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PERIODICALLY GROWING SOLUTIONS IN A CLASS OF STRONGLY MONOTONE SEMIFLOWS

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Dedicated to Professor Hiroshi Matano on the occasion of his 60th birthday

Abstract. We study the behavior of unbounded global orbits in a class of strongly monotone semiflows and give a criterion for the existence of orbits with periodic growth. We also prove the uniqueness and asymptotic stability of such orbits. We apply our results to a certain class of nonlinear parabolic equations including a weakly anisotropic curvature flow in a two-dimensional annulus and show the convergence of the solutions to a periodically growing solution which grows up in infinite time changing its profile time-periodically.

1. **Introduction.** Various types of differential equations enjoy a comparison principle, which has been a useful method of analysis for many problems — to show the existence of equilibria, to prove the convergence of solutions, and so on. Such equations, characterized as dynamical systems having an order-preserving property, include second-order scalar parabolic equations or weakly coupled systems of ordinary, parabolic and delay differential equations.

Hirsch [2, 3] and Matano [8] independently developed the theory of infinite-dimensional dynamical systems called *strongly monotone semiflows* or *strongly order-preserving semiflows* which have a stronger version of order-preserving property.

One of the most important result in the theory of order-preserving dynamical systems is the quasi-convergence (that is, convergence to the set of equilibrium points) or the convergence of the generic orbit. Hirsch [2, 3] proved that almost all of the bounded orbits of a strongly monotone semiflow are quasi-convergent. Matano [9] (and also Smith and Thieme [16]) extended this result using the strong order-preserving property which is more flexible than strong monotonicity used by Hirsch. The convergence result was given by Poláčik [15] for abstract semilinear parabolic evolution equations, and later extended by Smith and Thieme [17] to general strongly order-preserving semilflows. However, little is known about the behavior of the orbit which is defined globally in time but not bounded.

There are several earlier works on the asymptotic behavior of unbounded time-global solutions for nonlinear parabolic equations. Namah and Roquejoffre [12]

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studied the equation

$$u_t = \Delta u + f(x, u, \nabla u), \quad x \in \mathbb{R}^N, \ t > 0, \tag{1}$$

where the nonlinearity $f(x, u, p)$ is 1-periodic in x, u and satisfies $f(x, u, p) \geq m$ for some $m > 0$. The comparison principle yields that every solution $u(x, t)$ of (1) has no L^∞-bound. Under some growth conditions on f, they proved that there exist a unique $T > 0$ and a unique (up to time shift) $\varphi(x, t) \in C^2(\mathbb{R}^N \times \mathbb{R})$, 1-periodic in x and T-periodic in t such that

$$U(x, t) = \frac{t}{T} + \varphi(x, t) \tag{2}$$

is a solution of (1). Furthermore, they proved the asymptotic stability of the solution. In [14], the authors of the present paper studied the following semilinear parabolic equation related to a model of spiral crystal growth:

$$\begin{cases} u_t = \Delta u + f(u - \sigma\theta), & x \in \Omega, \ t > 0, \\ u_r = 0, & x \in \partial\Omega, \ t > 0, \end{cases} \tag{3}$$

where σ is a positive integer, Ω is a two-dimensional annulus, (r, θ) denotes the polar coordinates of $x \in \overline{\Omega}$ and $f \in C^1$ is a 2π-periodic function. Under the supposition that f has positive integral mean, every solution $u(x, t)$ of (3) has no L^∞-bound. They proved the existence and asymptotic stability of the spiral traveling wave solution of the form

$$U(x, t) = \varphi\Big(r, \theta - \frac{\omega}{\sigma} t\Big) + \omega t, \tag{4}$$

where $\varphi = \varphi(r, \theta)$ is $2\pi/\sigma$-periodic in θ and $\omega > 0$. Both solutions U in (2) and (4) have the following property: there exist positive constants T and L satisfying

$$U(x, t + T) = U(x, t) + L \quad \text{for all } \ x, t. \tag{5}$$

In other words, they grow up in infinite time varying their speeds and profiles time-periodically. We call such solutions *periodically growing solutions*.

In the present paper we will discuss the existence, uniqueness and asymptotic stability of periodically growing solutions in a more general framework, namely that of strongly monotone semiflows having a certain compactness property. This generalization applies to a much wider class of nonlinear parabolic problems including (1) and (3).

This paper is organized as follows: in Section 2 we state our main theorems, Theorems 2.1 to 2.3. Theorems 2.1 and 2.2 assert the existence and uniqueness of orbits with periodic growth for a class of strongly monotone semiflows. Theorem 2.3 asserts that the orbit with periodic growth is asymptotically stable. We prove Theorems 2.1 and 2.2 in Section 3, and Theorem 2.3 in Section 4. Some applications of the main theorems to nonlinear parabolic equations are given in Section 5. In Appendix we recall the results of Ogiwara and Matano [13] on the structure of a class of subsets in an ordered metric space, which will be helpful in proving our theorems.

