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OSCILLATORY DYNAMICS IN A REACTION-DIFFUSION SYSTEM IN THE PRESENCE OF 0:1:2 RESONANCE

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Abstract. Oscillatory dynamics in a reaction-diffusion system with spatially nonlocal effect under Neumann boundary conditions is studied. The system provides triply degenerate points for two spatially non-uniform modes and uniform one (zero mode). We focus our attention on the 0:1:2-mode interaction in the reaction-diffusion system. Using a normal form on the center manifold, we seek the equilibria and study the stability of them. Moreover, Hopf bifurcation phenomena is studied for each equilibrium which has a Hopf instability point. The numerical results to the chaotic dynamics are also shown.

1. **Introduction.** There has been a lot of studies about spatiotemporal patterns and their dynamics to systems of reaction-diffusion equations. Let us consider variations of bifurcations from a stationary solution. Suppose this stationary solution is spatially uniform. Then we linearize the equation about this equilibrium and we know a bifurcation occurs only if the linearized problem has zero or purely imaginary eigenvalues which we call critical eigenvalues. Moreover if this critical eigenvalue is 0 with spatially non-trivial eigenfunction, a stationary bifurcation to non-uniform steady state occurs. One of the typical examples for this is the well-known Turing instability. On the contrary, if the critical eigenvalues are a pair of purely imaginary numbers with spatially non-trivial eigenfunctions, spatially non-trivial oscillations may occur. The so-called wave instability corresponds to this (for instance, see [5, 14]).

Now how can we say about the bifurcations from a spatially non-uniform steady state? Since the linearization about non-uniform steady states is not easy in general, we need to restrict ourselves to some special cases. Bifurcation analysis about degenerate instability points is one of such examples. In fact it is easy to find the degenerate instability points even in the case of Turing instability where n and $n+1$ modes become critical at the same time by choosing the system size appropriately. And in this case we can see the n-mode stationary solution becomes unstable to the $(n+1)$-mode perturbation and the mixed mode stationary solution may bifurcate.

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Here, we focus our attention on the case where the system has the triple degeneracy with 0, 1 and 2-modes (0:1:2-mode interaction). There, we would like to introduce the case where the 1-mode stationary solution may become unstable with a pair of purely imaginary critical eigenvalues. And we will give the explicit condition for the Hopf bifurcation from the 1-mode stationary solutions.

This kind of triple degeneracy can be observed by considering the following types of 3-component reaction-diffusion system :

$$\begin{cases} u_t = D_1 u_{xx} + au + bv + sw + F(u,v), & x \in (0,L),\, t > 0, \\ v_t = D_2 v_{xx} + cu + dv + G(u,v), & x \in (0,L),\, t > 0, \\ \tau w_t = D_3 w_{xx} + u - w, & x \in (0,L),\, t > 0, \\ u_x = v_x = w_x = 0, & x = 0, L. \end{cases} \tag{1}$$

Here, D_1, D_2 and D_3 are diffusion coefficients a, b, c, d and s are constants, and F and G are higher order terms. Moreover, the time constant τ is supposed to be very small. This system consists of two component activator-inhibitor type reaction-diffusion equations and one scalar equation which has the feedback effect to the first component. We shall see the triple degeneracy for this system precisely later. It is, on the other hand, easy to understand the mechanism for the triple degeneracy by assuming that it has two small time constants as follows:

$$\begin{cases} u_t = D_1 u_{xx} + au + bv + sw + F(u,v), & x \in (0,L), t \geq 0, \\ \tau_1 v_t = D_2 v_{xx} + u - v + G(u,v), & x \in (0,L), t \geq 0, \\ \tau_2 w_t = D_3 w_{xx} + u - w, & x \in (0,L), t \geq 0, \\ u_x = v_x = w_x = 0, & \text{at } x = 0, L. \end{cases} \tag{2}$$

In fact, we can reduce this system to a scalar equation by setting $\tau_1 = \tau_2 = 0$. Let us consider only the linear terms in (2). Then we obtain the following by taking the Fourier transformation.

$$\begin{cases} \dfrac{d\hat{u}_k}{dt} = -Dk^2 \hat{u}_k + a\hat{u}_k + b\hat{v}_k + s\hat{w}_k, \\ 0 = -D_2 k^2 \hat{v}_k + \hat{u}_k - \hat{v}_k, \\ 0 = -D_3 k^2 \hat{w}_k + \hat{u}_k - \hat{w}_k. \end{cases}$$

The second and third equations of the above can be solved as

$$\hat{v}_k = \frac{\hat{u}_k}{1 + D_2 k^2}, \quad \hat{w}_k = \frac{\hat{u}_k}{1 + D_3 k^2}.$$

Therefore, we obtain

$$\frac{d\hat{u}_k}{dt} = \lambda_k \hat{u}_k,$$

where

$$\lambda_k = a - D_1 k^2 + \frac{b}{1 + D_2 k^2} + \frac{s}{1 + D_3 k^2}.$$

Now we define the neutral stability curve by $C = \{(k, a)\,;\, \lambda_k = 0\}$. We take the constants satisfying $b < 0$, $s > 0$, $-1/b < D_2/D_1$ and $D_2 \ll D_3$. Then it turns out that the curve C has the shapes as shown in Fig. 1.

Since we are considering the Neumann boundary conditions, the wave number k should be an integer multiple of the fundamental wave number $k_0 = \pi/L$: $k = mk_0\,(m = 0, 1, 2, \cdots)$. Let us define the neutral stability curves for each mode m by

$$C_m = \{(k_0, a)\,;\, \lambda_{mk_0} = 0\}.$$

It is now easy to conclude that 0:1:2-triple degeneracy occurs by choosing the constants appropriately (see Fig. 1).

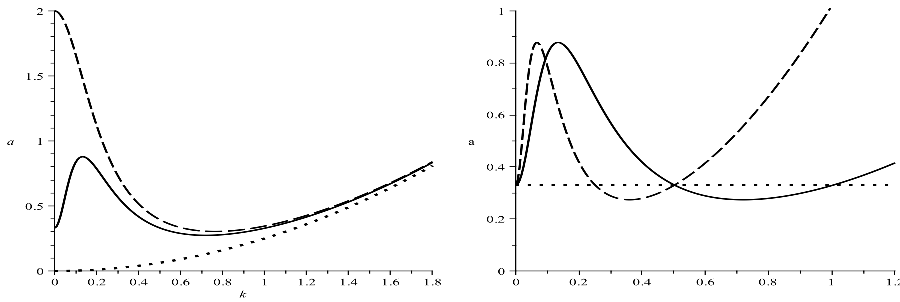

FIGURE 1. The neutral stability curves $C = \{(k, a); \lambda_k = 0\}$ and $C_m = \{(k_0, a); \lambda_{mk_0} = 0\}$ in the case of $D_1 = 0.25$, $D_2 = 20$, $D_3 = 100$. [(Left:) The neutral stability curves C. Dotted line: $b = s = 0$; Dashed line: $s = 0, b = -2$; Solid line: $b = -2, s \approx 1.67$.] [(Right:) The neutral stability curves C_0 (dotted line), C_1 (solid line) and C_2 (dashed line) in the case where triple degeneracy of 0:1:2-mode interaction occur at $(k_0, a) \approx (0.50, 0.33)$. Constants (critical values) are $b = 2$, $s \approx 1.67$. The horizontal axis correspond to k_0.]

Let us go back to the case (1). It can be also reduced to a simpler system of reaction-diffusion equations as follows: putting $\tau = 0$, we have

$$\begin{cases} u_t = D_1 u_{xx} + au + bv + sw + F(u, v), & x \in (0, L),\ t > 0, \\ v_t = D_2 v_{xx} + cu + dv + G(u, v), & x \in (0, L),\ t > 0, \\ 0 = D_3 w_{xx} + u - w, & x \in (0, L),\ t > 0, \\ u_x = v_x = w_x = 0, & x = 0, L. \end{cases} \tag{3}$$

Let $(u_m(t), v_m(t), w_m(t))$ be the Fourier coefficients so that the following holds:

$$u(t, x) = u_0(t) + \sum_{m \in \mathbb{N}} u_m(t) \cos(m\pi x/L),$$
$$v(t, x) = v_0(t) + \sum_{m \in \mathbb{N}} v_m(t) \cos(m\pi x/L),$$
$$w(t, x) = w_0(t) + \sum_{m \in \mathbb{N}} w_m(t) \cos(m\pi x/L).$$

Then, the third equation can be solved as

$$w_m(t) = \frac{1}{1 + (\pi/L)^2 m^2 D_3} u_m(t), \quad m \in \{0\} \cup \mathbb{N}.$$

It holds that $w_0 = u_0$, and taking $D_3 \to \infty$, then

$$w_m \equiv 0, \quad m \in \mathbb{N}.$$

Combining them, we obtain

$$\begin{aligned} w(t,x) &= w_0(t) + \sum_{m\in\mathbb{N}} w_m(t)\cos(m\pi x/L) \\ &= u_0(t) + \sum_{m\in\mathbb{N}} (1+(\pi/L)^2 m^2 D_3)^{-1} u_m(t)\cos(m\pi x/L) \\ &\to u_0(t) \qquad (\text{ as } D_3 \to \infty) \\ &= \frac{1}{L}\int_0^L u(t,x)\ dx. \end{aligned}$$

Substituting it into (1), we obtain the reaction-diffusion system as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_1 \dfrac{\partial^2 u}{\partial x^2} + au + bv + F(u,v) + \dfrac{s}{L}\int_0^L u(t,x)\ dx,\ \ x\in(0,L), t>0, \\ \dfrac{\partial v}{\partial t} = D_2 \dfrac{\partial^2 v}{\partial x^2} + cu + dv + G(u,v),\ \ x\in(0,L), t>0, \\ \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial x} = 0 \text{ at } x = 0, L. \end{cases} \tag{4}$$

Let (u_m, v_m) are Fourier coefficients of $(u(t,x), v(t,x))$. Then, the linearized matrix about a trivial solution $(u,v) = (0,0)$ is

$$M_m = \begin{cases} \begin{pmatrix} a+s & b \\ c & d \end{pmatrix} & (m=0), \\ \begin{pmatrix} a - D_1 m^2 k_0^2 & b \\ c & d - D_2 m^2 k_0^2 \end{pmatrix} & (m \neq 0), \end{cases} \tag{5}$$

Then, a triply degenerate point (D_2, k_0, s) of 0:1:2-modes is given by a solution of $\det M_0 = \det M_1 = \det M_2 = 0$ as follows:

$$\begin{aligned} k_0 = k_0^* &:= \left[\frac{1}{8dD_1}\left\{5\Delta - \sqrt{25\Delta^2 - 16ad\Delta}\right\}\right]^{1/2}, \\ D_2 &= \frac{\{dD_1(k_0^*)^2 - \Delta\}}{(k_0^*)^2\{D_1(k_0^*)^2 - a\}}, \\ s &= -\Delta/d, \end{aligned}$$

where $\Delta = ad - bc$. We also have a triple degeneracy of 0:1:2-mode interaction. It should be noted that the linear stability is also discussed in the next section.

If we have the triple degeneracy, the dynamics about the critical point can be *generically* analyzed by using the quadratic normal form by Smith, Moehlis and Holmes [11]. It also includes the 1:2-mode dynamics which is governed by the quadratic normal form. It had been studied by Armbruster and Guckenheimer [1]. They have periodic orbits (standing and traveling waves) as well as fixed points (pure and mixed mode stationary solutions), invariant tori (modulated traveling waves) and heteroclinic cycles. However, we focus our attention on the case where the problem has up-down symmetry. Then we don't have any quadratic terms in the normal form which is different from their studies. Therefore we need to study the cubic type normal form for 0:1:2-modes. This is the main contribution of this paper and by this we can conclude that the 1-mode stationary solution of these systems can exhibit the Hopf bifurcation and we can observe the limit cycle bifurcating from the non-uniform steady state. In fact, there have been several studies on reaction-diffusion systems where periodic motions around 1-mode stationary solutions are observed ([9, 12, 16]). This result could be one of the starting points to study the oscillatory dynamics around 1-mode stationary solutions.

As we imposed the Neumann boundary conditions for both ends of the interval, the periodic motions mentioned above are standing wave oscillations around 1-mode stationary solutions. Therefore, they are not traveling waves. In addition to the periodic motions, the normal form also suggests that the system exhibits chaotic behaviors.

This paper is organized as follows: in the next section, we show the mathematical settings and precise assumptions to (4). Under these settings, the normal form around the triply degenerate point is derived. In section 3, we seek the equilibria of the normal form. We also compute the linearized stability of them, and seek the Hopf instability points for each equilibrium. In section 4, we state our main result which is summarized in Theorem 4.3. The numerical results are also presented in this section. In section 5, we study the Hopf bifurcation phenomena around mixed mode stationary solutions. In section 6, we show the numerical results to the chaotic solutions in the normal form. The bifurcation problem of (2) is considered in section 7. A brief discussion and the numerical results to (4) are included in section 8. In the appendixes, we show a proof of lemma 4.2 which gives normal forms for the Hopf bifurcation from 1-mode stationary solutions. We also show constants appearing in the normal forms.

2. Formulation.

2.1. Mathematical formulation and assumptions to (4):

We start this section by introducing mathematical formulation and precise assumptions to the reaction-diffusion system (4). We consider the system (4) in a function space

$$X := \left\{(u,v) \in [H^2(\Omega)]^2; u_x = v_x = 0 \text{ at } x = 0, L\right\},$$

where Ω denotes an interval $(0,L) \subset \mathbb{R}$. And we assume the following:

- (A1) The functions (higher order terms) F and G are sufficiently smooth;
- (A2) $F(u,v) \equiv -F(-u,-v)$ and $G(u,v) \equiv -G(-u,-v)$ hold;
- (A3) The coefficients of linear parts satisfy $a, c > 0$, $b, d < 0$, $a + d < 0$ and $\Delta := ad - bc > 0$;
- (A4) $\dfrac{bc}{d} + d < 0$ holds.

