TOWARDS CLASSIFICATION OF MULTIPLE-END SOLUTIONS TO THE ALLEN-CAHN EQUATION IN $\mathbb{R}^2$

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ABSTRACT. An entire solution of the Allen-Cahn equation $\Delta u = f(u)$, where f is an odd function and has exactly three zeros at ± 1 and 0, e.g. $f(u) = u(u^2 - 1)$, is called a $2k$-ended solution if its nodal set is asymptotic to $2k$ half lines, and if along each of these half lines the function u looks (up to a multiplication by -1) like the one dimensional, odd, heteroclinic solution H, of $H'' = f(H)$. In this paper we present some recent advances in the theory of the multiple-end solutions. We begin with the description of the moduli space of such solutions. Next we move on to study a special class of these solutions with just four ends. A special example is the saddle solutions U whose nodal lines are precisely the straight lines $y = \pm x$. We describe the connected components of the moduli space of 4-ended solutions. Finally we establish a uniqueness result which gives a complete classification of these solutions. It says that all 4-ended solutions are continuous deformations of the saddle solution.

1. **Some entire solutions to the Allen-Cahn equation in $\mathbb{R}^2$.** In this paper we present some recent results in the theory of entire solutions to the Allen-Cahn equation:

$$\Delta u = F'(u) \text{ in } \mathbb{R}^2. \tag{1.1}$$

The function F is a smooth, double well potential, which means that we assume the following conditions for F : F is even, nonnegative and has only two zeros at ± 1, $F'(t) \neq 0, t \in (0,1)$. We also suppose $F''(1) \neq 0$, $F''(0) \neq 0$. A standard example is $F(u) = \frac{1}{4}(1 - u^2)^2$ and for simplification in this paper we will restrict our considerations to this nonlinearity. Our results apply as well in the most general setting but the presentation becomes somewhat cumbersome.

It is known that (1.1) has a solution whose nodal set is a straight line, it will be called *a planar solution*. It is simply obtained by taking the unique, odd, heteroclinic solution connecting -1 to 1:

$$H'' = F'(H), \quad H(\pm\infty) = \pm 1, \quad H(0) = 0, \tag{1.2}$$

which in the particular case of F we are considering is explicit $H(t) = \tanh\left(\frac{t}{\sqrt{2}}\right)$, and letting

$$u(\mathbf{x}) = \pm H(\mathbf{x} \cdot \mathbf{e}^\perp - r), \tag{1.3}$$

for some vector $\mathbf{e} \in S^1$, and $r \in \mathbb{R}$, where $\perp$ denotes the rotation by the angle $\pi/2$ in $\mathbb{R}^2$. The nodal set of the planar solution is the affine line

$$s \mapsto r\mathbf{e}^\perp + s\mathbf{e}, \quad s \in \mathbb{R}.$$

We note that $\partial_{\pm\mathbf{e}^\perp} u = H' > 0$. De Giorgi conjecture says that if u is any smooth and bounded solution of (1.1) such that $\partial_{\mathbf{v}} u > 0$ for certain fixed direction $\mathbf{v}$ then in fact u must be the planar solution of the form (1.3) with $\pm\mathbf{e}^\perp \cdot \mathbf{v} > 0$. Indeed this conjecture holds in $\mathbb{R}^N$, $N \leq 8$ ([12] when $N = 2$, [2] when $N = 3$, and [28], for $4 \leq N \leq 8$ under some additional limit condition), while a counterexample can be given when $N \geq 9$ [9]. It is worth mentioning that the De Giorgi conjecture is a direct analog of the famous Bernstein conjecture in the theory of minimal surfaces.

There are other examples of entire solutions of (1.1) that are of interest. The simplest one is the so called *saddle solution U* whose existence was established in [6]. The nodal set of this solution is the union of the two lines $y = \pm x$. Another important fact, also proven in [6] (see also [13]), is that up to a sign and a rigid motion, the *saddle solution* is the unique bounded solution whose nodal set coincides with the union of the two lines $y = \pm x$. The saddle solution satisfies $U(x,y) = U(-x,y) = U(x,-y)$, i.e. it is even with respect to the coordinate axis. Also, we have $\partial_y U > 0$, and $\partial_x U < 0$ in the upper right quadrant $Q^\llcorner$. More general examples include solutions whose nodal lines are precisely the affine half lines

$$s \mapsto se^{j\pi i/k + i\pi/4}, \quad s \geq 0, \quad j = 1, \ldots, 2k,$$

see [1]. Again, along each of the half lines these solutions are asymptotic to copies of $\pm H$, i.e the planar solutions.

This example do no exhaust all the possible entire solutions of (1.1). In [8] it is shown that for any $k \geq 2$ there exists a family of solutions to (1.1) with the following characteristics: (1) the nodal set of each solution consists of k nonintersecting curves; (2) outside of a large compact set each of these curves is asymptotic to a set of $2k$ affine half lines and along each of these half lines the solution looks like a copy of the planar solution; (3) the angles between the asymptotic half lines are small, or in other words the nodal lines are nearly parallel, outside of a compact. This family depends smoothly on $2k$ parameters.

From the above rough description we see that all the entire solutions we have considered so far share a common feature: their asymptotic behavior outside of a compact is that of a planar solution along a set of $2k$ affine half lines. This leads to the definition of the set of $2k$-*ended solution*: we define $\mathcal{M}_{2k}$ to be the set of

solutions of (1.1) whose nodal set outside of a compact is asymptotic to $2k$ affine half lines and which, along each of these half lines looks like copies of $\pm H$ (more precise definition will follow). The main question we will deal with in this paper is the description of the set of $2k$-ended solutions $\mathcal{M}_{2k}$. In general terms our goal is twofold: first we want to understand the local structure of $\mathcal{M}_{2k}$, and second we want to classify $2k$-ended solutions. As for the former question our main result says that locally near a nondegenerate solution $u \in \mathcal{M}_{2k}$, the set $\mathcal{M}_{2k}$ has a structure of $2k$ dimensional, smooth manifold. As for the second issue at this point we can only give a satisfactory answer in the case $k = 2$, i.e. for 4-ended solution. We will show that $\mathcal{M}_4$ consists of a single connected component that contains the saddle solution U and the 4-ended solutions with almost parallel ends.

2. **Existence of $2k$-ended solutions with almost parallel nodal lines.** The starting point of the program outlined above is the existence result which shows that for any $k > 1$ there exists a $2k$ parameter family of $2k$-ended solutions [8]. To state our result in precise way, we assume that we are given a solution $\mathbf{q} := (q_1, \ldots, q_k)$ of the *Toda system*

$$c_0 \, q_j'' = e^{\sqrt{2}(q_{j-1} - q_j)} - e^{\sqrt{2}(q_j - q_{j+1})}, \tag{2.1}$$

for $j = 1, \ldots, k$, where $c_0 = \frac{\sqrt{2}}{24}$ and we agree that

$$q_0 \equiv -\infty \qquad \text{and} \qquad q_{k+1} \equiv +\infty \,.$$

The Toda system (2.1) is a classical example of integrable system which has been extensively studied. It models the dynamics of finitely many mass points on the line under the influence of an exponential potential. We refer to [16] and [24] for the complete description of the theory. Of importance for us is the fact that solutions of (2.1) can be described (almost explicitly) in terms of $2k$ parameters. Moreover, if $\mathbf{q}$ is a solution of (2.1), then the long term behavior (i.e. long term scattering) of the q_j at $\pm\infty$ is well understood and it is known that, for all $j = 1, \ldots, k$, there exist $a_j^+, b_j^+ \in \mathbb{R}$ and $a_j^-, b_j^- \in \mathbb{R}$, all depending on the solution $\mathbf{q}$, such that

$$q_j(t) = a_j^\pm \, |t| + b_j^\pm + \mathcal{O}_{\mathcal{C}^\infty(\mathbb{R})}(e^{-\tau_0 \, |t|}), \tag{2.2}$$

as t tends to $\pm\infty$, for some $\tau_0 > 0$. Moreover, $a_{j+1}^\pm > a_j^\pm$ for all $j = 1, \ldots, k - 1$.