2. **Main Results.**

Let X be a strongly ordered Banach space with norm $\|\cdot\|$, namely, X is a real Banach space and a partial order relation on X is defined by a closed convex cone $X_+ \subset X$ with nonempty interior $\mathrm{int}(X_+)$. In what follows we use the following notation:

$$\begin{aligned} u \le v &\quad \text{if } v - u \in X_+, \\ u < v &\quad \text{if } u \le v \text{ and } u \ne v, \\ u \ll v &\quad \text{if } v - u \in \operatorname{int}(X_+). \end{aligned}$$

A set $B \subset X$ is called *order-bounded* if there exist $u_1, u_2 \in X$, $u_1 \le u_2$ satisfying $u_1 \le v \le u_2$ for all $v \in B$.

Remark 1. Since X is strongly ordered, any bounded set in X is order-bounded. In particular, for any $e \in \operatorname{int}(X_+)$ and $v \in X$, there exists a positive constant k satisfying $-ke \le v \le ke$. In fact, since $B_r(e) = \{u \in X \mid \|u - e\| < r\} \subset X_+$ for some $r > 0$, we have

$$e \pm \frac{r}{\|v\| + 1} v \in B_r(e),$$

hence

$$-\frac{\|v\| + 1}{r} e \le v \le \frac{\|v\| + 1}{r} e.$$

Let $\Phi = \{\Phi_t\}_{t \ge 0}$ be a global semiflow defined on a closed subset V of X. More precisely, Φ is a family of mappings $\Phi_t : V \to V$ with the following properties:

(i) $\Phi_t(u)$ is continuous in $(t, u) \in [0, +\infty) \times V$;
(ii) $\Phi_0(u) = u$ for all $u \in V$;
(iii) $\Phi_t \circ \Phi_s = \Phi_{t+s}$ for any $t, s \ge 0$.

In this paper, we assume that

(H1) For each $t > 0$, Φ_t is strongly monotone (that is, $u < v$ implies $\Phi_t(u) \ll \Phi_t(v)$ for all $u, v \in V$);
(H2) For each $t > 0$, Φ_t is order-compact (that is, $\Phi_t(B)$ is relatively compact whenever $B \subset V$ is order-bounded);
(H3) There exists an element $e \in \operatorname{int}(X_+)$ such that for any $u \in V$, $t \ge 0$, we have $u + e \in V$ and $\Phi_t(u + e) = \Phi_t(u) + e$;
(H4) There exist $u_0 \in V$, $\{t_j\}_j \subset \mathbb{R}$ and $\{k_j\}_j \subset \mathbb{N}$ with $t_j \to +\infty$, $k_j \to \infty$ such that the set $\{\Phi_{t_j}(u_0) - k_j e\}_{j \in \mathbb{N}}$ is order-bounded.

The condition (H3) implies that

$$\Phi_t(u + ke) = \Phi_t(u) + ke \quad \text{for all } u \in V,\ k \in \mathbb{Z},\ t \ge 0. \tag{6}$$

This can be regarded as an invariance of the semiflow Φ under some group action in the following way: let $\mathbb{Z}$ be the additive group of integers and define a continuous mapping $\gamma : \mathbb{Z} \times V \to V$ by

$$\gamma(k, u) := u + ke.$$

Then $k \mapsto \gamma(k, \cdot)$ is a group homomorphism of $\mathbb{Z}$ into $\operatorname{Hom}(V)$, the group of homeomorphism of V into itself, hence $\mathbb{Z}$ acts on V. Furthermore, (6) means that the action γ commutes with Φ_t for each $t \ge 0$, namely,

$$\Phi_t(\gamma(k, u)) = \gamma(k, \Phi_t(u)) \quad \text{for } k \in \mathbb{Z},\ u \in V,\ t \ge 0.$$

Some examples of nonlinear parabolic equations which satisfy all the assumptions (H1)-(H4) are given in Section 5.

Our results are stated as follows:

Theorem 2.1. *There exist $\varphi \in V$ and $T > 0$ such that*

$$\Phi_T(\varphi) = \varphi + e. \tag{7}$$

Furthermore, if there exists $\psi \in V$ satisfying $\Phi_S(\psi) = \psi + e$ for some $S > 0$, then $S = T$.

By (7) and (H3), we obtain $\Phi_{nT}(\varphi) = \varphi + ne$ for all $n \in \mathbb{N}$. This means that the orbit of φ is not order-bounded (hence unbounded) in V and grows in e-direction. Furthermore, modulo the growing direction the orbit behaves periodically in time. The latter statement implies the uniqueness of the growth speed.

