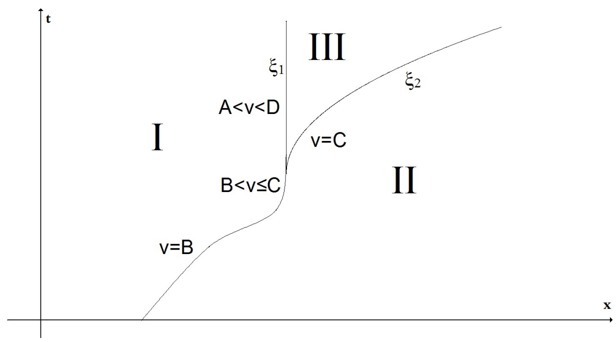

FIGURE 5. Assumption (A_1): Forward (1,3,2)-point.

3.2. **Proofs.** Without loss of generality we can assume $a < 0 < b$.

Proof of Theorem 3.1. We shall only prove claim (i), the proof of claims (ii) and (iii) being analogous.

We assume for convenience that the origin belongs to the first unstable branch S_{01} of the graph of ϕ, thus in particular $B < 0 < C$ (see Figure 6).

Let us prove claim (i). By assumption, there exist $\delta, \tau > 0$ such that

$$\bar{x} - \delta < \xi(t) < \bar{x} + \delta \quad \text{for any } t \in [\bar{t}, \bar{t} + \tau),$$

$$u(x,t) = s_1(v(x,t)) \quad \text{for any } x \in (\xi(t) - \delta, \xi(t)), \; t \in [\bar{t}, \bar{t} + \tau), \qquad (22)$$
$$u(x,t) = s_2(v(x,t)) \quad \text{for any } x \in (\xi(t), \xi(t) + \delta), \; t \in [\bar{t}, \bar{t} + \tau).$$

Since $v \in C(\bar{V})$ (see Definition 2.1-(i)), v is continuous across the interface γ. Then we have

$$v(\xi(t), t) \in [B, C] \quad \text{for any } t \in [\bar{t}, \bar{t} + \tau) \qquad (23)$$

(see Figure 2). To prove (18), let us show that:
(α) there holds $\xi'(\bar{t}) \geq 0$;
(β) if $v(\bar{P}) \in (B, C]$, then $\xi'(\bar{t}) = 0$.
(α) Set

$$\rho_0 := \frac{D - C}{2}$$

(observe that $\rho_0 > 0$ by assumption (A_1)). Since $v \in C(\bar{V})$ and $v(\bar{x}, \bar{t}) \leq C$ by (23), there exist $\delta_1 \in (0, \delta)$, $\tau_1 \in (0, \tau)$ such that

$$v(x,t) \leq C + \rho_0 \qquad (24)$$

for any $(x,t) \in [\bar{x} - \delta_1, \bar{x} + \delta_1] \times [\bar{t}, \bar{t} + \tau_1]$. Observe that

$$C + \rho_0 = \frac{C + D}{2} \in (C, D).$$

Suppose, contrary to the assertion, that $\xi'(\bar t) < 0$. Then there exists $\theta \in (0, \tau_1)$ such that

$$\xi(t) < \xi(\bar t) = \bar x \quad \text{for any } \; t \in (\bar t, \bar t + \theta] \,.$$

By equalities (22), this implies that there exists $\eta \in (0, \delta_1)$ such that

$$u(x, \bar t) = s_1(v(x, \bar t)) \,, \quad u(x, \bar t + \theta) = s_2(v(x, \bar t + \theta)) \quad \text{for any } \; x \in [\bar x - \eta, \bar x) \,. \tag{25}$$

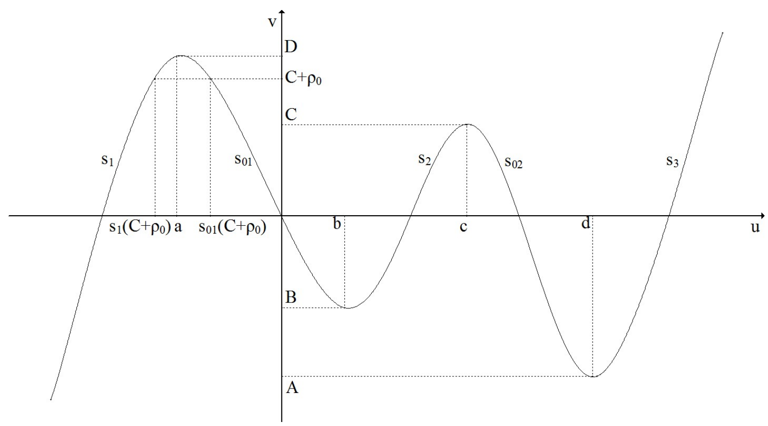

FIGURE 6. Proof of Theorem 3.1: Claim (i)-(α).

Now fix any $g_{\rho_0} \in C^1(\mathbb{R})$, $g'_{\rho_0} \geq 0$ such that

$$g_{\rho_0} \begin{cases} = 0 & \text{in } \; (-\infty, C + \rho_0] \,, \\ > 0 & \text{in } \; (C + \rho_0, \infty) \,, \end{cases} \tag{26}$$

and denote by G_{ρ_0} the function (11) with $g = g_{\rho_0}$ and $k = 0$. Then the entropy inequality (17) with $t_1 = \bar t$, $t_2 = \bar t + \theta$, $g = g_{\rho_0}$ and $\psi = \zeta \in C^1_c((\bar x - \eta, \bar x))$, $\zeta \geq 0$ reads

$$\begin{aligned} &\int_{\bar x - \eta}^{\bar x} G_{\rho_0}(u)(x, \bar t + \theta) \zeta(x) \, dx - \int_{\bar x - \eta}^{\bar x} G_{\rho_0}(u)(x, \bar t) \zeta(x) \, dx \leq \\ &\leq - \int_{\bar t}^{\bar t + \theta} \int_{\bar x - \eta}^{\bar x} g_{\rho_0}(v)(x,t) v_x(x,t) \zeta'(x) \, dx dt \,. \end{aligned} \tag{27}$$

