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ON A CLASS OF REVERSIBLE ELLIPTIC SYSTEMS

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Abstract. The existence of periodic and spatially heteroclinic solutions is studied for a class of semilinear elliptic partial differential equations.

1. Introduction.

Consider the family of semilinear elliptic systems of the form

$$-\Delta u + F_u(x,u) = 0, \qquad (PDE)$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$, and F satisfies

(F_1) $F \in C^2(\mathbb{R}^{n+m}/\mathbb{Z}^{n+m}, \mathbb{R})$

i.e. F is C^2 and $1-$periodic in each of its arguments. In $(PDE), F_u \equiv (\frac{\partial F}{\partial u_1}, \cdots, \frac{\partial F}{\partial u_m})$. When $n = 1$ and $m > 1$, (PDE) is a special case of the dynamical systems studied in Aubry-Mather Theory [1, 2, 3]. On the other hand, for $m = 1$ and $n \geq 1$, it is a special case of the class of problems initiated by Moser [4], and furthered by Bangert [5] and others [6] towards the development of a version of Aubry-Mather Theory for $PDEs$. Also for $m = 1$ and $n \geq 1, (PDE)$ arises in studying Allen-Cahn models of phase transitions by reducing the Allen-Cahn model equation to one of the form of (PDE) [6]. Some other work on (PDE) and Allen-Cahn models can be found e.g. in [7]–[16].

In this note, we begin a study of the extent to which the results obtained for the earlier cases persist for the system (PDE). Variational methods, especially minimization arguments, played a major role in the treatment of the one dimensional and one codimensional cases. E.g. for $m = 1$, a large number of locally minimal heteroclinic and homoclinic solutions have been obtained for (PDE) in [6]. Variational structure persists here. However the maximum principle, which among other tools, led to certain ordered sets of heteroclinic solutions when $m = 1$, is no longer available in general when $m > 1$. Therefore we will aim for the sort of results for (PDE) that are known when $n = 1$ and $m \geq 2$. These results are of two types. First is the existence of what we will call, *basic solutions*. Second, there are more complex solutions that are obtained by, roughly speaking, gluing together basic solutions. In this note, the focus will be on obtaining the basic solutions; the glued solutions will be treated in the future.

To be more precise, in §2, we begin by finding solutions of (PDE) that are periodic in $x_1, \cdots, x_n$. Set $|\nabla u|^2 = \Sigma_1^m |\nabla u_i|^2$ and $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. The periodics will

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be obtained by minimizing

$$J_0(u) = \int_{\mathbb{T}^n} (\frac{1}{2}|\nabla u|^2 + F(x,u))\,dx$$

over the class of functions that are 1-periodic in $x_1, \cdots, x_n$. Letting $\mathcal{M}_0$ denote the resulting set of minimizers, we can describe what we mean by *basic solutions.* Namely they are solutions of (PDE) that are heteroclinic in some x_i, say x_1, between distinct members of $\mathcal{M}_0$. One cannot expect to find such solutions even for $n = 1$ unless $\mathcal{M}_0$ is not too degenerate and more is required of F. Generically, $\mathcal{M}_0 = \{v + k \mid k \in \mathbb{Z}^m\}$ for any $v \in \mathcal{M}_0$. Thus a simple and convenient nondegeneracy condition to require is that $\mathcal{M}_0/\mathbb{Z}^m$ is finite, i.e. there are only finitely many functions $w = (w_1, \cdots, w_m) \in \mathcal{M}_0$ such that $[w] = ([w_1], \cdots, [w_m]) \in [0,1]^m$ where

$$[w_i] = \int_{[0,1]^n} w_i\,dx.$$

The further condition we impose on F is

(F_2) F is even in x_i, $1 \le i \le n$,

i.e. spatial reversibility for F. This is the condition that was useful when $n = 1$ in [20] and [24].

The properties of $\mathcal{M}_0$ will be studied under these hypotheses in §2. A unique continuation result for (PDE) will also be obtained. Then in §3, (F_2) will be used as in [6],[18] [24] to define a renormalized functional whose properties will be developed. These properties enable us to formulate a minimization problem to find heteroclinic solutions of (PDE) joining v to $\mathcal{M}_0 \setminus \{v\}$ for any $v \in \mathcal{M}_0$. Lastly in §4, we will show that for any $v, w \in \mathcal{M}_0$, there is a minimal heteroclinic chain of solutions of (PDE) joining v and w. This is an analogue of a result of Maxwell, [26], for $n = 1$.

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2. **Periodic solutions of** (PDE)**.** In this section, first assuming (F_1) and then (F_2), the existence and properties of periodic solutions of (PDE) that minimize J_0 will be studied.

Let $E_0 = W^{1,2}(\mathbb{T}^n, \mathbb{R}^m)$ under the norm

$$\|u\|^2 = \int_{\mathbb{T}^n} (|\nabla u|^2 + |u|^2)\,dx$$

where $u = (u_1, \cdots, u_m)$. For $u \in E_0$, set $L(u) = \frac{1}{2}|\nabla u|^2 + F(x,u)$. Define

$$J_0(u) = \int_{\mathbb{T}^n} L(u)\,dx$$

and let

$$c_0 = \inf_{u \in E_0} J_0(u) \tag{1}$$

Then we have:

Theorem 2.1. *If F satisfies (F_1),*

$$\mathcal{M}_0 = \{u \in E_0 \mid J_0(u) = c_0\} \neq \emptyset$$

and any $u \in \mathcal{M}_0$ is a classical solution of (PDE).