Set $W = \{u \in V \mid \Phi_T(u) = u + e\} \ni \varphi$. For each $t < 0$ we define a map $\Phi_t : W \to W$ by

$$\Phi_t(u) := \Phi_{t+m(t)T}(u) - m(t)e,$$

where $m(t)$ is the smallest nonnegative integer such that $t + m(t)T \geq 0$. One can easily see that $\Phi^* = \{\Phi_t\}_{t\in\mathbb{R}}$ is a continuous one-parameter group acting on W and is a flow extension of the semiflow Φ. Furthermore, by (H1) and (H3), Φ_t is monotone (that is, $u \leq v$ implies $\Phi_t(u) \leq \Phi_t(v)$) for each $t \in \mathbb{R}$. The Φ^*-orbit of u is denoted by $O(u) = \{\Phi_t(u) \mid t \in \mathbb{R}\}$.

Theorem 2.2. *The group Φ^* acts on W transitively. In other words, $W = O(\varphi)$. Furthermore, the orbit $O(\varphi)$ is strongly monotone increasing in t, namely, $\Phi_{t_1}(\varphi) \ll \Phi_{t_2}(\varphi)$ holds for any $t_1 < t_2$.*

The above theorem implies the uniqueness and monotonicity of the orbit with periodic growth.

Theorem 2.3. *Let $\varphi \in V$ be as in Theorem 2.1 and let $O(\varphi)$ is the orbit of φ. Then we have the following:*

(i) *$O(\varphi)$ is order-stable in the following sense: for any $v \in O(\varphi)$ and $\varepsilon > 0$ there exists some $\delta > 0$ such that $-\delta e \ll u - v \ll \delta e$ implies $-\varepsilon e \ll \Phi_t(u) - \Phi_t(v) \ll \varepsilon e$ for all $t \geq 0$.*

(ii) *$O(\varphi)$ is stable in the sense of Lyapunov, that is, for any $v \in O(\varphi)$ and $\varepsilon > 0$ there exists some $\delta > 0$ such that $\|u - v\| < \delta$ implies $\|\Phi_t(u) - \Phi_t(v)\| < \varepsilon$ for all $t \geq 0$.*

(iii) *For any $u \in V$, there exists some $\tau \in \mathbb{R}$ such that*

$$\lim_{t\to+\infty} \|\Phi_t(u) - \Phi_{t+\tau}(\varphi)\| = 0.$$

This theorem yields that the orbit $O(\varphi)$ is globally stable with asymptotic phase.

Remark 2. Theorems 2.1-2.3 remain true if we replace (H2) and (H4) by (H2′) and (H4′) below.

(H2′) For each $t > 0$, Φ_t is compact;

(H4′) There exist $u_0 \in V$, $\{t_j\}_j \subset \mathbb{R}$ and $\{k_j\}_j \subset \mathbb{N}$ with $t_j \to +\infty$, $k_j \to \infty$ such that the set $\{\Phi_{t_j}(u_0) - k_j e\}_{j\in\mathbb{N}}$ is bounded.

3. **Existence and uniqueness of orbits with periodic growth.** In this section we prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Let $u_0 \in V$, $\{t_j\}_{j\in\mathbb{N}} \subset \mathbb{R}$ and $\{k_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ be as in (H4). Fix a positive constant δ and put $w_j = \Phi_\delta(\Phi_{t_j}(u_0) - k_j e) = \Phi_{t_j+\delta}(u_0) - k_j e$. In view of (H2) and (H4), taking a subsequence if necessary, we see that $w_j \to \varphi$ as $j \to \infty$ for some $\varphi \in V$. For $t \geq 0$, we define

$$\tau(t) = \inf\{s \geq 0 \mid \Phi_t(u_0) + e \leq \Phi_{t+s}(u_0)\}.$$

We will show that $\tau(t)$ is well-defined for each $t \geq 0$. By (H4), we can find $u_1 \in V$ such that

$$u_1 \leq \Phi_{t_j}(u_0) - k_j e \quad \text{for } j \in \mathbb{N}.$$

In view of Remark 1, for each $t \geq 0$ we can take $m \in \mathbb{N}$ satisfying

$$\Phi_t(u_0) \leq u_1 + me.$$

Therefore, for sufficiently large j, we have

$$\Phi_t(u_0) + e \leq u_1 + (m+1)e \leq u_1 + k_j e \leq \Phi_{t_j}(u_0),$$

hence $\tau(t)$ is well-defined. Since $\Phi_t(u_0) \ll \Phi_t(u_0) + e$, the function $\tau(t)$ is positive. Furthermore, by the continuity of Φ, we have

$$\Phi_t(u_0) + e \leq \Phi_{t+\tau(t)}(u_0) \quad \text{for } t \geq 0. \tag{8}$$

In the case where $\Phi_{t_0}(u_0)+e = \Phi_{t_0+\tau(t_0)}(u_0)$ for some $t_0 > 0$, taking $\varphi = \Phi_{t_0}(u_0)$ and $T = \tau(t_0) > 0$, we obtain the assertion.

