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SELF-SIMILAR SOLUTIONS IN A SECTOR FOR A QUASILINEAR PARABOLIC EQUATION*

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Dedicated to Professor Hiroshi Matano on the occasion of his 60th birthday

Abstract. We study a two-point free boundary problem in a sector for a quasilinear parabolic equation. The boundary conditions are assumed to be spatially and temporally "self-similar" in a special way. We prove the existence, uniqueness and asymptotic stability of an expanding solution which is self-similar at discrete times. We also study the existence and uniqueness of a shrinking solution which is self-similar at discrete times.

1. **Introduction.**

Consider the problem

$$\begin{cases} u_t = a(u_x)u_{xx}, \quad -\xi_1(t) < x < \xi_2(t),\ t > 0, \\ u_x(x,t) = -k_1(t, u(x,t)), \quad u(x,t) = -x\tan\beta \quad \text{for} \quad x = -\xi_1(t),\ t > 0, \\ u_x(x,t) = k_2(t, u(x,t)), \quad u(x,t) = x\tan\beta \quad \text{for} \quad x = \xi_2(t),\ t > 0, \end{cases} \tag{1}$$

where $a \in C^2(\mathbb{R})$, $a(\cdot) > 0$, $\beta \in (0, \frac{\pi}{2})$ and $k_1, k_2 \in C^2([0,\infty) \times [0,\infty), \mathbb{R})$. In this problem, u, ξ_1, ξ_2 are unknown positive functions to be determined.

The equation in (1) includes the heat equation and the curvature flow equation as special examples. In [2, 3, 4, 8, 9, 12], the authors considered problem (1) with constant k_i $(i = 1, 2)$, that is,

$$\begin{cases} u_t = a(u_x)u_{xx}, \quad -\zeta_1(t) < x < \zeta_2(t),\ t > 0, \\ u_x(x,t) = -\gamma_1, \quad u(x,t) = -x\tan\beta \quad \text{for} \quad x = -\zeta_1(t),\ t > 0, \\ u_x(x,t) = \gamma_2, \quad u(x,t) = x\tan\beta \quad \text{for} \quad x = \zeta_2(t),\ t > 0, \end{cases} \tag{2}$$

where γ_1, γ_2 are constants. They proved the existence of solutions of (2) for some initial data. Moreover, in [2, 4, 12], they proved that when $\gamma_1 + \gamma_2 > 0$, any time-global solution u is expanding (that is, it moves upward to infinity) and it converges asymptotically to a *self-similar solution*: $\sqrt{2t}\,\varphi\left(x/\sqrt{2t}\right)$. In [3, 8, 9], they proved that when $\gamma_1 + \gamma_2 < 0$, any solution shrinks to 0 as $t \to T$ for some $T > 0$; if a is analytic, then the rescaled solution $u/\sqrt{2(T-t)}$ converges to a *shrinking/backward self-similar solution* with the form: $\psi\left(x/\sqrt{2(T-t)}\right)$ as $t \to T$.

Problem (2) arises in the model of flame propagation in combustion theory. It also arises in the study of the motion of interface moving with curvature in which the studied problem is confined in the conical region bounded by two straight lines

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and the interface has prescribed touching angles with these two straight lines (cf. [2, 3, 4, 7, 8, 9, 12] etc.). In this paper we will consider a more general problem (1). In this new problem, the boundary conditions are spatially and temporally inhomogeneous, which mean that the touching angles between the interface and the boundaries of the sector domain depend on the spatial and temporal variables. Clearly, self-similar functions like $\sqrt{2t}\,\varphi(x/\sqrt{2t})$ or $\sqrt{2(T-t)}\,\psi\big(x/\sqrt{2(T-t)}\big)$ is no longer a solution of (1). We have to adopt new concepts for the analogue of self-similar solutions. Our results in this paper show that problem (1) has an expanding solution which is self-similar at discrete times if k_1, k_2 have some special "self-similarity" (see (5) below) and if $\min k_1 + \min k_2 > 0$. On the other hand, problem (1) has a shrinking solution which is self-similar at discrete times if k_1, k_2 have some special "self-similarity" (see (11) or (14) below) and if $\max k_1 + \max k_2 < 0$.

Definition 1.1. Let $(u, \xi_1, \xi_2) = (U, \Xi_1, \Xi_2)$ be a solution of (1) defined for $t \in (0, \infty)$. It is called an *expanding self-similar solution* if

$$bU(x,t) \equiv U\left(bx, b^2 t\right) \quad \text{for } -\Xi_1(t) \leqslant x \leqslant \Xi_2(t),\ t > 0, \tag{3}$$

for some $b > 1$, and if

$$b\,\Xi_i(t) = \Xi_i\left(b^2 t\right) \quad \text{for } t > 0 \qquad (i = 1, 2). \tag{4}$$

From (3) we see that, for any $t_0 > 0$,

$$\cdots = bU(b^{-1}x, b^{-2}t_0) = U(x, t_0) = b^{-1}U(bx, b^2 t_0) = \cdots .$$

This means that $U(x,t)$ is similar to $U(x,t_0)$ only at *discrete* times: $t = b^{2m}t_0$ ($m \in \mathbb{Z}$). In this sense we may also say that (U, Ξ_1, Ξ_2) (or, just U) is a *discrete expanding self-similar solution* and $\sqrt{2t}\,\varphi(x/\sqrt{2t})$ is a *classical expanding self-similar solution*.

It is easily seen that a necessary condition for the existence of a discrete expanding self-similar solution is that k_1 and k_2 are *self-similar* in a special way:

$$k_i(t,u) = k_i\left(b^2 t, bu\right) \quad \text{for } t, u \geqslant 0. \tag{5}$$

We will give more explanation on this condition near the end of this section. For simplicity, we also impose another technical conditions on k_i: there exists $\sigma \in (0, \tan\beta)$ such that

$$|k_i(t,u)| \leqslant (\tan\beta) - \sigma \quad \text{for } t, u \geqslant 0 \qquad (i = 1, 2). \tag{6}$$

Theorem 1.2. *Assume that k_1, k_2 satisfy conditions (5) and (6). Assume also that*

$$\min k_1 + \min k_2 > 0 \tag{7}$$

holds. Then problem (1) has a discrete expanding self-similar solution (U, Ξ_1, Ξ_2).

In addition, if $k_i(t,u) \equiv k_i(u)$ $(i = 1, 2)$, then

(i) *the expanding self-similar solution is unique and $U_t > 0$, $\Xi_{1t} > 0$, $\Xi_{2t} > 0$ for all $t > 0$;*

(ii) *U is asymptotically stable in the sense that*

$$d_{\mathcal{H}}(\Gamma(t), \gamma(t)) \leqslant Ct^{-1/2} \qquad \text{as } t \to \infty, \tag{8}$$

where $\Gamma(t)$ is the graph of U, $\gamma(t)$ is the graph of any time-global solution u of (1) and $d_{\mathcal{H}}$ denotes the Hausdorff distance.

The existence, uniqueness and asymptotic stability conclusions in this theorem are proved in subsections 3.6, 3.8 and 3.7, respectively.

Next we consider self-similar solutions which shrink to 0 in finite time.

Definition 1.3. Given $T>0$. Let $(u,\xi_1,\xi_2)=(\widetilde{U},\widetilde{\Xi}_1,\widetilde{\Xi}_2)$ be a solution of (1) for $t\in[0,T)$. If $\|\widetilde{U}(\cdot,t)\|_{L^\infty}\to 0,\ \widetilde{\Xi}_1(t)\to 0,\ \widetilde{\Xi}_2(t)\to 0$ as $t\to T-0$,

$$\widetilde{U}(x,t)\equiv b\,\widetilde{U}\left(b^{-1}x,b^{-2}t+(1-b^{-2})T\right)\quad\text{for }-\widetilde{\Xi}_1(t)\leqslant x\leqslant\widetilde{\Xi}_2(t),\ 0\leqslant t<T,\ (9)$$

for some $b>1$, and if

$$\widetilde{\Xi}_i(t)=b\,\widetilde{\Xi}_i\left(b^{-2}t+(1-b^{-2})T\right)\quad\text{for }0\leqslant t<T\qquad(i=1,2),\qquad(10)$$

then $(\widetilde{U},\widetilde{\Xi}_1,\widetilde{\Xi}_2)$ (or, just $\widetilde{U}$) is called a *shrinking/backward self-similar solution* of (1) on time interval $[0,T)$.

Since $\widetilde{U}(x,t)$ is similar to $\widetilde{U}(x,t_0)$ only at discrete times: $t=b^{-2m}t_0+(1-b^{-2m})T$ ($m\in\mathbb{Z}$ and $2m\geqslant\log(\frac{T-t_0}{T})/\log b$), we may also say that $(\widetilde{U},\widetilde{\Xi}_1,\widetilde{\Xi}_2)$ is a *discrete shrinking self-similar solution* on $[0,T)$.

A necessary condition for the existence of such a solution is that

$$k_i(t,u)=k_i\left(b^{-2}t+(1-b^{-2})T,\ b^{-1}u\right)\quad\text{for }0\leqslant t<T,\ u\geqslant 0.\qquad(11)$$

Replacing t by $T-t'$ then we see that (9), (10) and (11) are equivalent to

$$\widetilde{U}(x,T-t')\equiv b\,\widetilde{U}\left(b^{-1}x,T-b^{-2}t'\right)\qquad(12)$$

for $-\widetilde{\Xi}_1(T-t')\leqslant x\leqslant\widetilde{\Xi}_2(T-t')$, $0<t'\leqslant T$,

$$\widetilde{\Xi}_i(T-t')=b\,\widetilde{\Xi}_i\left(T-b^{-2}t'\right)\quad\text{for }0<t'\leqslant T\qquad(i=1,2),\qquad(13)$$

and

$$k_i(T-t',u)=k_i(T-b^{-2}t',b^{-1}u)\quad\text{for }0<t'\leqslant T,\ u\geqslant 0\qquad(i=1,2),\qquad(14)$$

respectively.

Theorem 1.4. *Given $T>0$, assume that k_1,k_2 satisfy condition (11) or (14). Assume also that (6) and*

$$\max k_1+\max k_2<0\qquad(15)$$

hold. Then problem (1) has a discrete shrinking self-similar solution $(\widetilde{U},\widetilde{\Xi}_1,\widetilde{\Xi}_2)$ on $[0,T)$.

In addition, if $k_i(t,u)\equiv k_i(u)$ $(i=1,2)$, then the discrete shrinking self-similar solution is unique and $\widetilde{U}_t<0,\ \widetilde{\Xi}_{1t}<0,\ \widetilde{\Xi}_{2t}<0$ for $t\in[0,T)$.

The uniqueness for shrinking self-similar solutions is not necessary to be true, even for the special problem (2) (cf. [8, 9])). But the above theorem shows that it can be unique under certain assumptions.

Definition 1.3 and Theorem 1.4 deal with shrinking solutions on finite time interval $[0,T)$. If we take a time shift, these solutions can be regarded as solutions defined on $[-T,0)$. More precisely, let $(\widetilde{U},\widetilde{\Xi}_1,\widetilde{\Xi}_2)$ be a shrinking self-similar solution of (1) on $[0,T)$. Then

$$\widehat{U}(x,t;T):=\widetilde{U}(x,T+t)\quad\text{for }-\widehat{\Xi}_1(t;T)\leqslant x\leqslant\widehat{\Xi}_2(t;T),\ t\in[-T,0),$$

and

$$\widehat{\Xi}_i(t;T):=\widetilde{\Xi}_i(T+t)\quad\text{for }t\in[-T,0)\quad(i=1,2)$$

satisfy

$$\widehat{U}(x,t)\equiv b\,\widehat{U}(b^{-1}x,b^{-2}t)\quad\text{for }-\widehat{\Xi}_1(t)\leqslant x\leqslant\widehat{\Xi}_2(t),\ -T\leqslant t<0,\qquad(16)$$

and

$$\widehat{\Xi}_i(t)=b\,\widehat{\Xi}_i(b^{-2}t)\quad\text{for }-T\leqslant t<0\qquad(i=1,2).\qquad(17)$$

So $(\widehat{U}, \widehat{\Xi}_1, \widehat{\Xi}_2)$ (which is defined on $[-T, 0)$ and shrinks to 0 as $t \to 0-0$) is a self-similar solution of

$$\begin{cases} u_t = a(u_x)u_{xx}, \quad -\xi_1(t) < x < \xi_2(t),\ t < 0, \\ u_x(x,t) = -k_1(T+t, u(x,t)), \quad u(x,t) = -x\tan\beta \quad \text{for} \quad x = -\xi_1(t),\ t < 0, \\ u_x(x,t) = k_2(T+t, u(x,t)), \quad u(x,t) = x\tan\beta \quad \text{for} \quad x = \xi_2(t),\ t < 0. \end{cases} \tag{18}$$

We now consider shrinking self-similar solutions defined in $(-\infty, 0)$.