Proof. The existence of minimizers is almost exactly as e.g. in [4, 6]. If $u \in E_0$, so is $u+k$ for all $k \in \mathbb{Z}^m$ and $J_0(u+k) = J_0(u)$. Thus if (u_p) is a minimizing sequence for (1), and $u_p = (u_{p1}, \cdots, u_{pm})$, it can be assumed that $[u_p] \in [0,1]^m$. This normalization and the form of J_0 implies (u_p) is bounded in E_0 and due to (F_1), the functional, J_0, is weakly lower semicontinuous. Therefore along a subsequence (u_p) converges weakly in E_0 to a minimizer, u, of J_0 on E_0. This immediately implies that u is a weak solution of (PDE):

$$\int_{\mathbb{T}^n} (\nabla u \cdot \nabla \varphi + F_u(x,u) \cdot \varphi)\, dx = 0$$

for all $\varphi \in E_0$. Then (F_1) and standard regularity arguments [17] show u is a classical solution of (PDE). □

Remark 1. (i) If $f \in C^1(\mathbb{T}^n, \mathbb{R}^m)$ and $[f] = 0$, then for all $k \in \mathbb{Z}^m$,

$$\int_{\mathbb{T}^n} f(x) \cdot (u+k)\, dx = \int_{\mathbb{T}^n} f(x) \cdot u\, dx + (\int_{\mathbb{T}^n} f(x)\, dx) \cdot k = \int_{\mathbb{T}^n} f(x) \cdot u\, dx.$$

Hence the argument of Theorem 2.1 also yields periodic solutions of the system: $-\Delta u + F_u(x,u) = f(x)$. (ii) If $F = F(u)$, each critical point of $F|_{\mathbb{T}^m}$ is a solution of (PDE). Since there are at least $m+1$ such critical points, (PDE) has at least $m+1$ constant solutions. (iii) As was mentioned earlier, generically $\mathcal{M}_0 = \{v + k \mid k \in \mathbb{Z}^m\}$ for any $v \in \mathcal{M}_0$. See e.g. [6] for details.

Next some compactness properties of $\mathcal{M}_0$ and J_0 will be studied. They are analogues of related results in [6, 18, 19]. Let

$$\hat{\mathcal{M}} = \{u \in \mathcal{M}_0 \mid [u] \in [0,1]^m\}$$

so $\mathcal{M}_0 = \hat{\mathcal{M}} + \mathbb{Z}^m$. For $A \subset E_0$, set $N_\rho(A) = \{u \in E_0 \mid \|u - A\| \le \rho\}$.

Proposition 1. *If F satisfies (F_1), then*

1^o *$\hat{\mathcal{M}}$ is compact in E_0.*
2^o *Any minimizing sequence, (u_k), for (1) with $[u_k] \in [0,1]^m$ for $k \in \mathbb{N}$ has a convergent subsequence.*
3^o *For any $\rho > 0$, there is a $\beta = \beta(\rho) > 0$ such that if $u \in E_0 \setminus N_\rho(\mathcal{M}_0)$, then $J_0(u) - c_0 \ge \beta$.*

Proof. If $u \in \hat{\mathcal{M}}$, there is a constant $K_1 > 0$ such that $\|u\| \le K_1$. Then for any $\alpha \in (0,1)$, elliptic regularity theory gives a constant, $K_2 = K_2(\alpha)$ so that $\|u\|_{C^{2,\alpha}(\mathbb{T}^n)} \le K_2$, from which 1^o follows. Next let (u_k) be as in 2^o. The proof of Theorem 2.1 shows (u_k) is bounded in E_0. Therefore there is a $v \in E_0$ such that, along a subsequence, u_k converges to v weakly in E_0 and strongly in $L^2(\mathbb{T}^n, \mathbb{R}^m)$. If $u_k \not\to v$ along a subsequence in E_0, there is a $\delta > 0$ such that $\varphi_k = u_k - v$ satisfies $\|\nabla \varphi_k\|_{L^2(\mathbb{T}^n, \mathbb{R}^m)} \ge \delta$ and $\varphi_k \to 0$ weakly in E_0. Hence

$$J_0(u_k) = J_0(v + \varphi_k) = \int_{\mathbb{T}^n} (\frac{1}{2}|\nabla v|^2 + \nabla v \cdot \nabla \varphi_k + \frac{1}{2}|\nabla \varphi_k|^2 + F(x, v + \varphi_k))\, dx \quad (2)$$

$$= c_0 + \int_{\mathbb{T}^n} (\frac{1}{2}|\nabla \varphi_k|^2 + \nabla v \cdot \nabla \varphi_k + F(x, v+\varphi_k) - F(x,v))\, dx$$

$$\ge c_0 + \frac{\delta^2}{2} + \int_{\mathbb{T}^n} (\nabla v \cdot \nabla \varphi_k + F(x, v+\varphi_k) - F(x,v))\, dx$$

Letting $k \to \infty$, (2) shows

$$c_0 = \lim_{k\to\infty} J_0(u_k) \geq c_0 + \frac{\delta^2}{2},$$

a contradiction. Hence 2^o is satisfied. Lastly if 3^o does not hold, there is a sequence $(u_p) \subset E_0 \setminus N_\rho(\mathcal{M}_0)$ such that $J_0(u_p) \to c_0$ as $p \to \infty$. Without loss of generality, we can assume $(u_p) \subset \hat{\mathcal{M}}$. Therefore (u_p) is a minimizing sequence for (1) so by 2^o, there is a $v \in \mathcal{M}_0$ such that $u_p \to v$ in E_0 along a subsequence as $p \to \infty$. But then $u_p \in N_\rho(\mathcal{M}_0)$ for large p in the subsequence, a contradiction. □

Remark 2. For 2^o, that $[u_k] \in [0,1]^m$ for all $k \in \mathbb{N}$ was only used to prove that (u_k) was bounded in E_0. Thus 2^o is also true whenever this boundedness holds. The conclusion of 2^o is false for an arbitrary minimizing sequence. E.g. if $v \in \hat{\mathcal{M}}$, take $u_k = v + p_k$ where p_k is an unbounded sequence in $\mathbb{Z}^m$.