Next we consider the case where $\Phi_t(u_0) + e < \Phi_{t+\tau(t)}(u_0)$ for all $t \geq 0$. In this case, the function $\tau(t)$ is strictly monotone decreasing in t. Indeed, (H1) implies $\Phi_{t+s}(u_0) + e \ll \Phi_{t+s+\tau(t)}(u_0)$ for any $s > 0$, hence $\tau(t+s) < \tau(t)$. Therefore, the limit

$$T := \lim_{t\to+\infty} \tau(t) \geq 0$$

exists and satisfies $\tau(t) > T$ for all $t \geq 0$. By (8), we have $w_j + e \leq \Phi_{\tau(t_j+\delta)}(w_j)$ for any $j \in \mathbb{N}$, hence

$$\varphi + e \leq \Phi_T(\varphi). \tag{9}$$

This implies $T > 0$.

Suppose that $\varphi + e \neq \Phi_T(\varphi)$. Then (9) implies $\varphi + e < \Phi_T(\varphi)$. By (H1) and (H3), we have $\Phi_\delta(\varphi) + e \ll \Phi_{T+\delta}(\varphi)$. Therefore, $\Phi_\delta(w_j) + e \leq \Phi_{T+\delta}(w_j)$ for sufficiently large $j \in \mathbb{N}$, hence $\Phi_{t_j+2\delta}(u_0) + e \leq \Phi_{t_j+2\delta+T}(u_0)$. This means that $\tau(t_j + 2\delta) \leq T$, contradicting the fact that $\tau(t) > T$ for all $t \geq 0$. Thus we obtain $\varphi + e = \Phi_T(\varphi)$.

Next we assume that there exists a positive constant $S \neq T$ such that $\Phi_S(\psi) = \psi + e$ for some $\psi \in V$. We only consider the case $S < T$, since the other case $S > T$ can be treated in the same manner.

By Remark 1, we can find $m \in \mathbb{N}$ satisfying $-me \leq \psi - \varphi \leq me$. Then we have $\varphi - me \leq \Phi_{n(T-S)}(\psi) \leq \varphi + me$ for all $n \in \mathbb{N}$. Hence the set $\{\Phi_{n(T-S)}(\psi)\}_{n\in\mathbb{N}}$ is order-bounded.

On the other hand, since $S < T$,

$$\Phi_{n(T-S)}(\psi) = \Phi_{r_n}(\Phi_{q_n S}(\psi)) = \Phi_{r_n}(\psi) + q_n e, \tag{10}$$

where $n(T-S) = q_n S + r_n$ with $q_n \in \mathbb{N}$, $q_n \to \infty$ and $r_n \in [0, S)$. Since $\{\Phi_{r_n}(\psi)\}_{n\in\mathbb{N}}$ is bounded, (10) contradicts the fact that $\{\Phi_{n(T-S)}(\psi)\}_{n\in\mathbb{N}}$ is order-bounded.

The theorem is proved. □

In order to prove the uniqueness of orbits with periodic growth, we use the result of Ogiwara and Matano [13] on the structure of a certain class of subsets in an ordered metric space.

Proof of Theorem 2.2. Applying Proposition 1 in Appendix for $X = V$, $Y = W$, $G = \Phi^*$ and $\overline{\varphi} = \varphi$, we see that $W = O(\varphi)$ and that W is order-isomorphic to $\mathbb{R}$. The last statement of the theorem follows from (H1) and (H3). □

4. **Stability of orbits with periodic growth.**

Proof of Theorem 2.3. (i) For any $v \in O(\varphi)$ and $\varepsilon > 0$, we can take $\delta_0 > 0$ satisfying

$$\Phi_{\tau+\delta_0}(v) - \Phi_{\tau-\delta_0}(v) \ll \varepsilon e \quad \text{for all } \tau \in [0, T],$$

where $T > 0$ be the constant in Theorem 2.1. Fix $t \geq 0$ and let $q \in \mathbb{N}$, $r \in [0, T)$ be such that $t = qT + r$. Then, since $v \in O(\varphi)$,

$$0 \ll \Phi_{t+\delta_0}(v) - \Phi_{t-\delta_0}(v) = \Phi_{r+\delta_0}(v) - \Phi_{r-\delta_0}(v) \ll \varepsilon e. \tag{11}$$

On the other hand, by Theorem 2.2, we have

$$\Phi_{-\delta_0}(v) \ll v \ll \Phi_{\delta_0}(v). \tag{12}$$

This implies that $\Phi_{-\delta_0}(v) + \delta e \ll v \ll \Phi_{\delta_0}(v) - \delta e$ for some $\delta > 0$. Therefore, for any $u \in V$ satisfying $-\delta e \ll u - v \ll \delta e$, we obtain

$$\Phi_{-\delta_0}(v) \ll u \ll \Phi_{\delta_0}(v). \tag{13}$$

In view of (12) and (13), we see that

$$\Phi_{t-\delta_0}(v) - \Phi_{t+\delta_0}(v) \ll \Phi_t(u) - \Phi_t(v) \ll \Phi_{t+\delta_0}(v) - \Phi_{t-\delta_0}(v). \tag{14}$$

Combining this with (11), we obtain

$$-\varepsilon e \ll \Phi_t(u) - \Phi_t(v) \ll \varepsilon e$$

for all $t \geq 0$.

