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SINGULAR LIMIT OF AN ACTIVATOR-INHIBITOR TYPE MODEL

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Abstract. We consider a reaction-diffusion system of activator-inhibitor type arising in the theory of phase transition. It appears in biological contexts such as pattern formation in population genetics. The purpose of this work is to prove the convergence of the solution of this system to the solution of a free boundary Problem involving a motion by mean curvature.

1. **Introduction.** In physics, biology or chemistry many phenomena are modelled by the following large class of coupled reaction-diffusion system

$$u_t = d_1\Delta u + f(u,v)\ v_t = d_2\Delta v + rg(u,v).$$

For instance this system describes pattern formation in an activator-inhibitor model and it has been thoroughly studied by many authors. They highlighted in particular that the patterns observed are produced by the interaction between kinetics and diffusion effects. In activator-inhibitor models, u is the activator variable and v is the inhibitor variable. Moreover d_1 and d_2 are respectively the diffusion rates of u and v while r represents the ratio of the reaction rates of u and v. As a consequence, the order of magnitude between d_1, d_2 and r has a major impact on the formation and evolution of patterns.

In this paper, we consider the case where u diffuses very slowly and v diffuses and react slowly, namely $d_1 = \varepsilon^2$, $d_2 = r = \varepsilon$. More precisely, the system we study after rescaling in time is given by

$$(P^\varepsilon)\begin{cases} u_t^\varepsilon = \Delta u^\varepsilon + \dfrac{1}{\varepsilon^2} f(u^\varepsilon, \varepsilon v^\varepsilon) & \text{in } \Omega\times(0,T) \quad (1.1)\\ \varepsilon v_t^\varepsilon = \Delta v^\varepsilon - \gamma v^\varepsilon + u^\varepsilon & \text{in } \Omega\times(0,T) \quad (1.2)\\ \dfrac{\partial u^\varepsilon}{\partial n} = \dfrac{\partial v^\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega\times(0,T) \quad (1.3)\\ u^\varepsilon(x,0) = u_0(x),\ v^\varepsilon(x,0) = v_0(x) & \text{for } x\in\Omega, \quad (1.4)\end{cases}$$

where γ is a strictly positive constant and

$$f(s,w) := s(1-s^2) - w. \tag{1.5}$$

We also suppose that Ω is a smooth bounded domain of $\mathbf{R}^N$ with $N \geq 2$.

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The purpose of this paper is to prove that the solution of Problem (P^ε) converges, as ε tends to 0, to the solution of the following free boundary Problem

$$(P)\begin{cases} u(x,t) &= \begin{cases} 1 & x\in\Omega_+(t)\\ -1 & x\in\Omega_-(t)\cup\Gamma_t \end{cases} \\ \Delta v(x,t) &= \gamma v(x,t) - u(x,t) \qquad x\in\Omega_+(t)\cup\Omega_-(t),\ t\in(0,\overline{T}] \\ \dfrac{\partial v}{\partial n} &= 0 \qquad (x,t)\in\partial\Omega\times[0,\overline{T}] \\ V_n &= -(N-1)K - \dfrac{3}{\sqrt{2}}v(x,t) \qquad x\in\Gamma_t=\Omega\setminus(\Omega_+(t)\cup\Omega_-(t)) \\ \Gamma_{|t=0} &= \Gamma_0, \end{cases}$$

where $\Gamma_t \subset\subset \Omega$ is a smooth compact hypersurface without boundaries dividing Ω into two subdomains $\Omega_-(t)$ and $\Omega_+(t)$, such that $\partial\Omega_+(t) = \Gamma_t$ and $\Omega_-(t) = \Omega\setminus(\Omega_+(t)\cup\partial\Omega_+(t))$. Moreover V_n and K denote respectively the outward normal velocity and the mean curvature of Γ_t.

The existence and uniqueness locally in time of the solution of (P) have been provided in [2]. Moreover in [2] the authors claim the convergence of the solution of (P^ε) to the solution of (P) and here we will prove rigorously this conjecture.

Let us mention that the convergence of the solution of variant of problem (P^ε) has been extensively studied. For example E. Logak in [8] considered the case where the equation of v^ε is an elliptic equation and showed the convergence to the solution of (P), for some prepared initial data. In [1] the authors studied the Fitzhugh-Nagumo system, where the equation of v^ε is a parabolic equation and demonstrated the convergence to the associated free boundary problem. We also refer to X. Chen (see [4]) for the study of another scaling of the Allen-Cahn equation. In our case the main difficulty is that the parameter ε is also introduced in the parabolic equation (1.2) of v^ε, as a consequence v^ε satisfies a parabolic equation while its limit v satisfies an elliptic equation. To overcome this difficulty we construct a function, which satisfies an approximation of the elliptic equation of v.