Next the effect of (F_2) will be studied. See also [20, 18, 19] for similar results and arguments. Suppose that F satisfies (F_1)-(F_2). Consider

$$I(u) = \int_{[0,\frac{1}{2}]^n} L(u)\,dx \tag{3}$$

on the space $W^{1,2}([0,\frac{1}{2}]^n, \mathbb{R}^m)$. The argument of Theorem 2.1 gives a minimizer, w, of (3) which is a classical solution of (PDE). Extend w as an even function, first about $x_1 = \frac{1}{2}$, then about $x_2 = \frac{1}{2}$, etc., to obtain $\hat{w}$, on $[0,1]^n$. Further extend $\hat{w}$ as a $1-$periodic function in $x_i, 1 \leq i \leq n$, producing $W \in E_0$. Therefore $J_0(W) \geq c_0$. By (F_2) and the construction of $W, J_0(W) = 2^n I(w)$. Let Q be any sub-n-cube of $[0,1]^n$ of the form $\Pi_1^n[a_i, b_i]$ where $[a_i, b_i] = [0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$ for each i. For any such $Q, \int_Q L(W)\,dx = I(w)$ and W minimizes $\int_Q L(u)\,dx$ for $u \in W^{1,2}(Q, \mathbb{R}^m)$. Hence $J_0(W) \leq c_0$ so we have equality: $J(W) = c_0$ and $W \in \mathcal{M}_0$. Moreover we have found $W \in \mathcal{M}_0$ which is even in $x_i, 1 \leq i \leq n$. In fact we have:

Proposition 2. *If F satisfies (F_1)-(F_2), any $u \in \mathcal{M}_0$ is a minimizer of I on $W^{1,2}([0,\frac{1}{2}]^n, \mathbb{R}^m)$ and u is even in $x_i, 1 \leq i \leq n$.*

Proof. The first statement follows from the above paragraph. Set $w = u|_{[0,\frac{1}{2}]^n}$ and take W as above. The function $U = (U_1, \cdots, U_m) = u - W \in E_0,\ U \equiv 0$ in $[0,\frac{1}{2}]^n$, and U satisfies

$$-\Delta U = -F_u(x,u) + F_u(x,W) = \int_0^1 \frac{d}{ds} F_u(x, u+sU)\,ds \tag{4}$$
$$= \int_0^1 \Sigma_k F_{uu_k}(x, u+sU) U_k\,ds \equiv M(x)U$$

where the matrix, $M(x)$, is continuous and $1-$periodic in $x_1, \cdots, x_n$.

Now Proposition 2 follows immediately from:

Proposition 3. *The system (4) has the unique continuation property, i.e. if U is a classical solution of (4) and vanishes on an open set, then $U \equiv 0$.*

The Proposition will be justified at the end of this section. An immediate consequence of Proposition 2 and the remarks preceding it is:

Corollary 1. $c_0 = \inf_{u \in W^{1,2}([0,1]^n, \mathbb{R}^m)} J_0(u)$

This characterization of c_0 will be useful in §3.

Now a few further observations about properties of (PDE) which are analogues of those known for $n = 1$ or $m = 1$ will be made. Let $k \in \mathbb{N}^n$ and consider the question of the existence of solutions of (PDE) that are k_1-periodic in $x_1, \cdots, k_n-$periodic in x_n. Setting up a corresponding minimization problem for (1), the argument of Theorem 2.1 shows it possesses a family of minimizers that we denote by $\mathcal{M}_0(k)$ with a minimization value, $c_0(k)$. Then we have:

Corollary 2. *$\mathcal{M}_0(k) = \mathcal{M}_0$ and $c_0(k) = (\Pi_1^n k_i)c_0$.*

Proof. Any $u \in \mathcal{M}_0$ is admissible for the $'k'$ problem so $(\Pi_1^n k_i)c_0 \geq c_0(k)$. On the other hand, decomposing $\Pi[0, k_i]$ into $2^n \Pi k_i$ sets of the form of the sub-n-cube Q employed earlier, and using the construction of W and Proposition 2 shows $c_0(k) = (\Pi k_i)c_0$ and $\mathcal{M}_0(k) = \mathcal{M}_0$. □

Next set $\hat{x} = (x_2, \cdots, x_n)$.

Corollary 3. *Let $v, w \in \mathcal{M}_0$ with $v \not\equiv w$. Then $v(0, \hat{x}) \not\equiv w(0, \hat{x})$.*

Proof. If $v(0, \hat{x}) \equiv w(0, \hat{x}) = w(1, \hat{x})$, glue v to w at $x_1 = 1$ and extend the resulting function, u, $2-$periodically in x_1. Then $u \in \mathcal{M}_0(k)$ with $k = (2, 1, \cdots, 1)$. But by Corollary 2, $\mathcal{M}_0(k) = \mathcal{M}_0$ and then Proposition 3 implies $u \equiv v \equiv w$. □

As in Moser [4], a solution of (PDE) will be called *minimal* if for all $\varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^m)$,

$$\int_{\mathbb{R}^n} (L(u + \varphi) - L(U))\, dx \geq 0.$$

Following e.g. arguments of [6] and using Corollary 2, any $u \in \mathcal{M}_0$ is a minimal solution of (PDE).

To conclude this section, we will prove Proposition 3. The only result in the literature that we know of that contains it can be found in [21]. This much more general result relies strongly on a Carleman inequality from a reference that is not readily available. Therefore we prefer to give a more elementary argument.