(ii) Suppose that $O(\varphi)$ is not stable in the sense of Lyapunov. Then there exist $v \in O(\varphi)$, $\nu > 0$, $\{u_n\}_{n\in\mathbb{N}} \subset V$ and $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ satisfying

$$\lim_{n\to\infty} \|u_n - v\| = 0, \quad \inf_{n\in\mathbb{N}} \|\Phi_{t_n}(u_n) - \Phi_{t_n}(v)\| \geq \nu. \tag{15}$$

Fix $s > 0$ and let $q_n \in \mathbb{N}$, $r_n \in [0, T)$ be such that $t_n - s = q_nT + r_n$ for each $n \in \mathbb{N}$.

By (i), for any fixed $\varepsilon \in (0, 1)$ there exists $N(\varepsilon) \in \mathbb{N}$ such that if $n \geq N(\varepsilon)$, then

$$-\varepsilon e \ll \Phi_t(u_n) - \Phi_t(v) \ll \varepsilon e \quad \text{for } t \geq 0. \tag{16}$$

Hence we have

$$-\varepsilon e \ll \Phi_{t_n-s}(u_n) - \Phi_{t_n-s}(v) = \Phi_{t_n-s}(u_n) - \Phi_{r_n}(v) - q_ne \ll \varepsilon e, \quad n \geq N(\varepsilon).$$

Since $\{\Phi_{r_n}(v)\}_n$ is bounded, the above inequalities imply that $\{\Phi_{t_n-s}(u_n) - q_ne\}_n$ is order-bounded and hence $\{\Phi_{t_n}(u_n) - q_ne\}_n$ is relatively compact. Furthermore, $\{\Phi_{r_n}(v)\}_n = \{\Phi_{t_n-s}(v) - q_ne\}_n$ is order-bounded and hence $\{\Phi_{t_n}(v) - q_ne\}_n$ is relatively compact. Therefore, taking a subsequence if necessary, we see that $\Phi_{t_n}(u_n) - q_ne \to \overline{u}$ and $\Phi_{t_n}(v) - q_ne \to \overline{v}$ for some $\overline{u}, \overline{v} \in V$. Combining these with (15) and (16), we obtain

$$\|\overline{u} - \overline{v}\| \geq \nu, \quad -\varepsilon e \leq \overline{u} - \overline{v} \leq \varepsilon e.$$

However, they are inconsistent, since $\varepsilon > 0$ can be chosen arbitrarily small.

(iii) Define a strongly monotone map F on V by $F(u) = \Phi_T(u) - e$. Then the set of fixed points of F coincides with $O(\varphi)$ by Theorem 2.2. We remark that $F^n(u) = \Phi_{nT}(u) - ne$ for all $n \in \mathbb{N}$.

Fix $u \in V$ and let $k \in \mathbb{N}$ be such that $-ke \leq u - \varphi \leq ke$. This shows that $\varphi - ke \leq F^n(u) \leq \varphi + ke$, hence $\{F^n(u)\}_{n\in\mathbb{N}}$ is order-bounded. Therefore, the omega limit set of u

$$\omega(u) = \bigcap_{n\in\mathbb{N}} \overline{\{F^m(u) \mid m \geq n\}}$$

is nonempty and compact, where $\overline{M}$ denotes the closure of a set M. Note that $\omega(v) = \{v\}$ for $v \in O(\varphi)$.

Applying Proposition 2 in Appendix for $X = V$, $Y = \{\omega(u) \mid u \in V\} \ni \{\varphi\}$ and $G = \Phi^*$ acting on $O(\varphi)$, we obtain $Y = \{\{v\} \mid v \in O(\varphi)\}$. This means that for any $u \in V$ there exists some $v = \Phi_\tau(\varphi) \in O(\varphi)$ satisfying

$$\lim_{n\to\infty} \|F^n(u) - v\| = 0.$$

Let $\{t_j\}_{j\in\mathbb{N}}$ be any sequence with $t_j \to +\infty$ as $j \to \infty$. Then there exists a subsequence also denoted by $\{t_j\}_{j\in\mathbb{N}}$ such that $t_j = q_jT + r_j$ with $q_j \in \mathbb{N}$, $q_j \to \infty$, $r_j \in [0, T)$ and $r_j \to \overline{r}$ for some $\overline{r} \in [0, T]$. Since

$$\Phi_{t_j}(u) - \Phi_{t_j}(v) = \Phi_{r_j}(\Phi_{q_jT}(u)) - \Phi_{r_j}(\Phi_{q_jT}(v)) = \Phi_{r_j}(F^{q_j}(u)) - \Phi_{r_j}(v),$$

we have

$$\lim_{j\to\infty} \|\Phi_{t_j}(u) - \Phi_{t_j}(v)\| = 0.$$

The proof of the theorem is completed. □

5. **Application to quasilinear equations.** Let us consider an initial-boundary value problem for a quasilinear parabolic equation:

$$\begin{cases} u_t = d(x, u, u_x)u_{xx} + f(x, u, u_x), & x \in I := (a, b), \ \ t > 0, \\ u_x(a, t) = u_x(b, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), & x \in I. \end{cases} \tag{17}$$