Given $\varepsilon > 0$, we define the vector valued function $\mathbf{q}_\varepsilon$, whose components are given by

$$q_{j,\varepsilon}(x) := q_j(\varepsilon \, x) - \sqrt{2} \left(j - \frac{k+1}{2} \right) \log \varepsilon \,. \tag{2.3}$$

It is easy to check that the $q_{j,\varepsilon}$ are again solutions of (2.1).

Observe that, according to the description of the asymptotics of the functions q_j, the graphs of the functions $q_{j,\varepsilon}$ are asymptotic to oriented half lines at infinity. In addition, for $\varepsilon > 0$ small enough, these graphs are disjoint and in fact their mutual distance is given by $-\sqrt{2} \log \varepsilon + \mathcal{O}(1)$ as ε tends to 0.

It will be convenient to agree that χ^+ (resp. χ^-) is a smooth cutoff function defined on $\mathbb{R}$ which is identically equal to 1 for $x > 1$ (resp. for $x < -1$) and identically equal to 0 for $x < -1$ (resp. for $x > 1$) and additionally $\chi^- + \chi^+ \equiv 1$. With these cutoff functions at hand, we define the 4 dimensional space

$$D := \mathrm{Span} \left\{ x \longmapsto \chi^\pm(x), \, x \longmapsto x \, \chi^\pm(x) \right\}, \tag{2.4}$$

and, for all $\mu \in (0,1)$ and all $\tau \in \mathbb{R}$, we define the space $\mathcal{C}_\tau^{2,\mu}(\mathbb{R})$ of $\mathcal{C}^{2,\mu}$ functions h which satisfy

$$\|h\|_{\mathcal{C}_\tau^{2,\mu}(\mathbb{R})} := \|?(\cosh x)^\tau \, h\|_{\mathcal{C}^{2,\mu}(\mathbb{R})} < \infty \,.$$

Keeping in mind the above notations, we have the following theorem [8] :

Theorem 2.1. *For all $\varepsilon > 0$ sufficiently small, there exists an entire solution u_ε of the Allen-Cahn equation* (1.1) *whose nodal set is the union of k disjoint curves $\Gamma_{1,\varepsilon}, \ldots, \Gamma_{k,\varepsilon}$ which are the graphs of the functions*

$$x \longmapsto q_{j,\varepsilon}(x) + h_{j,\varepsilon}(\varepsilon \, x) \,,$$

for some functions $h_{j,\varepsilon} \in \mathcal{C}_\tau^{2,\mu}(\mathbb{R}) \oplus D$ satisfying

$$\|h_{j,\varepsilon}\|_{\mathcal{C}_\tau^{2,\mu}(\mathbb{R}) \oplus D} \leq C \, \varepsilon^\alpha,$$

for some constants $C, \alpha, \tau, \mu > 0$ independent of $\varepsilon > 0$.

In other words, for any solution of the Toda system of the form (2.3), with ε small, we can find a $2k$-ended solutions of (1.1) whose nodal lines are asymptotic to the graphs of the functions $q_{j,\varepsilon}$, as ε tends to 0. Also, as $x \to \pm\infty$ the nodal lines of the function u_ε approach some affine half lines with small slopes at an exponential rate. Since, as we have observed, the family of solutions of the Toda system depends smoothly on $2k$ parameters, the same apply for the family of solutions of the (1.1) we have constructed in [8].

Going through the proof, one can be more precise about the description of the solution u_ε. If $\Gamma \subset \mathbb{R}^2$ is a curve in $\mathbb{R}^2$ which is the graph over the x-axis of some function, we denote by $\mathrm{dist}^s(\cdot, \Gamma)$ the signed distance to Γ which is positive in the upper half of $\mathbb{R}^2 \setminus \Gamma$ and is negative in the lower half of $\mathbb{R}^2 \setminus \Gamma$. Then, we have the :

Proposition 2.1. [8] *The solution of* (1.1) *provided by Theorem* 2.1 *satisfies*

$$\|e^{\varepsilon \, \hat{\alpha} \, |\mathbf{x}|} \, (u_\varepsilon - u_\varepsilon^*)\|_{L^\infty(\mathbb{R}^2)} \leq C \, \varepsilon^{\bar{\alpha}} \,,$$

for some constants $C, \bar{\alpha}, \hat{\alpha} > 0$ independent of ε, where

$$u_\varepsilon^* := \sum_{j=1}^k (-1)^{j+1} \, H\big(\mathrm{dist}^s(\cdot, \Gamma_{j,\varepsilon})\big) - \frac{1}{2}((-1)^k + 1) \,. \tag{2.5}$$

It is interesting to observe that, when $k \geq 3$, there are solutions of (2.1) whose graphs have no symmetry and our result yields the existence of entire solutions of (1.1) without any symmetry provided the number of ends is larger than or equal to 6.

Let us also remark that when $k \geq 3$ is odd, the solutions constructed in Theorem 2.1 satisfy

$$\lim_{y \to \pm\infty} u_\varepsilon(x, y) = \pm 1. \tag{2.6}$$

This shows that the uniformity condition in Gibbons' conjecture is necessary. Let us recall that Gibbons' conjecture states that the level sets of any bounded solution to (1.1) satisfying

$$\lim_{x_N \to \pm\infty} u(x_1, ..., x_{N-1}, x_N) = \pm 1, \ \text{uniformly in } (x_1, ..., x_{N-1}), \tag{2.7}$$

must be hyperplanes. Gibbons' conjecture for all dimensions is proved independently in [3], [4] and [10].

3. **Moduli space of $2k$-ended solutions.** We start with a brief review of the material which is necessary for a precise statement of results proven in [7]. Let us recall that in the case of the bistable quartic nonlinearity $F(u) = \frac{1}{4}(1 - u^2)^2$ which we are considering here the heteroclinic solution $H(t) = \tanh\left(\frac{t}{\sqrt{2}}\right)$. We now collect some basic information about the spectrum of the operator

$$L := -\partial_x^2 + F''(H),$$

which arises as the linearized operator of one dimensional Allen-Cahn equation:

$$\partial_x^2 u = F'(u), \quad \text{in } \mathbb{R},$$

about H and which is acting on functions defined on $\mathbb{R}$. All the information we need is included in the :

Lemma 3.1. *All the the eigenvalues of L are known and are given by*

$$\mu_0 = 0, \quad \text{with associated eigenfunction} \quad w_0(x) = \frac{1}{\cosh^2(\frac{x}{\sqrt{2}})},$$

and

$$\mu_1 = \frac{3}{2}, \quad \text{with associated eigenfunction} \quad w_1(x) = \frac{\sinh(\frac{x}{\sqrt{2}})}{\cosh^2(\frac{x}{\sqrt{2}})},$$

while the bottom of the continuous spectrum is $\alpha^2 = 2$.

For a proof of this fact, we refer to [25].