Proof of Proposition 3: By (4), U satisfies

$$|\Delta U_i(x)| \leq M_1 |U(x)|, \quad 1 \leq i \leq m, \quad x \in \mathbb{R}^n \tag{5}$$

where the constant,M_1, depends on $\|F\|_{C^2(\mathbb{R}^n, \mathbb{R}^m)}$ and $U \equiv 0$ in $[0, \frac{1}{2}]^n$. Let $B_r(a)$ denote the open ball in $\mathbb{R}^n$ of radius r about $x = a$. Translating variables, it suffices to show that if U satisfies (5) and $U \equiv 0$ near 0, then $U \equiv 0$ in $B_{\frac{1}{2}}(0)$. We employ an inequality of Protter [22] which in turn is a simplification of one of Pederson [23].

Suppose $0 < R < 1$ and $v \in C^2(\mathbb{R}^n)$ with $v \equiv 0$ for x near 0 and for $|x| \geq R$. Then Protter showed there is a constant $\gamma = \gamma(R) > 0$ so that for all such v and sufficiently large β,

$$\gamma \beta^4 \int_{B_R(0)} r^{-2\beta-2} \exp(2r^{-\beta}) v^2\, dx \leq \int_{B_R(0)} r^{\beta+2} \exp(2r^{-\beta}) |\Delta v|^2\, dx \tag{6}$$

where $r = |x|$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi \equiv 1$ for $|x| \leq \frac{R}{2}$. Note that *supp* $\varphi \subset B_R(0)$. Applying (6) to φU_i and summing over i gives

$$\gamma \beta^4 \int_{B_R(0)} r^{-2\beta-2} \exp(2r^{-\beta}) \varphi^2 |U|^2\, dx \leq \int_{\{r \geq \frac{R}{2}\}} r^{\beta+2} \exp(2r^{-\beta}) \Sigma_1^m |\Delta(\varphi U_i)|^2\, dx \tag{7}$$

$$+\int_{\{r\leq\frac{R}{2}\}} r^{\beta+2}\exp(2r^{-\beta})\Sigma_1^m|\Delta U_i|^2\,dx$$

Substituting (5) into (7) yields

$$\int_{\{r\leq\frac{R}{2}\}}(\gamma r^{-2\beta-2}-\beta^{-4}M_1r^{\beta+2})\exp(2r^{-\beta})|U|^2\,dx \qquad (8)$$
$$\leq\int_{\{r\geq\frac{R}{2}\}}\beta^{-4}r^{\beta+2}\exp(2r^{-\beta})\Sigma_1^m|\Delta(\varphi U_i)|^2\,dx$$

Since $R<1$ and U vanishes near 0, for large β, (8) gives

$$\int_{\{r\leq\frac{R}{2}\}}\gamma r^{-2\beta-2}\exp(2r^{-\beta})|U|^2\,dx\leq\int_{\{r\geq\frac{R}{2}\}}2\beta^{-4}r^{\beta+2}\exp(2r^{-\beta})\Sigma_1^m|\Delta(\varphi U_i)|^2\,dx. \qquad (9)$$

Observing that the exponential term is monotone decreasing in R shows

$$\gamma\exp(2(\frac{R}{2})^{-\beta})\int_{\{r\leq\frac{R}{2}\}}r^{-2\beta-2}|U|^2\,dx$$
$$\leq 2\beta^{-4}\exp(2(\frac{R}{2})^{-\beta})\int_{\{r\geq\frac{R}{2}\}}r^{\beta+2}\Sigma_1^m|\Delta(\varphi U_i)|^2\,dx$$

or

$$\gamma\int_{\{r\leq\frac{R}{2}\}}r^{-2\beta-2}|U|^2\,dx\leq 2\beta^{-4}\int_{\{r\geq\frac{R}{2}\}}r^{\beta+2}\Sigma_1^m|\Delta(\varphi U_i)|^2\,dx. \qquad (10)$$

Letting $\beta\to\infty$, this shows $U\equiv 0$ in $B_{\frac{R}{2}}(0)$ for all $R<1$ and the proof is complete. □

3. **Heteroclinic solutions.** This section treats the existence of solutions of (PDE) that are heteroclinic between members of $\mathcal{M}_0$. For simplicity, only the case of heteroclinics in x_1 will be studied. Of course x_1 can be replaced by $x_i, 1\leq i\leq n$, and we suspect that as for $m=1$ in Chapter 5 of [6], a much more general case can be handled in the same fashion.

The natural functionals to work with for the existence of heteroclinics will generally be infinite on the class of admissible functions, so as in earlier research, a renormalized functional will be introduced. The approach here mainly exploits ideas from [24], [18], [19], and [6].

The space employed here is $E_1=W^{1,2}_{loc}(\mathbb{R}\times\mathbb{T}^{n-1},\mathbb{R}^m)$, i.e. $u\in E_1$ means $u\in W^{1,2}(D,\mathbb{R}^m)$ for all bounded $D\subset\mathbb{R}\times\mathbb{T}^{n-1}$. Also set $\hat{E}=W^{1,2}_{loc}(\mathbb{R}^n,\mathbb{R}^m)$. For each $p\in\mathbb{Z}$, let $T_p=[p,p+1]\times\mathbb{T}^{n-1}$ and for $u\in\hat{E}$, define

$$J_{1,p}(u)=\int_{T_p}L(u)\,dx-c_0.$$

By Proposition 2,

$$J_{1,p}(u)\geq 0,\quad u\in\hat{E}. \qquad (11)$$

As renormalized functional, we take

$$J_1(u)=\Sigma_{p\in\mathbb{Z}}J_{1,p}(u).$$

Therefore $J_1(u)\geq 0$ for $u\in\hat{E}$.