Definition 1.5. Assume that $k_i(t,u) \equiv k_i(u)$ $(i = 1, 2)$. Let $(\widehat{U}, \widehat{\Xi}_1, \widehat{\Xi}_2)$ be a solution of (18) defined for $t \in (-\infty, 0)$. If it satisfies (16) and (17) for some $b > 1$ and $t \in (-\infty, 0)$, then it is called a *discrete shrinking/backward self-similar solution* of (18) in $(-\infty, 0)$.

Theorem 1.6. *Assume that $k_1 \equiv k_1(u)$ and $k_2 \equiv k_2(u)$ satisfy (6), (15) and $k_i(u) = k_i(bu)$ for all $u \geqslant 0$ and some $b > 1$. Then problem (18) has a unique discrete shrinking self-similar solution $(\widehat{U}, \widehat{\Xi}_1, \widehat{\Xi}_2)$ in $(-\infty, 0)$, and $\widehat{U}_t < 0$, $\widehat{\Xi}_{1t} < 0$, $\widehat{\Xi}_{2t} < 0$ for $t \in (-\infty, 0)$.*

Our theorems extend the results about classical self-similar solutions in [2, 3, 4, 8, 9, 12] to problem (1) with nonlinear boundary conditions. Our approach is essentially different from theirs though we will use their classical self-similar solutions as lower and upper solutions to give the growth bound for the solution of (1). We will convert problem (1) by changing variables to a new problem in a fixed domain. Then we use a convergence result in [1] to show that the ω-limit of the unknown in the new problem is a periodic solution, which corresponds to a discrete self-similar solution of (1).

Our boundary conditions are given by functions k_1 and k_2 which are self-similar as in (5). We now give some examples and/or backgrounds on such kind of self-similarity. First, some reactions in chemistry occur in a media with obstacles (cf. [18]). When the obstacles arrange in a regular way, it is possible to be studied from a mathematical point of view. For example, if we consider a Belousov-Zhabotinsky (BZ) reaction in a media with obstacles arranging in columns, then the interface propagation in the BZ experiment can be studied through a curvature flow in a band domain with undulating boundaries (cf. [15, 16]). Similarly, if we consider the BZ reaction in a media with obstacles arranging in radial rays with center at origin O, and if the ratios of the sizes of adjacent obstacles are constant, then the interface propagation can be studied through a curvature flow in a sector with undulating boundaries, which is essentially a similar problem as our (1). Another example is the following. In geology, Liesegang rings are colored bands of cement observed in sedimentary rocks, which are often referred to as great examples of geochemical self-organization (cf. [17]). Generally, the Liesegang rings are arranged in a regular self-similar way: the ratios of the widths of adjacent annuluses are constant (cf. [10, 11, 17]). If we cut off a sector with apex at the center of the rings and consider the interface propagation in this notch, then the problem is reduced to one like (1).

In Section 2 we give some preliminaries, including the selection of the initial data, the local existence result and comparison principles. In Section 3 we consider the expanding case and prove Theorem 1.2. In Section 4 we consider the shrinking case and prove Theorems 1.4 and 1.6.

2. Preliminaries.

We use notation $S := \{(x,y) \mid y > |x|\tan\beta,\ x \in \mathbb{R}\}$, and use $\partial_1 S$ and $\partial_2 S$ to denote the left and right boundaries of S, respectively. For any

$t_0 > 0$, let $(u(x,t), \xi_1(t), \xi_2(t))$ be a classical solution of (1) on the time interval $[0, t_0]$ with some initial data. Then we write

$$Q_{t_0} := \{(x,t) \mid -\xi_1(t) < x < \xi_2(t) \text{ and } 0 < t \leqslant t_0\}.$$

2.1. **Initial data.** We will consider the problem (1) with initial data

$$u(x,0) = u_0(x), \quad -\xi_{01} \leqslant x \leqslant \xi_{02}, \tag{19}$$

where $u_0(x) > 0$, $\xi_{01} > 0$ and $\xi_{02} > 0$ satisfy

$$u_0(-\xi_{01}) = \xi_{01} \tan\beta, \quad u_0(\xi_{02}) = \xi_{02} \tan\beta \tag{20}$$

and the compatibility conditions:

$$(u_0)_x(-\xi_{01}) = -k_1(0, u_0(-\xi_{01})), \quad (u_0)_x(\xi_{02}) = k_2(0, u_0(\xi_{02})). \tag{21}$$

Since our main purpose in this paper is to construct self-similar solutions, we will not focus on general solutions of (1) and (19) for general u_0 as it was done in [2, 8, 12], but choose $u_0 \in C^{2+\mu}([-\xi_{01}, \xi_{02}])$ for some $\mu \in (0,1)$, and only consider classical solution u of (1) and (19) in $C^{2+\mu,1+\mu/2}(\overline{Q}_{t_0})$ for $t_0 > 0$. Moreover, we require that u_0 satisfies

$$|(u_0)_x(x)| \leqslant \tan\beta - \sigma \quad \text{for } -\xi_{01} \leqslant x \leqslant \xi_{02}, \tag{22}$$

where σ is as in (6). This inequality does not conflict with the compatibility conditions by (6).

In summary, in this paper we choose initial data from the following set of admissible functions:

$$C^{2+\mu}_{\text{ad}} := \left\{ u_0 \;\middle|\; \begin{array}{l} u_0 \in C^{2+\mu}([-\xi_{01}, \xi_{02}]) \text{ for some } \mu \in (0,1), \text{ where} \\ \xi_{01},\ \xi_{02} > 0 \text{ and } u_0(\cdot) > 0 \text{ satisfy (20), (21) and (22)} \end{array} \right\}. \tag{23}$$

2.2. **Gradient bound of u.**

Lemma 2.1. *Let $u_0 \in C^{2+\mu}_{\text{ad}}$ for some $\mu \in (0,1)$, $u(x,t) \in C^{2+\mu,1+\mu/2}(\overline{Q}_{t_0})$ be a solution of (1) and (19) on $[0, t_0]$. Then*

$$|u_x(x,t)| \leqslant \tan\beta - \sigma \quad \text{for } (x,t) \in \overline{Q}_{t_0}. \tag{24}$$

Proof. By (6) we have

$$u_x(\xi_2(t), t) = k_2(t, u(\xi_2(t), t)) \leqslant \tan\beta - \sigma,$$

and

$$u_x(-\xi_1(t), t) = -k_1(t, u(-\xi_1(t), t)) \leqslant \tan\beta - \sigma.$$

Combining these inequalities with (22) we obtain $u_x \leqslant \tan\beta - \sigma$ by maximum principle. $u_x \geqslant -\tan\beta + \sigma$ is proved similarly. □

Corollary 1. *Let u_0 and u be as in the previous lemma. Then, for any $\theta \in [-\theta_0, \theta_0]$ with $\theta_0 := \frac{\pi}{2} - \beta$, there holds*

$$\sigma \cos\beta \leqslant (1 \pm u_x(x,t) \tan\theta) \cos\theta \leqslant 2 - \sigma \cot\beta \quad \text{for } (x,t) \in \overline{Q}_{t_0}. \tag{25}$$

2.3. **Change of variables.** To study the local and global existence of solutions of the initial boundary value problem (1) and (19), it is convenient to introduce new coordinates that convert the sector domain S into a flat cylinder. More precisely, we will make a change of variables $(x, y, t) \mapsto (\theta, \rho, s)$, which gives a diffeomorphism $(\overline{S}\backslash\{0\}) \times [0, t_\infty) \to \overline{D} \times [s_0, s_\infty)$, where

$$D := \{(\theta, \rho) \in \mathbb{R}^2 \mid -\theta_0 < \theta < \theta_0,\ -\infty < \rho < \infty\}$$

with $\theta_0 := \frac{\pi}{2} - \beta$. The functions $\theta = \theta(x, y, t)$, $\rho = \rho(x, y, t)$ and $s = s(t)$ are to be specified below. With these new coordinates, the function $y = u(x, t)$ is expressed as $\rho = \omega(\theta, s)$, where the new unknown $\omega(\theta, s)$ is determined by the relation

$$\rho\left(x, u(x, t), t\right) = \omega\left(\theta(x, u(x, t), t), s(t)\right). \tag{26}$$

The function $\omega(\theta, s)$ is well-defined provided that the map $t \mapsto s(t)$ is strictly monotone for $t \in [0, t_\infty)$ and $x \mapsto \theta(x, u(x, t), t)$ is strictly monotone for each fixed $t \in [0, t_\infty)$. We will see later that these monotonicity conditions always hold for the class of solutions that we consider. Indeed we will prove

$$\frac{\partial}{\partial t} s(t) > 0, \quad \frac{\partial}{\partial x}\theta\left(x, u(x, t), t\right) = \theta_x + \theta_y u_x > 0. \tag{27}$$

Once $\omega(\theta, s)$ is defined, then substituting it into the relation $y = u(x, t)$ yields

$$Y\left(\theta, \omega(\theta, s), s\right) = u\left(X(\theta, \omega(\theta, s), s), T(s)\right), \tag{28}$$

where the map $(\theta, \rho, s) \mapsto (X(\theta, \rho, s), Y(\theta, \rho, s), T(s)) : \overline{D} \times [s_0, s_\infty) \to (\overline{S}\backslash\{0\}) \times [0, t_\infty)$ is the inverse map of $(x, y, t) \mapsto (\theta(x, y, t), \rho(x, y, t), s(t))$. The expression (28) gives a formula for recovering the original solution $u(x, t)$ from $\omega(\theta, s)$. In order for u to be smoothly dependent on ω, we need the map $\theta \mapsto X(\theta, \omega(\theta, s), s)$ to be one-to-one for each fixed s and that $s \mapsto T(s)$ is strictly monotone for $s \in [s_0, s_\infty)$. Indeed we will prove

$$\frac{\partial}{\partial s} T(s) > 0, \quad \frac{\partial}{\partial \theta} X\left(\theta, \omega(\theta, s), s\right) = X_\theta + X_\rho \omega_\theta > 0. \tag{29}$$

2.4. **Local existence.** To get the local existence we make the following change of variables.

$$\begin{cases} \theta = \arctan \dfrac{x}{y}, & (x, y) \in \overline{S}\backslash\{0\}, \\ \rho = \dfrac{1}{2}\log(x^2 + y^2), & (x, y) \in \overline{S}\backslash\{0\}, \\ s = t, & t \geqslant 0. \end{cases} \tag{30}$$

The inverse map is

$$\begin{cases} x = e^\rho \sin\theta, & (\theta, \rho) \in \overline{D}, \\ y = e^\rho \cos\theta, & (\theta, \rho) \in \overline{D}, \\ t = s, & s \in [0, \infty). \end{cases} \tag{31}$$

Clearly, $\theta = \theta_0$ and $\theta = -\theta_0$ correspond to $\partial_2 S$ and $\partial_1 S$, respectively.