Since

$$[\bar x - \eta, \bar x] \times [\bar t, \bar t + \theta] \subseteq [\bar x - \delta_1, \bar x + \delta_1] \times [\bar t, \bar t + \tau] \,,$$

by (24) and (26) there holds

$$\int_{\bar t}^{\bar t + \theta} \int_{\bar x - \eta}^{\bar x} \underbrace{g_{\rho_0}(v)(x,t)}_{=0} v_x(x,t) \zeta'(x) \, dx ds = 0 \,. \tag{28}$$

On the other hand, by equalities (25), the definition of G_{ρ_0} and (26), for any ζ as above we have

$$\int_{\bar{x}-\eta}^{\bar{x}} G_{\rho_0}(u)(x,\bar{t}+\theta)\zeta(x)\,dx = \\ = \int_{\bar{x}-\eta}^{\bar{x}} \underbrace{\left(\int_0^{s_2(v(x,\bar{t}+\theta))} g_{\rho_0}(\phi)(z)\,dz\right)}_{=0} \zeta(x)\,dx = 0\,, \tag{29}$$

$$\int_{\bar{x}-\eta}^{\bar{x}} G_{\rho_0}(u)(x,\bar{t})\zeta(x)\,dx = \\ = \int_{\bar{x}-\eta}^{\bar{x}} \left(\int_0^{s_1(v(x,\bar{t}))} g_{\rho_0}(\phi)(z)\,dz\right) \zeta(x)\,dx \leq \\ \leq \int_{\bar{x}-\eta}^{\bar{x}} \left(\int_{s_{01}(C+\rho_0)}^{s_1(C+\rho_0)} g_{\rho_0}(\phi)(z)\,dz\right) \zeta(x)\,dx \ \leq 0\,. \tag{30}$$

In fact, for any $x \in [\bar{x}-\eta, \bar{x}]$ there holds

$$0 < b \leq s_2(v(x,\bar{t}+\theta)) \leq c\,,$$

hence by (26)

$$B \leq \phi(z) \leq C \ \Rightarrow\ g_{\rho_0}(\phi)(z) = 0 \quad \text{for any } z \in (0, s_2(v(x,\bar{t}+\theta)))\ .$$

Moreover, by inequality (24) for any $x \in [\bar{x}-\eta, \bar{x}]$ there holds

$$s_1(v(x,\bar{t})) \leq s_1(C+\rho_0) \leq s_{01}(C+\rho_0) < 0$$

(where S_{01} denotes the first unstable branch of the graph of ϕ; see Figure 6), and by (26)

$$C+\rho_0 < \phi(z) \leq D \ \Rightarrow\ g_{\rho_0}(\phi)(z) > 0 \quad \text{for any } z \in (s_1(C+\rho_0), s_{01}(C+\rho_0))\ .$$

By the arbitrariness of the test function ζ, from (27)-(30) we obtain

$$0 = G_{\rho_0}(u)(x,\bar{t}+\theta) \leq G_{\rho_0}(u)(x,\bar{t}) \leq \int_{s_{01}(C+\rho_0)}^{s_1(C+\rho_0)} g_{\rho_0}(\phi)(z)\,dz < 0$$

for any $x \in (\bar{x}-\eta, \bar{x})$, a contradiction. This proves that $\xi'(\bar{t}) \geq 0$, as asserted.

(β) Let us now prove that $\xi'(\bar{t}) = 0$ if $v(\bar{P}) \in (B, C]$. Since $v \in C(\bar{V})$ and by assumption $v(\bar{P}) > B$, there exist $\delta_1 \in (0,\delta)$, $\tau_1 \in (0,\tau)$ such that

$$v(x,t) \geq B^* - \rho_1 \tag{31}$$

for any $(x,t) \in [\bar{x}-\delta_1, \bar{x}+\delta_1] \times [\bar{t}, \bar{t}+\tau_1]$ and some $B^* \in (B, 0)$; here

$$\rho_1 := \frac{B^*-B}{2} \ > \ 0\,.$$

Observe that

$$B^* - \rho_1 = \frac{B^*+B}{2} \ \in\ (B, 0)$$

(recall that by assumption there holds $B < 0$; see Figure 7).

To argue by reduction to an absurdity, suppose that $\xi'(\bar{t}) > 0$. Arguing as in (α) above, we see that there exist $\eta \in (0,\delta_1)$, $\theta \in (0,\tau_1)$ such that

$$u(x,\bar{t}+\theta) = s_1(v(x,\bar{t}+\theta))\,, \quad u(x,\bar{t}) = s_2(v(x,\bar{t})) \quad \text{for any } x \in (\bar{x}, \bar{x}+\eta]\,. \tag{32}$$

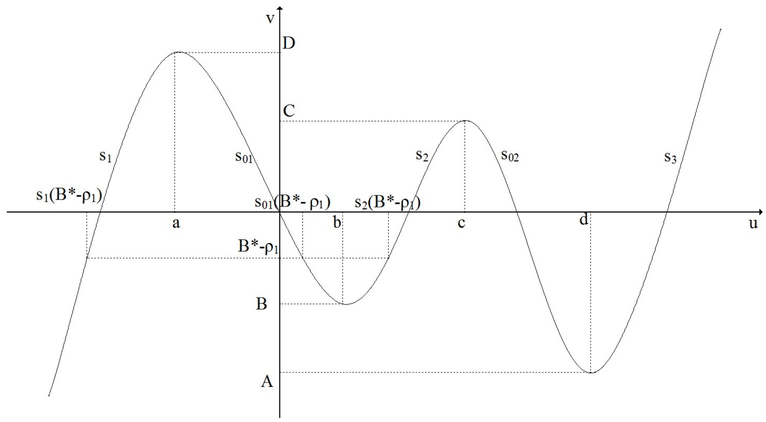

FIGURE 7. Proof of Theorem 3.1: Claim (i)-(β).