To find heteroclinic solutions of (PDE), we must require that $\mathcal{M}_0$ is not too degenerate. For $m=1$, Moser [4] showed that $\mathcal{M}_0$ is an ordered set and there are heteroclinics between v and $w \in \mathcal{M}_0$ whenever v, w is a *gap pair*, i.e. there are no other members of $\mathcal{M}_0$ between v and w. Working with such a gap pair serves as a nondegeneracy condition for the setting of [4]. Due to the lack of the Maximum Principle for (PDE), the current situation does not permit such a simple geometrical condition. Although one can proceed much more generally, for simplicity here we take as our nondegeneracy condition:

$$(N_0): \ \mathcal{M}_0/\mathbb{Z}^m \ \ is \ \ a \ \ finite \ set$$

i.e. $\hat{\mathcal{M}}$ is a finite set. When $m = 1$, some more degenerate situations were treated in [25]. A simple example of when (N_0) is satisfied occurs when $n = m = 1$ and $F(x, u) = a(x)(1 - \cos(\pi u))$. For this case, $\mathcal{M}_0 = \mathbb{Z}$.

To formulate our class of admissible functions, let $v \in \mathcal{M}_0$ and set

$$A(v) = \{u \in E_1 \mid \|u - v\|_{L^2(T_i)} \to 0,\ i \to -\infty; \|u - \mathcal{M}_0 \setminus \{v\}\|_{L^2(T_i)} \to 0, i \to \infty\}$$

Note that by (N_0),

$$r = \inf_{v \in \mathcal{M}_0} \|v - \mathcal{M}_0 \setminus \{v\}\|_{L^2(T_0)} > 0. \tag{12}$$

Now define

$$c_1(v) = \inf_{u \in A(v)} J_1(u) \tag{13}$$

Then we have:

Theorem 3.1. *Suppose F satisfies $(F_1) - (F_2), (N_0)$ holds, and $v \in \mathcal{M}_0$. Then*

1^o $\mathcal{M}_1(v) \equiv \{u \in E_1 \mid J_1(u) = c_1(v)\} \neq \emptyset$.

2^o *Any $U \in \mathcal{M}_1(v)$ is a classical solution of (PDE) and is even in $x_2, \cdots, x_n$.*

3^o $\|U - v\|_{C^2(T_p)} \to 0,\ p \to -\infty$.

4^o *There is a $w \in \mathcal{M}_0 \setminus v$ such that $\|U - w\|_{C^2(T_p)} \to 0, p \to \infty$.*

Proof. Let (u_k) be a minimizing sequence for (13). Therefore there is an $M > 0$ such that

$$J_1(u_k) \leq M, \ \ k \in \mathbb{N}. \tag{14}$$

Let $e_1 = (1, 0, \cdots, 0) \in \mathbb{R}^n$. Observing that $J_1(u_k) = J_1(u_k(\cdot - qe_1))$ and $u_k(\cdot - qe_1) \in A(v)$, the minimizing sequence can be normalized by requiring that

$$\|u_k - v\|_{L^2(T_p)} \leq r(v)/3,\ p < 0; \ \ \|u_k - v\|_{L^2(T_0)} \geq r(v)/3. \tag{15}$$

We claim (u_k) is bounded in E_1. Indeed observe first that by (14) and the form of J_1, $\|\nabla u_k\|^2_{L^2(T_i)} \leq 2(M + c_0 + \|F\|_{C^2(\mathbb{R}^n, \mathbb{R}^m)})$ for all $i \in \mathbb{Z}$. By (15), $\|u_k\|_{L^2(T_i)} \leq \|v\|_{L^2(T_0)} + r(v)/3$ for all $i < 0$, while for $i \geq 0$,

$$\|u_k\|^2_{L^2(T_i)} \leq 2(\|u_k\|^2_{L^2(T_{-1})} + (i + 2)\Sigma^i_{-1} \|u_{k,x_1}\|^2_{L^2(T_i)}).$$

Thus (u_k) is bounded in E_1. Therefore there is a $U \in E_1$ such that, along a subsequence, $u_k \to U$, weakly in E_1 and strongly in $L^2_{loc}(\mathbb{R} \times \mathbb{T}^{n-1}, \mathbb{R}^m)$ as $k \to \infty$. In fact, an argument almost exactly as in Proposition 3.4 of [19] shows that along our subsequence, for each $i \in \mathbb{Z}$, $u_k \to U$ in $W^{1,2}(T_i)$ as $k \to \infty$.

Using this stronger convergence, we will show (a) U is a solution of (PDE), (b) $U \in A(v)$, (c) $J_1(u) = c_1(v)$, (d) the C^2 asymptotics, and lastly, (e) the evenness in $x_2, \cdots, x_n$. Let $\varphi \in C^\infty_0(\mathbb{R} \times \mathbb{T}^{n-1}, \mathbb{R}^m)$. Then $u_k + t\varphi \in A(v)$ for all $t \in \mathbb{R}$. Writing $J_1(u_k) = c_1(v) + \epsilon_k$, where $\epsilon_k \to 0$ as $k \to \infty$, we have

$$c_1(v) \le J_1(u_k) = c_1(v) + \epsilon_k \le J_1(u_k + t\varphi) + \epsilon_k \tag{16}$$

for all $t \in \mathbb{R}$. Suppose the support of φ, $supp\ \varphi, \subset [p, q+1] \times \mathbb{T}^{n-1}$. Then (16) implies

$$\Sigma_p^q J_{1,i}(u_k) \le \Sigma_p^q J_{1,i}(u_k + t\varphi) + \epsilon_k$$

or

$$0 \le \int_{supp\ \varphi} (L(u_k + t\varphi) - L(u_k))\, dx + \epsilon_k. \tag{17}$$

Since

$$|F(x,z) - F(x,y)| \le \|F_u\|_{L^\infty(\mathbb{R}^n,\mathbb{R}^m)} |z - y|,$$

using (17) and the $W^{1,2}_{loc}$ convergence shows:

$$0 \le \int_{supp\ \varphi} (L(U + t\varphi) - L(U))\, dx \tag{18}$$

Taking $t > 0$ and letting $t \to 0$, (18) and the fact that φ is arbitrary show

$$\int_{\mathbb{R}\times\mathbb{T}^{n-1}} (\nabla U \cdot \nabla\varphi + F_u(x,U)\cdot\varphi)\, dx = 0. \tag{19}$$

Thus U is a weak solution of (PDE) and as earlier is a classical solution, i.e. 2^o of the Theorem holds.