Let $u(x, t) > 0$ be a classical solution of (1) and (19) for $t \geqslant 0$, then

$$\rho(x, u(x, t), t) = \omega(\theta(x, u(x, t), t), s) \quad \Leftrightarrow \quad e^{\omega(\theta, s)}\cos\theta = u\Big(e^{\omega(\theta, s)}\sin\theta, t\Big). \tag{32}$$

defines a new unknown $\rho = \omega(\theta, s)$ for $s \geqslant 0$. This function is well-defined since

$$\frac{\partial}{\partial x}\theta(x, u(x, t), t) = \frac{\partial}{\partial x}\left(\arctan\frac{x}{u(x, t)}\right) = \frac{1 - u_x \tan\theta}{u \cdot (1 + \tan^2\theta)} > 0$$

by (25).

Differentiating the expression $e^{\omega(\theta,s)}\cos\theta = u(e^{\omega(\theta,s)}\sin\theta, t)$ twice by θ and once by t we obtain

$$u_x = \frac{\omega_\theta\cos\theta - \sin\theta}{\cos\theta + \omega_\theta\sin\theta}, \quad u_{xx} = \frac{\omega_{\theta\theta} - \omega_\theta^2 - 1}{e^\omega(\cos\theta + \omega_\theta\sin\theta)^3}, \quad u_t = \frac{e^\omega\omega_s}{\cos\theta + \omega_\theta\sin\theta}.$$

Therefore, problem (1) with (19) is converted into the following problem

$$\begin{cases} \omega_s = a\left(\dfrac{\omega_\theta\cos\theta - \sin\theta}{\cos\theta + \omega_\theta\sin\theta}\right)\dfrac{\omega_{\theta\theta} - \omega_\theta^2 - 1}{e^{2\omega}(\cos\theta + \omega_\theta\sin\theta)^2}, & \theta \in (-\theta_0, \theta_0),\ s \in (0,\infty), \\ \omega_\theta(-\theta_0, s) = -h_1^0(s, \omega(-\theta_0, s)), & s \in [0,\infty), \\ \omega_\theta(\theta_0, s) = h_2^0(s, \omega(\theta_0, s)), & s \in [0,\infty), \\ \omega(\theta, 0) = \tilde{\omega}(\theta), & \theta \in [-\theta_0, \theta_0]. \end{cases} \tag{33}$$

where $\tilde{\omega}$ is defined by (32) at $t = s = 0$ and

$$h_i^0(s,\omega) = \frac{\sin\theta_0 + k_i(s, e^\omega\cos\theta_0)\cos\theta_0}{\cos\theta_0 - k_i(s, e^\omega\cos\theta_0)\sin\theta_0} \quad (i = 1,2). \tag{34}$$

Estimate (24) implies that

$$\sigma - \tan\beta \leqslant u_x = \frac{\omega_\theta\cos\theta - \sin\theta}{\cos\theta + \omega_\theta\sin\theta} \leqslant \tan\beta - \sigma \quad \text{for } \theta \in [-\theta_0, \theta_0],\ s \geqslant 0. \tag{35}$$

Thus $\cos\theta + \omega_\theta\sin\theta > 0$ since it is positive at $\theta = 0$ and it can not be zero by (35). Considering the second inequality in (35) we have

$$\omega_\theta[\cos\theta - \sin\theta(\tan\beta - \sigma)] \leqslant \sin\theta + (\tan\beta - \sigma)\cos\theta. \tag{36}$$

Note that, for $\theta \in [-\theta_0, \theta_0]$ ($\theta_0 = \frac{\pi}{2} - \beta$), we have

$$\cos\theta - \sin\theta(\tan\beta - \sigma) \geqslant \cos\theta_0[1 - \tan\theta_0(\tan\beta - \sigma)] \geqslant \sigma\cos\beta,$$

$$\cos\theta - \sin\theta(\tan\beta - \sigma) \leqslant \cos\theta[1 + \tan\theta_0(\tan\beta - \sigma)] \leqslant 2 - \sigma\cot\beta.$$

So

$$\omega_\theta \leqslant \frac{\sin\theta + (\tan\beta - \sigma)\cos\theta}{\cos\theta - \sin\theta(\tan\beta - \sigma)} \leqslant \Omega_1 := \frac{1 + \tan\beta - \sigma}{\sigma\cos\beta}.$$

Using the first inequality in this formula we have, for $\theta \leqslant 0$,

$$\cos\theta + \omega_\theta\sin\theta \geqslant \frac{1}{\cos\theta - \sin\theta(\tan\beta - \sigma)} \geqslant \varepsilon_1 := \frac{1}{2 - \sigma\cot\beta}.$$

Similarly, considering the first inequality in (35) we have

$$\omega_\theta \geqslant \frac{\sin\theta - (\tan\beta - \sigma)\cos\theta}{\cos\theta + \sin\theta(\tan\beta - \sigma)} \geqslant -\Omega_1,$$

and $\cos\theta + \omega_\theta\sin\theta \geqslant \varepsilon_1$ for $\theta \geqslant 0$.

Summarizing the above results we have

$$|\omega_\theta(\theta, s)| \leqslant \Omega_1 \quad \text{and} \quad \cos\theta + \omega_\theta\sin\theta \geqslant \varepsilon_1 > 0 \tag{37}$$

for $\theta \in [-\theta_0, \theta_0]$ and $s \geqslant 0$.

By the standard theory for parabolic equations, we see that (33) has a classical solution on time interval $s \in [0, 2\tau]$ for positive $\tau = \tau(k_1, k_2, \mu, \tilde{\omega})$.

The second inequality in (37) implies that, once the solution ω of (33) is obtained then we can recover it to the original solution u of (1). In fact,

$$\frac{\partial}{\partial\theta}X(\theta, \omega(\theta, s), s) = \frac{\partial}{\partial\theta}e^{\omega(\theta,s)}\sin\theta = e^{\omega(\theta,s)}(\cos\theta + \omega_\theta\sin\theta) > 0.$$

Consequently, we have the following local existence result.

Lemma 2.2. *Problem* (1) *with initial data* $u_0(x) \in C^{2+\mu}_{\rm ad}$ ($\mu \in (0,1)$) *has a classical solution* u *on time interval* $[0, 2\tau]$*, where* τ *depends only on* k_1, k_2, μ *and* u_0.

2.5. **Comparison principle.** Let $v_1(x), v_2(x)$ be two functions whose graphs lie in $\overline{S}$ and meet the two boundaries of S. Hereafter, when we write

$$v_1 \leqslant v_2 \qquad (\text{resp. } v_1 \ll v_2\ ),$$

we mean that $v_1(x) \leqslant v_2(x)$ (resp. $v_1(x) < v_2(x)$) for all x with $(x, v_i(x)) \in \overline{S}$ ($i = 1, 2$); when we write

$$v_1 \preceq v_2$$

we mean that $v_1(x) \leqslant v_2(x)$ and the "equality" holds at some x.

Assume further that $|v_{1x}|, |v_{2x}| < \tan\beta$. Then for each x with $(x, v_1(x)) \in \overline{S}\backslash\{O\}$, there exists a unique $Z(x)$ such that

$$x \cdot v_2(Z(x)) = Z(x) \cdot v_1(x),$$

that is, $(x, v_1(x))$ and $(Z(x), v_2(Z(x)))$ lie on the same line passing the origin. By a simple geometric observation we have

$$v_1 \leqslant v_2 \ \Leftrightarrow\ x^2 + v_1^2(x) \leqslant Z^2(x) + v_2^2(Z(x)) \text{ for } x \text{ with } (x, v_1(x)) \in \overline{S}\backslash\{O\} \tag{38}$$

and

$$v_1 \ll v_2 \ \Leftrightarrow\ x^2 + v_1^2(x) < Z^2(x) + v_2^2(Z(x)) \text{ for } x \text{ with } (x, v_1(x)) \in \overline{S}\backslash\{O\}. \tag{39}$$

For some $t_0 > 0$, let $u_1(x,t) \in C^{2,1}\big(\overline{Q^{(1)}_{t_0}}\big)$ and $u_2(x,t) \in C^{2,1}\big(\overline{Q^{(2)}_{t_0}}\big)$ be two positive functions, where, for $i = 1, 2$,

$$Q^{(i)}_{t_0} := \{(x,t) \mid -\xi^{(i)}_1(t) < x < \xi^{(i)}_2(t),\ 0 < t \leqslant t_0\},$$

$$u_i(-\xi^{(i)}_1(t), t) = \xi^{(i)}_1(t) \cdot \tan\beta, \quad u_i(\xi^{(i)}_2(t), t) = \xi^{(i)}_2(t) \cdot \tan\beta, \quad 0 < t \leqslant t_0,$$

and $|(u_i)_x(x,t)| < \tan\beta$.

Definition 2.3. Let $t_0 > 0$ and $u_i \in C^{2,1}\big(\overline{Q^{(i)}_{t_0}}\big)$ ($i = 1, 2$) be positive functions as above. Then u_1 is called a lower solution of (1) on $[0, t_0]$ if

$$\begin{cases} u_{1t} \leqslant a(u_{1x})u_{1xx} & \text{for } -\xi^{(1)}_1(t) < x < \xi^{(1)}_2(t),\ 0 \leqslant t \leqslant t_0, \\ u_{1x}(x,t) \geqslant -k_1(t, u_1(x,t)) & \text{for } x = -\xi^{(1)}_1(t),\ 0 \leqslant t \leqslant t_0, \\ u_{1x}(x,t) \leqslant k_2(t, u_1(x,t)) & \text{for } x = \xi^{(1)}_2(t),\ 0 \leqslant t \leqslant t_0. \end{cases} \tag{40}$$

Similarly, u_2 is called an upper solution of (1) if the opposite inequalities hold.

The following comparison principle holds.

Lemma 2.4. *Let* $t_0 > 0$*. Assume that* $u_1(x,t)$ *and* $u_2(x,t)$ *are lower solution and upper solution of* (1) *on* $[0, t_0]$*, respectively. If* $u_1(\cdot, 0) \leqslant u_2(\cdot, 0)$*, then* $u_1(\cdot, t) \leqslant u_2(\cdot, t)$ *for* $0 \leqslant t \leqslant t_0$*. If* $u_1(\cdot, 0) \leqslant u_2(\cdot, 0)$ *and* $u_1(x, 0) \not\equiv u_2(x, 0)$*, then* $u_1(\cdot, t) \ll u_2(\cdot, t)$ *for* $0 < t \leqslant t_0$.

Proof. We change variables by (30) and (31), that is, using

$$e^{\rho_i} \cos\theta = u_i(e^{\rho_i} \sin\theta, s),$$

we define implicit functions $\rho_i = \omega_i(\theta, s)$ $(i = 1, 2)$. Since $u_1(x, t)$ is a lower solution of (1), it is easily seen that $\omega_1(\theta, s)$ is a lower solution of (33):

$$\begin{cases} \omega_{1s} \leqslant a\left(\dfrac{\omega_{1\theta}\cos\theta - \sin\theta}{\cos\theta + \omega_{1\theta}\sin\theta}\right)\dfrac{\omega_{1\theta\theta} - \omega_{1\theta}^2 - 1}{e^{2\omega_1}(\cos\theta + \omega_{1\theta}\sin\theta)^2}, & \theta \in (-\theta_0, \theta_0),\ s \in (0, t_0], \\ \omega_{1\theta}(-\theta_0, s) \geqslant -h_1^0(s, \omega_1(-\theta_0, s)), & s \in [0, t_0], \\ \omega_{1\theta}(\theta_0, s) \leqslant h_2^0(s, \omega_1(\theta_0, s)), & s \in [0, t_0]. \end{cases}$$

Similarly, $\omega_2(\theta, s)$ is an upper solution of (33). By (38), $u_1(\cdot, t) \leqslant u_2(\cdot, t)$ is equivalent to $\omega_1(\theta, s) \leqslant \omega_2(\theta, s)$. The latter follows from the comparison principle for (33), which is a problem in a fixed domain. Similarly, the conclusion $u_1(\cdot, t) \ll u_2(\cdot, t)$ can be proved by using (39). □

3. Expanding self-similar solutions.

In this section we always assume that (7) holds.