Fix any $g_{\rho_1} \in C^1(\mathbb{R})$, $g'_{\rho_1} \geq 0$ such that

$$g_{\rho_1} \begin{cases} < 0 & \text{in } (-\infty, B^* - \rho_1)\,, \\ = 0 & \text{in } [B^* - \rho_1, \infty)\,, \end{cases} \tag{33}$$

and denote by G_{ρ_1} the function (11) with $g = g_{\rho_1}$ and $k = 0$. Now from the entropy inequality (17) we obtain

$$\begin{aligned} &\int_{\bar{x}}^{\bar{x}+\eta} G_{\rho_1}(u)(x, \bar{t} + \theta)\zeta(x)\, dx - \int_{\bar{x}}^{\bar{x}+\eta} G_{\rho_1}(u)(x, \bar{t})\zeta(x)\, dx \leq \\ &\leq - \int_{\bar{t}}^{\bar{t}+\theta} \int_{\bar{x}}^{\bar{x}+\eta} g_{\rho_1}(v)(x,t) v_x(x,t)\zeta'(x)\, dxdt \end{aligned} \tag{34}$$

for any $\zeta \in C^1_c((\bar{x}, \bar{x} + \eta))$, $\zeta \geq 0$. By (31) and (33) we have

$$\int_{\bar{t}}^{\bar{t}+\theta} \int_{\bar{x}}^{\bar{x}+\eta} \underbrace{g_{\rho_1}(v)(x,t)}_{=0} v_x(x,t)\zeta'(x)\, dxds = 0\,. \tag{35}$$

On the other hand, by (32) and (33) for any ζ as above there holds

$$\begin{aligned} &\int_{\bar{x}}^{\bar{x}+\eta} G_{\rho_1}(u)(x, \bar{t} + \theta)\zeta(x)\, dx = \\ &= \int_{\bar{x}}^{\bar{x}+\eta} \underbrace{\left(\int_0^{s_1(v(x,\bar{t}+\theta))} g_{\rho_1}(\phi)(z)\, dz \right)}_{=0} \zeta(x)\, dx = 0\,, \end{aligned} \tag{36}$$

$$\begin{aligned} &\int_{\bar{x}}^{\bar{x}+\eta} G_{\rho_1}(u)(x, \bar{t})\zeta(x)\, dx = \\ &= \int_{\bar{x}}^{\bar{x}+\eta} \left(\int_0^{s_2(v(x,\bar{t}))} g_{\rho_1}(\phi)(z)\, dz \right) \zeta(x)\, dx \leq \\ &\leq \int_{\bar{x}}^{\bar{x}+\eta} \left(\int_{s_{01}(B^*-\rho_1)}^{s_2(B^*-\rho_1)} g_{\rho_1}(\phi)(z)\, dz \right) \zeta(x)\, dx \leq 0\,. \end{aligned} \tag{37}$$

In fact, by inequality (31) there holds

$$s_1(B^* - \rho_1) \leq s_1(v(x,\bar{t}+\theta)) < 0\,,$$
$$0 < s_{01}(B^* - \rho_1) \leq s_2(B^* - \rho_1) \leq s_2(v(x,\bar{t}))$$

for any $x \in [\bar{x}, \bar{x}+\eta]$, and by (33)

$$B^* - \rho_1 \leq \phi(z) \leq D \;\Rightarrow\; g_{\rho_1}(\phi)(z) = 0 \quad \text{for any } z \in (s_1(v(x,\bar{t}+\theta)), 0)\;,$$
$$B \leq \phi(z) < B^* - \rho_1 \;\Rightarrow\; g_{\rho_1}(\phi)(z) < 0 \quad \text{for any } z \in (s_{01}(B^* - \rho_1), s_2(B^* - \rho_1))$$

(see Figure 7). By the arbitrariness of ζ, from (34)-(37) we obtain

$$0 = G_{\rho_1}(u)(x,\bar{t}+\theta) \leq G_{\rho_1}(u)(x,\bar{t}) \leq \int_{s_{01}(B^*-\rho_1)}^{s_2(B^*-\rho_1)} g_{\rho_1}(\phi)(z)\,dz < 0$$

for any $x \in (\bar{x}-\eta, \bar{x})$, a contradiction again. This proves that $\xi'(\bar{t}) = 0$ if $v(\bar{P}) \in (B, C]$, hence claim (i) follows. This completes the proof. □

Proof of Theorem 3.2. We shall only prove claim (i); the proof of claim (ii) is similar, thus we leave it to the reader.

Let $\xi_1(t^*) = \xi_2(t^*)$ for some $t^* \in [0,\infty)$. By the continuity of ξ_1, ξ_2 the claim will follow, if we show that no forward (1,2,3)-point $\tilde{P} \equiv (\tilde{x}, \tilde{t})$ with $\tilde{t} \in (t^*, \infty)$ exists.

Suppose, contrary to the assertion, that such a point exists. Then there exists $\hat{t} \in (\tilde{t}, \infty)$ such that for any $t \in (\tilde{t}, \hat{t})$ (see Definitions 2.2-2.3):

(a) there holds $\xi_1(t) < \xi_2(t)$;

(b) $(\xi_1(t), t)$ is a $(1,2)$-point of γ_1;

(c) $(\xi_2(t), t)$ is a (2,3)-point of γ_2.