Next we will show that $U \in A(v)$. Due to the weak lower semicontinuity of $\int_\Omega \cdot\, dx$ on bounded sets, for any $p \le q \in \mathbb{Z}$,

$$\Sigma_p^q J_{1,i}(U) \le \liminf_{k\to\infty} \Sigma_p^q J_{1,i}(u_k) \le \liminf J_1(u_k) = c_1(v).$$

Letting $p \to -\infty, q \to \infty$ shows

$$J_1(U) \le c_1(v). \tag{20}$$

Let $\tau_i^j u(x) = u(x - ie_j)$ where e_j is the unit vector in $\mathbb{R}^n$ in the 'j-th' direction. Therefore $J_{1,i}(U) = J_0(\tau_{-i}^1 U) - c_0 \to 0$ as $|i| \to \infty$. This and Corollary 1 show $(\tau_i^1 U)$ is a minimizing sequence for (1) as $|i| \to \infty$. We claim this sequence is bounded in E_0. For $i < 0$, this follows via (14) and (20). Hence by Proposition 1 and Remark 2, there is a $W^- \in E_0$ such that along a subsequence $\tau_i^1 U \to W^-$ in $W^{1,2}(T_0)$ as $i \to -\infty$ and $J_0(W^-) = c_0$. Moreover (14) and (N_0) imply the entire sequence converges and $W^- = v$.

For $i > 0$, establishing the boundedness of $(\tau_i^1 U)$ requires a different argument. Since $J_{1,i}(U) \to 0$ as $i \to \infty$, by Proposition 1, $\|\tau_i^1 U - \mathcal{M}_0\| \to 0$. Therefore, by (N_0), for each large i, there is a unique $v_i \in \mathcal{M}_0$ such that $\|\tau_i^1 U - \mathcal{M}_0\| = \|\tau_i^1 U - v_i\|$. We claim that there is a $p \in \mathbb{N}$ such that for all $i \ge p$, $v_i = v_p$. It then follows that $\tau_i^1 U \to v_p$ as $i \to \infty$. To verify the claim, we rewrite the expression for $J_1(U)$ in two different ways. Set

$$\hat{J}(U) = \Sigma_{-\infty}^{-1} J_{1,i}(U) + \Sigma_0^\infty [\int_{T_{2i}\cup T_{2i+1}} L(U)\, dx - 2c_0]$$

and

$$\tilde{J}(U) = \Sigma_{-\infty}^{0} J_{1,i}(U) + \Sigma_0^\infty [\int_{T_{2i+1}\bigcup T_{2i+2}} L(U)\, dx - 2c_0].$$

Then $J_1(U) = \hat{J}(U) = \tilde{J}(U)$ and the proof of Proposition 1 carries over to the current setting with $\mathcal{M}_0$ replaced by $\mathcal{M}_0(2,1,\cdots,1)$, J_0 by $\int_{T_0\bigcup T_1} L(U)\, dx$ or

$\int_{T_1 \bigcup T_2} L(U)\ dx$ and c_0 by $c_0(2,1,\cdots,1)$. Consequently by this observation and Corollary 2, for large i, there is a unique $\hat{v}_i \in \mathcal{M}_0$ such that $\|\hat{v}_i - U\|_{W^{1,2}(T_{2i}\cup T_{2i+1})} < r/3$. Therefore $v_{2i} = \hat{v}_i = v_{2i+1}$. Similarly there is a unique $\tilde{v}_i \in \mathcal{M}_0$ such that $\|\tilde{v}_i - U\|_{W^{1,2}(T_{2i+1}\cup T_{2i+2})} < r/3$. Hence $v_{2i+1} = \tilde{v}_i = v_{2i+2}$. It follows that there is a $p \in \mathbb{N}$ such that $v_i = v_p$ for all large i and $\tau_i^1 U \to v_p$ as $i \to \infty$.

Next it will be shown that $v_p \not\equiv v$. Suppose to the contrary that $v_p \equiv v$. Let $\epsilon > 0$. Since the minimizing sequence u_k converges to U in $W^{1,2}_{loc}(\mathbb{R}\times\mathbb{T}^{n-1},\mathbb{R}^m)$ (along a subsequence), there is a $q = q(\epsilon) \in \mathbb{N}$ such that for all large k in the subsequence,

$$\|u_k - v\|_{W^{1,2}(T_q)} \le \epsilon \tag{21}$$

and $q(\epsilon) \to \infty$ as $\epsilon \to 0$. Modify (u_k) obtaining a new sequence, (u_k^*) via

$$\begin{aligned} u_k^* &= v, \quad x_1 \le q, \\ &= (q+1-x_1)v + (x_1 - q)u_k, \quad q \le x_1 \le q+1, \\ &= u_k, \quad q+1 \le x_1. \end{aligned}$$