Using assumptions (A1) and (A2) , the functions F and G can be represented by the Taylor series around the origin:

$$\begin{aligned} F(u,v) &= \sum_{j+\ell=3} f_{j\ell} u^j v^\ell + o(\|(u,v)\|), \\ G(u,v) &= \sum_{j+\ell=3} g_{j\ell} u^j v^\ell + o\|(u,v)\|). \end{aligned}$$

Here, $\|\cdot\|$ stands for norm on $[H^2(\Omega)]^2$, and

$$f_{j\ell} = \frac{1}{j!\ell!}\frac{\partial^{j\ell} F}{\partial u^j \partial v^\ell}(0,0), \quad g_{j\ell} = \frac{1}{j!\ell!}\frac{\partial^{j\ell} G}{\partial u^j \partial v^\ell}(0,0), \quad j + \ell = 3,\, j, \ell \in \mathbb{N}.$$

2.2. **Dynamical system on Fourier space:** We note that solutions of (4) can be considered as those of periodic boundary conditions with period $2L$. Indeed, if $(u(t,x),v(t,x)) \in X$ is a solution of (4), then we can extend it for $x \in [0,2L]$ as follows:

$$\tilde{u}(t,x) = \begin{cases} u(t,x) & x \in [0,L], \\ u(t,2L-x) & x \in [L,2L], \end{cases} \quad \tilde{v}(t,x) = \begin{cases} v(t,x) & x \in [0,L], \\ v(t,2L-x) & x \in [L,2L]. \end{cases}$$

One can verify that $(\tilde{u}(t,x),\tilde{v}(t,x))$ is a solution of

$$\begin{cases} u_t = D_1 u_{xx} + au + bv + \frac{s}{2L}\int_0^{2L} u(t,x)\,dx + F(u,v), \ \ x \in \Omega_p, t>0, \\ v_t = D_2 v_{xx} + cu + dv + G(u,v), \ \ x \in \Omega_p, t > 0, \\ u(t,x) = u(t,x+2L), u_x(t,x) = u_x(t,x+2L) \ t>0, \\ v(t,x) = v(t,x+2L), v_x(t,x) = v_x(t,x+2L) \ , t>0. \end{cases} \tag{6}$$

Here, Ω_p denotes the interval $(0,2L)$. Thus, we consider the system (6) in a function space

$$X_p := \{(u,v) \in [H^2_{per}(\Omega_p)]^2; (u(x),v(x)) = (u(2L-x), v(2L-x))\} \tag{7}$$

instead of (4) on X. By Fourier series, we can describe the solution of (6) as

$$(u(t,x),v(t,x)) = (\alpha_0,\beta_0) + \sum_{m\in\mathbb{N}} (\alpha_m(t),\beta_m(t))\cos(mk_0 x),$$

or

$$(u(t,x),v(t,x)) = \sum_{m\in\mathbb{Z}} (u_m(t),u_m(t))e^{imk_0x}, \tag{8}$$

where $k_0 = \pi/L$ and

$$(\alpha_m,\beta_m) = \begin{cases} (u_0,v_0), & m=0, \\ (2u_m,2v_m), & m\in\mathbb{N}. \end{cases}$$

Using (8), system (6) is equivalent to

$$\frac{d}{dt}\begin{pmatrix} u_m \\ v_m \end{pmatrix} = M_m \begin{pmatrix} u_m \\ v_m \end{pmatrix} + \begin{pmatrix} f_m \\ g_m \end{pmatrix}, m \geq 0, \tag{9}$$

where

$$\begin{aligned} f_m = \sum_{\substack{m_1+m_2+m_3=m \\ m_1,m_2,m_3\in\mathbb{Z}}} & (f_{30}u_{m_1}u_{m_2}u_{m_3} + f_{21}u_{m_1}u_{m_2}v_{m_3} \\ & + f_{12}u_{m_1}v_{m_2}v_{m_3} + f_{03}v_{m_1}v_{m_2}v_{m_3}), \\ g_m = \sum_{\substack{m_1+m_2+m_3=m \\ m_1,m_2,m_3\in\mathbb{Z}}} & (g_{30}u_{m_1}u_{m_2}u_{m_3} + g_{21}u_{m_1}u_{m_2}v_{m_3} \\ & + g_{12}u_{m_1}v_{m_2}v_{m_3} + g_{03}v_{m_1}v_{m_2}v_{m_3}) \end{aligned}$$

and M_m $(m=0,1,2,\cdots)$ are defined in (5). It should be noted that since $(u_m,v_m) = (u_{-m},v_{-m})$ holds by the symmetry of (7), it is sufficient to consider the equations for $m \geq 0$. Moreover, here we take the terms up to third order, truncating the higher order terms. We consider the system (9) in a space

$$\begin{aligned} X_F \quad := \quad & \Big\{\{(u_m, v_m)\}_{m\in\mathbb{Z}}; (u_m, v_m) = (u_{-m}, v_{-m}), \\ & \qquad \|\{(u_m, v_m)\}_{m\in\mathbb{Z}}\|^2_{X_F} = \sum_{m\in\mathbb{Z}} (1+m^2)^2 |(u_m, v_m)|^2 < \infty\Big\} \end{aligned}$$

which is equivalent to X_p by the map $\mathcal{P}$:

$$\mathcal{P}(u, v) = \left\{ \frac{1}{2L} \int_0^{2L} (u(t,x), v(t,x)) e^{-imk_0 x}\, dx \right\}_{m\in\mathbb{Z}}.$$

Throughout this paper, we consider the dynamical system (9) on a Fourier space X_F instead of the system (6) on X_p. It should be noted that bifurcation parameters in (9) are k_0, D_2 and s.

2.3. **Linear stability:** Let us consider the linear stability about a trivial solution again. The matrix M_m has a zero eigenvalue if and only if $\det M_m = 0$. Moreover, the following holds:

Lemma 2.1. *For given two positive integers $j, \ell, (j \neq \ell)$ and constants D_1, a, b, c, d, the linearized eigenvalue problem of (9) about a trivial solution has strictly three zero eigenvalues for $m = 0, j$ and ℓ at $(k_0, D_2, s) = (k_0^{j,\ell}, D_2^{j,\ell}, s^*)$, where*

$$\begin{aligned} k_0^{j,\ell} &= \left[\frac{1}{2dD_1 j^2 \ell^2} \left\{ \Delta(j^2+\ell^2) - \sqrt{\Delta^2 (j^2+\ell^2)^2 - 4ad\Delta j^2 \ell^2} \right\} \right]^{1/2}, \\ D_2^{j,\ell} &= \frac{\{dD_1 j^2 (k_0^{j,\ell})^2 - \Delta\}}{j^2 (k_0^{j,\ell})^2 \{D_1 j^2 (k_0^{j,\ell})^2 - a\}}, \\ s^* &= -\Delta/d = -(ad-bc)/d. \end{aligned}$$

Here we define the neutral stability curves. Solving $\det M_m = 0$ for D_2, we have

$$D_2 = \mathcal{D}_2(k_0; m) := \frac{(dD_1 m^2 {k_0}^2 - \Delta)}{m^2 {k_0}^2 (D_1 m^2 {k_0}^2 - a)}.$$

We can define the neutral stability curves for $m \in \mathbb{N}$ as follows:

$$C_m := \{(D_2, k_0) \in \mathbb{R}^2_+; D_2 = \mathcal{D}_2(k_0; m)\}.$$

Lemma 2.1 provides us a general triple degeneracy of 0:j:ℓ -mode interaction ($j, \ell \in \mathbb{N}$). And it is easy to see that this yields a triple degeneracy of 0:1:2-mode interaction by choosing $j = 1$ and $\ell = 2$. We are going to focus on this case since the normal form has cubic resonance terms. We would like to study the dynamics around 0:1:2-degenerate point, especially, oscillatory dynamics around 1-mode stationary solutions.

2.4. **Normal form in the presence of 0:1:2 resonance:** Let us derive the normal form on the center manifolds around the triply degenerate point $(k_0^{1,2}, D_2^{1,2}, s^*)$ of (4). Here we apply the standard theory of center manifold reduction (for instance, see [2, 15]).

2.4.1. *Diagonalization:* To apply the center manifold theory, we diagonalize the equations in (9) for $m = 0, 1$ and 2. Set $(k_0, D_2, s) = (k_0^{1,2}, D_2^{1,2}, s^*)$. Then changing variables ${}^t(u_m, v_m) = T_m\, {}^t(\tilde{u}_m, \tilde{v}_m), (m = 0, 1, 2)$ by the matrix

$$T_0 = \begin{pmatrix} -d & bc/d \\ c & c \end{pmatrix}, T_m = \begin{pmatrix} -d + D_2 m^2 k_0^2 & a - D_1 m^2 k_0^2 \\ c & c \end{pmatrix}, m = 1, 2,$$

we have

$$\begin{pmatrix} \dot{\tilde{u}}_m \\ \dot{\tilde{v}}_m \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mu_m^- \end{pmatrix} \begin{pmatrix} \tilde{u}_m \\ \tilde{v}_m \end{pmatrix} + T_m^{-1} \begin{pmatrix} \tilde{f}_m \\ \tilde{g}_m \end{pmatrix}, m = 0, 1, 2.$$

Here,

$$\begin{aligned} \mu_0^- &:= d + bc/d, \\ \mu_m^- &:= (a + d) - m^2 (D_1 + D_2^{1,2})(k_0^{1,2})^2, \\ \tilde{f}_m &:= f_m|_{{}^t(u_{m_j}, v_{m_j}) = T_m\, {}^t(\tilde{u}_{m_j}, \tilde{v}_{m_j})}, \\ \tilde{g}_m &:= g_m|_{{}^t(u_{m_j}, v_{m_j}) = T_m\, {}^t(\tilde{u}_{m_j}, \tilde{v}_{m_j})}. \end{aligned}$$

2.4.2. *Center manifold reduction:* Let

$$\begin{aligned} \rho &:= (k_0^{1,2}, D_2^{1,2}, s^*) - (k_0, D_2, s), \\ \mu_m^+ &:= \left\{ \text{tr } M_m + \sqrt{(\text{tr } M_m)^2 - 4 \det M_m} \right\} /2. \end{aligned}$$

We define a neighborhood $\mathcal{U}_\varepsilon$ of $X_F \times \mathbb{R}^3$:

$$\mathcal{U}_\varepsilon := \left\{ (\{(u_m, v_m)\}_{m \in \mathbb{Z}}, \rho) \in X_F \times \mathbb{R}^3; ||\{(u_m, v_m)\}_{m \in \mathbb{Z}}||_{X_F} + |\rho| < \varepsilon \right\}.$$

Then we have the following theorem.

Theorem 2.2. *For given constants a, b, c, d, D_1, there exists a positive constant ε such that the local center manifold $\mathcal{W}_{loc}^c$ of (9) is contained in $\mathcal{U}_\varepsilon$. Moreover, the dynamics of (9) on the manifold $\mathcal{W}_{loc}^c$ is governed by the following system:*

$$\left\{ \begin{array}{l} \dot{z}_0 = (\mu_0^+ + a_1 z_0^2 + a_2 z_1^2 + a_3 z_2^2) z_0 + a_4 z_1^2 z_2 + o(3), \\ \dot{z}_1 = (\mu_1^+ + b_1 z_0^2 + b_2 z_1^2 + b_3 z_2^2) z_1 + b_4 z_0 z_1 z_2 + o(3), \\ \dot{z}_2 = (\mu_2^+ + c_1 z_0^2 + c_2 z_1^2 + c_3 z_2^2) z_2 + c_4 z_0 z_1^2 + o(3). \end{array} \right. \tag{10}$$

Here, $z_j(t) \in \mathbb{R}$ denote $\tilde{u}_j(t)\, (j = 0, 1, 2)$, and $o(3)$ denotes $o(|(z_0, z_1, z_2)|^3)$. In addition, the coefficients μ_j^+, a_j, b_j, c_j are dependent on the coefficients and parameters appearing in (9).

Proof. The first statement of the theorem follows from standard center manifold theory. It also states that for $m \neq 0, 1, 2$, there exist functions

$$\begin{aligned} &h_m^u(\tilde{u}_0, \tilde{u}_1, \tilde{u}_2; \rho), m \geq 3, \\ &h_m^v(\tilde{u}_0, \tilde{u}_1, \tilde{u}_2; \rho),\ m \geq 0, \end{aligned}$$

satisfying

$$\frac{\partial h_m^u}{\partial \tilde{u}_j}(0, 0, 0; 0) = \frac{\partial h_m^v}{\partial \tilde{u}_j}(0, 0, 0; 0) = 0,\ (j = 0, 1, 2)$$

and

$$\frac{\partial h_m^u}{\partial \rho}(0, 0, 0; 0) = \frac{\partial h_m^v}{\partial \rho}(0, 0, 0; 0) = 0$$

such that the local invariant manifold $\mathcal{W}^c_{loc}$ is expressed by

$$\begin{aligned}\mathcal{W}^c_{loc} \quad = \quad & \big\{\{(\tilde{u}_\ell,\tilde{v}_\ell),(u_m,v_m)\}_{|\ell|\le 2,\,|m|\ge 3}\in X_F;\tilde{v}_\ell=h^v_\ell(\tilde{u}_0,\tilde{u}_1,\tilde{u}_2;\rho),\\ & (u_m,v_m)=(h^u_m(\tilde{u}_0,\tilde{u}_1,\tilde{u}_2;\rho),h^v_m(\tilde{u}_0,\tilde{u}_1,\tilde{u}_2;\rho)),|\ell|\le 2,\,|m|\ge 3\big\}.\end{aligned}$$

By the simple computation, if $|(\tilde{u}_0,\tilde{u}_1,\tilde{u}_2;\rho)|<\varepsilon$ then $h^u_m=h^v_m=o(\varepsilon^3)$. Then, the cubic truncated equations for $\tilde{u}_m$, $(m=0,1,2)$ are given by the following:

$$\dot{\tilde{u}}_0=\mu_0^+\tilde{u}_0-\frac{1}{\mu_0^-}\left\{\tilde{f}_0-\frac{b}{d}\tilde{g}_0\right\},$$

$$\dot{\tilde{u}}_m=\mu_m^+\tilde{u}_m-\frac{1}{c\mu_m^-}\{c\tilde{f}_m+(-a+D_1m^2(k_0^{1,2})^2)\tilde{g}_m\},m=1,2,$$

where

$$\begin{aligned}\tilde{f}_m:=&\sum_{\substack{m_1+m_2+m_3=m\\ m_j\in\{0,\pm1,\pm2\}}}(f_{30}B_{m_1}B_{m_2}B_{m_3}\tilde{u}_{m_1}\tilde{u}_{m_2}\tilde{u}_{m_3}\\ &+cf_{21}B_{m_1}B_{m_2}\tilde{u}_{m_1}\tilde{u}_{m_2}\tilde{u}_{m_3}+c^2f_{12}B_{m_1}\tilde{u}_{m_1}\tilde{u}_{m_2}\tilde{u}_{m_3}+c^3f_{03}\tilde{u}_{m_1}\tilde{u}_{m_2}\tilde{u}_{m_3})\end{aligned}$$

and

$$\begin{aligned}\tilde{g}_m:=&\sum_{\substack{m_1+m_2+m_3=m\\ m_j\in\{0,\pm1,\pm2\}}}(g_{30}B_{m_1}B_{m_2}B_{m_3}\tilde{u}_{m_1}\tilde{u}_{m_2}\tilde{u}_{m_3}\\ &+cg_{21}B_{m_1}B_{m_2}\tilde{u}_{m_1}\tilde{u}_{m_2}\tilde{u}_{m_3}+c^2g_{12}B_{m_1}\tilde{u}_{m_1}\tilde{u}_{m_2}\tilde{u}_{m_3}+c^3g_{03}\tilde{u}_{m_1}\tilde{u}_{m_2}\tilde{u}_{m_3}).\end{aligned}$$

Here, $B_j=-d+j^2D_2^{1,2}(k_0^{1,2})^2$. This gives the cubic truncated system (10). We show the explicit form of coefficients in Appendix B. □

3. **Existence and stability of equilibria.** Let us consider the third order truncated system of (10):

$$\left\{\begin{array}{l}\dot{z}_0=(\mu_0+a_1z_0^2+a_2z_1^2+a_3z_2^2)z_0+a_4z_1^2z_2,\\ \dot{z}_1=(\mu_1+b_1z_0^2+b_2z_1^2+b_3z_2^2)z_1+b_4z_0z_1z_2,\\ \dot{z}_2=(\mu_2+c_1z_0^2+c_2z_1^2+c_3z_2^2)z_2+c_4z_0z_1^2,\end{array}\right. \tag{11}$$

where

$$\mu_m=\mu_m^+\quad(m=0,1,2)$$

for simplicity. Now we introduce a scale-invariance property of (11):

3.1. **Scale-invariance of the normal form:** The system (11) is invariant under the scaling:

$$\tilde{z}_j=\eta z_j,\quad \tilde{\mu}_j=\eta^2\mu_j,\quad \tilde{t}=\eta^2t,\quad {}^\forall\eta\in\mathbb{R}. \tag{12}$$

Therefore, if

$$\mathbf{z}(t;\mu_0,\mu_1,\mu_2,\mathbf{z}(\mathbf{0})):=(z_0,z_1,z_2)(t;\mu_0,\mu_1,\mu_2,z_0(0),z_1(0),z_2(0))$$

is a solution to (11) , then

$$(\tilde{z}_0,\tilde{z}_1,\tilde{z}_2)(\tilde{t};\tilde{\mu}_0,\tilde{\mu}_1,\tilde{\mu}_2,\tilde{\mathbf{z}}(0))=\eta\mathbf{z}(\eta^2t;\eta^2\mu_0,\eta^2\mu_1,\eta^2\mu_2,\eta\mathbf{z}(\mathbf{0}))$$

is also a solution to (11). Then, the following hold:

- Let us consider the system (11) with higher order terms $O(|(z_0, z_1, z_2)|^5)$. Using (12), it becomes

$$\tilde{z}_j' = \tilde{\mu}_j \tilde{z}_j + f_j(\tilde{z}_0, \tilde{z}_1, \tilde{z}_2; \tilde{\mu}_j) + \eta^2 O(|(\tilde{z}_0, \tilde{z}_1, \tilde{z}_2)|^5). \tag{13}$$

  Here f_j are cubic terms in (11). Taking $\eta \to 0$, the higher order terms vanish;
- Let $\mathbf{z}(t; \mu_0, \mu_1, \mu_2) = (z_0, z_1, z_2)(t; \mu_0, \mu_1, \mu_2)$ ($\tilde{t} \in (0, \eta^2 T]$) be a solution of (11). Then, there exists a similar solution of (11):

$$\tilde{\mathbf{z}}(\tilde{t}; \tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2) = \eta \mathbf{z}(\eta^2 t; \eta^2 \mu_0, \eta^2 \mu_1, \eta^2 \mu_2), \quad t \in (0, T].$$

  This implies that we can obtain a small amplitude solution from a (large amplitude) solution by taking η sufficiently small.