Since $\xi_1(\tilde{t}) = \xi_2(\tilde{t}) = \tilde{x}$ (see Definition 2.3), by elementary results there exist $\bar{t}_1, \bar{t}_2 \in (\tilde{t}, \hat{t})$ such that

$$\frac{\xi_i\left(\hat{t}\right) - \tilde{x}}{\hat{t} - \tilde{t}} = \xi_i'\left(\bar{t}_i\right) \qquad (i = 1, 2)\,. \tag{38}$$

Suppose $\xi_1\left(\hat{t}\right) < \tilde{x}$. Then equality (38) with $i = 1$ implies $\xi_1'\left(\bar{t}_1\right) < 0$, which by (18) contradicts property (b) above (observe that by the continuity of v, ξ_1, ξ_2 there holds $v(\xi_i(t), t) \in [B, C]$ for any $t \in (\tilde{t}, \hat{t})$; $i = 1, 2$). On the other hand, let $\xi_1\left(\hat{t}\right) \geq \tilde{x}$. Then by the above property (a) there holds $\xi_2\left(\hat{t}\right) > \tilde{x}$, whence $\xi_2'\left(\bar{t}_2\right) > 0$ by equality (38) with $i = 2$. However, by (19) this contradicts property (c) above. Therefore we have reached a contradiction, whence claim (i) follows. This completes the proof. □

Proof of Theorem 3.4. We only prove claim (i), the proof of the other claims being similar. By Definition 2.2 every point of the arc $\{(\xi(t), t) \,|\, t \in [t_1, t_2]\}$ is a $(1,2)$-point, hence by (18) there holds $\xi'(t) \geq 0$ for any $t \in [t_1, t_2]$. On the other hand, by Definition 2.1-(ii) in the interval $[t_1, t_2]$ there is only a finite number of subintervals where $\xi' > 0$. Then by (18) the conclusion follows. □

Proof of Theorem 3.5. Again we limit ourselves to prove claim (i). By assumption, there exists $\hat{t} \in (0, t^*)$ such that for any $t \in (\hat{t}, t^*)$:

(a) there holds $\xi_1(t) < \xi_2(t)$;

(b) $(\xi_1(t), t)$ is a $(1,2)$-point of γ_1;

(c) $(\xi_2(t), t)$ is a (2,3)-point of γ_2.

By the continuity of v there holds $v(\xi_i(t), t) \in [B, C]$ for any $t \in (\hat{t}, t^*)$ $(i = 1, 2)$, and $v(x^*, t^*) \in [B, C]$.

Were $v(x^*,t^*) \in (B,C)$, by the equality $\xi_1(t^*) = \xi_2(t^*) = x^*$ and the continuity of v, ξ_1, ξ_2 there would exist $\bar{t} \in (\hat{t}, t^*)$ such that $v(\xi_i(t),t) \in (B,C)$ $(i = 1,2)$ for any $t \in (\bar{t}, t^*)$. However, by the properties (b)-(c) above and (18)-(19) this would imply that $\xi_1'(t) = \xi_2'(t) = 0$, thus $\xi_1(t) = \xi_2(t) = x^*$ for any $t \in (\bar{t}, t^*)$, which contradicts property (a) above. Therefore, either $v(x^*,t^*) = B$ or $v(x^*,t^*) = C$ holds true.

Let $v(x^*,t^*) = B$. Then by continuity there exists $\tilde{t} \in (\hat{t}, t^*)$ such that $v(\xi_i(t),t) \in [B,C)$ $(i = 1,2)$, thus by (18)-(19) $\xi_1'(t) \geq 0$, $\xi_2'(t) = 0$ for any $t \in (\tilde{t}, t^*)$. On the other hand, in the interval $(\tilde{t}, t^*)$ there is a finite number of subintervals (t', t) where $\xi_1' > 0$ (see Definition 2.1-(ii)). Therefore there exists $\theta > 0$ such that $t^* - \theta > \tilde{t}$, and either $\xi_1'(t) = 0$, or $\xi_1'(t) > 0$ for any $t \in (t^* - \theta, t^*)$. However, the former possibility can be ruled out, since it would imply that $\xi_1(t) = \xi_2(t) = x^*$ for any $t \in (t^* - \theta, t^*)$, which again contradicts property (a). Therefore there holds $\xi_1'(t) > 0$, hence $v(\xi_1(t),t) = B$ by (18) for any $t \in (t^* - \theta, t^*)$.

This establishes the possibility (α); the proof of (β) is similar, hence we omit it. Finally, observe that in both cases there holds $v(x^*,t^*) \in [B,C] \subset (A,D)$. Hence by the continuity of v, possibly choosing θ smaller we have $v(\xi_1(t),t) = v(\xi_2(t),t) \in (A,D)$, whence by (20) we have $\xi_1'(t) = \xi_2'(t) = 0$ for any $t \in (t^*, t^* + \theta)$. Then by Theorem 3.2-$(i)$ claim (i) follows. This completes the proof. □

4. Different assumptions on ϕ.

4.1. The case (A_2). In this subsection we suppose that the function ϕ, instead of (A_1), satisfies the assumption (see Figure 8):

$$(A_2) \qquad B < A < C < D\,.$$

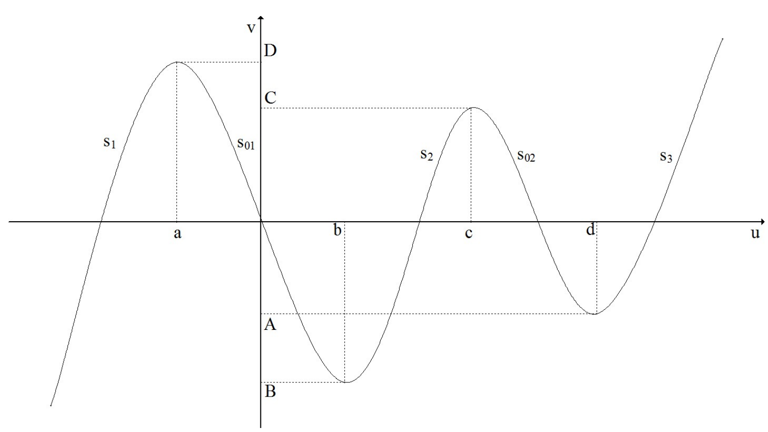

FIGURE 8. Assumption (A_2).