Concerning the asymptotic behavior of bounded solutions of (17) as $t \to +\infty$, extensive studies have been made in earlier works including Zelenyak [18] and Matano [7]. Applying the result in [18], we see that there exists a nontrivial Lyapunov functional of the form

$$E[u(\cdot, t)] := \int_a^b \Phi(x, u(x, t), u_x(x, t))dx$$

satisfying $\frac{d}{dt}E[u(\cdot, t)] \leq 0$ and that any solution with bounded $C^{2+\alpha}$-norm converges in $C^2(\overline{I})$ to an equilibrium solution of (17) as $t \to +\infty$. Matano [10] also constructed a Lyapunov functional for one-dimensional quasilinear parabolic equations including (17) by a different method. In [7], Matano discussed convergence of solutions of one-dimensional semilinear parabolic equations and proved that the ω-limit set of any bounded solution consists of a single equilibrium solution. Since the main tool in [7] is the so-called zero-number argument, the convergence result is given also for quasilinear equations by virtually the same argument.

In this section, we study the asymptotic behavior of the solution which has no L^∞-bound. Assume the following:

(A1) $d(x, u, p) > 0$ and $f(x, u, p)$ are smooth functions and are L-periodic in u;

(A2) there exists a positive constant M such that for any fixed $t_0 > 0$, if $u(x, t)$ is a solution of (17) on $[0, t_0]$ with initial data u_0 satisfying $\|u_0'\|_\infty \leq M$, then the gradient estimate $\|u_x(\cdot, t)\|_\infty \leq M$ holds for all $t \in [0, t_0]$;

By the theory of quasilinear parabolic equations [4, 5] and the optimal regularity theory of analytic semigroups [6], for every $u_0 \in h^{1+\beta}(\overline{I})$ with $\beta \in (0,1)$ satisfying $u_0'(a) = u_0'(b) = 0$, problem (17) has a solution $u(x,t)$ in some time interval $[0,T]$ with $u \in C([0,T]; h^{1+\beta}(\overline{I}))$. Here the space $h^{1+\beta}(\overline{I})$ is the so-called "little-Hölder space", which is the closure in the usual Hölder space $C^{1+\beta}(\overline{I})$ of the subspace of smooth functions.

Since f is L-periodic in u,

$$K := \sup_{x\in\overline{I}, u\in\mathbb{R}} |f(x,u,0)|$$

is finite and hence $u^+(t) := Kt + \|u_0\|_\infty$ is a supersolution of (17), while $u^-(t) := -u^+(t)$ is a subsolution of (17). Therefore, by the comparison principle we have

$$\|u(\cdot,t)\|_\infty \le Kt + \|u_0\|_\infty, \quad t \in [0,T]. \tag{18}$$

Furthermore, since u_x solves a linear parabolic equation of divergence form, the uniform gradient bound (A2) and the Hölder estimates for linear parabolic equations imply the Hölder gradient estimates

$$\|u_x\|_{C^{\alpha,\alpha/2}(\overline{I}\times[\tau,T])} \le C_\tau \tag{19}$$

for some constants $\alpha \in (0,1)$ and $C_\tau > 0$ independent of u and T. Consequently, under the conditions (A1) and (A2), equation (17) defines a global semiflow on V, where V is a closed subset of $X = h^{1+\alpha}(\overline{I})$ defined by

$$V := \{u \in X \mid u'(a) = u'(b) = 0, \ \|u'\|_\infty \le M\}.$$

We define an order relation in X by

$$u \le v \quad \text{if } u(x) \le v(x) \text{ for all } x \in \overline{I}.$$

Then the space X is strongly ordered with the interior of the positive cone being $\{u \in X \mid u(x) > 0 \text{ for all } x \in \overline{I}\}$.

We further assume the existence of an unbounded solution:

(A3) there exists a $\overline{u}_0 \in V$ such that the solution $\overline{u}(x,t)$ of (17) with initial value $\overline{u}_0$ satisfies

$$\limsup_{t\to+\infty} \max_{x\in\overline{I}} \overline{u}(x,t) = +\infty.$$

Condition (A3) and the periodicity of d and f in u ensure that for any $u_0 \in V$ the solution $u(x,t)$ diverges to $+\infty$ everywhere as $t \to +\infty$ and hence no equilibrium solution exists. A sufficient condition for (A2) and (A3) is that $f = f(u_x)$ with $f(0) > 0$. Conditions (A1)-(A3) are also fulfilled for a quasilinear parabolic equation related to a curvature-dependent motion of plane curves in a two-dimensional cylinder with periodically undulating boundary. See [11] for details.