Let us seek the equilibria of (11) , and study the stabilities of them with the assumptions:

$$\mu_0 \in \mathbb{R}, \quad a_j \neq 0, \quad b_j \neq 0 \quad \text{and} \quad c_j \neq 0.$$

Notice that μ_j $(j = 1, 2)$ are real by the assumption (A3).

3.2. **Existence of pure mode solutions:** By the simple computation, we obtain the following theorem:

Theorem 3.1. *If $\mu_0 a_1 < 0$, $\mu_1 b_2 < 0$, $\mu_2 c_3 < 0$ then the system (11) has equilibria $\pm e_0 := \pm(\sqrt{-\mu_0/a_1}, 0, 0)$, $\pm e_1 := \pm(\sqrt{-\mu_1/b_2}, 0, 0)$, $\pm e_2 := \pm(0, 0, \sqrt{-\mu_2/c_3})$, respectively. Moreover, these equilibria $\pm e_\ell$ correspond to the pure mode stationary solutions of (4):*

$$u(x) = U_\ell(x) := \pm \frac{2C_\ell}{B_\ell} \cos\left(\ell \frac{\pi}{L} x\right) + o(\varepsilon^3),$$
$$v(x) = V_\ell(x) := \pm \frac{2C_\ell}{c} \cos\left(\ell \frac{\pi}{L} x\right) + o(\varepsilon^3),$$

where

$$C_\ell = \begin{cases} \sqrt{-\mu_0/a_1}, & \ell = 0, \\ \sqrt{-\mu_1/b_2}, & \ell = 1, \\ \sqrt{-\mu_2/c_3}, & \ell = 2. \end{cases}$$

We call the solutions $(U_\ell(x), V_\ell(x))$ "ℓ-mode stationary solution (or ℓ-mode solution)". It is obvious that equilibria $\pm e_j, (j = 0, 1, 2)$ appear through the pitchfork bifurcation from the trivial solution at $\mu_j = 0$. It is quite a contrast to the case of [1] that the equilibria $\pm e_1$ corresponding to 1-mode stationary solution of (4) exist in our case.

Remark. Even though the equilibria $\pm e_0$ correspond to the spatially uniform solutions of (4), we call them "pure mode solutions".

By the simple computation, we have the following:
(i) Linearized eigenvalues about $\pm e_0$ are

$$-2\mu_0, \quad \mu_1 - b_1\mu_0/a_1, \quad \mu_2 - \mu_0 c_1/a_1;$$

(ii) Linearized eigenvalues about $\pm e_2$ are

$$\mu_0 - a_3\mu_2/c_2, \quad \mu_1 - \mu_2 b_3/c_2, \quad -2\mu_2;$$

(iii) Linearized matrix of $\pm e_1$ is given by

$$M_{\pm e_1} := \begin{pmatrix} \mu_0 - \frac{a_2}{b_2}\mu_1 & 0 & \frac{-a_4}{b_2}\mu_1 \\ 0 & -2\mu_1 & 0 \\ -\frac{c_4}{b_2}\mu_1 & 0 & \mu_2 - \frac{c_2}{b_2}\mu_1 \end{pmatrix}.$$

3.2.1. *Hopf instability of* $\pm e_1$*:* Now we put

$$\tilde{M}_{\pm e_1} := \begin{pmatrix} (M_{\pm e_1})_{11} & (M_{\pm e_1})_{13} \\ (M_{\pm e_1})_{31} & (M_{\pm e_1})_{33} \end{pmatrix} = \begin{pmatrix} \mu_0 - \frac{a_2}{b_2}\mu_1 & \frac{-a_4}{b_2}\mu_1 \\ -\frac{c_4}{b_2}\mu_1 & \mu_2 - \frac{c_2}{b_2}\mu_1 \end{pmatrix}.$$

If $\operatorname{tr}\tilde{M}_{\pm e_1} = 0$ and $\det \tilde{M}_{\pm e_1} > 0$ hold, then the matrix $M_{\pm e_1}$ has a pair of purely imaginary eigenvalues. That is, the following holds:

Lemma 3.2. *The linearized matrix $M_{\pm e_1}$ of equilibria $\pm e_1$ has a pair of purely imaginary eigenvalues at $\mu_2 = -\mu_0 + \mu_1(a_2 + c_2)/b_2$ if and only if $(\mu_0 - \mu_1 a_2/b_2)^2 + \mu_1^2 a_4 c_4/b_2^2 < 0$ holds.*

We consider the existence of periodic solutions by constructing a center manifold around $\pm e_1$ in the latter section. Here we note that the condition $a_4 c_4 < 0$ is necessary for the Hopf instability around $\pm e_1$.

3.3. **Existence of doubly mixed mode solutions:**

Theorem 3.3. *If $(a_3\mu_2 - c_3\mu_0)(a_1c_3 - c_1a_3) > 0$ and $(c_1\mu_0 - a_1\mu_2)(a_1c_3 - a_3c_1) > 0$, then the system (11) has equilibria*

$$(z_0, z_1, z_2) = \pm(z_0^*, 0, z_2^*) \quad \text{and} \quad (z_0, z_1, z_2) = \pm(z_0^*, 0, -z_2^*),$$

where

$$z_0^* := \sqrt{\frac{a_3\mu_2 - c_3\mu_0}{a_1c_3 - c_1a_3}} \quad \text{and} \quad z_2^* := \sqrt{\frac{c_1\mu_0 - a_1\mu_2}{a_1c_3 - a_3c_1}}.$$

Now we define

$$e_{02}^+ := (z_0^*, 0, z_2^*) \quad \text{and} \quad e_{02}^- := (z_0^*, 0, -z_2^*).$$

They correspond to doubly mixed mode stationary solutions of (4):

$$u(x) = \pm 2\left[\frac{z_0^*}{B_0} \pm \frac{z_2^*}{B_2}\cos\left(\frac{2\pi}{L}x\right)\right] + o(\varepsilon^3),$$
$$v(x) = \pm\frac{2}{c}\left[z_0^* \pm z_2^*\cos\left(\frac{2\pi}{L}x\right)\right] + o(\varepsilon^3).$$

Since it is easy to prove this theorem, we omit the proof. Let us study the stability of $\pm e_{02}^{\pm}$. Linearized matrix around them is given by

$$\begin{pmatrix} 2a_1(z_0^*)^2 & 0 & 2a_3z_0^*z_2^* \\ 0 & \mu_1 + b_1(z_0^*)^2 + b_3(z_2^*)^2 + b_4z_0^*z_2^* & 0 \\ 2c_1z_0^*z_2^* & 0 & 2c_3(z_2^*)^2 \end{pmatrix}.$$

Though it is hard to determine the stability of $\pm e_{02}^{\pm}$ in general, we can obtain the following sufficient condition.

3.3.1. *Hopf instability of doubly mixed mode solutions:*

Lemma 3.4. *If there exist constants μ_0, μ_2 such that $a_3\mu_2 - c_3\mu_0 > 0$ and $c_1\mu_0 - a_1\mu_2 > 0$ hold at*

$$\mu_2 = \frac{a_1c_3 + c_1c_3}{a_1a_3 + c_3a_1}\mu_0.$$

Then the linearized matrix around $\pm e_{02}^{\pm}$ has a pair of purely imaginary eigenvalues if and only if $a_1c_3 - c_1a_3 > 0$ holds.

Proof. One of the linearized eigenvalue about $\pm e_{02}^{\pm}$ is

$$\mu_1 + b_1(z_0^*)^2 + b_3(z_2^*)^2 + b_4 z_0^* z_2^*$$

and the others are given by the eigenvalues of matrix:

$$\begin{pmatrix} 2a_1(z_0^*)^2 & 2a_3 z_0^* z_2^* \\ 2c_1 z_0^* z_2^* & 2c_3(z_2^*)^2 \end{pmatrix}.$$

It has purely imaginary eigenvalues if and only if

$$4(a_1c_3 - a_3c_1)(z_0^* z_2^*)^2 > 0 \text{ and } a_1(z_0^*)^2 + c_3(z_2^*)^2 = 0.$$

Moreover, solving $a_1(z_0^*)^2 + c_3(z_2^*)^2 = 0$ for μ_2, we have

$$\mu_2 = \frac{a_1c_3 + c_1c_3}{a_1a_3 + c_3a_1}\mu_0.$$

This completes the proof. □

3.4. **Existence of triply mixed mode solutions:** The triply mixed mode equilibria of (11) are given by the roots of

$$\begin{cases} \mu_0 + a_1z_0^2 + a_2z_1^2 + a_3z_2^2 + a_4\dfrac{z_1^2 z_2}{z_0} = 0, \\ \mu_1 + b_1z_0^2 + b_2z_1^2 + b_3z_2^2 + b_4z_0z_2 = 0, \\ \mu_2 + c_1z_0^2 + c_2z_1^2 + c_3z_2^2 + c_4\dfrac{z_0z_1^2}{z_2} = 0. \end{cases} \tag{14}$$

We also note that if (z_{0*}, z_{1*}, z_{2*}) satisfying

$$z_{0*} \neq 0, \quad z_{1*} \neq 0 \quad \text{and} \quad z_{2*} \neq 0$$

is an equilibrium, then it correspond to the triply mixed mode stationary solution of (4):

$$u(x) = 2\left[\frac{z_{0*}}{B_0} + \frac{z_{1*}}{B_1}\cos\left(\frac{\pi}{L}x\right) + \frac{z_{2*}}{B_2}\cos\left(\frac{2\pi}{L}x\right)\right] + o(\varepsilon^3),$$
$$v(x) = \frac{2}{c}\left[z_{0*} + z_{1*}\cos\left(\frac{\pi}{L}x\right) + z_{2*}\cos\left(\frac{2\pi}{L}x\right)\right] + o(\varepsilon^3).$$

To study the dynamics around this equilibrium, we reduce the system (11) to the system on $\mathbb{R}^2$. Now we put

$$\rho_1 := (\mu_1, \mu_2) - (b_1\mu_0/a_1, c_1\mu_0/a_1),$$
$$z_{p0} := \sqrt{-\mu_0/a_1},$$
$$\xi_0(t) := z_0(t) - z_{p0}.$$

Then we have the following lemma.

Lemma 3.5. *There exists a positive constant ε_1 such that if*

$$|(\xi_0, z_1, z_2)| + |\rho_1| < \varepsilon_1$$

then there exists a function $h_0(z_1, z_2; \rho_1)$ satisfying

$$h_0(0,0;0) = \frac{\partial h_0}{\partial z_j}(0,0;0) = \frac{\partial h_0}{\partial \rho_1}(0,0;0) = 0$$

such that the center manifold of (11) is represented as

$$\mathcal{W}_0^c := \{(z_0, z_1, z_2) \in \mathbb{R}^3; z_0 = z_{p0} + h_0(z_1, z_2; \rho_1)\}.$$

Moreover, the dynamics of (11) on the center manifold is locally equivalent to the dynamics of the following system:

$$\left\{\begin{array}{l} \dot{z}_1 = P_1(z_{p0} + h_0(z_1, z_2), z_1, z_2) + o(|(z_1, z_2)|^3), \\ \dot{z}_2 = P_2(z_{p0} + h_0(z_1, z_2), z_1, z_2) + o(|(z_1, z_2)|^3), \end{array}\right. \tag{15}$$

where

$$P_1(z_0, z_1, z_2) := (\mu_1 + b_1 z_0^2 + b_2 z_1^2 + b_3 z_2^2) z_1 + b_4 z_0 z_1 z_2,$$
$$P_2(z_0, z_1, z_2) := (\mu_2 + c_1 z_0^2 + c_2 z_1^2 + c_3 z_2^2) z_2 + c_4 z_0 z_1^2.$$

This lemma immediately follows from the center manifold reduction to the system (11) around e_0. In addition, the system (11) is invariant under the mapping:

$$(z_0, z_1, z_2) \to (-z_0, z_1, -z_2).$$

This yields that the center manifold about $-e_0$ is represented as

$$\{(z_0, z_1, z_2) \in \mathbb{R}^3; z_0 = -z_{p0} - h_0(z_1, -z_2; \rho_1)\}.$$

Therefore, if $(z_1(t), z_2(t))$ is a solution to (15) then

$$(-z_{p0} - h_0(z_1(t), -z_2(t); \rho_1), z_1(t), -z_2(t))$$

is a solution to (11) near $-e_0$. We also have the following Lemma.

Lemma 3.6.
(i) : If $c_1 < 0$ then the center manifold $\mathcal{W}_0^c$ is locally attractive;
(ii) : If $c_1 > 0$ then the center manifold $\mathcal{W}_0^c$ is unstable.
Proof. Since if $c_1 < 0$ (resp. $c_1 > 0$) then $\pm e_0$ exist if and only if $\mu_0 > 0$ (resp. $\mu_0 < 0$). That is, linearized matrix about $\pm e_0$ has two zero eigenvalues and a negative (resp. positive) eigenvalue: $-2\mu_0$. □

It should be noted that the solutions to system (15) correspond to the solutions to (11) in the case when

$$|\rho_1| = |(\mu_1, \mu_2) - (b_1\mu_0/a_1, c_1\mu_0/a_1)| < 2|\mu_0|$$

and

$$z_0(t) = z_{p0} + O(z_1(t)^2 + z_2(t)^2).$$

We also note that if system (15) has an equilibrium (z_{1*}, z_{2*}) with $z_{1*} \neq 0$ and $z_{2*} \neq 0$, then it correspond to the triply mixed mode equilibrium of (11).