In this case the branch S_3 is visible from the extremum (c,C) and S_2 is visible from the extremum (d,A). Therefore, the transition

$$\text{phase 2} \; - \; \text{phase 3}$$

is allowed in both directions, thus a hysteresis loop between the branches S_1 and S_3 arises. Clearly, the above transition occurs at (2,3)-points $\bar{P}$ where either $v(\bar{P}) = A$, or $v(\bar{P}) = C$.

On the other hand,
- the branch S_1 is visible, whereas S_3 is not visible from the extremum (b, B);
- the branch S_3 is visible, whereas S_2 is not visible from the extremum (a, D).

Therefore, the transitions

$$\text{phase } 2 \rightsquigarrow \text{phase } 1\,, \qquad \text{phase } 1 \rightsquigarrow \text{phase } 3$$

are allowed only in one direction.

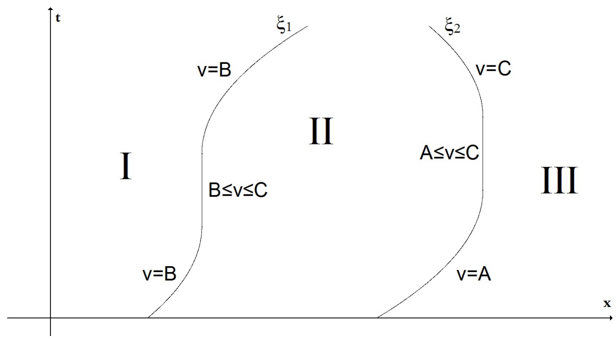

FIGURE 9. Assumption (A_2): (1,2,3)-solution without bifurcation.

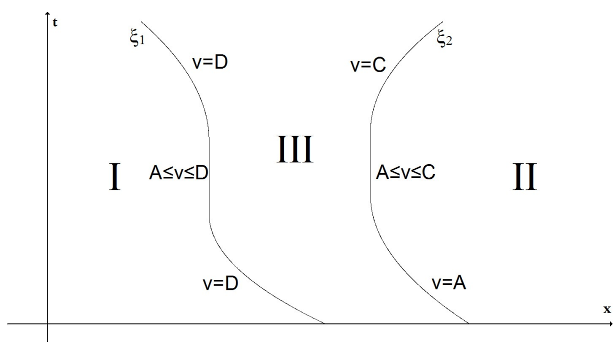

FIGURE 10. Assumption (A_2): (1,3,2)-solution without bifurcation.

The global structure of the interfaces is more complicated in the present case than in case (A_1), since both phases 2 and 3 can disappear and appear again. Therefore, both backward and forward (1,2,3)- and (1,3,2)-points of the interface can exist, at variance with Theorems 3.2-3.3 above. On the other hand, the local behaviour of the interfaces can be described as in Theorems 3.4 and 3.5. The behaviour of interfaces at regular points is qualitatively depicted in Figures 9 and 10, that in a small neighbourhood of a bifurcation point is represented in Figures 11 and 12.

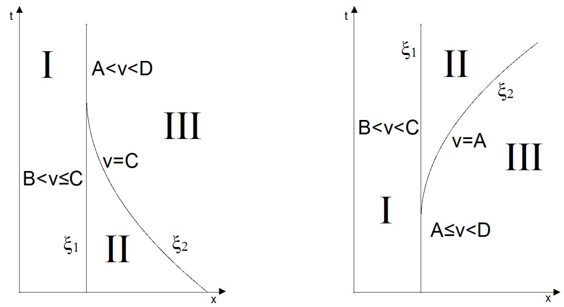

FIGURE 11. Assumption (A_2): (1,2,3)-points.

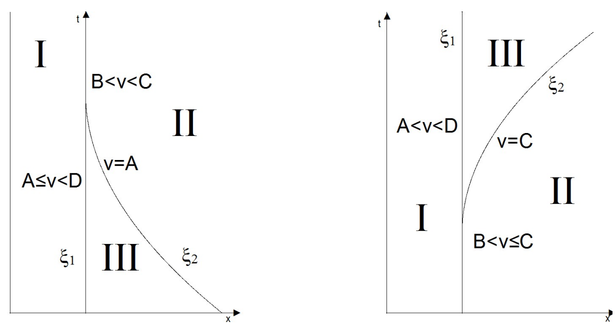

FIGURE 12. Assumption (A_2): (1,3,2)-points.

4.2. **The case** (A_3)**.** Consider the case (see Figure 13):

$$(A_3) \qquad B < A < D < C\,.$$

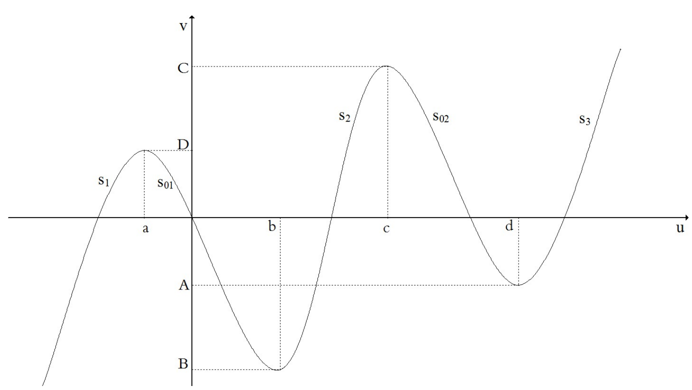

FIGURE 13. Assumption (A_3)

In this case:
- the branch S_1 is only visible from the extremum (b, B);

- the branch S_2 is visible from both extrema (a, D) and (d, A);
- the branch S_3 is only visible from the extremum (c, C).