Then by (21), there exists a $\kappa(\epsilon) \to 0$ as $\epsilon \to 0$ such that

$$|J_{1,q}(u_k^*)| \le \kappa(\epsilon). \tag{22}$$

Observe that $u_k^* \in A(v)$. Consequently

$$c_1(v) - (c_1(v) + \epsilon_k) \le J_1(u_k^*) - J_1(u_k) = J_{1,q}(u_k^*) - \Sigma_{-\infty}^{q} J_{1,i}(u_k)$$

or for any $i \le q$,

$$J_{1,i}(u_k) \le \kappa(\epsilon) + \epsilon_k. \tag{23}$$

Letting $k \to \infty$ along our $W^{1,2}_{loc}$ convergent subsequence, (23) yields

$$J_{1,i}(U) \le \kappa(\epsilon) \tag{24}$$

for all $i \le q(\epsilon)$. Thus letting $\epsilon \to 0$ shows $J_{1,i}(U) = 0$ for all $i \in \mathbb{Z}$. Hence $U|_{T_i} \in \mathcal{M}_0$ for all $i \in \mathbb{Z}$. But then $U = v$ for $i < 0$. Therefore by Proposition 3, $U \equiv v$, contrary to (15). Thus $v_p \in \mathcal{M}_0 \setminus \{v\}$ which in turn implies $U \in A(v)$ and $J_1(U) \ge c_1(v)$. This combined with (20) gives $J_1(U) = c_1(v)$ and 1^o of the Theorem is proved.

Since we already have $W^{1,2}(T_0)$ convergence, the first half of 2^o and standard interpolation inequalities, e.g. the Gagliardo-Nirenberg inequality [27], yield $3^o - 4^o$ of the Theorem. Lastly to get the second half of 2^o, we argue as in §2. Let P be a subset of $[0,1]^n$ of the form $[0,1]\times\Pi_2^n[a_i,b_i]$ again with $[a_i,b_i] = [0,\frac{1}{2}]$ or $[\frac{1}{2},1]$. Then by Proposition 2, for any $u \in A(v)$,

$$0 \le \Sigma_{i\in\mathbb{Z}} \int_P (L(\tau_i^1 u)\ dx - c_0/2^{n-1}) \le J_1(u)$$

Reflecting u evenly successively about $x_2 = \frac{1}{2},\cdots,x_n = \frac{1}{2}$ yields $\hat{u} \in \hat{A}(v) \equiv \{w \in A(v) \mid w \textit{ is even in } x_2,\cdots,x_n\}$. Choosing the set $P = P(u)$ which gives the smallest value to the above integral shows $0 \le J_1(\hat{u}) \le J_1(u)$. Thus $c_1(v)$ is achieved on $\hat{A}(v)$. Moreover this argument, with that of Propostion 2 and Proposition 3 shows any $U \in \mathcal{M}_1(v)$ must belong to $\hat{A}(v)$ and the proof is complete. □

Remark 3. Suppose $u \in A_1(v)$ and $J_1(u) < \infty$. Then $J_{1,i}(u) \to 0$ as $|i| \to \infty$ so by Proposition 1, $\|\tau_i^1 u - \mathcal{M}_0\| \to 0$ as $|i| \to \infty$. Since we already know $\|u - v\|_{L^2(T_i)} \to 0$ as $i \to -\infty$, it follows that $\|u - v\|_{W^{1,2}(T_i)} \to 0$ as $i \to -\infty$. Similarly arguing as in the paragraph after that containing (20), there is a $w \in \mathcal{M}_0$ such that $\|u - w\|_{W^{1,2}(T_i)} \to 0$ as $i \to \infty$. These observations will be needed in §4.

4. **Heteroclinic chains.** A generalization of Theorem 3.1 will be given in this section. First some notation. It was shown in Theorem 3.1 that for any $v \in \mathcal{M}_0$, there is a $w \in \mathcal{M}_0 \setminus \{v\}$ and a $U \in \mathcal{M}_1(v)$ having v as its asymptote as $x_1 \to -\infty$, w as its asymptote as $x_1 \to \infty$, and $J_1(U) = c_1(v)$. Set

$$A_1(v,w) = \{u \in E_1 \mid \|u - v\|_{L^2(T_i)} \to 0,\ i \to -\infty; \|u - w\|_{L^2(T_i)} \to 0\ , i \to \infty\},$$

$$C_1(v,w) = \inf_{u \in A_1(v,w)} J_1(u), \tag{25}$$

and

$$\mathcal{M}_1(v,w) = \{u \in A_1(v,w) \mid J_1(u) = c_1(v,w)\}.$$

Then we have $C_1(v,w) = c_1(v)$ and $\emptyset \neq \mathcal{M}_1(v,w) \subset \mathcal{M}_1(v)$.

More generally we can take any $v \in \mathcal{M}_0$, $w \in \mathcal{M}_0 \setminus \{v\}$, define $C_1(v,w)$ by (25) and ask for $U \in A_1(v,w)$ such that $J_1(U) = C_1(v,w)$. This question was studied by Maxwell [26] for the case of $n = 1$. In general there may not be a $U \in A_1(v,w)$ such that $J_1(U) = C_1(v,w)$. However Maxwell showed there is a heteroclinic chain of solutions of the corresponding system of ordinary differential equations that join v and w. We will prove that the same result holds for (PDE).

By a *heteroclinic chain of solutions of* (PDE) *joining* v *and* w, we mean there is a $q \in \mathbb{N}$, functions $v_i \in \mathcal{M}_0$, $0 \le i \le q$ with $v_0 = v$, $v_q = w$ and functions $U_i \in \mathcal{M}_1(v_{i-1}, v_i), 1 \le i \le q$. The heteroclinic chain will be called *minimal* if

$$C_1(v,w) = \Sigma_1^q C_i(v_{i-1}, v_i) \ \ (= \Sigma_1^q J_1(U_i) = \Sigma_1^q c_1(v_{i-1})). \tag{26}$$

Remark 4. Note that it is not possible for $v_i = v_j$ for $i \neq j$ in (26) since if there were such an equality with say, $i < j$, removing $\Sigma_{i+1}^j C_i(v_{i-1}, v_i)$ from the sum decreases the right hand side of (26) while the left hand side remains unchanged.