Let us compute the approximation of $h_0(z_1, z_2; \rho)$. The functions $(\xi_0(t), z_1(t), z_2(t))$ satisfies the following equations:

$$\left\{\begin{array}{l} \dot{\xi}_0 = -2\mu_0\xi_0 + (3a_1\xi_0^2 + a_2 z_1^2 + a_3 z_2^2) z_{p0} + (a_1\xi_0^2 + a_2 z_1^2 + a_3 z_2^2)\xi_0 + a_4 z_1^2 z_2, \\ \dot{z}_1 = \tilde{\mu}_1 z_1 + (2b_1 z_1 \xi_0 + b_4 z_1 z_2) z_{p0} + (b_1\xi_0^2 + b_2 z_1^2 + b_3 z_2^2) z_1 + b_4 \xi_0 z_1 z_2, \\ \dot{z}_2 = \tilde{\mu}_2 + (2c_1\xi_0 z_2 + c_4 z_1^2) z_{p0} + (c_1\xi_0^2 + c_2 z_1^2 + c_3 z_2^2) z_2 + c_4 \xi_0 z_1^2. \end{array}\right. \tag{16}$$

Here $\tilde{\mu}_1 := \mu_1 - b_1\mu_0/a_1$ and $\tilde{\mu}_2 := \mu_2 - c_1\mu_0/a_1$. Then, the graph of $h_0(z_1, z_2; \rho_1)$ satisfies the following equation:

$$\begin{aligned}\frac{\partial h_0}{\partial z_1}\dot{z}_1 + \frac{\partial h_0}{\partial z_2}\dot{z}_2 &= -2\mu_0 h_0(z_1, z_2; \rho_1) + a_2(z_1^2 + a_3 z_2^2)z_{p0} \\ &+ 3z_{p0}a_1(h_0(z_1, z_2; \rho_1))^2 + \cdots .\end{aligned}$$

This yields

$$h(z_1, z_2) = \frac{-z_{p0}}{2\mu_0}(a_2 z_1^2 + a_3 z_2^2) + o(\varepsilon_1^2).$$

Thus, the dynamics on the center manifold $\mathcal{W}_0^c$ of (11) can be approximated by the following system:

$$\begin{cases} \dot{z}_1 = \mu_1 z_1 + d_{10} z_1 z_2 + (d_{11} z_1^2 + d_{12} z_2^2) z_1, \\ \dot{z}_2 = \mu_2 z_2 + d_{20} z_1^2 + (d_{21} z_1^2 + d_{22} z_2^2) z_2, \end{cases} \tag{17}$$

where $\mu_j = \tilde{\mu}_j$ and

$$d_{10} := b_4\sqrt{-\mu_0/a_1}, \quad d_{11} := \left(b_2 - \frac{a_2 b_1}{a_1}\right), \quad d_{12} := \left(b_3 - \frac{a_3 b_1}{a_1}\right),$$

$$d_{20} := c_4\sqrt{-\mu_0/a_1}, \quad d_{21} := \left(c_2 - \frac{a_2 c_1}{a_1}\right), \quad d_{22} := \left(c_3 - \frac{a_3 c_1}{a_1}\right).$$

This is a typical case of 1:2 resonance with O(2) symmetries (for instance, see [1, 6, 7]) restricted to the real subspace.

Remark. If (u^*, v^*) is a non-zero spatially uniform steady state of (4), then by changing variables $(\tilde{u}, \tilde{v}) = (u, v) - (u^*, v^*)$, we can obtain a reaction-diffusion system which has quadratic nonlinearity, and moreover, it yields a normal form with 1:2 resonance. However, for the sake of simplicity, we consider the system (17) to compute existence and stability of mixed mode solutions.

Let us solve the stationary problem of (17). Using the scaling

$$z_j = \eta\tilde{z}_j,\ \mu_j = \eta^2\tilde{\mu}_j, t = \eta\tilde{t}$$

and drop the tildes, we have

$$d_{10} z_1 z_2 + \eta(\mu_1 + d_{11} z_1^2 + d_{12} z_2^2) z_1 = 0, \tag{18}$$

$$d_{20} z_1^2 + \eta(\mu_2 + d_{21} z_1^2 + d_{22} z_2^2) z_2 = 0. \tag{19}$$

Then we obtain

$$z_1^2 = \frac{-1}{d_{11}}(\mu_1 + d_{12} z_2^2) - \frac{d_{10}}{\eta d_{11}} z_2. \tag{20}$$

Substituting (20) into (19) and using the power series method:

$$z_2 = \eta z_2^{(1)} + \eta^2 z_2^{(2)} + \eta^3 z_2^{(3)} + \cdots ,$$

we obtain

$$z_2 = -\frac{\mu_1}{d_{10}}\eta - \frac{\mu_1(d_{11}d_{10}\mu_2 + d_{20}d_{12}\mu_1)}{d_{20}d_{10^3}}\eta^3 + o(\eta^3)$$

and

$$z_1^2 = \frac{\mu_1\mu_2}{d_{20}d_{10}}\eta^2 - \left(\frac{2d_{12}\mu_1^2\mu_2}{d_{10}^3 d_{20}^3} + \frac{2d_{12}^2\mu_1^3}{d_{11}d_{10}^4}\right)\eta^3 + o(\eta^3).$$

This implies that if $\mu_1\mu_2 d_{20}d_{10} > 0$ then the triply mixed mode equilibrium exists for small η. Further calculation gives the linearized eigenvalues about these equilibria as follows:

$$\left(\frac{\mu_2}{2} \pm \sqrt{\mu_2^2 + 2\mu_1\mu_2}\right)\eta + o(\eta).$$

3.4.1. *Hopf instability of triply mixed mode solutions:* Here we consider the Hopf instability around the equilibria. Suppose that (z_{1*}, z_{2*}), ($z_{1*} \neq 0$ and $z_{2*} \neq 0$) is an equilibrium of (17), and M is a linearized matrix around (z_{1*}, z_{2*}). Then, the matrix M has purely imaginary eigenvalues if and only if $\det M > 0$ and trace $M = 0$ hold. Solving the stationary problem of (17) with $z_1 = \rho z_2$, $(\rho \in \mathbb{R})$ and trace $M = 0$ for (z_1, z_2, μ_1, μ_2), we have

$$\begin{aligned} z_2 = z_{2*} &:= \frac{\rho^2 d_{20}}{2(d_{11} + d_{22})}, \\ z_1 = z_{1*} &:= \rho z_{2*} \end{aligned}$$

and the bifurcation point is

$$\begin{aligned} \mu_1 = \mu_{1*} &:= -\frac{\rho^2 d_{20}\{2d_{10}(\rho^2 d_{11} + d_{22}) + \rho^2 d_{20}(\rho^2 d_{11} + d_{12})\}}{4(\rho^2 d_{11} + d_{22})^2}, \\ \mu_2 = \mu_{2*} &:= -\frac{\rho^4 d_{20}^2(2\rho^2 d_{11} + 3d_{22} + \rho^2 d_{21})}{4(\rho^2 d_{11} + d_{22})^2}. \end{aligned}$$

Thus, if $\det M > 0$ at $(\mu_1, \mu_2, z_1, z_2) = (\mu_{1*}, \mu_{2*}, z_{1*}, z_{2*})$, then the matrix M has a pair of purely imaginary eigenvalues at the point.

4. **Existence of time-periodic solutions around the 1-mode stationary solutions .** Let us compute the normal form for the Hopf bifurcation around e_1 in the case of

$$\text{tr } \tilde{M}_{\pm e_1} = 0 \text{ and } \det \tilde{M}_{\pm e_1} > 0.$$

More precisely,

$$\mu_2 = -\mu_0 + \mu_1(a_2 + c_2)/b_2 \quad \text{and} \quad (\mu_0 - \mu_1 a_2/b_2)^2 + \mu_1^2 a_4 c_4/b_2^2 < 0.$$

Now we put

$$\xi_1 = z - z_{p1},\ z_{p1} = \sqrt{-\mu_1/b_2},$$

and introduce a new parameter ρ_2 satisfying

$$2\rho_2 = (\mu_2 + \mu_0 + (a_2 + c_2)z_{p1}^2) = \text{tr } \tilde{M}_{\pm e_1}$$

as follows:

$$\begin{aligned} \begin{pmatrix} \dot{\xi}_1 \\ \dot{z}_0 \\ \dot{z}_2 \end{pmatrix} &= \begin{pmatrix} -2\mu_1 & 0 & 0 \\ 0 & \mu_0 + a_2 z_{p1}^2 + \rho_2 & a_4 z_{p1}^2 \\ 0 & c_4 z_{p1}^2 & -\mu_0 - a_2 z_{p1}^2 + \rho_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ z_0 \\ z_2 \end{pmatrix} \\ &\quad + \begin{pmatrix} N_1(z_0, \xi_1, z_2) \\ N_0(z_0, \xi_1, z_2) \\ N_2(z_0, \xi_1, z_2) \end{pmatrix}, \end{aligned} \tag{21}$$

where $N_j(z_0, \xi_1, z_2)$ are higher order terms. Let

$$\vec{v} = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} + i \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}, \quad (v_{jk} \in \mathbb{R})$$

be an eigenvector corresponding to an eigenvalue of $\tilde{M}_{\pm e_1}$:

$$\rho_2 + i\sqrt{-\rho_2^2 + \det \tilde{M}_{\pm e_1}}.$$

The system (21) can be diagonalized by the following:

$$\begin{pmatrix} \xi_0 \\ \xi_2 \end{pmatrix} = T^{-1} \begin{pmatrix} z_0 \\ z_2 \end{pmatrix}, T = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}.$$

It should be noted that the matrix $T = T(\rho_2)$ satisfies the following

$$T(0) = \begin{pmatrix} \nu & \omega \\ c_4 z_{p1}^2 & 0 \end{pmatrix}.$$

where

$$\nu = \mu_0 + a_2 z_{p1}^2$$
$$\omega^2 = -\nu^2 - a_4 c_4 z_{p1}^4.$$

Here we note that $\omega = \sqrt{-\nu^2 - a_4 c_4 z_{p1}^4}$ is a real value. The system of the new variables ξ_j is given by the following:

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_0 \\ \dot{\xi}_2 \end{pmatrix} = \begin{pmatrix} -2\mu_1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_0 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \tilde{N}_1(\xi_0, \xi_1, \xi_2) \\ \rho_2\xi_0 + \tilde{N}_0(\xi_0, \xi_1, \xi_2) \\ \rho_2\xi_2 + \tilde{N}_2(\xi_0, \xi_1, \xi_2) \end{pmatrix}, \tag{22}$$

where

$$\tilde{N}_1(\xi_0, \xi_1, \xi_2) = N_1(\nu\xi_0 + \omega\xi_2, \xi_1, c_4 z_{p1}^2 \xi_0),$$
$$\begin{pmatrix} \tilde{N}_0(\xi_0, \xi_1, \xi_2) \\ \tilde{N}_2(\xi_0, \xi_1, \xi_2) \end{pmatrix} = T^{-1} \begin{pmatrix} N_0(\nu\xi_0 + \omega\xi_2, \xi_1, c_4 z_{p1}^2 \xi_0) \\ N_2((\nu\xi_0 + \omega\xi_2, \xi_1, c_4 z_{p1}^2 \xi_0) \end{pmatrix}.$$

Then we can apply the center manifold theory for (22).

Lemma 4.1. *There exists a positive constant ε_2 such that if*

$$|(\xi_0, \xi_1, \xi_2)| + |\rho_2| < \varepsilon_2,$$

then, there exists a function $h_1(\xi_0, \xi_2; \rho_2)$ satisfying

$$h_1(0, 0; 0) = \frac{\partial h_1}{\partial \xi_j}(0, 0; 0) = \frac{\partial h_1}{\partial \rho_2}(0, 0; 0) = 0, \ (j = 0, 2)$$

such that the center manifold of (22) is represented as

$$\mathcal{W}_1^c := \{(\xi_0, \xi_1, \xi_2) \in \mathbb{R}^3; \xi_1 = h_1(\xi_0, \xi_2; \rho_2)\}.$$

Moreover, the dynamics of (22) on $\mathcal{W}_1^c$ is locally equivalent to the dynamics of

$$\begin{pmatrix} \dot{\xi}_0 \\ \dot{\xi}_2 \end{pmatrix} = \begin{pmatrix} \rho_2 & \omega \\ -\omega & \rho_2 \end{pmatrix} \begin{pmatrix} \xi_0 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \tilde{N}_0(\xi_0, h_1(\xi_0, \xi_2; \rho_2), \xi_2) \\ \tilde{N}_2(\xi_0, h_1(\xi_0, \xi_2; \rho_2), \xi_2) \end{pmatrix}.$$

4.1. **Approximation of h_1.** Let us compute the approximation of h_1. Differentiating $\xi_1(t) = h_1(\xi_0(t), \xi_2(t); \rho_2)$ with respect to t, we have

$$-2\mu_1 h_1(\xi_0, \xi_2) + \tilde{N}_1(\xi_0, \xi_2) = \omega\xi_2 \frac{\partial h_1}{\partial xi_0}(\xi_0, \xi_2) + \frac{\partial h_1}{\partial \xi_2}(\xi_0, \xi_2).$$

Substituting $h_1 = \gamma_{20}\xi_0^2 + \gamma_{11}\xi_0\xi_2 + \gamma_{02}\xi_2^2$, and equating quadratic terms, we obtain

$$\begin{aligned}\gamma_{20} &= \frac{-1}{4b_2 z_{p1}(\omega^2 + b_2^2 z_{p1}^4)}(-2b_2 z_{p1}^2 b_1 \omega^2 \nu - b_2 z_{p1}^4 b_4 \omega^2 c_4 + \omega^4 b_1 \\ &\quad + \omega^2 b_3 c_4^2 z_{p1}^4 + \omega^2 c_4 \nu b_4 z_{p1}^2 + \omega^2 b_1 \nu^2 + 2b_2^2 z_{p1}^4 b_1 \nu^2 \\ &\quad + 2b_2^2 z_{p1}^8 b_3 c_4^2 + 2b_2^2 z_{p1}^6 c_4 \nu b_4),\end{aligned}$$

$$\gamma_{11} = \frac{-\omega z_{p1}}{2(\omega^2 + b_2^2 z_{p1}^4)}(b_3 c_4^2 z_{p1}^4 + c_4 \nu b_4 z_{p1}^2 + 2b_2 z_{p1}^2 b_1 \nu + b_2 z_{p1}^4 b_4 c_4 - \omega^2 b_1 + b_1 \nu^2),$$

$$\begin{aligned}\gamma_{02} &= \frac{-\omega^2}{4b_2 z_{p1}(\omega^2 + b_2^2 z_{p1}^4)}(b_3 c_4^2 z_{p1}^4 + c_4 \nu b_4 z_{p1}^2 + 2b_2 z_{p1}^2 b_1 \nu \\ &\quad + b_2 z_{p1}^4 b_4 c_4 + \omega^2 b_1 + b_1 \nu^2 + 2b_2^2 z_{p1}^4 b_1).\end{aligned}$$

Then the dynamics of (22) on the center manifold $\mathcal{W}_1^c$ is approximated by the dynamics of the following:

$$\begin{pmatrix} \dot{\xi}_0 \\ \dot{\xi}_2 \end{pmatrix} = \begin{pmatrix} \rho_2 & \omega \\ -\omega & \rho_2 \end{pmatrix} \begin{pmatrix} \xi_0 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \tilde{N}_0(\xi_0, h_1(\xi_0, \xi_2), \xi_2) \\ \tilde{N}_2(\xi_0, h_1(\xi_0, \xi_2), \xi_2) \end{pmatrix}. \quad (23)$$

4.2. **Normal form for the Hopf bifurcation around 1-mode solutions.** Finally, we obtain the following result:

Lemma 4.2. *The system (23) can be transformed by a parameter-dependent change of complex coordinate and a nonlinear time re-parameterization, into a single equation of the form*

$$\frac{dz}{d\tau} = (\beta + i)z + \varsigma|z|^2 z + o(|z|^3), \quad (24)$$

where z and τ are a new complex coordinate and a new time, respectively, and β is a new parameter satisfying $d\beta/d\rho_2 = 0$ at $\rho_2 = 0$, and moreover, ς is given by the following:

$$\begin{aligned}\varsigma &= \text{sign}\Big[\frac{1}{2c_4 z_{p1}^2}(2c_4 z_{p1}^2 c_1 \omega^2 + 6c_4 z_{p1}^2 c_1 \nu^2 + 12c_4 z_{p1}^3 c_2 \gamma_{20} \\ &\quad + 4c_4 z_{p1}^3 c_2 \gamma_{02} + 6c_4^3 z_{p1}^6 c_3 + 12c_4 z_{p1} \nu \gamma_{20} + 4c_4 z_{p1} \nu \gamma_{02} + 4c_4 z_{p1} \omega \gamma_{11}) \\ &\quad + \frac{1}{z_{p1}\omega}(3z_{p1} a_1 \omega^3 + 3z_{p1} \omega a_1 \nu^2 + 2\omega a_2 z_{p1}^2 \gamma_{20} + 6\omega a_2 z_{p1}^2 \gamma_{02} \\ &\quad + \omega a_3 c_4^2 z_{p1}^5 + 2\nu a_2 z_{p1}^2 \gamma_{11} + 2a_4 z_{p1}^4 c_4 \gamma_{11} \\ &\quad - 2z_{p1} c_1 \omega \nu^2 - 2\nu z_{p1}^2 c_2 \gamma_{11} - 2\nu\omega\gamma_{20} - 6\nu\omega\gamma_{02} - 2\nu^2\gamma_{11})\Big].\end{aligned} \quad (25)$$

The constants in the general normal form (24) are depend on a parameter μ_0 even though μ_1 fixed so that the equilibria $\pm e_1$ exist. This implies that the stabilities of the periodic solutions around $\pm e_1$ are dependent on the parameters. Now it should be noted that if we consider the case when

$$(\mu_0, \mu_2) = (a_2\mu_1/b_2, c_2\mu_1/b_2), \quad (26)$$

then the constant ς is independent of the parameters ν_j, and (sign of) it can be simply determined. In fact, $\mu_0 = a_2\mu_1/b_2$ yields $\nu = 0$, and ς is computed as follows:

$$\varsigma = \text{sign} \ \left[a_4^2(3P_1 + P_2) - a_4c_4(P_3 + 3P_4)\right], \tag{27}$$

where

$$\begin{aligned}
P_1 &= \frac{1}{2b_2(a_4c_4 - b_2^2)}(2a_1b_2c_4a_4 - 2a_1b_2^3 - a_2c_4a_4b_1 \\
&\quad + a_2c_4b_4b_2 + a_2c_4^2b_3 + 2a_2b_1b_2^2), \\
P_2 &= \frac{1}{2b_2(a_4c_4 - b_2^2)}(-2c_1b_2^3 + 2c_1b_2c_4a_4 - c_2c_4a_4b_1 + c_2c_4b_4b_2 \\
&\quad + c_2c_4^2b_3 + 2c_2b_1b_2^2 + 2b_2c_4a_4b_1 + 2b_2^2c_4b_4 + 2c_4^2b_3), \\
P_3 &= -\frac{1}{2b_2(a_4c_4 - b_2^2)}(a_2a_4^2b_1 + a_2a_4b_4b_2 - a_2a_4c_4b_3 + 2a_2b_3b_2^2 \\
&\quad + 2a_3b_2^3 - 2a_3b_2c_4a_4 - 2b_2a_4^2b_1 - 2b_2^2a_4b_4 - 2b_2a_4c_4b_3), \\
P_4 &= -\frac{1}{2b_2(a_4c_4 - b_2^2)}(c_2a_4^2b_1 + c_2a_4b_4b_2 \\
&\quad - c_2a_4c_4b_3 + 2c_2b_3b_2^2 + 2c_3b_2^3 - 2c_3b_2c_4a_4).
\end{aligned}$$

The proof of Lemma 4.2 is obtained directly by applying the elementary normal form transformation for the Hopf bifurcation to (23) (see Lemma 3.3–3.7 in [8]). We give the proof in the appendix A.

Let us summarize the obtained results by the following theorem which is the main results of this paper.

Theorem 4.3.
Consider the system (4) with assumptions (A1) – (A4). Take the parameters in (4) so that the 1-mode stationary solutions: $\pm(U_1(x), V_1(x))$ obtained in theorem 2 exist. Then, the following hold:

(i) *If $(\mu_0 - \mu_1a_2/b_2)^2 + a_4c_4 < 0$ (μ_j, a_j, b_2 and c_4 are coefficients of (11)) and if $\varsigma \neq 0$ (ς is given in (26)), then the system (4) has time periodic solutions bifurcated from 1-mode stationary solutions through the Hopf bifurcation;*

(ii) *The time periodic solutions obtained in* (i) *are locally asymptotically stable if and only if $b_2 < 0$ and $\varsigma = -1$.*

Remark. If $a_2 > 0$ and the same hypotheses in Theorem 4.3 hold, then the time periodic solutions around 1-mode stationary solutions exist even though the eigenvalues corresponding to the spatially uniform eigenfunction (0-mode) are negative.

4.3. **Numerical results:** Here we present the numerical results to (4) corresponding to the Hopf bifurcation studied in the previous subsection. Therefore, we focus our attention on behaviors of solutions, especially, around the 1-mode stationary solutions corresponding to equilibria $\pm e_1$ of (11). Let us study the case when

$$\begin{aligned} &D_1 = 1/4, \quad a = 1, \quad b = -10, \quad c = 2, \quad d = -5, \\ &F(u,v) = -u^3 \quad \text{and} \quad G(u,v) = -0.9u^3. \end{aligned} \tag{28}$$

Then we have $s^* = 3$ and

$$(k_0^{1,2}, D_2^{1,2}) \approx (0.87, 25.88).$$

The coefficients of the normal form (11) are listed as follows:

$$\begin{array}{llll} a_1 \approx 100.00, & a_2 \approx 14644.17, & a_3 \approx 1.69 \times 10^5, & a_4 \approx 2.45 \times 10^5, \\ b_1 \approx -49.29, & b_2 \approx -1203.01, & b_3 \approx -27695.10, & b_4 \approx -1652.32, \\ c_1 \approx -67.14, & c_2 \approx -3277.22, & c_3 \approx -18861.66, & c_4 \approx -97.761. \end{array} \tag{29}$$

It follows that $a_4 c_4 < 0$ (one of the necessary conditions for the Hopf bifurcation is satisfied). Now the special case (26) and the explicit form (27) is convenient to determine the type of Hopf bifurcation (super, or sub critical) since it is independent of the parameters. In the case (28), we can compute $\varsigma = -1$ by using (27). Therefore, time periodic solutions around 1-mode stationary solutions exist, and moreover, they are locally asymptotically stable in this case (see Figure 2).

Let us see the numerical results to (4). Now we set

$$k_0 = 0.857490\,(\Leftrightarrow L \approx 3.663708), \;\; s = 2.978084.$$

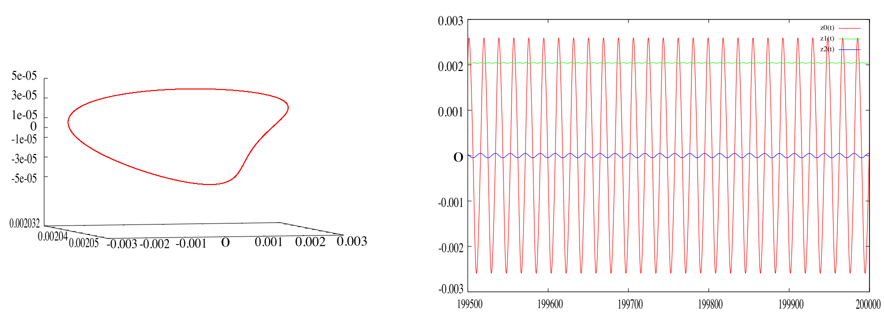

FIGURE 2. Numerical results to the normal form (11) in the case of (29) near a Hopf bifurcation point. The parameters are $\mu_0 = -0.609489$, $\mu_1 = 0.0052$ and $\mu_2 = 0.013621$. [Left: Periodic orbits of $(z_0(t), z_1(t), z_2(t))$.] [Right: The graph of $z_0(t)$ (blue line), $z_1(t)$ (green line) and $z_2(t)$ (blue line).]

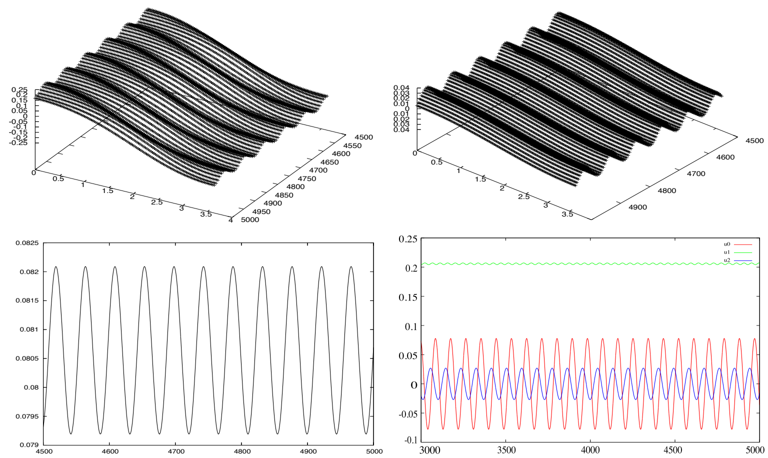

FIGURE 3. Numerical results for the reaction-diffusion system (4) with $D_2 = 27.13$ in the case of (28). The initial values are stable 1-mode stationary solutions at $D_2 = 27.0$. [Above: The left and right figures correspond to the graph of $u(t,x)$ and $v(t,x)$, ($t \in [4500, 5000], x \in [0, L]$), respectively.] [Below left: The graph of $\|(u,v)\|_{L^2}(t)$, $t \in [4500, 5000]$. Vertical axis : L^2 norm of u, horizontal axis: t.] [Below right: Graph of $u_0(t)$ (blue line), $u_1(t)$ (green line) and $u_2(t)$ (blue line). $u_j(t)$ are j-th mode Fourier coefficients of $u(t,x)$]

5. **Hopf bifurcation phenomena around the other equilibria.** In this section, we compute the normal forms for the Hopf bifurcation around the equilibria: $\pm e_{02}^{\pm}$ and triply mixed mode solutions of (11). Since the results in this section can be obtained by the similar calculation in the previous section, we omit the proofs. It should also be noted that the system (11) is invariant under the mappings:

$$(z_0, z_1, z_2) \to -(z_0, z_1, z_2) \quad \text{and} \quad (z_0, z_1, z_2) \to (z_0, -z_1, z_2).$$

This implies that if $(z_0(t), z_1(t), z_2(t))$ is a solution of (11), then

$$-(z_0(t), z_1(t), z_2(t)) \text{ and } \pm(z_0(t), -z_1(t), z_2(t))$$

are also solutions of (11). Therefore, it is sufficient to consider the Hopf bifurcation phenomena around the equilibrium (z_{1*}, z_{2*}) of (17), and around the equilibria $e_{02}^{\pm}$ of (11).

5.1. **Hopf bifurcation around doubly mixed mode equilibria $e_{02}^{\pm}$:** We show the normal form around an equilibrium e_{02}^{+} of (11). Here we assume that $a_1c_3 - c_1a_3 > 0$. We put

$$\xi_0(t) := z_0(t) - z_0^*,\ \xi_2(t) := z_2(t) - z_2^*,$$
$$\nu_1 := \mu_1 + b_1(z_0^*)^2 + b_3(z_2^*)^2 + b_4 z_0^* z_2^*.$$

Then we have

$$\dot{\mathbf{z}} = A\mathbf{z} + N(\mathbf{z}), \tag{30}$$

where

$$\begin{aligned}
\mathbf{z} &= {}^t(z_1, \xi_0, \xi_2), \\
A &= \begin{pmatrix} \nu_1 & 0 & 0 \\ 0 & 2a_1(z_0^*)^2 & 2a_3 z_0^* z_2^* \\ 0 & 2c_1 z_0^* z_2^* & 2c_3(z_2^*)^2 \end{pmatrix}, \\
N(\mathbf{z}) &= {}^t(N_1(\xi_0, z_1, \xi_2), N_0(\xi_0, z_1, \xi_2), N_2(\xi_0, z_1, \xi_2)) \\
&= \begin{pmatrix} \sum_{j+\ell+m=2} \beta_{j\ell m}\xi_0 z_1 \xi_2 \\ \sum_{j+\ell+m=2} \alpha_{j\ell m}\xi_0 z_1 \xi_2 \\ \sum_{j+\ell+m=2} \gamma_{j\ell m}\xi_0 z_1 \xi_2 \end{pmatrix} + \begin{pmatrix} (b_1\xi_0^2 + b_2 z_1^2 + b_3\xi_2^2)z_1 + b_4\xi_0 z_1 \xi_2 \\ (a_1\xi_0^2 + a_2 z_1^2 + a_3\xi_2^2)\xi_0 + a_4 z_1^2 \xi_2 \\ (c_1\xi_0^2 + c_2 z_1^2 + c_3\xi_2^2)\xi_2 + c_4\xi_0 z_1^2 \end{pmatrix}.
\end{aligned}$$

Here,

$$\begin{aligned}
&\alpha_{200} = 3a_1 z_0^*,\ \alpha_{020} = a_2 z_0^* + a_4 z_2^*, \\
&\alpha_{002} = a_3 z_0^*,\ \alpha_{101} = 2a_3 z_2^*, \\
&\alpha_{110} = \alpha_{011} = 0, \\
&\beta_{110} = 2b_1 z_0^* + b_4 z_2^*,\ \beta_{011} = 2b_3 z_2^* + b_4 z_0^*, \\
&\beta_{200} = \beta_{002} = \beta_{101} = 0, \\
&\gamma_{200} = c_1 z_2^*,\ \gamma_{020} = c_2 z_2^* + c_4 z_0^*, \\
&\gamma_{002} = 3c_3 z_2^*,\ \gamma_{101} = 2c_1 z_0^*, \\
&\gamma_{110} = \gamma_{011} = 0.
\end{aligned}$$

Let us consider the case when $a_1(z_0^*)^2 + c_3(z_2^*)^2 = 0$. In this case, the system (30) can be transformed to the following system:

$$\dot{\tilde{\mathbf{z}}} = \tilde{\mathbf{A}}\tilde{\mathbf{z}} + \mathbf{T}^{-1}\mathbf{N}(\tilde{\mathbf{z}}), \tag{31}$$

where

$$\begin{aligned}
&\tilde{\mathbf{z}} = {}^t(z_1, \tilde{\xi}_0, \tilde{\xi}_2) = T^{-1}\mathbf{z}, \\
&\tilde{A} = \begin{pmatrix} \nu_1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & -\omega & 0 \end{pmatrix},\ T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2a_3 z_0^* z_2^* & 0 \\ 0 & -2a_1(z_0^*)^2 & \omega \end{pmatrix}.
\end{aligned}$$

Here, $\omega := 2\sqrt{a_1 c_3 - a_3 c_1}|z_0^* z_2^*| > 0$. Then the following hold:

Lemma 5.1. *There exists a positive constant ε_3 such that if*

$$|(\tilde{\xi}_0, z_1, \tilde{\xi}_2)| + |a_1 z_0^* + c_3 z_2^*| < \varepsilon_3,$$

then there exists a function $h_{z_1}(\tilde{\xi}_0, \tilde{\xi}_2)$ satisfying

$$h_{z_1}(0,0) = \frac{\partial h_{z_1}}{\partial \tilde{\xi}_j}(0,0) = 0$$

such that the center manifold of (31) is represented as

$$\mathcal{W}_2^c := \{(\tilde{\xi}_0, \tilde{\xi}_2) \in \mathbb{R}^2; z_1 = h_{z_1}(\tilde{\xi}_0, \tilde{\xi}_2)\}.$$

Moreover, there exists a constant δ satisfying $|\delta| < \varepsilon_3$ such that the dynamics of (11) on $\mathcal{W}_2^c$ is locally equivalent to the dynamics of the following system:

$$\begin{pmatrix} \dot{\tilde{\xi}}_0 \\ \dot{\tilde{\xi}}_2 \end{pmatrix} = \begin{pmatrix} \delta & \omega \\ -\omega & \delta \end{pmatrix} \begin{pmatrix} \tilde{\xi}_0 \\ \tilde{\xi}_2 \end{pmatrix} + \begin{pmatrix} N_0(\tilde{\xi}_0, h_{z_1}(\tilde{\xi}_0, \tilde{\xi}_2), \tilde{\xi}_2) \\ N_2(\tilde{\xi}_0, h_{z_1}(\tilde{\xi}_0, \tilde{\xi}_2), \tilde{\xi}_2) \end{pmatrix}. \tag{32}$$

Lemma 5.2.
(i) : If $\nu_1 < 0$ *then the center manifold* $\mathcal{W}_1^c$ *is locally attractive;*
(ii) : If $\nu_1 > 0$ *then the center manifold* $\mathcal{W}_1^c$ *is unstable.*

We can also compute the normal form around $(\tilde{\xi}_0, \tilde{\xi}_2) = (0, 0)$. Here we present the following results without proof.