3.1. **Geometric description of the solutions.** As promised, we give a precise definition of the concept of $2k$-ended solutions as it was introduced in [7] (the set of such solutions is denoted by $\mathcal{M}_{2k}$). This requires some preliminary definitions. At the heart of the description of the nodal set of the solutions is the set Λ of *oriented affine lines* in $\mathbb{R}^2$. Any element $\lambda \in \Lambda$ can be uniquely written as

$$\lambda := r\, \mathsf{e}^\perp + \mathbb{R}\, \mathsf{e},$$

for some $r \in \mathbb{R}$ and some unit vector $\mathsf{e} \in S^1$, which defines the orientation of the line. Recall that we denote by $\perp$ the rotation of angle $\pi/2$ in $\mathbb{R}^2$. Clearly, Λ is diffeomorphic to $\mathbb{R} \times S^1$ and writing $\mathsf{e} = (\cos\theta, \sin\theta)$, we get local coordinates (r, θ) in Λ. Observe that the affine lines are oriented and hence we do not identify the line corresponding to (r, θ) and the line corresponding to $(-r, \theta + \pi)$. There is also a natural symplectic structure on Λ which, in these local coordinates, is given by

$$\omega := dr \wedge d\theta.$$

Note that the map J defined by

$$J\, \partial_\theta = -\partial_r \qquad \text{and} \qquad J\, \partial_r = \partial_\theta,$$

(which corresponds to the rotation by $\pi/2$ in the tangent space) induces an almost complex structure on Λ. This map, together with the 2-form ω induces the natural metric on Λ

$$g = dr^2 + d\theta^2.$$

More generally, for all $k' \geq 1$, let us denote by $\Lambda^{k'}$ the set of k'-tuples of oriented affine lines in $\mathbb{R}^2$. This set is clearly diffeomorphic to $\mathbb{R}^{k'} \times (S^1)^{k'}$ and,

again, there exists a natural symplectic structure on $\Lambda^{k'}$ which, in local coordinates $(r_1, \ldots, r_{k'}, \theta_1, \ldots, \theta_{k'})$, can be written as

$$\omega_{k'} := dr_1 \wedge d\theta_1 + \ldots + dr_{k'} \wedge d\theta_{k'}. \tag{3.1}$$

The almost complex structure and the metric on $\Lambda^{k'}$ can be introduced in the same way this was done for Λ.

We have the definition :

Definition 3.1. A k'-tuple of oriented affine lines $\lambda = (\lambda_1, \ldots, \lambda_{k'}) \in \Lambda^{k'}$ is said to be *ordered* if each λ_j can be written as

$$\lambda_j := r_j \, \mathsf{e}_j^\perp + \mathbb{R} \, \mathsf{e}_j, \tag{3.2}$$

for some $r_j \in \mathbb{R}$ and some unit vector $\mathsf{e}_j \in S^1$ which can be written as $\mathsf{e}_j = (\cos\theta_j, \sin\theta_j)$ with

$$\theta_1 < \theta_2 < \ldots < \theta_{k'} < 2\pi + \theta_1.$$

We denote by $\Lambda_{ord}^{k'}$ the set of k'-tuples of ordered, oriented affine lines and we denote by

$$\theta_\lambda := \frac{1}{2} \min\{\theta_2 - \theta_1, \ldots, \theta_{k'} - \theta_{k'-1}, 2\pi + \theta_1 - \theta_{k'}\},$$

the half of the minimum of the angles between any two consecutive oriented affine lines $\lambda_1, \ldots, \lambda_{k'}$.

Assume that we are given a k'-tuple of oriented affine lines $\lambda = (\lambda_1, \ldots, \lambda_{k'})$ as in (3.2). It is easy to check that for all $R > 0$ large enough and for all $j = 1, \ldots, k'$, there exists $s_j \in \mathbb{R}$ such that :

(i) The point $\mathsf{x}_j := r_j \, \mathsf{e}_j^\perp + s_j \, \mathsf{e}_j$ belongs to the circle ∂B_R, with $R > 0$.

(ii) The half lines

$$\lambda_j^+ := \mathsf{x}_j + \mathbb{R}^+ \, \mathsf{e}_j, \tag{3.3}$$

are disjoint and included in $\mathbb{R}^2 \setminus B_R$.

(iii) The minimum of the distance between two distinct half lines λ_i^+ and λ_j^+ is larger than 4.

The set of half affine lines $\lambda_1^+, \ldots, \lambda_{k'}^+$ together with the circle ∂B_R induce a decomposition of $\mathbb{R}^2$ into $k' + 1$ slightly overlapping connected components

$$\mathbb{R}^2 = \Omega_0 \cup \Omega_1 \cup \ldots \cup \Omega_{k'},$$

where

$$\Omega_0 := B_{R+1},$$

and where, for $j = 1, \ldots, k'$,

$$\Omega_j := \left\{ \mathsf{x} \in \mathbb{R}^2 : |\mathsf{x}| > R - 1 \quad \text{and} \quad \mathrm{dist}(\mathsf{x}, \lambda_j^+) < \mathrm{dist}(\mathsf{x}, \lambda_i^+) + 2, \quad \forall i \neq j \right\}, \tag{3.4}$$

where $\mathrm{dist}(\mathsf{x}, \lambda_j)$ denotes the distance to λ_j^+. Observe that, for all $j = 1, \ldots, k'$, the set Ω_j contains the half line λ_j^+.

We define $\mathbb{I}_0, \mathbb{I}_1, \ldots, \mathbb{I}_{k'}$, a smooth partition of unity of $\mathbb{R}^2$ which is subordinate to the above decomposition of $\mathbb{R}^2$. Hence

$$\sum_{j=0}^{k'} \mathbb{I}_j \equiv 1,$$

and the support of $\mathbb{I}_j$ is included in Ω_j, for $j = 0, \ldots, k'$. Without loss of generality, we can also assume that $\mathbb{I}_0 \equiv 1$ in

$$\Omega'_0 := B_{R-1},$$

and $\mathbb{I}_j \equiv 1$ in

$$\Omega'_j := \left\{ \mathbf{x} \in \mathbb{R}^2 \, : \, |\mathbf{x}| > R + 1 \quad \text{and} \quad \text{dist}(\mathbf{x}, \lambda_j^+) < \text{dist}(\mathbf{x}, \lambda_i^+) - 2, \quad \forall i \neq j \right\},$$

for $j = 1, \ldots, k'$. Finally, we assume that

$$\|\mathbb{I}_j\|_{\mathcal{C}^2(\mathbb{R}^2)} \leq C.$$

We now take $k' = 2k$, for some $k \geq 1$ and

$$\lambda = (\lambda_1, \ldots, \lambda_{2k}) \in \Lambda_{ord}^{2k},$$

we write $\lambda_j^+ = \mathbf{x}_j + \mathbb{R}^+ \, \mathbf{e}_j$ and we define

$$u_\lambda := \sum_{j=1}^{2k} (-1)^j \, \mathbb{I}_j \, H(\text{dist}^s(\,\cdot\,, \lambda_j)), \tag{3.5}$$

where

$$\text{dist}^s(\mathbf{x}, \lambda_j) := (\mathbf{x} - \mathbf{x}_j) \cdot \mathbf{e}_j^\perp, \tag{3.6}$$

denotes the *signed distance* from a point $\mathbf{x} \in \mathbb{R}^2$ to λ_j.

Observe that, by construction, the function u_λ is, away from a compact, asymptotic to copies of the model solutions whose nodal set are the half affine lines $\lambda_1^+, \ldots, \lambda_{2k}^+$. A simple computation shows that u_λ is not far from being a solution of (1.1) in the sense that $\Delta \, u_\lambda - F'(u_\lambda)$ is a function which decays exponentially to 0 at infinity (this uses the fact that $\theta_\lambda > 0$).