3.1. Classical expanding self-similar solutions.

We will use classical self-similar solutions of (2) as upper and lower solutions of (1) to give the growth bound for the solution u of (1) and (19). For any $\gamma_1, \gamma_2 \in \mathbb{R}$, consider the problem

$$\begin{cases} a(\varphi'(z))\varphi''(z) = \varphi(z) - z\varphi'(z), \quad z \in \mathbb{R}, \\ \varphi'(-p_1) = -\gamma_1,\ \varphi(-p_1) = p_1 \tan\beta, \\ \varphi'(p_2) = \gamma_2,\ \varphi(p_2) = p_2 \tan\beta. \end{cases} \tag{41}$$

In [2, 4, 12], the authors obtained the following result.

Lemma 3.1. *For any given γ_1, γ_2 with $\gamma_1 + \gamma_2 > 0$, there exists a unique pair $p_1, p_2 > 0$ such that problem (41) has a solution $\varphi(z; \gamma_1, \gamma_2)$, which is positive on $[-p_1, p_2]$.*

It is easily seen that the function

$$\sqrt{2t}\,\varphi\left(\frac{x}{\sqrt{2t}}; \gamma_1, \gamma_2\right) \quad \text{for } -\zeta_1(t) < x < \zeta_2(t),\ t > 0,$$

with $\zeta_i(t) = p_i\sqrt{2t}$ $(i = 1, 2)$ is an expanding self-similar solution of (2). Set

$$k_i^0 := \min k_i(t, u) \quad \text{and} \quad K_i^0 := \max k_i(t, u) \qquad (i = 1, 2), \tag{42}$$

and define

$$\varphi^-(z) := \varphi(z; k_1^0, k_2^0), \quad \varphi^+(z) := \varphi(z; K_1^0, K_2^0).$$

Then $\sqrt{2t}\,\varphi^-(x/\sqrt{2t})$ and $\sqrt{2t}\,\varphi^+(x/\sqrt{2t})$ (both are expanding self-similar solutions of (2)) are lower and upper solutions of (1), respectively.

Since the initial data $u_0 > 0$, there exist $t^+, t^- > 0$ such that

$$\sqrt{2t^-}\,\varphi^-\left(\frac{\cdot}{\sqrt{2t^-}}\right) \leqslant u_0(\cdot) \leqslant \sqrt{2t^+}\,\varphi^+\left(\frac{\cdot}{\sqrt{2t^+}}\right). \tag{43}$$

The comparison principle implies that

$$\sqrt{2(t+t^-)}\,\varphi^-\left(\frac{\cdot}{\sqrt{2(t+t^-)}}\right) \leqslant u(\cdot, t) \leqslant \sqrt{2(t+t^+)}\,\varphi^+\left(\frac{\cdot}{\sqrt{2(t+t^+)}}\right). \tag{44}$$

3.2. **Changes of variables.** In subsection 2.4 we gave a local existence result. One difficulty for deriving the global existence is the lack of the growth bound for ω. To give the global existence we adopt another change of variables.

Let τ be the constant in Lemma 2.2 and let $t^- > 0$ be as in (43). For any $n \in \mathbb{N}$ satisfying

$$n > \frac{1}{t^-} \quad \text{and} \quad n > \frac{1}{\tau}, \tag{45}$$

we introduce new variables by

$$\begin{cases} \theta = \arctan \dfrac{x}{y}, & (x,y) \in \overline{S}\backslash\{0\}, \\ \rho = \dfrac{1}{2} \log \dfrac{n(x^2+y^2)}{nt+1}, & (x,y) \in \overline{S}\backslash\{0\},\ t \geqslant 0, \\ s = \dfrac{1}{2} \log \left(t + \dfrac{1}{n}\right), & t \geqslant 0. \end{cases} \tag{46}$$

The inverse map is

$$\begin{cases} x = e^s e^\rho \sin\theta, & (\theta,\rho) \in \overline{D},\ s \in [-\frac{1}{2}\log n, \infty), \\ y = e^s e^\rho \cos\theta, & (\theta,\rho) \in \overline{D},\ s \in [-\frac{1}{2}\log n, \infty), \\ t = e^{2s} - \frac{1}{n}, & s \in [-\frac{1}{2}\log n, \infty). \end{cases} \tag{47}$$

Clearly, $\theta = \theta_0$ and $\theta = -\theta_0$ correspond to $\partial_2 S$ and $\partial_1 S$, respectively.

Let $u(x,t) > 0$ be a solution of (1) and (19) for $t \geqslant 0$, then a similar discussion as in subsection 2.4 shows that

$$\rho(x, u(x,t), t) = v(\theta(x, u(x,t), t), s(t)) \ \Leftrightarrow\ e^s e^{v(\theta,s)} \cos\theta = u\Big(e^s e^{v(\theta,s)} \sin\theta, e^{2s} - \frac{1}{n}\Big) \tag{48}$$

defines a new unknown $\rho = v(\theta, s)$ for $s \in [-\frac{1}{2}\log n, \infty)$. Differentiating the second equality twice by θ and once by s we obtain

$$u_x = \frac{v_\theta \cos\theta - \sin\theta}{\cos\theta + v_\theta \sin\theta}, \quad u_{xx} = \frac{v_{\theta\theta} - v_\theta^2 - 1}{e^s e^v (\cos\theta + v_\theta \sin\theta)^3}, \quad u_t = \frac{e^v(1+v_s)}{2e^s(\cos\theta + v_\theta\sin\theta)}.$$

Therefore, problem (1) is converted into the following problem

$$\begin{cases} v_s = 2a\left(\dfrac{v_\theta\cos\theta - \sin\theta}{\cos\theta + v_\theta \sin\theta}\right) \dfrac{v_{\theta\theta} - v_\theta^2 - 1}{e^{2v}(\cos\theta + v_\theta\sin\theta)^2} - 1, & |\theta| < \theta_0,\ s > -\frac{1}{2}\log n, \\ v_\theta(-\theta_0, s) = -g_1(s, v(-\theta_0, s)), & s \geqslant -\frac{1}{2}\log n, \\ v_\theta(\theta_0, s) = g_2(s, v(\theta_0, s)), & s \geqslant -\frac{1}{2}\log n, \end{cases} \tag{49}$$

where

$$g_i(s,v) = \frac{\sin\theta_0 + k_i\Big(e^{2s} - \frac{1}{n}, e^s e^v \cos\theta_0\Big)\cos\theta_0}{\cos\theta_0 - k_i\Big(e^{2s} - \frac{1}{n}, e^s e^v\cos\theta_0\Big)\sin\theta_0} \quad (i = 1,2). \tag{50}$$

3.3. **Gradient bound of** v**.** In a similar way as deriving (37) in subsection 2.4 one can obtain

$$|v_\theta(\theta,s)| \leqslant \Omega_2(\sigma,\beta) \quad \text{and} \quad \cos\theta + v_\theta\sin\theta \geqslant \varepsilon_2(\sigma,\beta) > 0 \tag{51}$$

for $\theta \in [-\theta_0, \theta_0]$ and $s \in [-\frac{1}{2}\log n, \infty)$.

The second inequality in (51) implies that, once the solution v of (49) is obtained then we can recover it to the original solution u of (1), since

$$\frac{\partial}{\partial\theta} X(\theta, v(\theta,s), s) = \frac{\partial}{\partial\theta} e^s e^{v(\theta,s)} \sin\theta = e^s e^{v(\theta,s)} (\cos\theta + v_\theta\sin\theta) > 0.$$

3.4. **Bound of v.** The local existence result Lemma 2.2 implies that v exists on $s \in [-\frac{1}{2}\log n, \frac{1}{2}\log(2\tau + \frac{1}{n})]$. We have changed $u(x,t)$ to a new unknown $v(\theta, s)$. Similarly, we define $v^{\pm}$ by $\varphi^{\pm}$ in the following way

$$e^s e^{v^{\pm}} \cos\theta = \sqrt{2\Big(e^{2s} - \frac{1}{n} + t^{\pm}\Big)}\, \varphi^{\pm} \left(\frac{e^s e^{v^{\pm}} \sin\theta}{\sqrt{2\Big(e^{2s} - \frac{1}{n} + t^{\pm}\Big)}} \right).$$

By (44) we have $e^s e^{v^+} \cos\theta \geqslant u(e^s e^{v^+} \sin\theta, e^{2s} - \frac{1}{n})$. Noting $e^s e^v \cos\theta = u(e^s e^v \sin\theta, e^{2s} - \frac{1}{n})$ we have

$$(e^{v^+} - e^v)\cos\theta \geqslant u_x\Big(\vartheta, e^{2s} - \frac{1}{n}\Big) \cdot [(e^{v^+} - e^v)\sin\theta],$$

where $\vartheta = e^s \sin\theta(\varsigma e^v + (1-\varsigma)e^{v^+})$ for some $\varsigma \in [0,1]$. Therefore $v^+ \geqslant v$ by (25).

On the other hand, by the definition of v^+ we have

$$e^{v^+} \cos\theta = \varphi^+(\cdot)\sqrt{2 + \Big(t^+ - \frac{1}{n}\Big)e^{-2s}} \leqslant \sqrt{2 + t^+/\tau}\ \max\varphi^+ \quad \text{for } s \geqslant \frac{1}{2}\log\tau. \tag{52}$$

So

$$v(\theta, s) \leqslant v^+(\theta, s) \leqslant \log\left[\frac{\sqrt{2 + t^+/\tau}\ \max\varphi^+}{\sin\beta}\right] \quad \text{for } \theta \in [-\theta_0, \theta_0],\ s \geqslant \frac{1}{2}\log\tau.$$

A similar discussion as above shows that

$$v(\theta, s) \geqslant v^-(\theta, s) \geqslant \log\big[2\ \min\varphi^-\big] \quad \text{for } \theta \in [-\theta_0, \theta_0],\ s \geqslant \frac{1}{2}\log\tau.$$

Remark 1. The definition of v depends on n, it is not easy to give a uniform (in n) bound for v on $[-\frac{1}{2}\log n, \infty)$, but the above results show that a uniform bound for v is possible on $[\frac{1}{2}\log\tau, \infty)$.

3.5. **Global existence.** Now we consider problem (49) with initial data $v(\theta, -\frac{1}{2}\log n) = v_0(\theta)$, which is defined by (48) at $s = -\frac{1}{2}\log n$. Using the growth bound and gradient bound in the previous subsections and using the standard theory of parabolic equations (cf. [5, 6, 13, 14]) we can get the following conclusions.

Lemma 3.2. *Problem (49) with initial data $v(\theta, -\frac{1}{2}\log n) = v_0(\theta)$ has a unique, time-global solution $v(\theta, s) \in C^{2+\mu, 1+\mu/2}([-\theta_0, \theta_0] \times [-\frac{1}{2}\log n, \infty))$ and*

$$\|v(\theta, s)\|_{C^{2+\mu,1+\mu/2}([-\theta_0,\theta_0]\times[\frac{1}{2}\log\tau,\infty))} \leqslant C_0 < \infty, \tag{53}$$

where C_0 depends on k_1, k_2, μ and u_0 but not on s and n.

This lemma implies the global existence of u.

Lemma 3.3. *Problem (1) and (19) has a unique, time-global solution $u(x,t)$. Moreover, $u \in C^{2+\mu,1+\mu/2}(\overline{Q}_\infty)$, where $Q_\infty := \{(x,t) \mid -\xi_1(t) < x < \xi_2(t), t > 0\}$. For any $t_0 > \tau$,*

$$\|u(x,t)\|_{C^{2+\mu,1+\mu/2}(\overline{Q}_{t_0}\setminus Q_\tau)} \leqslant C_1(t_0, k_1, k_2, \mu, u_0, \tau) < \infty.$$

Indeed, studying the relations between v and u more precisely, it is not difficult to see that C_1 in this lemma can be replaced by $C_2\sqrt{t_0} + C_3$ for some C_2, C_3 depending on k_1, k_2, μ, u_0 and τ.