Applying the theorems in Section 2 to (17), we obtain the following:

Theorem 5.1. *Assume* (A1), (A2) *and* (A3). *Then the following hold:*

(i) *There exists a periodically growing solution $U(x,t)$ of* (17) *satisfying*

$$U(x,t+T) = U(x,t) + L \quad \text{for all } (x,t) \in \overline{I}\times\mathbb{R} \tag{20}$$

for some $T > 0$. Furthermore, such a periodically growing solution is unique up to time shift.

(ii) $\inf_{x\in\overline{I},\, t\in\mathbb{R}} U_t(x,t) > 0$.

(iii) *U is globally stable with asymptotic phase, that is, for any $u_0 \in V$, there exists a constant $\tau \in \mathbb{R}$ such that the solution $u(x,t)$ of (17) with initial value u_0 satisfies*

$$\lim_{t\to+\infty} \|u(\cdot,t) - U(\cdot,t+\tau)\|_{C^2(\overline{I})} = 0.$$

Proof. Let $\Phi = \{\Phi_t\}_{t\geq 0}$ be the global semiflow on V defined by (17). First we shall show that Φ satisfies all the conditions (H1)-(H4).

The condition (H1) immediately follows from the strong comparison principle for (17).

For any order-bounded set B in V, there exists a constant $K > 0$ such that $\|u_0\|_\infty \leq K$ for all $u_0 \in B$. Therefore, by (18), (19) and the a priori estimates for linear parabolic equations, for each $t > 0$ the set $\Phi_t(B)$ is bounded in $C^{2+\alpha}(\overline{I})$, hence relatively compact in V. Thus (H2) holds.

Since d and f are L-periodic in u, if $u(x,t)$ is a solution of (17), then so is $u(x,t) + L$. This implies (H3) with $e(x) \equiv L > 0$.

By (H3), we may assume that $\overline{u}_0 \in V$ in (A3) satisfies $0 \leq \overline{u}_0(x) \leq M(b-a) + L$ for all $x \in \overline{I}$. Then we can find a sequence $0 \leq t_1 < t_2 < \cdots \to +\infty$ satisfying

$$\max_{x\in\overline{I}} \overline{u}(x,t_j) = M(b-a) + jL$$

for $j \in \mathbb{N}$. This means that $0 \leq \Phi_{t_j}(\overline{u}_0) - je \leq M(b-a)$ for all $j \in \mathbb{N}$, hence (H4) holds.

(i) By Theorem 2.1, we can find $\varphi \in V$ and $T > 0$ satisfying (7). Let $\Phi^* = \{\Phi_t\}_{t\in\mathbb{R}}$ be the flow extension of Φ defined as in Section 2 and define $U(\cdot,t) = \Phi_t(\varphi)$ for $t \in \mathbb{R}$. Then U is a periodically growing solution satisfying (20). The uniqueness (up to time shift) of U follows from Theorems 2.1 and 2.2.

(ii) Theorem 2.2 implies that U is strictly increasing in $t \in \mathbb{R}$. Hence we have $U_t(x,t) \geq 0$ for all $(x,t) \in \overline{I} \times \mathbb{R}$. Furthermore, the strict inequality $U_t(x,t) > 0$ follows from the strong comparison principle. In view of this and (20), we see that U_t is bounded away from 0 for all $(x,t) \in \overline{I} \times \mathbb{R}$.

(iii) Let $u_0 \in V$ and define $u(\cdot,t) = \Phi_t(u_0)$ for $t \geq 0$. Then it follows from Theorem 2.3 (iii) that

$$\lim_{t\to+\infty} \|u(\cdot,t) - U(\cdot,t+\tau)\|_{h^{1+\alpha}(\overline{I})} = 0.$$

Furthermore, by the a priori estimates, $\{u(\cdot,t) - U(\cdot,t+\tau)\}_{t\geq\tau}$ remains bounded in $C^{2+\alpha}(\overline{I})$ for some fixed $\tau > 0$. Therefore, the above convergence takes place in the C^2 topology.