We are interested in solutions of (1.1) which are asymptotic to a function u_λ for some choice of $\lambda \in \Lambda_{ord}^{2k}$. More precisely, we have the :

Definition 3.2. Let $\mathcal{S}_{2k}$ denote the set of functions u which are defined in $\mathbb{R}^2$ and which satisfy

$$u - u_\lambda \in W^{2,2}(\mathbb{R}^2), \tag{3.7}$$

for some $\lambda \in \Lambda_{ord}^{2k}$. We also define the decomposition operator $\mathcal{J}$ by

$$\mathcal{J}: \quad \mathcal{S}_{2k} \quad \longrightarrow \quad W^{2,2}(\mathbb{R}^2) \times \Lambda_{ord}^{2k}$$

$$u \quad \longmapsto \quad (u - u_\lambda, \lambda).$$

The topology on $\mathcal{S}_{2k}$ is the one for which the operator $\mathcal{J}$ is continuous (the target space being endowed with the product topology). We define $\mathcal{M}_{2k}$ to be the set of solutions u of (1.1) which belong to $\mathcal{S}_{2k}$.

The set $\mathcal{M}_2$ is non empty since it contains the planar solutions. As we have already discussed in the previous section, the result of [8] provides infinitely many solutions of (1.1) whose nodal set decomposes into $2k$ nearly parallel half lines. This result, together with the results in [6] and [1], imply that $\mathcal{M}_{2k} \neq \emptyset$ for any $k \geq 1$. Investigation of the structure of $\mathcal{M}_{2k}$ is then a natural question. At this point, the questions which are relevant are the following :

1. Is the space $\mathcal{M}_{2k}$ a smooth submanifold of $\mathcal{S}_{2k}$? If so, what is the dimension of $\mathcal{M}_{2k}$?

2. There exists a natural map

$$\mathcal{P} : \mathcal{M}_{2k} \longrightarrow \Lambda_{ord}^{2k},$$

defined by

$$\mathcal{P}(u) := \lambda, \tag{3.8}$$

if $u - u_\lambda \in W^{2,2}(\mathbb{R}^2)$. What can be said about this map ? Is it surjective ? If not, can one characterize its image ? In other words, what sets of half affine lines are asymptotic to nodal sets of solutions of (1.1) ?

3.2. **Local structure of the moduli space.** We keep the notations introduced above. Given $k \geq 1$ and

$$\lambda = (\lambda_1, \ldots, \lambda_{2k}) \in \Lambda_{ord}^{2k},$$

we write $\lambda_j^+ = \mathsf{x}_j + \mathbb{R}^+ \, \mathsf{e}_j$ as in (3.3). We denote by $\Omega_0, \ldots, \Omega_{2k}$ the decomposition of $\mathbb{R}^2$ associated to this $2k$-tuple of half affine lines and $\mathbb{I}_0, \ldots, \mathbb{I}_{2k}$ the partition of unity subordinate to this partition. Given $\gamma, \delta \in \mathbb{R}$, we define a *weight function* $\Gamma_{\gamma,\delta}$ such that

$$\Gamma_{\gamma,\delta}(\mathsf{x}) \sim 1 \, ,$$

in Ω_0 and, for $j = 1, \ldots, 2k$,

$$\Gamma_{\gamma,\delta}(\mathsf{x}) \sim e^{\gamma s} \, (\cosh r)^\delta.$$

in Ω_j, where we have written

$$\mathsf{x} = \mathsf{x}_j + r \, \mathsf{e}_j^\perp + s \, \mathsf{e}_j,$$

for some $r \in \mathbb{R}$ and $s > 0$ (observe that (r, s) are local coordinates which are well defined in each Ω_j). As usual, the notation $f \sim g$ means that there exists some constant $C > 1$ such that

$$\frac{1}{C} \, |g| \leq |f| \leq C \, |g|.$$

The explicit definition of the weight function $\Gamma_{\gamma,\delta}$ is given by

$$\Gamma_{\gamma,\delta}(\mathsf{x}) := \mathbb{I}_0(\mathsf{x}) + \sum_{j=1}^{2k} \mathbb{I}_j(\mathsf{x}) \, e^{\gamma \, (\mathsf{x} - \mathsf{x}_j) \cdot \mathsf{e}_j} \, \left(\cosh((\mathsf{x} - \mathsf{x}_j) \cdot \mathsf{e}_j^\perp) \right)^\delta, \tag{3.9}$$

so that, by construction, γ is the rate of decay or blow up along the half lines λ_j^+ and δ is the rate of decay or blow up in the direction orthogonal to λ_j^+.

With this definition in mind, we define the weighted Lebesgue space

$$L_{\gamma,\delta}^2(\mathbb{R}^2) := \Gamma_{\gamma,\delta} \, L^2(\mathbb{R}^2), \tag{3.10}$$

and the weighted Sobolev space

$$W_{\gamma,\delta}^{2,2}(\mathbb{R}^2) := \Gamma_{\gamma,\delta} \, W^{2,2}(\mathbb{R}^2). \tag{3.11}$$

Observe that, even though this does not appear in the notations, the partition of unity, the weight function and the induced weighted spaces all depend on the choice of $\lambda \in \Lambda_{ord}^{2k}$.

Our first result shows that, if u is a solution of (1.1) which is close to u_λ (in $W^{2,2}$ topology) then $u - u_\lambda$ tends to 0 exponentially fast at infinity.

Theorem 3.1 (Refined Asymptotics). [7] *Assume that $u \in \mathcal{S}_{2k}$ is a solution of (1.1) and define $\lambda = \mathcal{P}(u) \in \Lambda^{2k}_{ord}$, so that*

$$u - u_\lambda \in W^{2,2}(\mathbb{R}^2).$$

Then, there exist $\delta \in (0, \alpha)$ and $\gamma > 0$ such that

$$u - u_\lambda \in W^{2,2}_{-\gamma, -\delta}(\mathbb{R}^2). \tag{3.12}$$

More precisely, $\delta > 0$ and $\gamma > 0$ can be chosen so that

$$\gamma \in (0, \sqrt{\mu_1}), \qquad \gamma^2 + \delta^2 < \alpha^2 \qquad and \qquad \alpha > \delta + \gamma \cot \theta_\lambda, \tag{3.13}$$

where θ_λ is equal to the half of the minimum of the angles between two consecutive oriented affine lines $\lambda_1, \ldots, \lambda_{2k}$ (see Definition 3.1).

In particular, this implies that, given $u \in \mathcal{M}_{2k}$, there exists $\bar{\delta} > 0$ such that

$$\mathcal{J}(u) \in e^{-\bar{\delta}|\mathbf{x}|^2} W^{2,2}(\mathbb{R}^2) \times \Lambda^{2k}_{ord}.$$

In fact more is true. Namely, the choice of $\bar{\delta}$ can be made uniform in any neighborhood of $u \in \mathcal{M}_{2k}$ in $\mathcal{S}_{2k}$. More, precisely, given $u \in \mathcal{M}_{2k}$, there exists $\delta_u > 0$ and there exists a neighborhood $\mathcal{U}$ of u in $\mathcal{S}_{2k}$ such that

$$\begin{aligned} \mathcal{J}: \quad \mathcal{U} \cap \mathcal{M}_{2k} \quad &\longrightarrow \quad e^{-\delta_u |\mathbf{x}|^2} W^{2,2}(\mathbb{R}^2) \times \Lambda^{2k}_{ord} \\ \bar{u} \quad &\longmapsto \quad \mathcal{J}\,\bar{u}, \end{aligned},$$

is well defined and continuous (observe that continuity of this mapping is not a straightforward consequence of the definition of $\mathcal{J}$).

Before we state the next result, we have to introduce the notion of *nondegeneracy* in this context.