Now we have

Theorem 4.1. *Let F satisfy $(F_1) - (F_2)$ and (N_0) hold. Then for any $v \in \mathcal{M}_0$ and $w \in \mathcal{M}_0 \setminus \{v\}$, there exists a minimal heteroclinic chain of solutions of (PDE) joining v and w.*

Proof. It is straightforward to show that there is an $M > 0$ such that $C_1(v,w) \le M$. Let (u_k) be a minimizing sequence for (25). Normalizing (u_k) as in the proof of Theorem 3.1, it again follows that (u_k) is bounded in E_1, there is a $v_1 \in \mathcal{M}_0 \setminus \{v\}$ and a $U_1 \in \mathcal{M}_1(v, v_1)$ such that along a subsequence, $u_k \to U_1$ in E_1. If $v_1 = w$, we are through. If $v_1 \neq w$, we claim

$$C_1(v,w) = C_1(v,v_1) + C_1(v_1,w). \tag{27}$$

Indeed, choose $\epsilon > 0$. For $k, q \in \mathbb{N}$, set

$$W_k = \begin{cases} u_k, & x_1 \le q, \\ v_1, & q+1 \le x_1 \le q+2, \\ u_k, & q+3 \le x_1. \end{cases} \tag{28}$$

and define W_k for $x_1 \in [q, q+1] \cup [q+2, q+3]$ by interpolating as in (22). Choose $q = q(\epsilon)$ so large that $\|U_1 - v_1\|_{W^{1,2}(\cup_q^{q+2} T_i)} \le \epsilon$. Since $u_k \to U_1$ in E_1 as $k \to \infty$

along our subsequence, we can select $k = k(\epsilon)$ so large that $\|u_k - v_1\|_{W^{1,2}(\cup_q^{q+2} T_i)} \leq 2\epsilon$. Therefore

$$|J_1(u_k) - J_1(W_k)| \leq \kappa(\epsilon) \tag{29}$$

where $\kappa(\epsilon) \to 0$ as $\epsilon \to 0$. Further set

$$\Phi_k = \begin{cases} W_k, & x_1 \leq q+1 \\ v_1, & q+1 \leq x_1, \end{cases} \tag{30}$$

and

$$\Psi_k = \begin{cases} v_1, & x_1 \leq q+2;, \\ W_k, & q+2 \leq x_1. \end{cases} \tag{31}$$

Thus $W_k \in A_1(v, w),\ \Phi_k \in A_1(v, v_1), \Psi_k \in A_1(v_1, w)$ and

$$J_1(W_k) = J_1(\Phi_k) + J_1(\Psi_k). \tag{32}$$

Hence by (29)-(32), for k possibly still larger,

$$C_1(v, v_1) + C_1(v_1, w) - \kappa(\epsilon) \leq J_1(W_k) - \kappa(\epsilon) \leq J_1(u_k) \leq C_1(v, w) + \epsilon.$$

or

$$C_1(v, v_1) + C_1(v_1, w) \leq C_1(v, w) + \epsilon + \kappa(\epsilon). \tag{33}$$

Since ϵ is arbitrary, (33) implies

$$C_1(v, v_1) + C_1(v_1, w) \leq C_1(v, w). \tag{34}$$

To get the reverse inequality, let (f_k) be a minimizing sequence for $C_1(v_1, w)$. Again let $\epsilon > 0$. For $p, k, l \in \mathbb{N}$, define w_k via

$$w_k = \begin{cases} U_1, & x_1 \leq p, \\ v_1, & p+1 \leq x_1 \leq p+2, \\ f_k(\cdot - le_1), & p+3 \leq x_1. \end{cases} \tag{35}$$

and define w_k for $x_1 \in [p, p+1] \cup [p+2, p+3]$ by interpolating as in (22). Further set

$$\varphi_k = \begin{cases} w_k, & x_1 \leq p+1 \\ v_1, & p+1 \leq x_1, \end{cases} \tag{36}$$

and

$$\psi_k = \begin{cases} v_1, & x_1 \leq p+2, \\ w_k, & p+2 \leq x_1. \end{cases} \tag{37}$$

Thus $w_k \in A_1(v, w),\ \varphi_k \in A_1(v, v_1),\ \psi_k \in A_1(v_1, w)$ and

$$J_1(w_k) = J_1(\varphi_k) + J_1(\psi_k). \tag{38}$$

For $p = p(\epsilon)$ sufficiently large, independently of k,

$$J_1(\varphi_k) \leq C_1(v, v_1) + \epsilon. \tag{39}$$

For $k = k(\epsilon)$ sufficiently large,

$$J_1(f_k) \leq C_1(v_1, w) + \epsilon. \tag{40}$$

Using Remark 3 and (39), for $l = l(\epsilon)$ sufficiently large,

$$J_1(\psi_k) \leq C_1(v_1, w) + 2\epsilon. \tag{41}$$

Combining (38), (39), and (41) shows

$$C_1(v, w) \leq C_1(v, v_1) + C_1(v_1, w) + 3\epsilon. \tag{42}$$

Now (42) and (34) yield (27).

With (f_k) as above, as for (u_k), there is a $v_2 \in \mathcal{M}_0 \setminus \{v_1\}$ such that a subsequence of (f_k) converges in E_1 to $U_2 \in \mathcal{M}_1(v_1, v_2)$. If $v_2 = w$, we are through. If not, as above,

$$C_1(v_1, w) = C_1(v_1, v_2) + C_1(v_2, w).$$

Since each of the C_1 terms are positive, this process ends in a finite number of steps yielding (26) and the Theorem is proved. □

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