3.6. **Existence of self-similar solution.** Since the solution v in Lemma 3.2 is defined for $s \geqslant -\frac{1}{2}\log n$, we write it as v_n. By Cantor's diagonal argument, one can find a function $V \in C^{2+\mu,1+\mu/2}([-\theta_0,\theta_0]\times[\frac{1}{2}\log\tau,\infty))$ and a subsequence $\{n_i\}\subset\{n\}$ such that, as $i\to\infty$,

$$v_{n_i}(\theta,s)\to V(\theta,s) \qquad \text{in } C^{2,1}_{\mathrm{loc}}\Big([-\theta_0,\theta_0]\times\Big[\frac{1}{2}\log\tau,\infty\Big)\Big) \text{ topology.} \tag{54}$$

Moreover, V satisfies the estimate

$$\|V(\theta,s)\|_{C^{2+\mu,1+\mu/2}([-\theta_0,\theta_0]\times[\frac{1}{2}\log\tau,\infty))}\leqslant C_0<\infty, \tag{55}$$

and V is a solution of

$$\begin{cases} v_s = 2a\left(\dfrac{v_\theta\cos\theta-\sin\theta}{\cos\theta+v_\theta\sin\theta}\right)\dfrac{v_{\theta\theta}-v_\theta^2-1}{e^{2v}(\cos\theta+v_\theta\sin\theta)^2}-1, & |\theta|<\theta_0,\ s\geqslant\frac{1}{2}\log\tau,\\ v_\theta(-\theta_0,s)=-G_1(s,v(-\theta_0,s)), & s\geqslant\frac{1}{2}\log\tau,\\ v_\theta(\theta_0,s)=G_2(s,v(\theta_0,s)), & s\geqslant\frac{1}{2}\log\tau, \end{cases} \tag{56}$$

where

$$G_i(s,v)=\frac{\sin\theta_0+k_i\Big(e^{2s},e^se^v\cos\theta_0\Big)\cos\theta_0}{\cos\theta_0-k_i\Big(e^{2s},e^se^v\cos\theta_0\Big)\sin\theta_0} \quad (i=1,2). \tag{57}$$

Here G_1 and G_2 are $\log b$-periodic functions in s by (5).

Now we use a result in [1].

Lemma 3.4. *Let u be a time-global solution of*

$$\begin{cases} u_t=d(t,x,u,u_x)u_{xx}+f(t,x,u,u_x), & t>0,\ 0<x<1,\\ u_x(i,t)=g_i^*(t,u(i,t)), & i=0,1,\ t>0, \end{cases} \tag{58}$$

where d,f,g_i^* are C^2 functions, T-periodic in t. If $u(\cdot,t)$ is bounded in $H^2(0,1)$, then there exists a T-periodic solution p of (58) such that $\lim_{t\to\infty}\|u(\cdot,t)-p(\cdot,t)\|_{H^2(0,1)}=0$.

By this lemma, problem (56) has a solution $P(\theta,s)\in H^2(-\theta_0,\theta_0)$, which is $\log b$-periodic in s, and

$$\lim_{s\to\infty}\|V(\cdot,s)-P(\cdot,s)\|_{H^2(-\theta_0,\theta_0)}=0.$$

By (55) we indeed have

$$\|P(\theta,s)\|_{C^{2+\mu,1+\mu/2}([-\theta_0,\theta_0]\times\mathbb{R})}\leqslant C_0 \tag{59}$$

and

$$\lim_{k\to\infty}\|V(\theta,s+k\log b)-P(\theta,s)\|_{C^{2,1}([-\theta_0,\theta_0]\times[0,\log b])}=0. \tag{60}$$

We now recover P to a solution of (1), the corresponding change of variables should be the limiting version as $n\to\infty$ of (46) and (47), or for $s\in(-\infty,\infty)$. Using these variables we define U by

$$e^se^P\cos\theta=U(e^se^P\sin\theta,e^{2s}) \quad \text{for } \theta\in[-\theta_0,\theta_0],\ s\in(-\infty,\infty). \tag{61}$$

By (51), (54) and (60) we have

$$\frac{\partial}{\partial\theta}X(\theta,P(\theta,s),s)=\frac{\partial}{\partial\theta}\Big(e^se^{P(\theta,s)}\sin\theta\Big)=e^se^{P(\theta,s)}(P_\theta\sin\theta+\cos\theta)>0,$$

so the function $U(x,t)$ is well-defined for all $t>0$. Moreover, by the definition of U and the periodicity of P we have

$$\begin{aligned} bU(e^s e^{P(\theta,s)}\sin\theta, e^{2s}) &= be^s e^{P(\theta,s)}\cos\theta = e^{s+\log b}e^{P(\theta,s+\log b)}\cos\theta \\ &= U(e^{s+\log b}e^{P(\theta,s+\log b)}\sin\theta, e^{2s+2\log b}) \\ &= U(be^s e^{P(\theta,s)}\sin\theta, b^2e^{2s}). \end{aligned}$$

Hence

$$bU(x,t) = U(bx, b^2t) \quad \text{for } -\Xi_1(t)\leqslant x\leqslant \Xi_2(t),\ t>0,$$

where $-\Xi_1(t)$ and $\Xi_2(t)$ are the x-coordinate of the end points of the graph of U. Since $t=e^{2s}$ we have

$$\Xi_2(t) = e^s e^{P(\theta_0,s)}\sin\theta_0 = \sqrt{t}e^{P(\theta_0,\frac12\log t)}\sin\theta_0 \quad \text{for } t>0.$$

So

$$\Xi_2(b^2t) = b\sqrt{t}e^{P(\theta_0,\frac12\log t+\log b)}\sin\theta_0 = b\sqrt{t}e^{P(\theta_0,\frac12\log t)}\sin\theta_0 = b\Xi_2(t) \quad \text{for } t>0.$$

Similarly we have $\Xi_1(b^2t)=b\Xi_1(t)$ for $t>0$. Consequently, we obtain a discrete expanding self-similar solution of (1) and this proves the existence part of Theorem 1.2.

3.7. **Asymptotic stability.** In this subsection we assume that $k_i(t,u)\equiv k_i(u)$ ($i=1,2$) and prove the asymptotic stability result in Theorem 1.2:

$$d_{\mathcal{H}}(\Gamma(t),\gamma(t)) \leqslant Ct^{-1/2}, \quad t\to\infty,$$

where $\Gamma(t)$ is the graph of the discrete expanding self-similar solution U, $\gamma(t)$ is the graph of any solution u of (1) with some initial data, and $d_{\mathcal{H}}$ is the Hausdorff distance.

When k_1 and k_2 are constants satisfying (7), in [2, 4, 12], the authors proved similar results for problem (2) by constructing precise lower and upper solutions. We will use the change of variables and the a priori estimates but do not construct lower and upper solutions. So our approach is different from those in [2, 4, 12].

For any initial data $u_0\in C^{2+\mu}_{\rm ad}$, there exists $t_1>0$ such that

$$0 = U(\cdot,0)\leqslant u_0(\cdot)\leqslant U(\cdot,t_1).$$

By comparison principle Lemma 2.4 we have

$$U(\cdot,t)\leqslant u(\cdot,t)\leqslant U(\cdot,t+t_1), \quad t>0. \tag{62}$$

For any given $t>0$ and any $\bar{x}$ with $(\bar{x},u(\bar{x},t))\in\overline{S}$, denote $\bar\theta=\arctan\frac{\bar{x}}{u(\bar{x},t)}$. The line $\theta=\bar\theta$ contacts $\Gamma(t)$ (resp. $\Gamma(t+t_1)$) at exactly one point A (resp. A_1). Denote the x-coordinate of A (resp. A_1) by

$$Z = Z(\bar\theta) = Z(\bar{x},t) \qquad (\text{resp. } \ Z_1 = Z_1(\bar\theta) = Z_1(\bar{x},t)\ ).$$

It follows from (62) and the equivalence in (38) that

$$[Z^2+U^2(Z,t)]^{1/2} \leqslant [\bar{x}^2+u^2(\bar{x},t)]^{1/2} \leqslant [Z_1^2+U^2(Z_1,t+t_1)]^{1/2}. \tag{63}$$

Set $s=\frac12\log t$ and $s(t_1)=\frac12\log(t+t_1)$. By the definition of U in (61) we have

$$Z = e^s e^{P(\bar\theta,s)}\sin\bar\theta, \qquad U(Z,t) = e^s e^{P(\bar\theta,s)}\cos\bar\theta,$$

and

$$Z_1 = e^{s(t_1)}e^{P(\bar\theta,s(t_1))}\sin\bar\theta, \qquad U(Z_1,t+t_1) = e^{s(t_1)}e^{P(\bar\theta,s(t_1))}\cos\bar\theta,$$

where P is that in (59). Since

$$s(t_1) - s = \frac{t_1}{2t} + o\Big(\frac{1}{t}\Big) \quad (\text{as } t \to \infty),$$

by (59) we have

$$\begin{aligned} [Z_1^2 + U^2(Z_1, t+t_1)]^{1/2} - [Z^2 + U^2(Z,t)]^{1/2} &= e^{s(t_1)} e^{P(\bar\theta, s(t_1))} - e^s e^{P(\bar\theta, s)} \\ &\leqslant C_1 t^{-1/2}, \quad \text{as } t \to \infty, \end{aligned}$$

where C_1 depends on t_1 and C_0 in (59), but not on $\bar\theta$ and t. Therefore, we have

$$d_{\mathcal{H}}(\Gamma(t), \gamma(t)) \leqslant d_{\mathcal{H}}(\Gamma(t), \Gamma(t+t_1)) \leqslant C_1 t^{-1/2} \quad \text{as } t \to \infty.$$

3.8. **Uniqueness of self-similar solutions.** In this subsection we still assume $k_i(t,u) \equiv k_i(u)$ $(i = 1, 2)$ and to prove the uniqueness conclusion in Theorem 1.2. The uniqueness for general $k_i(t,u)$ is still open.

We begin with choosing a convex initial data u_0, that is,

$$a(u_{0x}) u_{0xx} \geqslant \epsilon$$

for some $\epsilon > 0$. Such a choice is possible. For example, draw a line ℓ_1 from $A_1 := (-1, \tan\beta)$ with slope $-k_1(\tan\beta)$. Denote the contacting point between ℓ_1 and $\partial_2 S$ by A_2', then $A_2' := (x', x' \tan\beta)$ with

$$x' := \frac{\tan\beta - k_1(\tan\beta)}{\tan\beta + k_1(\tan\beta)}.$$

Choose $A_2 := (x' + \epsilon', (x'+\epsilon')\tan\beta) \in \partial_2 S$ for some small $\epsilon' > 0$, then A_2 is above A_2'. Draw a line ℓ_2 from A_2 with slope $k_2((x'+\epsilon')\tan\beta)$. Since

$$k_2((x'+\epsilon')\tan\beta) \geqslant k_2^0 > -k_1^0 \geqslant -k_1(\tan\beta),$$

ℓ_2 must contact ℓ_1 at some point $A_3 \in S$ provided $\epsilon' > 0$ is small enough. Now we smoothen $\overline{A_1A_3} + \overline{A_3A_2}$ such that the smoothened curve $\mathcal{C}$ is strictly convex, it is tangent to $\overline{A_1A_3}$ at A_1, tangent to $\overline{A_3A_2}$ at A_2. Now the corresponding function u_0 of $\mathcal{C}$ is a desired initial data.