Definition 3.3. A function $u \in \mathcal{M}_{2k}$ is said to be *nondegenerate* if the linearized operator

$$-\Delta + F''(u),$$

is injective in the space $L^2_{-\gamma, \delta}(\mathbb{R}^2)$, for *some* $\gamma \in (0, \sqrt{\mu}_1)$ and *some* $\delta \in \mathbb{R}$ satisfying

$$\gamma^2 + \delta^2 < \alpha^2.$$

As already mentioned, the existence of a family of solutions of (1.1) which belongs to $\mathcal{M}_{2k}$ is guarantied by the result in [8]. We prove in [7] that the solutions obtained in [8] are nondegenerate, this implies that :

Proposition 3.1. [7] *For each $k \geq 1$, $\mathcal{M}_{2k}$ contains nondegenerate elements.*

Checking whether a given solution is nondegenerate or not is a hard problem. For example, non-degeneracy of the saddle solution $U \in \mathcal{M}_4$ constructed in [6] is shown in [17]. Anticipating a little bit further developments we should add that in fact all solutions in $\mathcal{M}_4$ are nondegenerate [19].

The second result of this paper is the following :

Theorem 3.2 (Dimension of the moduli space). [7] *Assume that $u \in \mathcal{M}_{2k}$ is nondegenerate. Then, in a neighborhood of u in $\mathcal{S}_{2k}$, the set of solutions of (1.1) is a smooth manifold of dimension $2k$.*

Near any nondegenerate elements of $\mathcal{M}_{2k}$, we also have some information about the mapping $\mathcal{P}$.

Theorem 3.3. [7] *Assume that $u \in \mathcal{M}_{2k}$ is nondegenerate. Then, there exists an open neighborhood of u in $\mathcal{M}_{2k}$ whose image by $\mathcal{P}$ is a Lagrangian submanifold of Λ^{2k} for the symplectic structure defined in (3.1).*

Geometrically the meaning of the mapping $\mathcal{P}$ is clear : Given any solution $u \in \mathcal{M}_{2k}$, $\mathcal{P}(u) \in \Lambda^{2k}$ corresponds to the choice of $2k$ oriented affine lines which determine the asymptotics of the nodal set of u at infinity. Theorem 3.2 and Theorem 3.3 show that there is in reality less *freedom* than what might be initially expected in selecting the half lines which are the asymptotes of the nodal sets of the solutions of (1.1). Indeed, at regular points of $\mathcal{M}_{2k}$, the image of $\mathcal{P}$ is a $2k$-dimensional submanifold of Λ^{2k} which is $4k$-dimensional. By definition the Lagrangian submanifold of a symplectic manifold (M, ω) of dimension $2n$ is a submanifold of M of half of the dimension of the ambient manifold (i.e. dimension $= n$), and such that on its tangent space the two form ω is degenerate. Let us explain what this means in the case at hand i.e. $M = \Lambda^{2k}$, $\omega = \omega_{2k}$. First we introduce the natural representation of the tangent space $T_\lambda \Lambda^{2k}$, $\lambda = (\lambda_1, \ldots, \lambda_{2k})$. To each λ_j we associate its representation

$$\lambda_j := r_j \, \mathsf{e}_j^\perp + \mathbb{R} \, \mathsf{e}_j, \quad r_j \in \mathbb{R}, \quad \mathsf{e}_j = (\cos \theta_j, \sin \theta_j).$$

Each affine half line λ_j can be translated in the direction orthogonal to e_j and rotated about the the point $\mathsf{x}_j = r_j \mathsf{e}^\perp$. In local coordinates $(r_1, \ldots, r_{2k}, \theta_1, \ldots, \theta_{2k})$ of λ these transformations correspond to vectors

$$\mathring{X}_j := (0, \ldots, 1, \ldots, 0, \ldots, 0),$$

and

$$\mathring{Y}_j(\mathsf{x}) := (0, \ldots, 0, 0, \ldots, 1, \ldots, 0),$$

respectively. Let us consider vectors $w^{(\ell)} \in T_\lambda \Lambda^{2k}$, $\ell = 1, 2$, where:

$$w^{(\ell)} = \sum_{j=1}^{2k} a_j^{(\ell)} \mathring{X}_j + b_j^{(\ell)} \mathring{Y}_j.$$

Then the condition $\omega_{2k}(w^{(1)}, w^{(2)}) = 0$ is equivalent to:

$$\sum_{j=1}^{2k} \left(a_j^{(1)} b_j^{(2)} - a_j^{(2)} b_j^{(1)} \right) = 0.$$

Observe that the image of $\mathcal{P}$ is naturally constrained. In fact it follows from [13] that

$$\sum_{j=1}^{2k} \mathsf{e}_j = 0. \tag{3.14}$$

Moreover, pushing further the analysis in [13], we can also prove that

$$\sum_{j=1}^{2k} r_j = 0. \tag{3.15}$$

These conservation laws stem from the fact that (1.1) is invariant under the action of isometries of $\mathbb{R}^2$ and, for the sake of completeness, we give a simple proof of these equalities below. Observe that (3.14) implies that the angle between two consecutive half lines is always less than or equal to π and that it can only be equal to π when $k = 1$. Therefore, if $u \in \mathcal{M}_{2k}$ and $\lambda = \mathcal{P}(u)$, we always have

$$0 \le \theta_\lambda \le \pi/2.$$

To prove (3.14)–(3.15) we will derive more general identity called *the balancing formula*. Actually, the balancing formula is a fundamental tool of our theory and its applications are crucial for most of the results we present here. To proceed we assume that u is a solution of (1.1) which is defined in $\mathbb{R}^2$. Assume that X and Y are two vector fields also defined in $\mathbb{R}^2$. In coordinates, we can write

$$X = \sum_j X^j \partial_{x_j}, \qquad Y = \sum_j Y^j \partial_{x_j},$$

and, if f is a smooth function, we use the following notations

$$X(f) := \sum_j X^j \, \partial_{x_j} f, \qquad \nabla f := \sum_j \partial_{x_j} f \, \partial_{x_j},$$

$$\operatorname{div} X := \sum_i \partial_{x_i} X^i,$$

and

$$\mathrm{d}^* X := \frac{1}{2} \sum_{i,j} (\partial_{x_i} X^j + \partial_{x_j} X^i) \, \mathrm{d}x_i \otimes \mathrm{d}x_j,$$

so that

$$\mathrm{d}^* X \, (Y, Y) = \sum_{i,j} \partial_{x_i} X^j \, Y^i \, Y^j.$$

Computing directly we get:

Lemma 3.2 (Balancing formula). *The following identity holds*

$$\operatorname{div}\left(\left(\frac{1}{2} |\nabla u|^2 + F(u) \right) X - X(u) \nabla u \right)$$
$$= \left(\frac{1}{2} |\nabla u|^2 + F(u) \right) \operatorname{div} X - \mathrm{d}^* X (\nabla u, \nabla u).$$

Translations of $\mathbb{R}^2$ correspond to the constant vector field

$$X := X_0$$

where X_0 is a fixed vector, while rotations correspond to the vector field

$$X := x \, \partial_y - y \, \partial_x.$$

In either case, we have $\operatorname{div} X = 0$ and $\mathrm{d}^* X = 0$. Therefore, we conclude that

$$\operatorname{div}\left(\left(\frac{1}{2} |\nabla u|^2 + F(u) \right) X - X(u) \nabla u \right) = 0,$$

for these two vector fields. The divergence theorem implies that

$$\int_{\partial \Omega} \left(\left(\frac{1}{2} |\nabla u|^2 + F(u) \right) X - X(u) \nabla u \right) \cdot \nu \, ds = 0, \tag{3.16}$$

where ν is the (outward pointing) unit normal vector field to $\partial \Omega$.