The theorem is proved. □

Remark 3. Giga, Ishimura and Kohsaka [1] studied a weakly anisotropic curvature flow in an annulus $\{x \in \mathbb{R}^2 \mid \rho < |x| < R\}$ and considered spiral shaped solutions of the form

$$\Gamma(t) = \{(r\cos\theta(r,t), r\sin\theta(r,t)) \mid \rho \leq r \leq R\}. \tag{21}$$

Then the function $\theta(r,t)$ satisfies the following equation:

$$\begin{cases} \theta_t = M(\boldsymbol{n}) \left(\dfrac{a(\boldsymbol{n})(r\theta_{rr} + r^2\theta_r^3 + 2\theta_r)}{r(1+r^2\theta_r^2)} + \dfrac{V_0(1+r^2\theta_r^2)^{1/2}}{r} \right), \\ \theta_r(\rho,t) = \theta_r(R,t) = 0. \end{cases} \tag{22}$$

Here, $M(\boldsymbol{n})$ is called the mobility which depends on the unit vector $\boldsymbol{n}$ represented as

$$\boldsymbol{n} = \boldsymbol{n}(r, \theta, \theta_r) = \frac{1}{(1 + r^2\theta_r^2)^{1/2}} \begin{pmatrix} -\sin\theta - r\theta_r\cos\theta \\ \cos\theta - r\theta_r\sin\theta \end{pmatrix},$$

$a(\boldsymbol{n})$ is a positive coefficient which comes from the anisotropic curvature of $\Gamma(t)$ and V_0 is a positive constant corresponding to a driving force term. They proved that the conditions (A2) and (A3) are satisfied for (22) and that there exists a unique (up to translation of time) spiral solution $\Gamma(t)$ of the form (21) satisfying

$$\theta(r, t + T) = \theta(r, t) + 2\pi, \quad r \in [\rho, R], \ \ t > 0$$

for some $T > 0$. Concerning the stability they proved the Lyapunov stability of the spiral solution $\Gamma(t)$, but the asymptotic stability of $\Gamma(t)$ was not discussed in [1]. Applying our Theorem 5.1 (iii) to (22), we see that $\Gamma(t)$ is stable with asymptotic phase. Note that the usual Hölder space $C^{1+\alpha}$ was used in [1] as state space V; while we adopt the little-Hölder space $h^{1+\alpha}$ to assure the continuity of Φ_t at $t = 0$.

Remark 4. Similar results to Theorem 5.1 also hold for a class of semilinear or more generally quasilinear parabolic equations in higher space dimension including (1) and (3).

Appendix A. **Structure of subsets under a group action.** In this appendix we recall two propositions in the paper of Ogiwara and Matano [13]. Proposition 1 is concerned with the structure of subsets of an ordered metric space satisfying certain conditions. Proposition 2 is a set-valued version of Proposition 1.

Let G be a metrizable topological group acting on an ordered metric space X. In other words, X is a metric space on which a closed partial order relation $\leq$ is defined and there exists a continuous mapping $\gamma : G \times X \to X$ such that $g \mapsto \gamma(g, \cdot)$ is a group homomorphism of G into the group of homeomorphism of X onto itself. For brevity, we write $\gamma(g, u) = gu$ and identify the element $g \in G$ with its action $\gamma(g, \cdot)$. Furthermore, we assume that

(G1) γ is order-preserving (that is, $u \leq v$ implies $gu \leq gv$ for any $g \in G$);
(G2) G is connected.

As in the previous sections, we write $u < v$ if $u \leq v$ and $u \neq v$. Similarly, for subsets $Y, Z \subset X$, we write $Y < Z$ if $u \leq v$ for all $u \in Y$, $v \in Z$ and $Y \neq Z$.

Proposition 1. [13, Proposition B1] *Let Y be a subset of X and $\overline{u}$ be an element of Y such that*

(Y1) *$g\overline{u} \in Y$ for any $g \in G$;*
(Y2) *for any $v \in Y$, there exist $g_1, g_2 \in G$ satisfying $g_1\overline{u} < v < g_2\overline{u}$;*
(Y3) *for any $v \in Y$ with $v < h\overline{u}$ for some $h \in G$, there exists a neighborhood B of the identity element $e \in G$ such that $v < gh\overline{u}$ for any $g \in B$.*

Then Y is totally-ordered connected set and satisfies

$$Y = G\overline{u} := \{g\overline{u} \mid g \in G\}.$$

If, in addition, Y is locally precompact, then Y is homeomorphic and order-isomorphic to $\mathbb{R}$.

Proposition 2. [13, Proposition B1] *Let Y be a subset of the power set of X and $\{\overline{u}\}$ be an element of Y such that*

(Y1) *$\{g\overline{u}\} \in Y$ for any $g \in G$;*

(Y2) *for any* $V \in Y$, *there exist* $g_1, g_2 \in G$ *satisfying* $\{g_1\overline{u}\} < V < \{g_2\overline{u}\}$.
(Y3) *for any* $V \in Y$ *with* $V < \{h\overline{u}\}$ *for some* $h \in G$, *there exists a neighborhood* B *of the identity element* $e \in G$ *such that* $V < \{gh\overline{u}\}$ *for any* $g \in B$.

Then

$$Y = G\{\overline{u}\} := \{\{g\overline{u}\} \mid g \in G\}.$$

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