Let u be the solution of (1) with the above constructed initial data u_0. Denote $\eta = u_t$. Differentiating the problem (1) by t we have

$$\begin{cases} \eta_t = a(u_x)\eta_{xx} + a'(u_x) u_{xx} \eta_x, \quad -\xi_1(t) < x < \xi_2(t),\ t > 0, \\ \eta_x(-\xi_1(t), t) = f_1(t)\eta(-\xi_1(t), t), \quad \eta_x(\xi_2(t), t) = f_2(t)\eta(\xi_2(t), t), \qquad t > 0, \\ \eta(x, 0) = a(u_{0x})u_{0xx} \geqslant \epsilon > 0, \end{cases}$$

where f_1 and f_2 are continuous functions. Maximum principle implies that $\eta = u_t > 0$ for $t > 0$. In a similar way as in the previous subsection 3.6 we can get a discrete expanding self-similar solution (U, Ξ_1, Ξ_2). Moreover, using the same notions as above we have $1 + v_s > 0$ and so $1 + V_s \geqslant 0$ for $s \geqslant \frac{1}{2}\log\tau$. Using (60) one has $1 + P_s \geqslant 0$ for all $s \in \mathbb{R}$, this implies that $U_t \geqslant 0$. Finally, the strong maximum principle implies that $U_t > 0$ for all $t > 0$, and so $\Xi_{1t}(t) > 0$, $\Xi_{2t}(t) > 0$ for $t > 0$.

Suppose that (U^*, Ξ_1^*, Ξ_2^*) is another discrete self-similar solution of (1). We want to prove that $U(x,t) \equiv U^*(x,t)$. Otherwise, either

$$U^*(\cdot, t) \leqslant U(\cdot, t) \quad \text{and} \quad U(x,t) \not\equiv U^*(x,t) \quad \text{for all } t > 0, \tag{64}$$

or

$$U(\cdot, t) \leqslant U^*(\cdot, t) \quad \text{and} \quad U(x,t) \not\equiv U^*(x,t) \quad \text{for all } t > 0, \tag{65}$$

holds. We now derive a contradiction from (64).

Since $U(x,t) \to 0$ as $t \to 0$, for any $t_0 > 0$, there exists $\tau \in (0, t_0)$ such that $U(\cdot, t_0 - \tau) \ll U^*(\cdot, t_0)$. Set

$$\tau(t_0) := \inf\{\tau \mid U(\cdot, t_0 - \tau) \ll U^*(\cdot, t_0)\}.$$

Then either

(a) $U(x, t_0 - \tau(t_0)) \equiv U^*(x, t_0)$, or
(b) $U(x, t_0 - \tau(t_0)) \not\equiv U^*(x, t_0)$ and $U(\cdot, t_0 - \tau(t_0)) \preceq U^*(\cdot, t_0)$

holds. By (64) and by $U_t > 0$ it is easily seen that $\tau(t_0) > 0$, and so $\tau(t_0) \in (0, t_0)$.

We first show that (b) is impossible. Otherwise, by Lemma 2.4 we have

$$U(\cdot, t_0 - \tau(t_0) + (b^2-1)t_0) \ll U^*(\cdot, t_0 + (b^2-1)t_0) = U^*(\cdot, b^2 t_0).$$

On the other hand, by the self-similarity we have

$$U(bx, b^2(t_0 - \tau(t_0))) \equiv bU(x, t_0 - \tau(t_0)) \preceq bU^*(x, t_0) \equiv U^*(bx, b^2 t_0).$$

Combining these inequalities with the fact that $U_t > 0$ we have

$$b^2 t_0 - \tau(t_0) < b^2(t_0 - \tau(t_0)).$$

This is a contradiction.

Now, (a) implies that, for any $t > 0$, there exists $\tau(t) \in (0, t)$ such that $U(x, t - \tau(t)) \equiv U^*(x, t)$. Hence, for any $t, s > 0$ we have

$$U(x, t - \tau(t) + s) \equiv U^*(x, t + s) \equiv U(x, t + s - \tau(t + s)).$$

So $\tau(t+s) = \tau(t)$ for all $t, s > 0$. Thus, $\tau(t)$ is a constant $\tau^* > 0$ and $U(x, t - \tau^*) \equiv U^*(x, t)$ for all $t > 0$. Taking limits as $t \to \tau^* + 0$ we have $U^*(x, \tau^*) = U(x, 0) = 0$, and so $\tau^* = 0$, a contradiction.

The above discussion shows that (64) is impossible. In a similar way, one can show that (65) is also impossible. So $U(x,t) \equiv U^*(x,t)$, and (U, Ξ_1, Ξ_2) is the unique discrete expanding self-similar solution of (1). This proves the uniqueness result in Theorem 1.2.

4. **Shrinking self-similar solutions.**

In this section we always assume that (15) holds.

4.1. **Classical shrinking/backward self-similar solutions.**

First, recall the classical self-similar solutions of (2). For any $\gamma_1, \gamma_2 \in \mathbb{R}$, consider the problem

$$\begin{cases} a(\psi'(z))\psi''(z) = z\psi'(z) - \psi(z), \quad z \in \mathbb{R}, \\ \psi'(-q_1) = -\gamma_1, \ \psi(-q_1) = q_1 \tan\beta, \\ \psi'(q_2) = \gamma_2, \ \psi(q_2) = q_2 \tan\beta. \end{cases} \tag{66}$$

In [8, 9], the authors obtained the following result.

Lemma 4.1. *For any γ_1, γ_2 satisfying $\gamma_1 + \gamma_2 < 0$, there exists a pair $q_1, q_2 > 0$ such that problem (66) has solution $\psi(z; \gamma_1, \gamma_2)$, which is positive on $[-q_1, q_2]$.*

It is easily seen that the function

$$\sqrt{2(T-t)}\, \psi\left(\frac{x}{\sqrt{2(T-t)}}\right) \quad \text{for } -\zeta_1(t) < x < \zeta_2(t),\ t > 0,$$

with $\zeta_i(t) = q_i\sqrt{2(T-t)}$ $(i = 1, 2)$ is a classical shrinking/backward self-similar solution of (2).

We use $(\psi^-(z), q_1^-, q_2^-)$ to denote the solution of (66) with $\gamma_i = k_i^0$ $(i = 1, 2)$, use $(\psi^+(z), q_1^+, q_2^+)$ to denote the solution of (66) with $\gamma_i = K_i^0$ $(i = 1, 2)$, where k_i^0 and K_i^0 are those in (42).

By (15), $K_1^0 + K_2^0 < 0$ and so $(\psi^\pm)'' < 0$. For any $T_0 > 0$, the function $\sqrt{2(T_0 - t)}\, \psi^-(x/\sqrt{2(T_0 - t)})$ and the function $\sqrt{2(T_0 - t)}\, \psi^+(x/\sqrt{2(T_0 - t)})$ (both are shrinking/backward self-similar solutions of (2)) are lower and upper solutions of (1), respectively. Since the initial data $u_0 > 0$, there exists $T^+, T^- > 0$ such that

$$\sqrt{2T^-}\, \psi^-\left(\frac{\cdot}{\sqrt{2T^-}}\right) \preceq u_0(\cdot) \preceq \sqrt{2T^+}\, \psi^+\left(\frac{\cdot}{\sqrt{2T^+}}\right). \tag{67}$$

Comparison principle implies that

$$\sqrt{2(T^- - t)}\, \psi^-\left(\frac{\cdot}{\sqrt{2(T^- - t)}}\right) \ll u(\cdot, t) \ll \sqrt{2(T^+ - t)}\, \psi^+\left(\frac{\cdot}{\sqrt{2(T^+ - t)}}\right) \tag{68}$$

on the time interval where these three functions are defined.

Lemma 4.2. *Let $\psi^\pm$ and $T^\pm$ be as in (67). Then $T^+ > T^- \geqslant \delta T^+$, where*

$$\delta = \delta(\beta, \sigma, k_i^0, K_i^0) := \frac{\sigma^4}{(2\tan\beta - \sigma)^4} \cdot \frac{(q_2^+)^2}{(q_2^-)^2} > 0.$$

Proof. We first prove $T^+ > T^-$. The areas $D^\pm(t)$ of the regions enclosed by the graph of $\sqrt{2(T^\pm - t)}\, \psi^\pm(x/\sqrt{2(T^\pm - t)})$, $\partial_1 S$ and $\partial_2 S$ are given by

$$D^\pm(t) = \int_{-\zeta_1(t)}^{\zeta_2(t)} \sqrt{2(T^\pm - t)}\, \psi^\pm\left(\frac{x}{\sqrt{2(T^\pm - t)}}\right) dx - \frac{1}{2}[\zeta_1^2(t) + \zeta_2^2(t)] \tan\beta.$$

A simple computation shows that

$$(D^+)'(t) = \int_{-K_1^0}^{K_2^0} a(p)dp, \quad (D^-)'(t) = \int_{-k_1^0}^{k_2^0} a(p)dp.$$

Since $D^+(T^+) = D^-(T^-) = 0$ we have

$$D^+(0) = \int_0^{T^+} dt \int_{K_2^0}^{-K_1^0} a(p)dp, \quad D^-(0) = \int_0^{T^-} dt \int_{k_2^0}^{-k_1^0} a(p)dp. \tag{69}$$

By (15) we have $k_2^0 < K_2^0 < -K_1^0 < -k_1^0$, and by (67) we have $D^+(0) \geqslant D^-(0)$. Therefore, we have $T^+ > T^-$ by (69).

Next we prove $T^- \geqslant \delta T^+$. For $i = 1, 2$, denote Q_i^+ (resp. Q_i^-) the end points of the graph of $\sqrt{2T^+}\, \psi^+(x/\sqrt{2T^+})$ (resp. the graph of $\sqrt{2T^-}\, \psi^-(x/\sqrt{2T^-})$) on $\partial_i S$, respectively.

Connecting Q_1^+ and Q_2^+ we get a line segment $\overline{Q_1^+ Q_2^+}$. It is below the graph of $\sqrt{2T^+}\, \psi^+(x/\sqrt{2T^+})$ since $(\psi^+)'' < 0$. Draw a line from Q_1^+ (resp. Q_2^+) with slope $-\tan\beta + \sigma$ (resp. $\tan\beta - \sigma$). Assume that it contacts $\partial_2 S$ (resp. $\partial_1 S$) at A_2 (resp. A_1).

By (67) the graph of u_0 contacts the line segment $\overline{Q_1^+ Q_2^+}$. Since $|u_{0x}| \leqslant \tan\beta - \sigma$, we see that the graph of u_0 is above the line segment $\overline{A_1 A_2}$.

If for some $T_0 > 0$, the graph of $\sqrt{2T_0}\ \psi^-(x/\sqrt{2T_0})$ is tangent to $\overline{A_1 A_2}$ from above, then (note $|(\psi^-)'| \leqslant \tan\beta - \sigma$) the graph of $\sqrt{2T_0}\ \psi^-(x/\sqrt{2T_0})$ lies above the line segment $\overline{B_1 B_2}$, where B_1 (resp. B_2) is the contacting point between the line passing A_2 (resp. A_1) with slope $\tan\beta - \sigma$ (resp. $-\tan\beta + \sigma$) and the left boundary $\partial_1 S$ (resp. right boundary $\partial_2 S$). Therefore, Q_i^- are above B_i for $i = 1, 2$.

Using the coordinates of $Q_i^+ = ((-1)^i r_i \cos\beta, r_i \sin\beta)$, where

$$r_i = \frac{\sqrt{2T^+}q_i^+}{\cos\beta} \quad (i = 1, 2),$$

one can easily calculate the coordinates of B_1 and B_2:

$$B_i = \Big(\frac{(-1)^i\sigma^2 r_i\cos\beta}{(2\tan\beta-\sigma)^2},\ \frac{\sigma^2 r_i \sin\beta}{(2\tan\beta-\sigma)^2}\Big) \quad (i = 1, 2).$$

The fact that Q_2^- is above B_2 implies that

$$\sqrt{2T^-}q_2^- \geqslant \frac{\sigma^2\cos\beta}{(2\tan\beta-\sigma)^2}\cdot\frac{\sqrt{2T^+}q_2^+}{\cos\beta}.$$

So

$$\frac{T^-}{T^+} \geqslant \left[\frac{\sigma^2 q_2^+}{q_2^-(2\tan\beta-\sigma)^2}\right]^2.$$

This proves the lemma. □

4.2. **Shrinking time for solutions of (1) and (19).** In this subsection we consider the shrinking time for the solution $u = u(x, t; u_0)$ of (1) with initial data u_0. We give two results. The first one is about the shrinking time of $u = u(x, t; u_0)$ for any given u_0, the second one is about the existence of u_0 for given shrinking time T.