We define

$$c_0 := \int_{-\infty}^{+\infty} \left(\frac{1}{2} (\partial_x u_0)^2 + F(u_0) \right) dx.$$

We now assume that $u \in \mathcal{M}_{2k}$ and we keep the notations introduced above. We use (3.16) with Ω which is equal to the ball of radius R centered at the origin and then, we let R tend to ∞. Taking $X = X_0$, we find with little work

$$c_0 \sum_{j=1}^{2k} \mathsf{e}_j \cdot X_0 = 0.$$

Taking $X = x \, \partial_y - y \, \partial_x$, we find as well

$$\sum_{j=1}^{2k} r_j = 0.$$

This completes the proof of (3.14) and (3.15).

The proofs of Theorem 3.2 and Theorem 3.3 follow from the application of the implicit function theorem in a suitably designed weighted function space. The results and the arguments are very much in the spirit of what has already been done in the study of the moduli spaces of complete non compact constant mean curvatures surfaces in Euclidean space or complete constant scalar curvature metrics in conformal geometry [20], [22] and [21]. We refer the reader to [7] for detailed proofs of the theorems.

4. Classification of 4-ended solutions.

4.1. **The space of 4-ended solutions.** In general understanding the structure of the space of $2k$-ended solutions is quite a difficult question. However in case of 4-ended solutions it is possible to describe all connected components of $\mathcal{M}_4$. This is the subject of the present section in which we summarize the results obtained in [19] and [18]. We have already mentioned two important examples of solutions to (1.1) that belong to $\mathcal{M}_4$: the saddle solution constructed in [6] and the solutions with almost parallel ends constructed in [8]. As far as the structure of the set of 4-ended solutions is concerned, the main result of the previous section asserts that if $u \in \mathcal{M}_4$ is *nondegenerate*, then, close to u, $\mathcal{M}_4$ is a 4-dimensional smooth manifold. Observe that, given $u \in \mathcal{M}_4$, translations and rotations of u are also elements of $\mathcal{M}_4$ and this accounts for 3 of the 4 formal dimensions of $\mathcal{M}_4$, moreover, if $u \in \mathcal{M}_4$ then $-u \in \mathcal{M}_4$.

All the 4-ended solutions constructed so far have two axes of symmetry and in fact, it follows from a result of C. Gui [14] that all 4-ended solutions are symmetric up to an isometry:

Theorem 4.1. [14] *Assume that $u \in \mathcal{M}_4$. Then, there exists a rigid motion g such that $\bar{u} := u \circ g$ is even with respect to the x-axis and the y-axis, namely*

$$\bar{u}(x, y) = \bar{u}(-x, y) = \bar{u}(x, -y). \tag{4.1}$$

In addition, $\bar{u}$ is a monotone function of both the x and y variables in the upper right quadrant $Q^{\llcorner}$ defined by

$$Q^{\llcorner} := \{(x, y) \in \mathbb{R}^2 \, : \, x > 0 \quad y > 0\},$$

and, changing the sign of $\bar{u}$ if this is necessary, we can assume that

$$\partial_x \bar{u} < 0 \qquad and \qquad \partial_y \bar{u} > 0,$$

in $Q^{\llcorner}$.

Thanks to this result, we can define the moduli space of 4-ended even solutions by :

Definition 4.1. The set $\mathcal{M}_4^{even}$ is defined to be the set of $u \in \mathcal{S}_4$ which are solutions of (1.1), are even with respect to the x-axis and the y-axis and which tend to $+1$ at infinity along the y-axis (and tend to -1 at infinity along the x-axis). In particular,

$$\partial_x u < 0 \qquad \text{and} \qquad \partial_y u > 0,$$

in the upper right quadrant $Q^{\llcorner}$.

Thus, when studying $\mathcal{M}_4$, we restrict our attention to functions which are even with respect to the x-axis and the y-axis and, in this case, a solution $u \in \mathcal{M}_4^{even}$ is said to be *even-nondegenerate* if there is no $w \in W^{2,2}(\mathbb{R}^2) - \{0\}$, which is symmetric with respect to the x-axis and the y-axis, belongs to the kernel of

$$L := -\Delta + F''(u),$$

and which decays exponentially at infinity.

In the equivariant case (namely solutions which are invariant under both the symmetry with respect to the x-axis and the y-axis), Theorem 3.2 reduces to :

Theorem 4.2. *Assume that $u \in \mathcal{M}_4^{even}$ is even-nondegenerate, then, close to u, $\mathcal{M}_4^{even}$ is a 1-dimensional smooth manifold.*

Any solution $u \in \mathcal{M}_4^{even}$ has a nodal set which is asymptotic to 4 half oriented affine lines and, given the symmetries of u, these half oriented affine lines are images of each other by the symmetries with respect to the x-axis and the y-axis. In particular, there is at most one of these half oriented affine line

$$\lambda := r\,\mathsf{e}^{\perp} + \mathbb{R}\,\mathsf{e},$$

which is included in the upper right quadrant $Q^{\llcorner}$. Writing $\mathsf{e} = (\cos\theta, \sin\theta)$ where $\theta \in (0, \pi/2)$, we define

$$\mathcal{F}: \quad \mathcal{M}_4^{even} \quad \to \quad (-\pi/4, \pi/4) \times \mathbb{R},$$

$$u \quad \mapsto \quad (\theta - \pi/4, r).$$

For example, the image by $\mathcal{F}$ of the *saddle solution* defined in [6] is precisely $(0,0)$, while the images by $\mathcal{F}$ of the solutions constructed in [8] correspond to parameters (θ, r) where θ is close to $\pm\pi/4$ and r is close to $\mp\infty$.

Remark 4.1. Let us observe that, if $u \in \mathcal{M}_4^{even}$, then $\bar{u}$ defined by

$$\bar{u}(x, y) = -u(y, x),$$

also belongs to $\mathcal{M}_4^{even}$ and

$$\mathcal{F}(\bar{u}) = -\mathcal{F}(u).$$

In this paper, we are interested in understanding the structure of $\mathcal{M}_4^{even}$. To begin with, we prove that :

Theorem 4.3 (Nondegeneracy)**.** [19] *Any $u \in \mathcal{M}_4$ is nondegenerate and hence any $u \in \mathcal{M}_4^{even}$ is even-nondegenerate.*

As a consequence of this result, we find that all connected components of $\mathcal{M}_4^{even}$ are one-dimensional smooth manifolds. Moreover, as a byproduct of the proof of this result, we also obtain that the image by $\mathcal{F}$ of any connected component of $\mathcal{M}_4^{even}$ is a smooth immersed curve in $(-\pi/4, \pi/4) \times \mathbb{R}$. Thanks to Remark 4.1, we find that the image of $\mathcal{M}_4^{even}$ by $\mathcal{F}$ is invariant under the action of the symmetry with respect to $(0,0)$.