Lemma 5.3. *The system (32) can be transformed by a parameter-dependent change of complex coordinate and a nonlinear time re-parameterization, into a single equation of the form*

$$\frac{dz}{d\tau} = (\beta + i)z + \varsigma_4 |z|^2 z + o(|z|^3), \tag{33}$$

where z *and* τ *are a new complex coordinate and the new time, respectively, and* $\beta = \beta(\delta)$ *is a new parameter satisfying* $\beta(0) = 0$ *and* $\partial\beta/\partial\delta \neq 0$ *at* $\delta = 0$*, and moreover,* ς_4 *is given by the following:*

$$\varsigma_4 := \text{sign}\left[\text{Re}\left\{\frac{-g_{20}g_{11}(2\bar{\lambda} + \lambda)}{2|\lambda|^2} - \frac{|g_{11}|}{\bar{\lambda}} + \frac{|g_{02}|^2}{2(\lambda - 2\bar{\lambda})} + \frac{g_{21}}{2}\right\}\right], \tag{34}$$

where λ *denotes* $i\omega$*, and*

$$\begin{aligned}
g_{20} &= \frac{-1}{\omega^2 a_3 z_0^* z_2^*}[(\omega + 2ia_1(z_0^*)^2)(-\alpha_{200} + \alpha_{002} - i\alpha_{101}) \\
&\quad + 2a_3 z_0^* z_2^*(-i\gamma_{200} + i\gamma_{002} + \gamma_{101})], \\
g_{11} &= \frac{1}{\omega^2 a_3 z_0^* z_2^*}[(\omega + 2ia_1(z_0^*)^2)(\alpha_{200} + \alpha_{002}) + 2ia_3 z_0^* z_2^*(\gamma_{200} + \gamma_{002})], \\
g_{02} &= \frac{-1}{\omega^2 a_3 z_0^* z_2^*}[(\omega + 2ia_1(z_0^*)^2)(-\alpha_{200} + \alpha_{002} + i\alpha_{101}) \\
&\quad + 2a_3 z_0^* z_2^*(-i\gamma_{200} + i\gamma_{002} - \gamma_{101})], \\
g_{21} &= \frac{1}{\omega^3 a_3 z_0^* z_2^*}[(\omega + 2ia_1(z_0^*)^2)(3a_1 + a_3) + 2ia_3 z_0^* z_2^*(3c_1 + c_3)].
\end{aligned}$$

We can conclude that the system (4) has a time periodic solution bifurcated from a doubly mixed mode solution corresponding to the equilibrium $(z_0^*, 0, z_2^*)$ of (11), moreover, it is locally asymptotically stable if and only if $\nu_1 < 0$ and $\varsigma_4 = -1$. Replacing (z_0^*, z_2^*) with $\pm(z_0^*, -z_2^*)$(or $-(z_0^*, z_2^*)$), a constant ς_4 in the normal form is computable for the all doubly mixed mode solutions: $(z_0, z_1, z_2) = \pm e_{02}^{\pm}$.

5.2. **Hopf bifurcation around the triply mixed mode solutions:** We also consider the Hopf bifurcation phenomena around the equilibrium (z_{1*}, z_{2*}) of (17). The corresponding triply mixed mode stationary solution of (11) is

$$(z_0, z_1, z_2) = (h_0(z_{1*}, z_{2*}), z_{1*}, z_{2*}).$$

Putting

$$\tilde{z}_1 = z_1 - z_{1*}, \tilde{z}_2 = z_2 - z_{2*},$$

the system (17) can be transformed to the system

$$\begin{cases} \dot{\tilde{z}}_1 = \displaystyle\sum_{j+\ell\leq 3} \alpha_{j\ell}\tilde{z}_1^j \tilde{z}_2^\ell, \\ \dot{\tilde{z}}_2 = \displaystyle\sum_{j+\ell\leq 3} \beta_{j\ell}\tilde{z}_1^j \tilde{z}_2^\ell, \end{cases} \tag{35}$$

where

$$\begin{array}{ll}
\alpha_{10} = \mu_1 + d_{10}z_{2*} + 3d_{11}z_{1*} + d_{12}z_{2*}, & \alpha_{01} = d_{10}z_{1*} + 2d_{12}z_{1*}z_{2*},\\
\alpha_{20} = 3d_{11}z_{1*}, & \alpha_{11} = d_{10} + d_{12}(2z_{2*}+1)z_{1*}z_{2*},\\
\alpha_{02} = d_{12}z_{2*}, & \alpha_{30} = d_{11},\\
\alpha_{12} = d_{12}, & \alpha_{21} = \alpha_{03} = 0,\\
 & \\
\beta_{10} = 2d_{20}z_{1*} + 2d_{21}z_{1*}z_{2*}, & \beta_{01} = \mu_2 + 3d_{22}z_{2*} + d_{21}z_{1*},\\
\beta_{20} = d_{20} + d_{21}z_{2*}, & \beta_{11} = 2d_{21}z_{1*},\\
\beta_{02} = 3d_{22}z_{2*}, & \beta_{21} = d_{21},\\
\beta_{03} = d_{22}, & \beta_{12} = \beta_{30} = 0.
\end{array}$$

We consider the case when α_{10} and β_{01} are small, that is,

$$\mu_1 \approx -(d_{10}z_{2_*} + 3d_{11}z_{1_*} + d12z_{2_*}), \quad \mu_2 \approx -(3d_{22}z_{2_*} + d_{21}z_{1_*}).$$

Let $\omega = \sqrt{-\alpha_{01}\beta_{10}}$ and $\lambda = i\omega$. We have similarly the following lemma.

Lemma 5.4. *The system (35) can be transformed by a parameter-dependent change of complex coordinate and a nonlinear time re-parameterization, into a single equation of the form*

$$\frac{dz}{d\tau} = (\beta + i)z + \varsigma_5|z|^2 z + o(|z|^3), \tag{36}$$

where z and τ are a new complex coordinate and a new time, respectively, and $\beta = \beta(\alpha_{10}, \beta_{01})$ is a new parameter satisfying $\beta(0,0) = 0$ and $\partial\beta/\partial\alpha_{10} \neq 0$, $\partial\beta/\partial\beta_{01} \neq 0$ at $\alpha_{10} = \beta_{01} = 0$, and moreover, ς_5 is given by the following:

$$\varsigma_5 := \mathrm{sign}\left[\mathrm{Re}\left\{\frac{-g_{20}g_{11}(2\bar{\lambda}+\lambda)}{2|\lambda|^2} - \frac{|g_{11}|}{\bar{\lambda}} + \frac{|g_{02}|^2}{2(\lambda - 2\bar{\lambda})} + \frac{g_{21}}{2}\right\}\right]. \tag{37}$$

Here,

$$\begin{aligned}
g_{20} &= (\alpha_{20}\alpha_{01} + i\omega\alpha_{11} + i\beta_{10}\alpha_{02}) - i\left(\beta_{20}\alpha_{01}^2/\omega + i\beta_{11}\alpha_{01} - i\omega\beta_{02}\right),\\
g_{11} &= (\alpha_{20}\alpha_{01} - \beta_{10}\alpha_{02}) - i(\beta_{20}\alpha_{01}^2/\omega + i\omega\beta_{02}),\\
g_{02} &= (\alpha_{02}\alpha_{01} - i\omega\alpha_{11} + i\beta_{10}\alpha_{02}) - i(\beta_{02}\alpha_{01}^2/\omega - i\alpha_{01}\beta_{11} - i\omega\beta_{02}),\\
g_{21} &= 3\alpha_{30}\alpha_{01}^2 + i\alpha_{21}\alpha_{01}\omega + \alpha_{12}\omega^2 - 3i\alpha_{03}\beta_{10}\omega\\
&\quad -i(3\beta_{30}\alpha_{01}^3/\omega + i\beta_{21}\alpha_{01}^2 + \beta_{12}\alpha_{01}\omega + 3i\beta_{03}\omega^2).
\end{aligned}$$

It should be noted that the time periodic-solutions to (4) bifurcated from triply mixed mode solutions are asymptotically locally stable if and only if $a_2 < 0$ and $\varsigma_5 = -1$.

Remark. Since $|z_{j_*}| < \varepsilon_1$, $(j = 1, 2)$ is assumed, we have

$$\alpha_{01} = d_{10}z_{1*} + O(\varepsilon_1^2),\ \beta_{10} = 2d_{20}z_{1*} + O(\varepsilon_1^2).$$

This implies that

$$\mathrm{sign}\{d_{10}d_{20}\} = \mathrm{sign}\{c_4 b_4\} = -1$$

is necessary to the Hopf bifurcation phenomena around triply mixed mode solutions for small ε_1.

6. **Numerical studies to chaotic dynamics in the normal form.** We have shown the normal forms with 0:1:2 resonance with odd symmetry in reaction terms F and G. They give us the existence and stability of spatially non-uniform stationary solutions, and time periodic solutions in (4). We would also like to introduce chaotic behaviors in another set of parameter values. We only consider the normal form in the case of (11) throughout this section.

6.1. **Results of numerical studies to the normal form.** Let us see numerical results. We can observe complex oscillatory dynamics in the normal form (11) (see Figure 4 – 7 and Table 1). They suggest that there is a homoclinic orbit which connect a triply mixed mode fixed point and itself (since of symmetries of normal form (11), there could be four homoclinic orbits), and they could induce the complex oscillation as shown in figures.

It should be noted that since the parameter s is real-valued, it is impossible that $\mu_0(s)$ attains the values listed in the caption of the figures. However, the

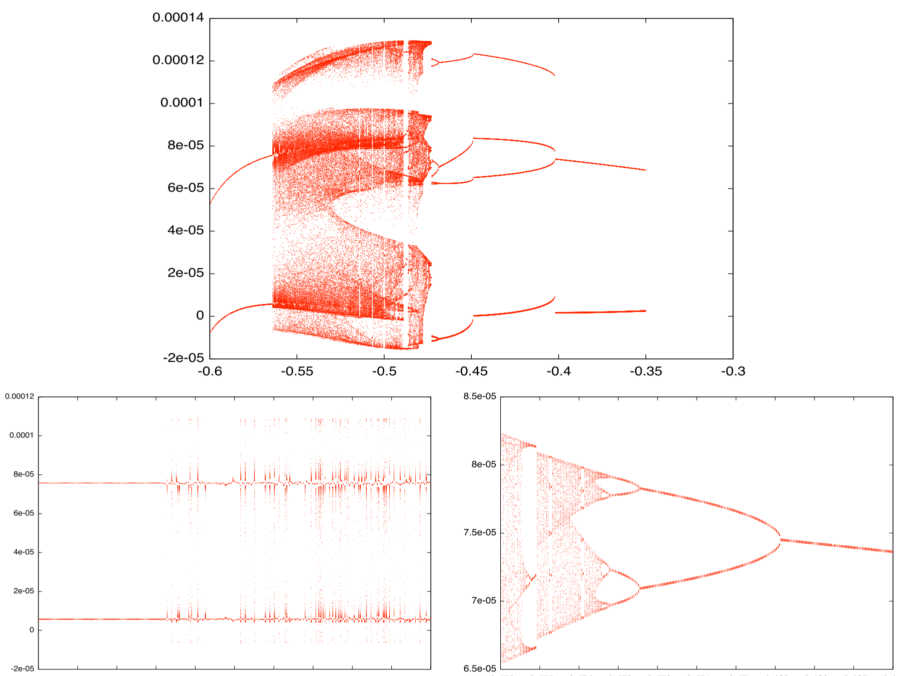

FIGURE 4. Bifurcation diagram of the Poincaré map of (11) on the section $z_0 = z_{0_*}$ (z_{0*} is a coordinate of the equilibrium: $(z_0(t), z_1(t), z_2(t)) = (z_{0*}, z_{1*}, z_{2*})$). Parameters are $\mu_1 = 0.0052, \mu_2 = -0.002$. The vertical and horizontal axes correspond to z_3 and μ_0, respectively. [Above : $\mu_0 \in [-0.6, -0.35]$ [Below left : Close-up view of the above in $\mu_0 \in [-0.5644, -0.5643]$. The periodic orbit disappear around $\mu_0 \approx 0.56437$.] [Below right : Close-up view of the above in $\mu_0 \in [-0.476, -0.466], z_2 \in [6.5 \times 10^{-5}, 8.5 \times 10^{-5}]$]

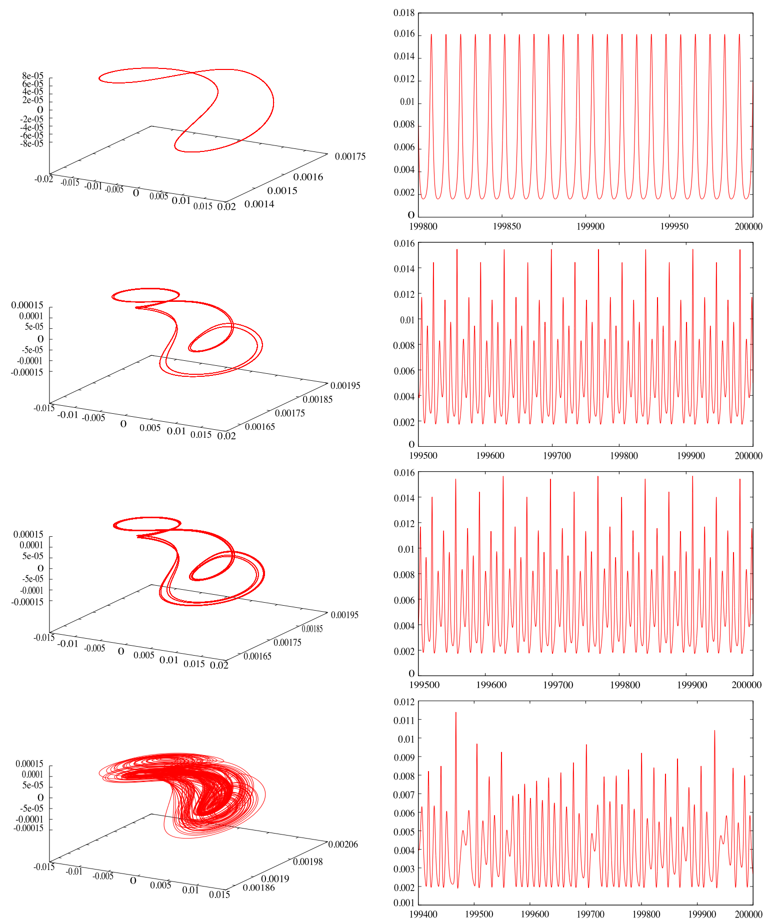

FIGURE 5. The orbits (left) and $\sqrt{z_0(t)^2 + z_1(t)^2 + z_2(t)^2}$ (right) for typical parameter values $\mu_1 = 0.0052$ and $\mu_2 = -0.002$. We change μ_0 as $\mu_0 = -0.38, -0.471, -0.473, -0.55$ from the above.

system (11) is invariant under the scaling (12), we can choose a parameter s so that the normal form (11) has small amplitude similar solutions by a suitable choice of scaling parameter η.