Therefore, the transitions

$$\text{phase } 1 \;-\; \text{phase } 2\,, \qquad \text{phase } 2 \;-\; \text{phase } 3$$

are allowed in both directions, whereas the transition

$$\text{phase } 1 \;-\; \text{phase } 3$$

is forbidden in both directions. It is easily seen that

- if the phase 2 appears at some positive time, it will not disappear;
- if the phase 3 disappears at some positive time, it will not reappear.

Comparing the above result with Theorem 3.2 shows that in the present case the behaviour of (1,2,3)-solutions is the same as that of (1,3,2)-solutions in the case (A_1), and conversely.

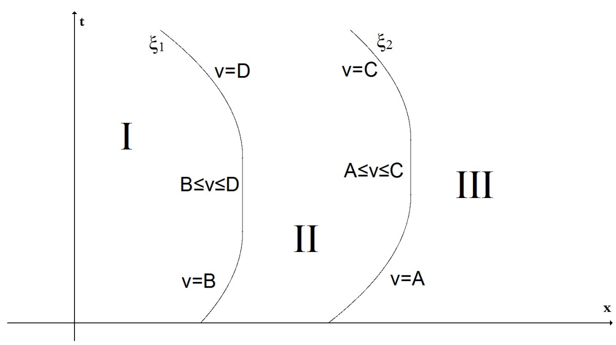

FIGURE 14. Assumption (A_3): (1,2,3)-solution without bifurcation.

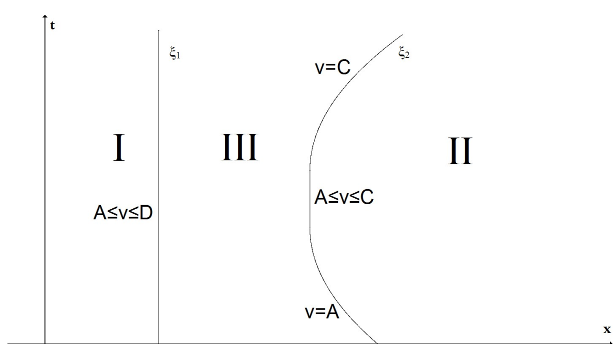

FIGURE 15. Assumption (A_3): (1,3,2)-solution without bifurcation.

On the other hand, since no transition between phases 1 and 3 is allowed, if (u, v, ξ_1, ξ_2) is a (1,3,2)-solution and $\xi_1(t) < \xi_2(t)$ for any $t \in [0, \infty)$, there holds $\xi_1'(t) = 0$ for any $t \in [0, \infty)$ (see Figures 14 and 15). The local structure of the interface near bifurcation points is represented in Figures 16 and 17.

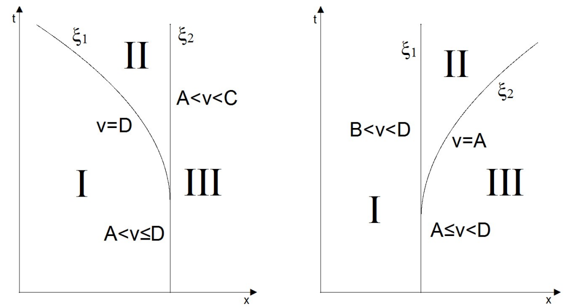

FIGURE 16. Assumption (A_3): Forward (1,2,3)-points.

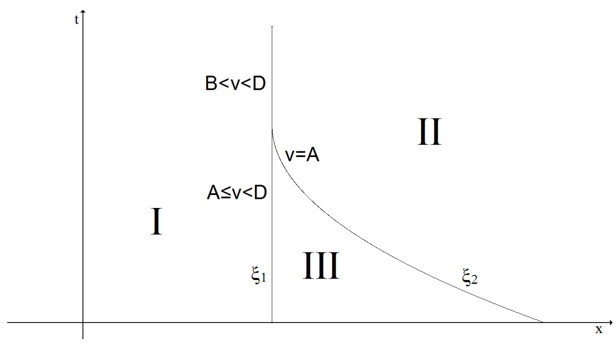

FIGURE 17. Assumption (A_3): Backward (1,3,2)-points.

4.3. **The case** (A_4)**.** The case

$$(A_4) \qquad A < B < D < C$$

(see Figure 18) can be reduced to case (A_2) by a suitable transformation.

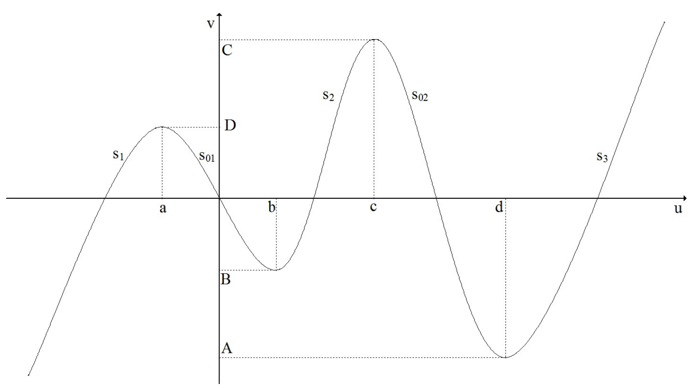

FIGURE 18. Assumption (A_4).