To proceed, we define the *classifying map* to be the projection of $\mathcal{F}$ onto the first variable

$$\mathcal{P}: \quad \mathcal{M}_4^{even} \quad \to \quad (-\pi/4, \pi/4),$$

$$u \quad \mapsto \quad \theta - \pi/4.$$

Our second result reads :

Theorem 4.4 (Properness)**.** [19] *The mapping $\mathcal{P}$ is proper, i.e. the pre-image of a compact in $(-\pi/4, \pi/4)$ is compact in $\mathcal{M}_4^{even}$ (endowed with the topology induced by the one of $\mathcal{S}_4$).*

Let M_0 be the connected component of $\mathcal{M}_4^{even}$ which contains the saddle solution. We claim that the properness of $\mathcal{P}$ implies that the image by $\mathcal{P}$ of M_0 is the entire interval $(-\pi/4, \pi/4)$. The proof of this claim goes as follows : we argue by contradiction and assume that $\mathcal{P}: M_0 \to (-\pi/4, \pi/4)$ is not onto. Recall that if $u \in \mathcal{M}_4^{even}$, then $\bar{u}$ defined by

$$\bar{u}(x,y) := -u(y,x),$$

also belongs to $\mathcal{M}_4^{even}$ and M_0 is also invariant under this transformation. We will write $\bar{u} = J\,u$. The properness of $\mathcal{P}$ implies that M_0 is compact and one dimensional. Hence, it must be diffeomorphic to S^1. Obviously $J: M_0 \to M_0$ is a diffeomorphism and the saddle solution is a fixed point of J. Since M_0 is diffeomorphic to S^1, there must be at least another fixed element $v \in M_0$ which is a fixed point of J. Then, the zero set of v is union of the two lines $y = \pm x$. But, according to [6] or [13], a solution of (1.1) having as zero set the two lines $y = \pm x$ is the saddle solution. This is a contradiction and the proof of the claim is complete. Note that this argument does not guarantee that there are no other components in $\mathcal{M}_4$, and in particular there may exist compact connected components in $\mathcal{M}_4^{even}$. To exclude this last possibiility we have:

Theorem 4.5. [19] *All connected components of $\mathcal{M}_4^{even}$ are diffeomorphic to $\mathbb{R}$, i.e. there is no closed loop in $\mathcal{M}_4^{even}$.*

Looking at the image by $\mathcal{P}$ of the connected component of $\mathcal{M}_4^{even}$ which contains the saddle solution, we conclude from the above results that :

Theorem 4.6 (Surjectivity of $\mathcal{P}$)**.** [19] *The mapping $\mathcal{P}$ is onto.*

As a consequence, for any $\theta \in (0, \pi/2)$, there exists a solution $u \in \mathcal{M}_4^{even}$ whose nodal set at infinity is asymptotic to the half oriented affine lines whose angles with the x-axis are given by $\theta, \pi - \theta, \pi + \theta$ and $2\pi - \theta$.

Given all the evidence we have, it is tempting to conjecture that $\mathcal{M}_4^{even}$ has only one connected component and that the image of $\mathcal{M}_4^{even}$ by $\mathcal{F}$ is a smooth embedded curve. Moreover, it is very likely that $\mathcal{P}$ is a diffeomorphism from $\mathcal{M}_4^{even}$ onto $(-\pi/4, \pi/4)$. Observe that Theorem 4.6 already proves that $\mathcal{P}$ is onto. To prove that $\mathcal{P}$ is a diffeomorphism is so far beyond our reach. But in the next section we

will show that $\mathcal{M}_4$ consist of just one connected component which contains both the saddle solution and the solutions with almost parallel ends.

To complete this list of results, we mention an interesting by-product of the proof of Theorem 4.3. Assume that u is a solution of (1.1) and denote by

$$L := -\Delta + F''(u),$$

the linearized operator about u. Recall that, if Ω is a bounded domain in $\mathbb{R}^2$, then the index of L in Ω is given by the number of negative eigenvalues of the operator L which belong to $W_0^{1,2}(\Omega)$. Following [11], we have the :

Definition 4.2. The function u, solution of (1.1), has finite Morse index if the index of every bounded domain $\Omega \subset \mathbb{R}^2$ has a uniform upper bound.

In [19], we prove the :

Theorem 4.7 (Morse index). [19] *Any $2k$-ended solution of (1.1) has finite Morse index.*

Since the Morse index of a $2k$-ended solution u is finite (equal to m), we know from [11], that there exists a finite dimensional subspace $E \subset L^2(\mathbb{R}^2)$, with $\dim E = m$, which is spanned by the eigenfunctions $\phi_1, \ldots, \phi_m$ of the operator L, corresponding to the negative eigenvalues $\mu_1, \ldots, \mu_m$ of L.

4.2. **Uniqueness.** Our goal in this section is to show that $\mathcal{M}_4$ consists of just one connected component, as we have conjectured above. We refer the reader to [18] for the details of the proofs of the results we describe below.

Our point of departure is the existence result in [8], where we have shown that there exists a small number ε_0 such that for all $0 < \theta$, with $\tan \theta < \varepsilon_0$ there exists a 4-ended solution with corresponding angles of the half lines λ_j^+, $j = 1, \ldots, 4$ given by

$$\theta_1 = \theta, \quad \theta_2 = \pi - \theta, \quad \theta_3 = \theta + \pi, \quad \theta_4 = 2\pi - \theta.$$

Observe that the fact that θ is small implies that the ends of this solution are almost parallel and their slopes given by $\pm \varepsilon$, $\varepsilon = \tan \theta$, are small as well. Clearly, by symmetry, it is easy to see that there exist also solutions with parallel ends whose angles are given by:

$$\theta_1 = \pi/2 - \theta, \quad \theta_2 = \pi/2 + \theta, \quad \theta_3 = -\theta + 3\pi/2, \quad \theta_4 = 3\pi/2 + \theta.$$

In this case we have $\tan \theta > \frac{1}{\varepsilon_0}$.

In principle the value of the classifying map $\mathcal{P}$ map is not enough to identify in a unique way a solution to (1.1) in $\mathcal{M}_4^{even}$. However for solutions with almost parallel ends we have the following:

Theorem 4.8. [18] *There exists a small number ε_0 such that for any two solutions $u_1, u_2 \in \mathcal{M}_4^{even}$ satisfying $\tan \theta(u_1) = \tan \theta(u_2) < \varepsilon_0$, we have necessarily $u_1 \equiv u_2$.*

This result gives in some sense classification of the subfamily of the family of 4-ended solutions which contains solutions with almost parallel ends. It says that this subfamily consists precisely of the solutions constructed in [8]. Let us explain the importance of this statement from the point of view of classification of all 4-ended solutions. We recall that by Theorem 4.5 the classifying map is surjective. Consider for example the connected component $M_0 \subset \mathcal{M}_4^{even}$ which contains the saddle solution U. Theorem 4.5 implies that U can be deformed along M_0 to a

solution with the value of the classifying map arbitrarily close to $-\frac{\pi}{4}$ or to $\frac{\pi}{4}$, thus yielding a solution in the subfamily of the solutions with almost parallel ends. But these solutions are uniquely determined by the values of $\mathcal{P}$, which follows from the uniqueness statement in Theorem 4.8. As a result we obtain the following classification theorem:

Theorem 4.9. [18] *Any solution $u \in \mathcal{M}_4^{even}$ belongs to M_0 and is a continuous deformation of the saddle solution U.*

We observe that according to the conjecture of De Giorgi in two dimensions any bounded solution u which is monotonic in one direction must be one dimensional and equal to $u(\mathbf{x}) = H(\mathbf{a} \cdot \mathbf{x} + b)$, i.e. it is a planar solution. In the language of multiple-end solutions, this solution has *two* (heteroclinic, planar) ends. Theorem 4.9 gives on the other hand the classification of the family of solutions with four planar ends. Since the number of ends of a solution to (1.1) must be even, the family of 4-ended solutions is the natural object to study. In this context, one may wonder if it is possible to classify solutions to (1.1) assuming for instance that the nodal sets of u_x, and u_y have just one component. This question is beyond the scope of this paper, however since partial derivatives of 4-ended solution satisfy this assumption it seems reasonable to conjecture that a result similar to Theorem 4.9 should hold in this more general setting. We should mention here that it is in principle possible to study the problem of classification of solutions assuming for example that their Morse index is 1. This is natural since the Morse index of u and the number of the nodal domains of u_x and u_y are related. We recall here that the heteroclinic is stable, and from [5] we know that in dimension $N = 2$ stability of a solution implies that it is necessarily a one dimensional solution (for the related minimality conjecture, see for example [28] and [26] and the reference therein). We expect that in fact the family of 4-ended solutions should contain all multiple-end solutions with Morse index 1. We recall here that the Morse index of the saddle solution is 1 [29].