Lemma 4.3. *Let $u_0 \in C^{2+\mu}_{\rm ad}$ ($\mu \in (0, 1)$) be an initial data. If it satisfies (67), then there exists $T \in (T^-, T^+)$ such that the solution $u(x, t; u_0)$ of (1) and (19) exists on time interval $[0, T)$, and*

$$\|u(\cdot, t)\|_{L^\infty} \to 0, \quad \xi_1(t) \to 0, \quad \xi_2(t) \to 0 \quad \textit{as } t \to T. \tag{70}$$

Proof. We use polar coordinates $x = r\sin\theta,\ y = r\cos\theta$ for $\theta \in [-\theta_0, \theta_0]$, that is,

$$r\cos\theta = u(r\sin\theta, t)$$

defines an implicit function $r = r(\theta, t)$ by (25). Problem (1) is then converted into

$$\begin{cases} r_t = a\left(\dfrac{r_\theta\cos\theta - r\sin\theta}{r_\theta\sin\theta + r\cos\theta}\right)\dfrac{rr_{\theta\theta} - 2r_\theta^2 - r^2}{r(r_\theta\sin\theta + r\cos\theta)^2}, & \theta \in [-\theta_0, \theta_0],\ t > 0,\\ r_\theta(-\theta_0, t) = -\tilde h_1(t, r(-\theta_0, t)), & t \geqslant 0,\\ r_\theta(\theta_0, t) = \tilde h_2(t, r(\theta_0, t)), & t \geqslant 0, \end{cases} \tag{71}$$

where

$$\tilde h_i(t, r) := \frac{\sin\theta_0 + k_i(t, r\cos\theta_0)\cos\theta_0}{\cos\theta_0 - k_i(t, r\cos\theta_0)\sin\theta_0} r \quad (i = 1, 2).$$

By (68) we have $r\cos\theta = u(r\sin\theta, t) \leqslant \sqrt{2T^+}\max\psi^+$. So $0 \leqslant r(\theta, t) \leqslant \sqrt{2T^+}\max\psi^+/\sin\beta$. By (24) and (25) we have

$$|r_\theta| = \left|\frac{\sin\theta + u_x\cos\theta}{\cos\theta - u_x\sin\theta} r\right| \leqslant \frac{1 + \tan\beta - \sigma}{\sigma\cos\beta}\cdot\frac{\sqrt{2T^+}\max\psi^+}{\sin\beta}.$$

Thus the standard a priori estimates (cf. [5, 6, 13, 14]) show that the solution of (71) with initial data $r(\theta, 0)$ will not develop singularity till $\min r(\cdot, t) \to 0$ as $t \to T$ for some $T > 0$. Moreover, $T^- < T < T^+$ follows from (68) and Lemma 4.2.

Now we prove (70). If $u(0, T+0) = 0$ but $u(\bar{x}, T+0) > 0$ for some $\bar{x} \neq 0$, then there exists $\hat{x}$ lying between 0 and $\bar{x}$ such that

$$|u_x(\hat{x}, T+0)| = \frac{|u(\bar{x}, T+0) - u(0, T+0)|}{|\bar{x}|} \geqslant \tan\beta.$$

This contradicts Lemma 2.1 and so the first limit in (70) holds. The last two limits in (70) follow from the first one. This proves the lemma. □

Lemma 4.4. *For any given $T > 0$, there exists an initial data u_0 such that the solution $u(x, t; u_0)$ of (1) and (19) exists on $[0, T)$ and it shrinks to 0 just at time T.*

Proof. Choose two initial data $u_{01}, u_{02} \in C^{2+\mu}_{\text{ad}}$. Moreover, u_{01} is chosen so large such that (67) holds for $u_0 = u_{01}$ and for some $T^- > T$. By Lemma 4.3, the solution $u(x, t; u_{01})$ shrinks to 0 as $t \to \widetilde{T}_1$ for some $\widetilde{T}_1 > T^- > T$. On the other hand, we choose u_{02} small such that (67) holds for $u_0 = u_{02}$ and for some $T^+ < T$. By Lemma 4.3 again, the solution $u(x, t; u_{02})$ shrinks to 0 as $t \to \widetilde{T}_2$ for some $\widetilde{T}_2 < T^+ < T$.

Now we modify the initial data from u_{02} to u_{01} little by little such that the modified initial data is still in $C^{2+\mu}_{\text{ad}}$. Since the solution $u(x, t; u_0)$ of (1) and (19) depends on the initial data u_0 continuously, we finally have an initial data u_0 such that $u(x, t; u_0)$ shrinks to 0 at time $T \in (\widetilde{T}_2, \widetilde{T}_1)$. □

In the following, we fix $T > 0$ and choose the initial data as in Lemma 4.4.

4.3. **Change of variables.** Lemma 4.3 gives the existence and boundedness of r (and so, of u), but the time interval is finite: $[0, T)$. So Lemma 3.4 can not be applied to give a periodic solution. To get a shrinking self-similar solution of (1), we introduce new coordinates. Set

$$\begin{cases} \theta = \arctan \dfrac{x}{y}, & (x, y) \in \overline{S} \backslash \{0\}, \\ \rho = -\dfrac{1}{2} \log \dfrac{x^2 + y^2}{T - t}, & (x, y) \in \overline{S} \backslash \{0\},\ 0 \leqslant t < T, \\ s = -\dfrac{1}{2} \log(T - t), & 0 \leqslant t < T. \end{cases} \tag{72}$$

The inverse map is

$$\begin{cases} x = e^{-s} e^{-\rho} \sin\theta, & (\theta, \rho) \in \overline{D},\ s \in [-\frac{1}{2} \log T, \infty), \\ y = e^{-s} e^{-\rho} \cos\theta, & (\theta, \rho) \in \overline{D},\ s \in [-\frac{1}{2} \log T, \infty), \\ t = T - e^{-2s}, & s \in [-\frac{1}{2} \log T, \infty). \end{cases} \tag{73}$$

A similar discussion as in subsections 2.4 and 3.2 shows that in these new variables, the original function $y = u(x, t)$ is converted into a new function $\rho = w(\theta, s)$. Differentiating the expression

$$e^{-s} e^{-w(\theta, s)} \cos\theta = u(e^{-s} e^{-w(\theta, s)} \sin\theta, T - e^{-2s}) \tag{74}$$

twice by θ and once by s we obtain

$$u_x = \frac{w_\theta \cos\theta + \sin\theta}{w_\theta \sin\theta - \cos\theta}, \quad u_{xx} = \frac{e^s e^w (w_{\theta\theta} + w_\theta^2 + 1)}{(w_\theta \sin\theta - \cos\theta)^3}, \quad u_t = \frac{e^s (1 + w_s)}{2 e^w (w_\theta \sin\theta - \cos\theta)}. \tag{75}$$

Therefore, problem (1) is converted into the following problem

$$\begin{cases} w_s = 2e^{2w} a \left(\dfrac{w_\theta \cos\theta + \sin\theta}{w_\theta \sin\theta - \cos\theta} \right) \dfrac{w_{\theta\theta} + w_\theta^2 + 1}{(w_\theta \sin\theta - \cos\theta)^2} - 1, \\ \qquad -\theta_0 < \theta < \theta_0, \ s \in [-\frac{1}{2}\log T, \infty), \\ w_\theta(-\theta_0, s) = \tilde{g}_1(s, w(-\theta_0, s)), \quad s \in [-\frac{1}{2}\log T, \infty), \\ w_\theta(\theta_0, s) = -\tilde{g}_2(s, w(\theta_0, s)), \quad s \in [-\frac{1}{2}\log T, \infty), \end{cases} \tag{76}$$

where

$$\tilde{g}_i(s, w) = \frac{\sin\theta_0 + k_i(T - e^{-2s}, e^{-s}e^{-w}\cos\theta_0)\cos\theta_0}{\cos\theta_0 - k_i(T - e^{-2s}, e^{-s}e^{-w}\cos\theta_0)\sin\theta_0} \quad (i = 1, 2). \tag{77}$$

4.4. **Bound of w.** We derive the boundedness of w in a series time intervals: $[0, \delta^2 T]$, $[\delta^2 T, T - (1-\delta^2)^2 T]$, $[T - (1-\delta^2)^2 T, T - (1-\delta^2)^3 T], \cdots$, where $\delta \in (0, 1)$ is that in Lemma 4.2.

In the first step, we choose $\psi^\pm$ and $T^\pm$ as in (67), and consider (1) on time interval $t \in [0, \delta^2 T]$ (note that $\delta^2 T < \delta^2 T^+ \leqslant \delta T^- < \delta T < \delta T^+ \leqslant T^-$), or, equivalently, consider (76) on time-interval $s \in [-\frac{1}{2}\log T, -\frac{1}{2}\log T - \frac{1}{2}\log(1-\delta^2)]$. In this period,

$$e^{-2s} = T - t \geqslant (1-\delta^2)T \quad \Rightarrow \quad e^{2s} T \leqslant \frac{1}{1-\delta^2}.$$

Thus,

$$\begin{aligned} e^{2s}(T^+ - t) &\leqslant e^{2s}[(T^+ - T^-) + (T - t)] \leqslant e^{2s}\left[\frac{1-\delta}{\delta} T^- + (T - t)\right] \\ &\leqslant \frac{1-\delta}{\delta} e^{2s} T + 1 \leqslant \delta_1 := 1 + \frac{1}{\delta(1+\delta)}. \end{aligned}$$

By (68) we have, for $t \in [0, \delta^2 T]$ or $s \in [-\frac{1}{2}\log T, -\frac{1}{2}\log T - \frac{1}{2}\log(1-\delta^2)]$,

$$e^{-s}e^{-w}\cos\theta = u(\cdot, T - e^{-2s}) \leqslant \sqrt{2(T^+ - t)}\max\psi^+.$$

So

$$e^{-w} \leqslant \frac{\max\psi^+}{\cos\theta_0} e^s \sqrt{2(T^+ - t)} \leqslant \frac{\max\psi^+}{\cos\theta_0}\sqrt{2\delta_1}.$$

On the other hand, in the same time interval $t \in [0, \delta^2 T]$ we have

$$\frac{T - T^-}{T - t} \leqslant \frac{T - T^-}{T - \delta^2 T} \leqslant \frac{T - \delta T}{T - \delta^2 T} = \frac{1}{1+\delta}.$$

So

$$e^{2s}(T^- - t) = \frac{(T^- - T) + (T - t)}{T - t} \geqslant 1 - \frac{1}{1+\delta} = \frac{\delta}{1+\delta}.$$

Thus by

$$e^{-s}e^{-w}\cos\theta = u(\cdot, T - e^{-2s}) \geqslant \sqrt{2(T^- - t)}\min\psi^-,$$

we have

$$e^{-w} \geqslant \min\psi^- e^s\sqrt{2(T^- - t)} \geqslant \min\psi^-\sqrt{\frac{2\delta}{1+\delta}}.$$

Therefore we obtain the bound of w for $s \in [-\frac{1}{2}\log T, -\frac{1}{2}\log T - \frac{1}{2}\log(1-\delta^2)]$:

$$-\log\left[\frac{\max\psi^+}{\cos\theta_0}\sqrt{2\delta_1}\right] \leqslant w \leqslant -\log\left[\min\psi^-\sqrt{\frac{2\delta}{1+\delta}}\right]. \tag{78}$$

Note that the lower and upper bounds do not depend on s.