TABLE 1. Equilibriums of (11) with the constants (29) and their linearized eigenvalues. Parameters are $\mu_0 = -0.55$, $\mu_1 = 0.0052$ and $\mu_2 = -0.002$ (This case correspond to Figure 5).

Equilibria	Eigenvalues		
$(z_0, z_1, z_2) = (0, 0, 0)$	-0.550000	0.005200	-0.00200
$(\pm 0.074162, 0, 0)$	-0.37127	-0.265895	1.100000
$(0, \pm 0.002079, 0)$	0.079174	-0.011472	-0.010400
$\pm(-0.004570, 0.001968, 0.000106)$	-0.014227	$0.007193 \pm 0.035272i$	
$\pm(-0.004570, -0.001968, 0.000106)$	-0.014227	$0.007193 \pm 0.035272i$	

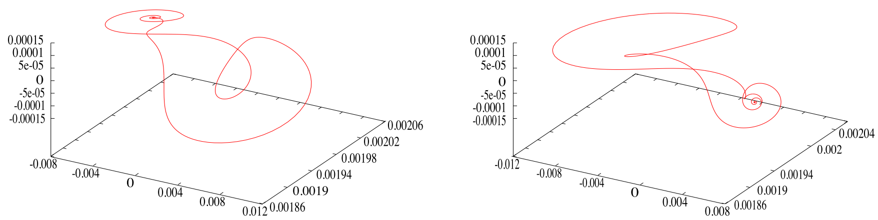

FIGURE 6. Left and right figures show the orbits from the initial conditions which are close to the triply mixed mode fixed points ($\mu_0 = -0.55, \mu_1 = 0.0052, \mu_2 = -0.002.\ \ 0 \le t \le 210$). These numerical results suggest that there is a pair of homoclinic orbits in the half of the phase space $\{(z_0, z_1, z_2) \in \mathbb{R}^3; z_1 > 0\}$.

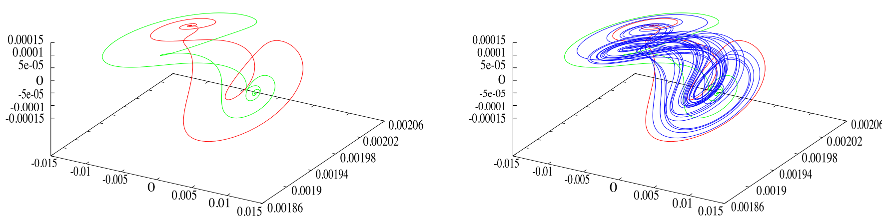

FIGURE 7. [Left: The orbits in fig. 6 are shown.] [Right: The orbits in fig. 6 and the attractor are shown as well.]

6.2. **Lyapunov characteristic exponents and Lyapunov dimension.** In order to see that there are chaotic dynamics, we compute the Lyapunov characteristic exponents by using the algorithm in [10]: more precisely, let

$$\dot{\mathbf{z}} = F(\mathbf{z}) \tag{38}$$

be a system of differential equation in $\mathbb{R}^3$, and let $\{\mathbf{e}_j\}$, $j = 1, 2, 3$ be a set of basis of tangent space at $\mathbf{z} = \mathbf{z}_0 := \mathbf{z}(0)$. Consider the variational equation around the flow $\mathbf{z}(t; \mathbf{z}_0)$:

$$\dot{\mathbf{y}}(t) = DF(\mathbf{z}(t; \mathbf{z}_0))\mathbf{y}(t). \tag{39}$$

Then, the solution of (39) can be written as $\mathbf{y}(t) = U^t\mathbf{y}(0)$, where U^t is the fundamental matrix. We define

$$\lambda(e^1, \mathbf{z}_0) = \lim_{t\to\infty} t^{-1} \log\left(|U^t\mathbf{e}_1|/|\mathbf{e}_1|\right),$$
$$\lambda(e^2, \mathbf{z}_0) = \lim_{t\to\infty} t^{-1} \log\left(|U^t\mathbf{e}_1 \times U^t\mathbf{e}_2|/|\mathbf{e}_1 \times \mathbf{e}_2|\right),$$
$$\lambda(e^3, \mathbf{z}_0) = \lim_{t\to\infty} t^{-1} \log\left(|U^t\mathbf{e}_1 \cdot (U^t\mathbf{e}_2 \times U^t\mathbf{e}_3)|/|\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)|\right),$$

where e^j are j-dimensional space defined by $e^j = span\{\mathbf{e}_1, \ldots, \mathbf{e}_j\} \subset \mathbb{R}^3$, moreover $\circ \cdot \circ$ and $\circ \times \circ$ denote inner product and exterior product, respectively. We can compute the Lyapunov characteristic exponents λ_j, $j = 1, 2, 3$ by using $\lambda(e^j, z_0)$.

Now we show the results to our problem. We solve the equations (11) by Runge-Kutta-Fehlberg method checking the difference of the fourth order and the fifth order approximation is smaller than 10^{-6}. Here we also choose the scaling parameters as $\eta = 4$. The typical time difference is taken as 10^{-2}, and we compute $0 \le t \le T$. The vectors $\mathbf{e}_j$ are normalized in each step as follows: $\mathbf{e}_1$ is taken as a tangent vector of the trajectory ($\lambda(e^1, \mathbf{z_0})$ must attain zero) ; The bases $\{\mathbf{e}_j\}$ are normalized by the Gram-Schmidt orthonormalization. We also set

$$\mu_0 = -0.559489, \quad \mu_1 = 0.0052, \quad \mu_2 = -0.002.$$

Then, we obtain the numerical results as shown in Table 2 for each T.

TABLE 2. The Lyapunov characteristic exponents for each T.

T	λ_1	λ_2	λ_3
5×10^3	0.014125	0.000286	−0.136404
10^4	0.021547	0.000112	−0.136693
5×10^4	0.025032	−0.000014	−0.137087
10^5	0.023560	−0.000000	−0.137040
5×10^5	0.023984	−0.000028	−0.137153
10^6	0.023906	−0.000024	−0.137146
5×10^6	0.023768	−0.000025	−0.137136

And we estimate

$$\lambda_1 \approx 0.0240, \quad \lambda_2 \approx 0.0000, \quad \lambda_3 \approx -0.1371.$$

We can also compute Lyapunov dimension of the attractor as follows (for instance, see [4]): let j be an integer satisfying

$$\sum_{\ell=1}^{j} \lambda_\ell > 0 \text{ and } \sum_{\ell=1}^{j+1} \lambda_\ell < 0,$$

then the Lyapunov dimension d_f is defined by

$$d_f = j + \frac{\sum_{\ell=1}^{j} \lambda_\ell}{\lambda_{j+1}}.$$

In our case, we obtain

$$d_f \approx 2.175.$$

These numerical results suggest that the normal form (11) yields chaotic dynamics by a suitable choice of parameters and coefficients.

7. Reaction-diffusion equation with two positive and negative global feedback.

As we showed in the introduction, the following 3-component system (2):

$$\begin{cases} u_t = D_1 u_{xx} + au + bv + sw + F(u, v), & x \in \Omega, t \ge 0, \\ \tau_1 v_t = D_2 v_{xx} + u - v + G(u, v), & x \in \Omega, t \ge 0 \\ \tau_2 w_t = D_3 w_{xx} + u - w, & x \in \Omega, t \ge 0 \\ u_x = v_x = w_x = 0, & \text{at } x = 0, L \end{cases}$$

is convenient to check the condition of the Hopf bifurcation. In this section, we set the higher order terms $F(u, v) = F_1(u)$ and $G(u, v) = F_2(u)$ (namely, they are independent of second variable v). And we also assume the following:

- $F_j(u) = -F_j(-u)$ $(j = 1, 2)$ hold, and they can be represented in Taylor series $F_j(u) = \alpha_j u^3 + h.o.t.$, $\alpha_j \in \mathbb{R}$;
- The coefficients in (2) satisfy the assumptions (A3) and (A4) in the section 2 (with $c = 1$, $d = -1$);
- $-1/b < D_2/D_1$ and $D_2 \ll D_3$ hold.

It should also be noted that the results shown in this section holds under the appropriate settings of function space similarly to section 2. One of the advantage in the system (2) is that we can obtain a coefficient of normal form for the Hopf bifurcation around 1-mode stationary solutions more simply than (4). This is essentially because the system can be reduced to the scalar equation with two global feedback effects as follows. Here we note again that if $\mathbf{u}(t, x) = (u, v, w)(t, x)$ is a solution to (2) then

$$\tilde{\mathbf{u}}(t, x) := \begin{cases} \mathbf{u}(t, x) & x \in (0, L), \\ \mathbf{u}(t, 2L - x) & x \in (L, 2L) \end{cases}$$

is a solution of the same equations under periodic boundary condition with period $2L$ satisfying even symmetries. Thus, it is sufficient to consider the equations in (2) with periodic boundary conditions on a interval $(0, 2L)$ with even symmetry.

Let $\tau_j = 0$, $s_1 = -b > 0$ and $s = s_2 > 0$. We have the equation in a Fourier space which is equivalent to (2) as follows:

$$\dot{u}_m = \lambda_m u_m + [F_1(u)]_m - s_1 \beta_2^{(m)} [F_2(u)]_m, \quad m \in \mathbb{N} \cup \{0\}, \tag{40}$$

where

$$\lambda_m := \lambda_{mk_0} = a - D_1 m^2 k_0^2 - s_1 \beta_2^{(m)} + s_2 \beta_3^{(m)}, \quad m \in \mathbb{N} \cup \{0\},$$

$$\beta_j^{(m)} := (1 + D_j m^2 k_0^2)^{-1} \quad j = 2, 3, \, m \in \mathbb{N} \cup \{0\},$$

$$[F_j(u)]_m := \alpha_j \sum_{m_1 + m_2 + m_3 = m} u_{m_1} u_{m_2} u_{m_3},$$
$$j = 1, 2, \quad m_\ell \in \mathbb{Z} \ (\ell = 1, 2, 3), \quad m \in \mathbb{N} \cup \{0\}.$$

We also consider the dynamics of the equation (40) around a critical point, especially, triply degenerate point. It holds that $\lambda_0 = \lambda_1 = \lambda_2 = 0$ if $(k_0, a, s_2) = (k_0^*, a^*, s_2^*)$ and

$$s_1 > \frac{D_1 D_3}{D_2 (D_3 - D_2)}, \tag{41}$$

where

$$k_0^{*2} = \frac{-5 D_1 D_3 \pm \sqrt{9 D_1^2 D_3^2 + 16 s_1 D_1 D_2 D_3 (D_3 - D_2)}}{8 D_1 D_2 D_3},$$

$$s_2^* = \frac{(s_1 D_2 - D_1 - D_1 D_2 k_0^{*2})(1 + D_3 k_0^{*2})}{(1 + D_2 k_0^{*2}) D_3}$$

and

$$a^* = D_1 k_0^{*2} + \frac{s_1}{1 + D_2 k_0^{*2}} - \frac{s_2^*}{1 + D_3 k_0^{*2}}.$$

We note that k_0^* attains the real value if and only if the inequality (41) holds, and a^* is positive if $s_1 - s_2^* > 0$ holds.

Applying the center manifold theorem around this critical point, we have the reduced equation on the local center manifold $\mathcal{W}^c_{loc}$ of (40). Moreover, we obtain the following third order truncated system which approximate the dynamics on the manifold $\mathcal{W}^c_{loc}$:

$$\begin{cases} \dot{u}_0 = \lambda_0 u_0 + \gamma_0[\ (u_0^2 + 6u_1^2 + 6u_2^2)u_0 + 3u_1^2 u_2\ ], \\ \dot{u}_1 = \lambda_1 u_1 + \gamma_1[\ (3u_0^2 + 3u_1^2 + 6u_2^2)u_1 + 6u_0 u_1 u_2\ ], \\ \dot{u}_2 = \lambda_2 u_2 + \gamma_2[\ (3u_0^2 + 6u_1^2 + 3u_2^2)u_2 + 3u_0 u_1^2\ ], \end{cases} \tag{42}$$

where

$$\gamma_j = \alpha_1 - s_1\alpha_2\beta_2^{(j)}.$$

Now we seek a sufficient condition for the super critical Hopf bifurcation around the equilibria of (42): $(0, \pm\sqrt{-\lambda_1/(3\gamma_1)}, 0)$ which correspond to the 1-mode stationary solutions of (40). It should be noted again that if $(u_0(t), u_1(t), u_2(t))$ is a solutions of (40), then $-(u_0(t), u_1(t), u_2(t))$ is also a solution of (40). Therefore, it is sufficient to consider the Hopf bifurcation around $(0, \sqrt{-\lambda_1/(3\gamma_1)}, 0)$. Now we assume that $\alpha_j < 0$ $(j = 1, 2)$. In this case, it holds that if

$$s_1\alpha_2 < \alpha_1 < \frac{s_1\alpha_2}{1 + D_2 k_0^{*2}} \tag{43}$$

then

$$\gamma_0 > 0 \text{ and } \gamma_j < 0,\ j = 1, 2.$$

This implies that the equilibrium of (42) corresponding to 1-mode solution can be destabilized with purely imaginary eigenvalues. Using the lemmas in section 4, we can compute a coefficient ς of normal form for the Hopf bifurcation around an equilibrium $(0, \sqrt{-\lambda_1/(3\gamma_1)}, 0)$ as follows:

$$\varsigma = \text{sign}\left(\frac{27\gamma_0}{\gamma_1^2 - \gamma_0\gamma_2}\mathcal{F}(\gamma_0, \gamma_1, \gamma_2)\right),$$

where

$$\mathcal{F}(\gamma_0, \gamma_1, \gamma_2) = -(5\gamma_0^2 + 7\gamma_0\gamma_2 + 15\gamma_2^2)\gamma_1^2 - 8\gamma_0\gamma_2(\gamma_0 + \gamma_2)\gamma_1 + \gamma_0^3\gamma_2 + 7\gamma_0\gamma_2^3 - 5\gamma_0^2\gamma_2^2.$$

Here, we using the formula (27) for the shake of simplicity. In fact, the constants ς computed here can be determined as follows: we can see that

$$5\gamma_0^2 + 7\gamma_0\gamma_2 + 15\gamma_2^2 = 5\left[\left(\gamma_0 + \frac{7}{10}\gamma_2\right)^2 + \frac{251}{100}\gamma_2^2\right] > 0$$

and

$$(\text{Discriminant of } \mathcal{F}) = 4\gamma_0\gamma_2(5\gamma_0^4 - 2\gamma_0^3\gamma_2 + 47\gamma_0^2\gamma_2^2 - 10\gamma_0\gamma_2^3 + 105\gamma_2^4).$$

Then, if the inequality (43) holds then $\mathcal{F}(\gamma_0, \gamma_1, \gamma_2)$ is negative. This yields $\varsigma = -1$, namely, there exist the periodic solutions around the equilibria of (42) corresponding to 1-mode stationary solutions bifurcated from the super critical Hopf bifurcation.

This result is obtained in the case when the higher order terms $F(u, v) = F_1(u)$ and $G(u, v) = F_2(u)$ are independent of v. In this case, if $F_1(u) \equiv 0$ (or $F_2(u) \equiv 0$), then the Hopf bifurcation cannot occur, namely, the both cubic terms $F(u, v) = F_1(u) = \alpha_1 u^3 + h.o.t.$ and $G(u, v) = F_2(u) = \alpha_2 u^3 + h.o.t.$, $(\alpha_j \neq 0)$ are necessary to the Hopf bifurcation around the 1-mode solutions.