In fact, consider the function (see Figure 19):

$$\phi^*(u) := -\phi(-u) \qquad (u \in \mathbb{R})\,. \tag{39}$$

Now the previous definitions $A := \phi(d)$, $B := \phi(b)$, $C := \phi(c)$ and $D := \phi(a)$ suggest to define

$$a^* := -d\,,\ \ b^* := -c\,,\ \ c^* := -b\,,\ \ d^* := -a\,,$$

$$A^* := \phi^*(d^*) = -D\,,\ \ B^* := \phi^*(b^*) = -C\,,$$
$$C^* := \phi^*(c^*) = -B\,,\ \ D^* := \phi^*(a^*) = -A\,.$$

Then (A_4) reads

$$B^* < A^* < C^* < D^*\,,$$

namely assumption (A_2) with B replaced by B^*, and so on. Also observe the following correspondence between stable branches of the graphs of ϕ and ϕ^*:

$$S_1 \ \longleftrightarrow\ S_3^*\,, \qquad S_2 \ \longleftrightarrow\ S_2^*\,, \qquad S_3 \ \longleftrightarrow\ S_1^*\,. \tag{40}$$

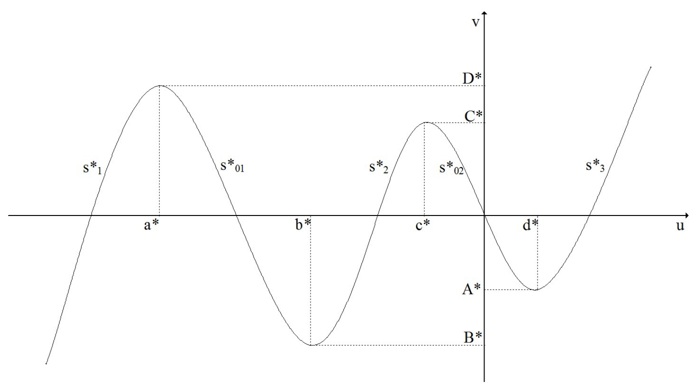

FIGURE 19. The function ϕ^*.

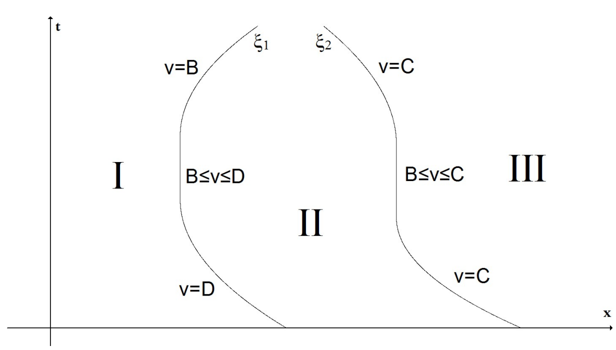

FIGURE 20. Assumption (A_4): (1,2,3)-solution without bifurcation.

On the other hand, if (u, v, ξ_1, ξ_2) is any three-phase solution of problem (12), by setting

$$u^*(x,t) := -u(x,t)\,,\ \ v^*(x,t) := -v(x,t)\,, \tag{41}$$
$$\xi_1^*(t) := \xi_1(t)\,,\ \ \xi_2^*(t) := \xi_2(t) \qquad ((x,t) \in V)$$

we obtain a three-phase solution of the new Cauchy problem:

$$\begin{cases} w_t = [\phi^*(w)]_{xx} & \text{in } S \\ w = w_0 := -u_0 & \text{in } \mathbb{R} \times \{0\} . \end{cases}$$

In fact, there holds $u^* \in L^\infty(V)$, $v^* \in C(\bar{V}) \cap L^2((0,T); H^1_{loc}(\mathbb{R}))$, and

$$v^*(x,t) = -v(x,t) = -\phi(u(x,t)) = -\phi(-u^*(x,t)) = \phi^*(u^*(x,t)) \qquad ((x,t) \in V)$$

(see (39) and (41)); hence equalities (16) and the entropy inequality (17) (with u and v replaced by u^* and v^*, respectively) hold true. Observe that under the transformation (40) a (2,1)-point of any interface γ of (u, v, ξ_1, ξ_2) corresponds to a $(2^*, 3^*)$-point of γ^*, γ^* denoting the corresponding interface of $(u^*, v^*, \xi_1^*, \xi_2^*)$. Similarly, (1,3)-points and (2,3)-points of γ correspond to $(3^*,1^*)$-points and $(2^*,1^*)$-points of γ^*, respectively.

By the above considerations, the results concerning the present case are easily derived from those proven in case (A_2). They are depicted in Figures 20 (for $(1, 2, 3)$-solutions; the situation is similar for $(1, 3, 2)$-solutions) and 21.

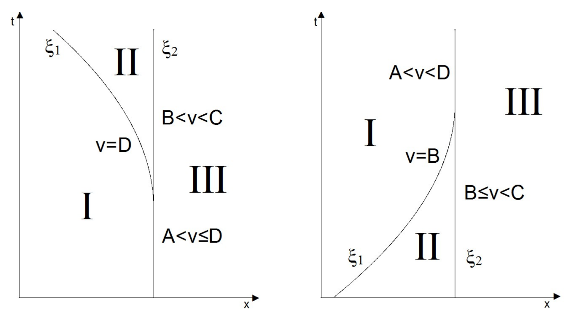

FIGURE 21. Assumption (A_4): (1,2,3)-points.