Take a another pair $T_*^+, T_*^- > 0$ such that

$$\sqrt{2T_*^-}\ \psi^-\left(\frac{\cdot}{\sqrt{2T_*^-}}\right) \preceq u(\cdot,\delta^2 T) \preceq \sqrt{2T_*^+}\ \psi^+\left(\frac{\cdot}{\sqrt{2T_*^+}}\right).$$

Lemma 4.2 implies that $T_*^+ > T_*^- \geqslant \delta T_*^+$. From time $\delta^2 T$, u will shrink to 0 in time $T_2 := (1-\delta^2)T$, we consider another time interval: $t \in [\delta^2 T, \delta^2 T + \delta^2 T_2] = [\delta^2 T, T - (1-\delta^2)^2 T]$, or $s \in [-\frac{1}{2}\log T - \frac{1}{2}\log(1-\delta^2), -\frac{1}{2}\log T - \log(1-\delta^2)]$. Replacing $T^\pm$ by $T_*^\pm$ in the above discussion we see that (78) holds on this time interval.

Repeat such processes infinite times we obtain the estimate (78) for w on $[0,T)$.

4.5. **A priori estimate for w.** The gradient bound of w is similar as that for ω in subsection 2.4 and that for v in subsection 3.3. Using the standard theory of parabolic equations (cf. [5, 6, 13, 14]) we can get the following conclusions.

Lemma 4.5. *Problem (76) with initial data $w(\theta, -\frac{1}{2}\log T)$ (which is defined by (74) at $s = -\frac{1}{2}\log T$) has a unique, time-global solution $w(\theta,s) \in C^{2+\mu,1+\mu/2}([-\theta_0,\theta_0]\times[-\frac{1}{2}\log T,\infty))$ and*

$$\|w(\theta,s)\|_{C^{2+\mu,1+\mu/2}([-\theta_0,\theta_0]\times[-\frac{1}{2}\log T,\infty))} \leqslant C < \infty,$$

where C depends only on μ, k_i, σ and β but not on t, T and u_0.

The global existence of w is not new, it has been obtained from the existence of r on $[0,T)$ in subsection 4.2. The estimate is important and will be used below.

4.6. **Proof of Theorem 1.4.** In this subsection we prove the existence and uniqueness of a discrete shrinking self-similar solution on $[0,T)$. Conditions (11) and (14) imply that $\tilde g_1(s,w)$ and $\tilde g_2(s,w)$ are $\log b$-periodic in s. A similar discussion as in subsection 3.6 shows that (76) has a solution $\widetilde P(\theta,s)$, which is $\log b$-periodic in s,

$$\|\widetilde P(\theta,s)\|_{C^{2+\mu,1+\mu/2}([-\theta_0,\theta_0]\times\mathbb{R})} \leqslant C(\mu,k_1,k_2,\sigma,\beta), \tag{79}$$

and $\|w(\cdot,s) - \widetilde P(\cdot,s)\|_{C^2([-\theta_0,\theta_0])} \to 0$ as $s\to\infty$.

Now we recover $\widetilde P(\theta,s)$ back to a corresponding solution of (1), that is, define $\widetilde U, \widetilde\Xi_i$ $(i=1,2)$ by

$$\begin{aligned} &e^{-s}e^{-\widetilde P}\cos\theta = \widetilde U(e^{-s}e^{-\widetilde P}\sin\theta, T-e^{-2s}),\\ &\widetilde\Xi_i(T-e^{-2s}) = (-1)^i e^{-s}e^{-\widetilde P(\theta_0,s)}\sin\theta_0, \qquad \text{for } s\geqslant -\tfrac{1}{2}\log T. \end{aligned}\tag{80}$$

They are well-defined as in previous subsections. Moreover,

$$\begin{aligned} b^{-1}\widetilde U(e^{-s}e^{-\widetilde P(\theta,s)}\sin\theta, T-e^{-2s}) &= b^{-1}e^{-s}e^{-\widetilde P(\theta,s)}\cos\theta\\ &= e^{-s-\log b}e^{-\widetilde P(\theta,s+\log b)}\cos\theta\\ &= \widetilde U(b^{-1}e^{-s}e^{-\widetilde P(\theta,s)}\sin\theta, T-b^{-2}e^{-2s}) \end{aligned}$$

and for $i=1,2$,

$$b^{-1}\widetilde\Xi_i(T-e^{-2s}) = (-1)^i e^{-s-\log b}e^{-\widetilde P(\theta_0,s+\log b)}\sin\theta_0 = \widetilde\Xi_i(T-e^{-2s-2\log b}).$$

Hence, for $t' := e^{2s}\in(0,T]$ and $-\widetilde\Xi_1(T-t')\leqslant x\leqslant\widetilde\Xi_2(T-t')$ we have

$$b^{-1}\widetilde U(x,T-t') = \widetilde U(b^{-1}x, T-b^{-2}t'), \quad b^{-1}\widetilde\Xi_i(T-t') = \widetilde\Xi_i(T-b^{-2}t')\ \ (i=1,2). \tag{81}$$

This means that $(\widetilde{U}, \widetilde{\Xi}_1, \widetilde{\Xi}_2)$ is a discrete shrinking self-similar solution of (1) on $[0, T)$.

We now prove the uniqueness result under the assumption that

$$k_i(t,u) \equiv k_i(u) \quad (i = 1, 2).$$

First, it is convenient to take a time shift and consider the problem on $[-T, 0)$. More precisely, as in section 1, we define

$$\widehat{U}(x,t;T) := \widetilde{U}(x, T+t) \quad \text{for } -\widehat{\Xi}_1(t;T) \leqslant x \leqslant \widehat{\Xi}_2(t;T),\ t \in [-T, 0), \tag{82}$$

with

$$\widehat{\Xi}_i(t;T) := \widetilde{\Xi}_i(T+t) \quad \text{for } t \in [-T, 0) \quad (i = 1, 2).$$

Then (81) implies that $\widehat{U}$, $\widehat{\Xi}_1$ and $\widehat{\Xi}_2$ satisfy (16) and (17). So $(\widehat{U}, \widehat{\Xi}_1, \widehat{\Xi}_2)$ is a discrete self-similar solution of (18) on time-interval $t \in [-T, 0)$, and $\widehat{U}, \widehat{\Xi}_1, \widehat{\Xi}_2$ all converge to 0 as $t \to 0 - 0$.

Next, we construct self-similar solutions which decrease monotonically. Indeed, as in subsection 3.8, under the assumption (15), we can choose a concave initial data u_0^* such that $a(u_{0x}^*)u_{0xx}^* < 0$. Let $u^*(x,t)$ be the solution of (1) with initial data u_0^*, then $u_t^*(x,t) < 0$ by maximum principle, and $u^*(x,t)$ shrinks to 0 as $t \to T^*$ for some $T^* > 0$. Moreover, for any given $T > 0$, we can choose u_0^* sufficiently large such that $T^* > T$.

Converting this $u^*(x,t)$ to a new unknown $w^*(\theta, s)$ as in subsection 4.3 we have $1 + w_s^*(\theta, s) < 0$ by (75), and so the ω-limit $\widetilde{P}^*$ of w^* as in (79) satisfies $1 + \widetilde{P}_s^*(\theta, s) \leqslant 0$. Consequently, the corresponding functions $\widetilde{U}^*$, $\widetilde{\Xi}_1^*$ and $\widetilde{\Xi}_2^*$ defined by $\widetilde{P}^*$ as in (80) satisfy $\widetilde{U}_t^* \leqslant 0$. In addition, $\widetilde{U}^*$ is a shrinking self-similar solution of (1) on $[0, T^*)$. Hence,

$$\widehat{U}^*(x,t;T^*) := \widetilde{U}^*(x, T^* + t), \tag{83}$$

which is defined for $t \in [-T^*, 0)$ and

$$-\widehat{\Xi}_1^*(t) := -\widetilde{\Xi}_1(T^* + t) \leqslant x \leqslant \widetilde{\Xi}_2(T^* + t) =: \widehat{\Xi}_2^*(t),$$

is a shrinking self-similar solution of (18) on $[-T^*, 0)$. By $\widetilde{U}_t^* \leqslant 0$ we have $\widehat{U}_t^* \leqslant 0$. By the strong maximum principle we even have

$$\widehat{U}_t^* < 0, \quad \widehat{\Xi}_{1t}^* < 0, \quad \widehat{\Xi}_{2t}^* < 0 \quad \text{for } t \in [-T^*, 0). \tag{84}$$

Lemma 4.6. *Let $\widehat{U}$ be as in (82) and $\widehat{U}^*$ be as in (83). Then $\widehat{U}(x,t) \equiv \widehat{U}^*(x,t)$ for $t \in [-T, 0)$.*

Proof. Suppose on the contrary that, there exists $t_0 \in [-T, 0)$ such that

$$\widehat{U}(\cdot, t_0) \leqslant \widehat{U}^*(\cdot, t_0) \quad \text{but} \quad \widehat{U}(x, t_0) \not\equiv \widehat{U}^*(x, t_0), \tag{85}$$

or

$$\widehat{U}^*(\cdot, t_0) \leqslant \widehat{U}(\cdot, t_0) \quad \text{but} \quad \widehat{U}^*(x, t_0) \not\equiv \widehat{U}(x, t_0). \tag{86}$$

Then we derive a contradiction from (85) (In a similar way one can derive a contradiction from (86)).

By (85) and by $\widehat{U}_t^* < 0$, there exists $\tau(t_0) \in [0, -t_0)$ such that

$$\widehat{U}(\cdot, t_0) \preceq \widehat{U}^*(\cdot, t_0 + \tau(t_0)). \tag{87}$$

By comparison result Lemma 2.4 we have

$$\widehat{U}\Big(\cdot, \frac{t_0}{b^2}\Big) = \widehat{U}\Big(\cdot, t_0 + \frac{t_0}{b^2} - t_0\Big) \ll \widehat{U}^*\Big(\cdot, t_0 + \tau(t_0) + \frac{t_0}{b^2} - t_0\Big) = \widehat{U}^*\Big(\cdot, \tau(t_0) + \frac{t_0}{b^2}\Big).$$

On the other hand, by (87) and the self-similarity (16) we have

$$\widehat{U}\Big(\cdot, \frac{t_0}{b^2}\Big) \preceq \widehat{U}^*\Big(\cdot, \frac{t_0+\tau(t_0)}{b^2}\Big).$$

Combining these inequalities with the fact that $\widehat{U}^*_t < 0$ we have $\tau(t_0) > b^2\tau(t_0)$, a contradiction. This proves the lemma. □

This completes the proof of Theorem 1.4.

4.7. **Proof of Theorem 1.6.** Under the assumption $k_i(t,u) \equiv k_i(u)$ $(i = 1,2)$, Theorem 1.6 follows from the previous subsection easily.

Indeed, we can define positive functions $\Xi_1^\natural(t)$ and $\Xi_2^\natural(t)$ for $t \in (-\infty, 0)$ by

$$\Xi_i^\natural(t) := \widehat{\Xi}_i^*(t) \quad \text{for } t \in [-T^*, 0),\ i = 1,2,$$

and define $U^\natural(x,t)$ on $(x,t) \in [-\Xi_1^\natural(t), \Xi_2^\natural(t)] \times (-\infty, 0)$ by

$$U^\natural(x,t) := \widehat{U}^*(x,t) \quad \text{for } x \in [-\Xi_1^\natural(t), \Xi_2^\natural(t)],\ t \in [-T^*, 0).$$

Lemma 4.6 implies that the functions $U^\natural(x,t)$, $\Xi_1^\natural(t)$ and $\Xi_2^\natural(t)$ are well-defined (i.e., they do not depend on T^*). The triple $(U^\natural,\ \Xi_1^\natural,\ \Xi_2^\natural)$ is unique on time interval $[-T, 0)$ for any $T > 0$, and so is unique in $(-\infty, 0)$. Moreover, we have

$$U_t^\natural(x,t) < 0,\ \Xi_{1t}^\natural(t) < 0,\ \Xi_{2t}^\natural(t) < 0 \quad \text{for } t < 0$$

by (84). This proves Theorem 1.6.

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