Let us now explain the analogy of Theorem 4.9 with some aspects of the theory of minimal surfaces in $\mathbb{R}^3$. In 1834, Scherk discovered an example of singly-periodic, embedded, minimal surface in $\mathbb{R}^3$ which, in a complement of a vertical cylinder, is asymptotic to 4 half planes with angle $\frac{\pi}{2}$ between them. This surface, after a rigid motion, has two planes of symmetry, say $\{x_2 = 0\}$ plane and $\{x_1 = 0\}$, and it is periodic, with period 1 in the x_3 direction. If θ is the angle between the asymptotic end of the Scherk surface contained in $\{x_1 > 0, x_2 > 0\}$ and the $\{x_2 = 0\}$ plane then $\theta = \frac{\pi}{4}$. This is the so called second Scherk's surface and it will be denoted here by $S_{\frac{\pi}{4}}$. In 1988 Karcher [15] found Scherk surfaces other than the original example in the sense that the corresponding angle between their asymptotic planes and the $\{x_2 = 0\}$ plane can be any $\theta \in (0, \frac{\pi}{2})$. The one parameter family $\{S_\theta\}_{\{0<\theta<\frac{\pi}{2}\}}$ of these surfaces is the family of Scherk singly periodic minimal surfaces. Thus, accepting that the saddle solution of the Allen-Cahn equation U corresponds to the Scherk surface $S_{\frac{\pi}{4}}$ Theorem 4.5, can be understood as an analog of the result of Karcher. We note that, unlike in the case of the Allen-Cahn equation, the Scherk family is given explicitly, for example it can be represented as the zero level set of the function:

$$F_\theta(x_1, x_2, x_3) = \cos^2 \theta \cosh \left(\frac{x_1}{\cos \theta} \right) - \sin^2 \theta \cosh \left(\frac{x_2}{\cos \theta} \right) - \cos x_3.$$

From this it follows immediately that the angle map (which is the classifying map in this context) $S_\theta \mapsto \theta$ is a diffeomorphism. A corresponding result for the family $\mathcal{M}_4^{even}$ is of course more difficult since no explicit formula is available in this case.

We will explore further the analogy of our result with the theory of minimal surfaces in $\mathbb{R}^3$, now in the context of the classification of the 4-ended solutions in Theorem 4.9. The corresponding problem can be stated as follows: if S is an embedded, singly periodic, minimal surface with 4 Scherk ends, what can be said about this surface? It is proven by Meeks and Wolf [23] that S must be one of the Scherk surfaces S_θ described above (similar result is proven in [27] assuming additionally that the genus of S in the quotient $\mathbb{R}^3/\mathbb{Z}$ is 0). The key results to prove this general statement are in fact the counterparts of Theorem 4.8 and Theorem 4.5.

We now sketch the basic elements in the proofs of Theorem 4.8. First of all let us discuss the existence result in [8] in the particular case of 4-ended solutions. The point of departure of the construction is the following Toda system

$$\begin{cases} c_0 q_1'' = -e^{\sqrt{2}(q_1 - q_2)} \\ c_0 q_2'' = e^{\sqrt{2}(q_1 - q_2)} \end{cases}, \tag{4.2}$$

for which $q_1 < 0 < q_2$ and $q_1(x) = -q_2(x)$, as well as $q_j(x) = q_j(-x)$, $j = 1, 2$. Here $c_0 = \frac{\sqrt{2}}{24}$. Any solution of this system is asymptotically linear, namely:

$$q_j(x) = (-1)^j (m|x| + b) + \mathcal{O}(e^{-2\sqrt{2}m|x|}), \quad x \to \infty,$$

where $m > 0$ is the slope of the asymptotic straight line in the first quadrant. On the other hand, given that we only consider solutions whose trajectories are symmetric with respect to the x-axis, the value of the slope m determines the unique solution of (4.2). When the asymptotic lines become parallel then $m \to 0$ or $m \to \infty$. By symmetry it suffices to consider the case $m \to 0$ and in this paper we will denote small slopes by $m = \varepsilon$ and the corresponding solutions by $q_{\varepsilon,j}$. Note that if by $q_{1,j}$ we denote a solution with slope $m = 1$ then

$$q_{\varepsilon,j}(x) = q_{1,j}(\varepsilon x) + \frac{(-1)^j}{\sqrt{2}} \log \frac{1}{\varepsilon}.$$

Then, the existence result in [8] implies that given a small ε, there exists a 4-ended solution u to (1.1) whose nodal set $N(u)$ is close to the trajectories of the Toda system given by the graphs $y = q_{\varepsilon,j}(x)$. Although we do not use directly this result in [18] but the idea of relating solutions of the Toda system and the 4-ended solutions of (1.1) that comes from [8] is very important. In fact, what we want to achieve is to parameterize the manifold of 4-ended solutions with almost parallel ends using corresponding solutions of the Toda system as parameters. To do this we first obtain a very precise control of the nodal sets of the 4-ended solutions. The key observation is that in every quadrant the nodal set $N(u)$ of any 4-ended solution is a bigraph, and if we assume that the slope of its asymptotic lines is small then it is a graph of a smooth function, both in the lower and in the upper half plane. We have then

$$N(u) = \{(x, y) \in \mathbb{R}^2 \mid y = f_{\varepsilon,j}(x), \quad j = 1, 2, \quad f_{\varepsilon,1}(x) < 0, \quad f_{\varepsilon,2}(x) = -f_{\varepsilon,1}(x)\},$$

for any $u \in \mathcal{M}_4^{even}$, with $\varepsilon = \tan \theta(u)$. Then we prove that for each ε small there exists a constant v_ε, with $|v_\varepsilon| = \mathcal{O}(\varepsilon^\alpha)$ such that

$$f_{\varepsilon,1}(x) + v_\varepsilon - q_{\varepsilon,1}(x) = \mathcal{O}(\varepsilon^\alpha e^{-\varepsilon\beta|x|}), \quad x \to \infty$$

with some positive constants α, β. Next, we define a suitable approximate 4-ended solution based on the solution of the Toda system with slope ε. To explain this, by $\tilde{\mathcal{N}}_{\varepsilon,1}$ we denote the graph of the function $y = q_{\varepsilon,1}(x)$, which is contained in the lower half plane. In a suitable neighborhood of the curve $\tilde{\mathcal{N}}_{\varepsilon,1}$ we introduce Fermi coordinates $\mathbf{x} = (x,y) \mapsto (x_1, y_1)$, where y_1 denotes the signed distance to $\tilde{\mathcal{N}}_{\varepsilon,1}$, and x_1 is the x coordinate of the projection of the point $\mathbf{x}$ onto $\tilde{\mathcal{N}}_{\varepsilon,1}$. With this notation we write locally the solution u, with $\varepsilon = \tan\theta(u)$ in the form

$$u(\mathbf{x}) = H(y_1 - h_\varepsilon(x_1) - w_\varepsilon) + \phi.$$

This definition is suitably adjusted to yield a globally defined function. Then it is proven that w_ε is a small constant, and $h_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ and $\phi \colon \mathbb{R}^2 \to \mathbb{R}$ are small functions, of order $\mathcal{O}(\varepsilon^\alpha)$ in some weighted norms.

Finally we prove the Lipschitz dependence of the solution u on w_ε and the function h_ε and conclude the proof of Theorem 4.8 using the mapping property of the linearized Toda system.

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