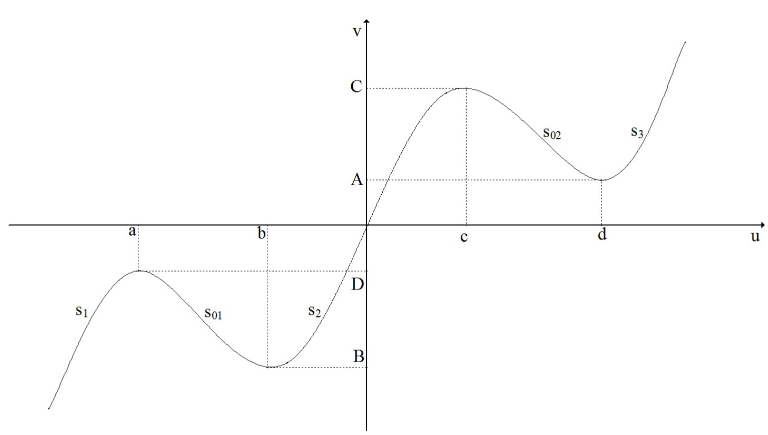

FIGURE 22. Assumption (A_5).

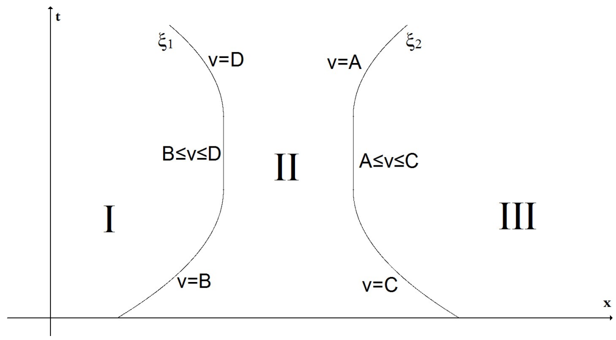

FIGURE 23. Assumption (A_5): (1,2,3)-solution.

4.4. **The case** (A_5)**.** Now assume (see Figure 22):

$$(A_5) \qquad B < D < A < C\,.$$

In this case the branch S_1 is visible only from the extremum (b, B) and S_3 only from the extremum (c, C). On the other hand, the branch S_2 is visible from both extrema (a, D) and (d, A). Therefore, the transitions

$$\text{phase } 1 \;-\; \text{phase } 2\,, \qquad \text{phase } 2 \;-\; \text{phase } 3$$

are allowed in both directions, whereas the transition

$$\text{phase } 1 \;-\; \text{phase } 3$$

is forbidden in both directions. Therefore only (1,2,3)- and (3,2,1)-solutions exist, and no bifurcation of the interfaces arises. The situation arising in this simple case is represented in Figure 23.

Acknowledgments. The authors would like to thank the referees for their useful comments and suggestions, which greatly helped improving the original manuscript.

REFERENCES

[1] G. I. Barenblatt, M. Bertsch, R. Dal Passo and M. Ughi, *A degenerate pseudoparabolic regularization of a nonlinear forward-backward heat equation arising in the theory of heat and mass exchange in stably stratified turbulent shear flow*, SIAM J. Math. Anal., **24** (1993), 1414–1439.

[2] M. Brokate and J. Sprekels, "Hysteresis and Phase Transitions," Appl. Math. Sci. **121**, Springer-Verlag, New York, 1996.

[3] L. T. T. Bui, F. Smarrazzo and A. Tesei, forthcoming.

[4] L. C. Evans and M. Portilheiro, *Irreversibility and hysteresis for a forward-backward diffusion equation*, Math. Mod. Meth. Appl. Sci., **14** (2004), 1599–1620.

[5] O. A. Ladyzhenskaja, V. A. Solonnikov and N. N. Ural'ceva, "Linear and Quasi-linear Equations of Parabolic Type," Amer. Math. Soc., Providence, 1991.

[6] C. Mascia, A. Terracina and A. Tesei, *Evolution of stable phases in forward-backward parabolic equations*, in: "Asymptotic Analysis and Singularities" (edited by H. Kozono, T. Ogawa, K. Tanaka, Y. Tsutsumi and E. Yanagida), 451–478, Advanced Studies in Pure Mathematics, **47-2** (Math. Soc. Japan, 2007).

[7] C. Mascia, A. Terracina and A. Tesei, *Two-phase entropy solutions of a forward-backward parabolic equation*, Arch. Rational Mech. Anal., **194** (2009), 887–925.

[8] A. Novick-Cohen and R. L. Pego, *Stable patterns in a viscous diffusion equation*, Trans. Amer. Math. Soc., **324** (1991), 331–351.

[9] V. Padrón, *Sobolev regularization of a nonlinear ill-posed parabolic problem as a model for aggregating populations*, Comm. Partial Differential Equations, **23** (1998), 457–486.

[10] P. Perona and J. Malik, *Scale space and edge detection using anisotropic diffusion*, IEEE Trans. Pattern Anal. Mach. Intell., **12** (1990), 629–639.

[11] P. I. Plotnikov, *Passing to the limit with respect to viscosity in an equation with variable parabolicity direction*, Diff. Equ., **30** (1994), 614–622.

[12] D. Serre, "Systems of Conservation Laws, Vol. 1: Hyperbolicity, Entropies, Shock Waves," Cambridge University Press, Cambridge, 1999.

[13] F. Smarrazzo and A. Terracina, *Local existence and uniqueness of two-phase entropy solutions*, Discrete Contin. Dyn. Syst. (to appear).

[14] A. Terracina, *Qualitative behaviour of the two-phase entropy solution of a forward-backward parabolic equation*, SIAM J. Math. Anal., **43** (2011), 228–252.

[15] M. Valadier, *A course on Young measures*, Rend. Ist. Mat. Univ. Trieste, **26** (1995), 349–394.

Received March 2012; revised October 2012.

E-mail address: flavia.smarrazzo@gmail.com
E-mail address: tesei@mat.uniroma1.it