8. **Discussion.** Here, we show the numerical results to the reaction-diffusion system (4) with the constants and higher order terms shown in (28). Setting

$$D_2 = 26.5, \quad k_0 = 0.874919,$$

then we can also observe "chaotic" solutions as Figure 8 (see also Figure 9). It should be noted that μ_0 is dependent on only one parameter s, and it increases as s approaches to 3 from the left.

To check the sensitivity to initial conditions, we also show a result of the following numerical experiment: let $(u_0^{(1)}(x), v_0^{(1)}(x))$ belong to a function space X, and put

$$(u_0^{(2)}(x), v_0^{(2)}(x)) := (u_0^{(1)}(x), v_0^{(1)}(x)) + (10^{-6}, 0).$$

Let $(u^{(1)}, v^{(1)})$ and $(u^{(2)}, v^{(2)})$ be solutions of (4) satisfying $(u^{(j)}(0,x), v^{(j)}(0,x)) = (u_0^{(j)}(x), v_0^{(j)})$, $j = 1, 2$. We show the logarithmic plot of the difference of the two solutions in the right of Figure 9. These results agree with the results in the section 6.

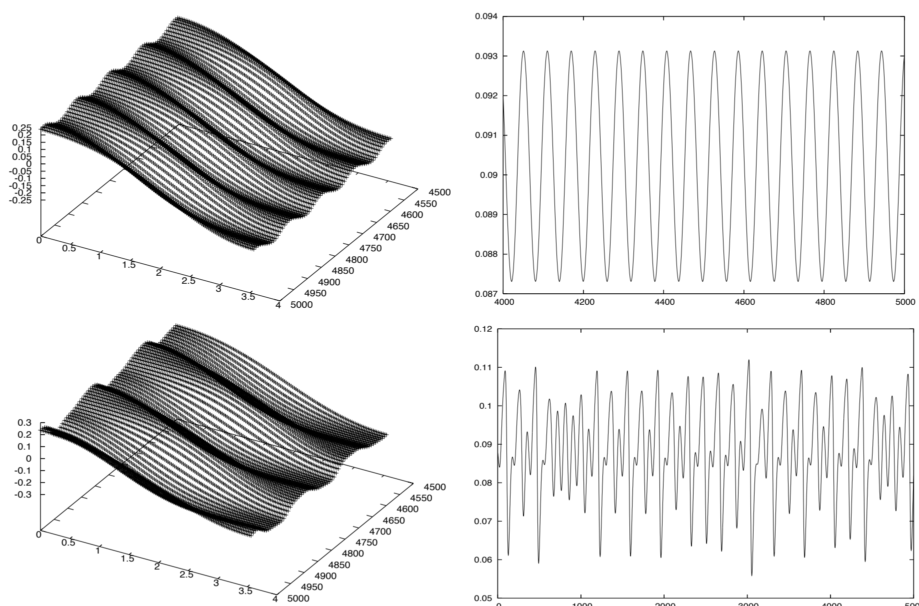

FIGURE 8. Numerical results of (4) for $s = 2.979$ (above) and $s = 2.98212$ (below). The left and right figures correspond to the graph of $u(t,x)$ and $\|u\|_{L^2}(t)$, for each s, respectively.

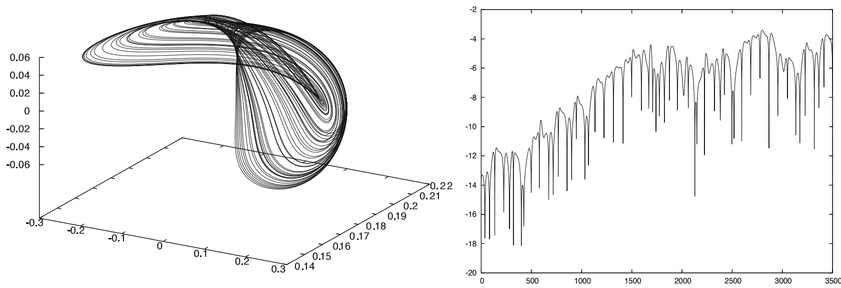

FIGURE 9. [Left: Dynamics of Fourier coefficients $(u_0(t), u_1(t), u_2(t))$ of the solution of (4) for $s = 2.98212$ (This corresponds to the PDE simulation in Figure 8 (below).] [Right:Logarithmic scale plot of the difference of the L^2 norm of the two solutions $u^{(1)}(t)$ and $u^{(2)}(t)$. The vertical and horizontal axes correspond to $\log |\ \|u^{(1)}\|_{L^2} - \|u^{(2)}\|_{L^2}\ |$ and t, respectively.]

Appendix A: Proof of Lemma 4.2. Here we give the proof of Lemma 4.2:

Proof. Now we put $\sigma := 2\rho_2$ and

$$A = \begin{pmatrix} \rho_2 & \omega \\ -\omega & \rho_2 \end{pmatrix}.$$

Then eigenvalues of the matrix A are λ and $\bar{\lambda}$, where

$$\lambda = \frac{\sigma}{2} + i\omega.$$

Let $q = (q_1, q_2) \in \mathbb{C}^2$ be an eigenvector of A corresponding to the eigenvalue λ: $Aq = \lambda q$, and let $p = (p_1, p_2) \in \mathbb{C}^2$ be an eigenvector of the transposed matrix ${}^tA$ corresponding to its eigenvalue $\bar{\lambda}$. It can be normalized with respect to q: $< p, q >= 1$, where

$$< p, q >:= \bar{p}_1 q_1 + \bar{p}_2 q_2.$$

Then the vector (z_0, z_2) can be uniquely represented as follows:

$$(z_0, z_2) = zq + \bar{z}\bar{q}.$$

Moreover, we have

$$z =< p, (z_0, z_2) > .$$

Then the system (23) can be transformed into the single equation:

$$\dot{z} = \lambda z + < p, N(zq + \bar{z}\bar{q}) >, \tag{44}$$

where $N =^t (N_0(z_1, z_2), N_2(z_0, z_2))$. It also follows that $< p, N >$ is represented as polynomial formula:

$$< p, N >= \frac{R_{30}}{6} z^3 + \frac{R_{21}}{2} z^2 \bar{z} + \frac{R_{12}}{2} z\bar{z}^2 + \frac{R_{03}}{6} \bar{z}^3.$$

Using a near-identity transformation:

$$z = \tilde{z} + \frac{h_{30}}{6}\tilde{z}^3 + \frac{h_{21}}{2}\tilde{z}^2\bar{\tilde{z}} + \frac{h_{12}}{2}\tilde{z}\bar{\tilde{z}}^2 + \frac{h_{03}}{6}\bar{\tilde{z}}^3,$$

where

$$h_{30} = \frac{R_{30}}{2\lambda}, h_{12} = \frac{R_{12}}{2\bar{\lambda}}, h_{03} = \frac{R_{03}}{3\bar{\lambda} - \lambda}.$$

Then (44) becomes

$$\dot{\tilde{z}} = \lambda z + \varsigma_1 |\tilde{z}|^2 \tilde{z} + o(|z|^3).$$

Here,

$$\varsigma_1 := \frac{R_{21}}{2}.$$

By time scaling $\tilde{t} = \omega t$, we have

$$\frac{d\tilde{z}}{d\tilde{t}} = (\beta + i)\tilde{z} + \varsigma_2 |\tilde{z}|^2 \tilde{z},$$

where

$$\beta = \frac{\sigma}{2\omega}, \ \varsigma_2 = \frac{\varsigma_1}{\omega}.$$

Change the time parameterization by:

$$d\tau = (1 + \varsigma_3 |\tilde{z}|^2) d\tilde{t},$$

where $\varsigma_3 = \operatorname{Im} \varsigma_2$ and using new parameterization of the time, we have

$$\frac{d\tilde{z}}{d\tau} = (\beta + i)\tilde{z} + l_1 |\tilde{z}|^2 \tilde{z} + o(|\tilde{z}|^3),$$

where

$$l_1 = \frac{\operatorname{Re} \varsigma_1}{\omega_0}.$$

Finally, by changing variable

$$\tilde{z} = \frac{z}{\sqrt{|l_1|}},$$

we have

$$\frac{dz}{d\tau} = (\beta + i)z + \varsigma |z|^2 z + o(|z|^3).$$

with $\varsigma = l_1/|l_1| = \operatorname{sign} \operatorname{Re} \varsigma_1$. Direct computation yields the explicit form of ς as shown in the lemma. □

Appendix B: Explicit form of the coefficients of normal form (11). We show the coefficients of reduced system (11) explicitly. We put $A_m := -a + D_1^{1,2} m^2 (k_0^{1,2})^2$, then we have the following.

$$a_j = -\frac{1}{\mu_0^-} P_f^{a_j} + \frac{b}{d\mu_0^-} P_g^{a_j}, \ j = 1 \dots 4,$$

$$b_j = -\frac{1}{\mu_1^-} P_f^{b_j} - \frac{A_1}{c\mu_1^-} P_g^{b_j}, \ j = 1 \dots 4,$$

$$c_j = -\frac{1}{\mu_2^-} P_f^{c_j} - \frac{A_2}{d\mu_2^-} P_g^{c_j}, \ j = 1 \dots 4.$$

Here,

$$\begin{pmatrix} P_f^{a_1} \\ P_g^{a_1} \end{pmatrix} := B_0^3 \begin{pmatrix} f_{30} \\ g_{30} \end{pmatrix} + cB_0^2 \begin{pmatrix} f_{21} \\ g_{21} \end{pmatrix} + c^2 B_0 \begin{pmatrix} f_{12} \\ g_{12} \end{pmatrix} + c^3 \begin{pmatrix} f_{03} \\ g_{03} \end{pmatrix},$$

$$\begin{aligned} \begin{pmatrix} P_f^{a_2} \\ P_g^{a_2} \end{pmatrix} &:= 6B_1^2 B_0 \begin{pmatrix} f_{30} \\ g_{30} \end{pmatrix} + 2cB_1(B_1 + 2B_0) \begin{pmatrix} f_{21} \\ g_{21} \end{pmatrix} \\ &\quad + 2c^2(B_0 + 2B_1) \begin{pmatrix} f_{12} \\ g_{12} \end{pmatrix} + 6c^3 \begin{pmatrix} f_{03} \\ g_{03} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} P_f^{a_3} \\ P_g^{a_3} \end{pmatrix} &:= 6B_2^2 B_0 \begin{pmatrix} f_{30} \\ g_{30} \end{pmatrix} + 2cB_2(B_2 + 2B_0) \begin{pmatrix} f_{21} \\ g_{21} \end{pmatrix} \\ &\quad + 2c^2(B_0 + 2B_2) \begin{pmatrix} f_{12} \\ g_{12} \end{pmatrix} + 6c^3 \begin{pmatrix} f_{03} \\ g_{03} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} P_f^{a_4} \\ P_g^{a_4} \end{pmatrix} &:= 6B_1^2 B_2 \begin{pmatrix} f_{30} \\ g_{30} \end{pmatrix} + 2cB_1(B_1 + 2B_2) \begin{pmatrix} f_{21} \\ g_{21} \end{pmatrix} \\ &\quad + 2c^2(B_2 + 2B_1) \begin{pmatrix} f_{12} \\ g_{12} \end{pmatrix} + 6c^3 \begin{pmatrix} f_{03} \\ g_{03} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} P_f^{b_1} \\ P_g^{b_1} \end{pmatrix} &:= 3B_0^2 B_1 \begin{pmatrix} f_{30} \\ g_{30} \end{pmatrix} + cB_0(B_0 + 2B_1) \begin{pmatrix} f_{21} \\ g_{21} \end{pmatrix} \\ &\quad + c^2(B_1 + 2B_0) \begin{pmatrix} f_{12} \\ g_{12} \end{pmatrix} + 3c^3 \begin{pmatrix} f_{03} \\ g_{03} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} P_f^{b_2} \\ P_g^{b_2} \end{pmatrix} &:= 3B_1^3 \begin{pmatrix} f_{30} \\ g_{30} \end{pmatrix} + 3cB_1^2 \begin{pmatrix} f_{21} \\ g_{21} \end{pmatrix} \\ &\quad + 3c^2 B_1 \begin{pmatrix} f_{12} \\ g_{12} \end{pmatrix} + 3c^3 \begin{pmatrix} f_{03} \\ g_{03} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} P_f^{b_3} \\ P_g^{b_3} \end{pmatrix} &:= 6B_2^2 B_1 \begin{pmatrix} f_{30} \\ g_{30} \end{pmatrix} + 2cB_2(B_2 + 2B_1) \begin{pmatrix} f_{21} \\ g_{21} \end{pmatrix} \\ &\quad + 2c^2(B_1 + 2B_2) \begin{pmatrix} f_{12} \\ g_{12} \end{pmatrix} + 6c^3 \begin{pmatrix} f_{03} \\ g_{03} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} P_f^{b_4} \\ P_g^{b_4} \end{pmatrix} &:= 6B_0 B_1 B_2 \begin{pmatrix} f_{30} \\ g_{30} \end{pmatrix} + 2c(B_0 B_1 + B_1 B_2 + B_2 B_0) \begin{pmatrix} f_{21} \\ g_{21} \end{pmatrix} \\ &\quad + 2c^2(B_0 + B_1 + B_2) \begin{pmatrix} f_{12} \\ g_{12} \end{pmatrix} + 6c^3 \begin{pmatrix} f_{03} \\ g_{03} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} P_f^{c_1} \\ P_g^{c_1} \end{pmatrix} &:= 3B_0^2 B_2 \begin{pmatrix} f_{30} \\ g_{30} \end{pmatrix} + cB_0(B_0 + 2B_2) \begin{pmatrix} f_{21} \\ g_{21} \end{pmatrix} \\ &\quad + c^2(B_2 + 2B_0) \begin{pmatrix} f_{12} \\ g_{12} \end{pmatrix} + 3c^3 \begin{pmatrix} f_{03} \\ g_{03} \end{pmatrix}, \end{aligned}$$

$$\begin{pmatrix} P_f^{c_2} \\ P_g^{c_2} \end{pmatrix} := 6B_1^2 B_2 \begin{pmatrix} f_{30} \\ g_{30} \end{pmatrix} + 2cB_1(B_1+2B_2) \begin{pmatrix} f_{21} \\ g_{21} \end{pmatrix} + 2c^2(B_2+2B_1) \begin{pmatrix} f_{12} \\ g_{12} \end{pmatrix} + 6c^3 \begin{pmatrix} f_{03} \\ g_{03} \end{pmatrix},$$

$$\begin{pmatrix} P_f^{c_3} \\ P_g^{c_3} \end{pmatrix} := 3B_2^3 \begin{pmatrix} f_{30} \\ g_{30} \end{pmatrix} + 3cB_2^2 \begin{pmatrix} f_{21} \\ g_{21} \end{pmatrix} + 3c^2 B_2 \begin{pmatrix} f_{12} \\ g_{12} \end{pmatrix} + 3c^3 \begin{pmatrix} f_{03} \\ g_{03} \end{pmatrix},$$

$$\begin{pmatrix} P_f^{c_4} \\ P_g^{c_4} \end{pmatrix} := 3B_1^2 B_0 \begin{pmatrix} f_{30} \\ g_{30} \end{pmatrix} + cB_1(B_1+2B_0) \begin{pmatrix} f_{21} \\ g_{21} \end{pmatrix} + c^2(B_0+2B_1) \begin{pmatrix} f_{12} \\ g_{12} \end{pmatrix} + 3c^3 \begin{pmatrix} f_{03} \\ g_{03} \end{pmatrix}.$$

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