Denoting by $dist(x,\Gamma_t)$ the signed distance from x to Γ_t such that

$$\begin{cases} dist(x,\Gamma_t) > 0 & \text{if } x\in\Omega_+(t)\\ dist(x,\Gamma_t) < 0 & \text{if } x\in\Omega_-(t), \end{cases}$$

we make the following hypotheses about the initial data:

(H1) u_0 is bounded in $C^2(\overline{\Omega})$ and satisfies the compatibility condition

$$\frac{\partial u_0}{\partial n} = 0\ \forall x\in\partial\Omega. \tag{1.6}$$

(H2) $\Gamma_0 := \{x\in\Omega,\ u_0(x)=0\}$ is a $C^{2+\alpha}$ compact hypersurface with $\alpha\in(0,1)$

(H3) The open set $\Omega_+(0)$ defined by

$$\Omega_+(0) := \{x\in\Omega,\ u_0(x) > 0\}$$

is connected and $\Omega_+(0)\subset\subset\Omega$.

(H4) v_0 satisfy the elliptic problem

$$\Delta v_0 = \gamma v_0 - u_0 \text{ for all } x\in\Omega \text{ and } \frac{\partial v_0}{\partial n} = 0 \text{ for all } x\in\partial\Omega.$$

(H5) There exists a positive constant, η_0, such that

$$\begin{cases} u_0(x) \geq \eta_0 dist(x,\Gamma_0) \text{ if } x\in\Omega_+(0)\\ u_0(x) \leq \eta_0 dist(x,\Gamma_0) \text{ if } x\in\Omega_-(0). \end{cases}$$

Under this assumptions we will prove the convergence of $(u^\varepsilon, v^\varepsilon)$ to (u, v), as ε tends to 0, namely

Theorem 1.1. *Assume the hypotheses $(H1) - (H5)$. Let $(u^\varepsilon, v^\varepsilon)$ be the solution of Problem (P^ε) and let (u, v, Γ) with $\Gamma = (\Gamma_t \times \{t\})_{t\in[0,\overline{T}]}$ be the solution of the free boundary Problem (P) on $[0, \overline{T}]$. Then $(u^\varepsilon, v^\varepsilon)$ converges to (u, v), as $\varepsilon \downarrow 0$. More precisely*

$$u^\varepsilon \text{ converges to } u \text{ everywhere in } \cup_{0<t\le\overline{T}}(\Omega_t^\pm \times \{t\}),$$

$$v^\varepsilon \text{ converges to } v \text{ uniformly on } \overline{\Omega} \times [0, \overline{T}].$$

In the first step, we will prove that for any initial data u_0 the solution u^ε becomes, at time of order $O(\varepsilon^2|\ln\varepsilon|)$, close to ± 1 except in a small neighborhood of the initial interface Γ_0. Moreover for short time the function v^ε stay close to its initial data v_0. Precisely the Theorem of Generation of Interface reads as follow.

Theorem 1.2. *Assume $(H1) - (H4)$ then there exist positive constants ε_0, $\tilde{M}_0$ and τ_0 such that for all $\varepsilon \in (0, \varepsilon_0]$ we have*

$$-1 - \tilde{M}_0\varepsilon \le u^\varepsilon(x, t_0^\varepsilon) \le 1 + \tilde{M}_0\varepsilon, \text{ for all } x \in \Omega$$

with $t_0^\varepsilon := \tau_0\varepsilon^2|\ln\varepsilon|$ and

$$|u^\varepsilon(x, t_0^\varepsilon) - 1| \le \tilde{M}_0\varepsilon, \text{ for all } x \in \tilde{\Omega}_+^\varepsilon,$$

$$|u^\varepsilon(x, t_0^\varepsilon) + 1| \le \tilde{M}_0\varepsilon, \text{ for all } x \in \tilde{\Omega}_-^\varepsilon,$$

where

$$\tilde{\Omega}_+^\varepsilon := \{x \in \Omega,\ u_0(x) \ge \tilde{M}_0\sqrt{\varepsilon}|\ln\varepsilon|\},$$
$$\tilde{\Omega}_-^\varepsilon := \{x \in \Omega,\ u_0(x) \le -\tilde{M}_0\sqrt{\varepsilon}|\ln\varepsilon|\}.$$

Moreover v^ε satisfies

$$|v^\varepsilon(x, t) - v_0(x)| \le \tilde{M}_0\varepsilon|\ln\varepsilon|, \text{ for all } x \in \Omega \text{ and } t \in [0, t_0^\varepsilon]. \tag{1.7}$$

In a second step we will establish the propagation of the interface, namely

Theorem 1.3. *Assume $(H1) - (H5)$ and let $\overline{T}$ be the time existence of the smooth solution (Γ, v) of the free boundary problem (P), such that $\Gamma = \cup_{t\in[0,\overline{T}]}\Gamma_t \subset\subset \Omega \times [0, \overline{T}]$. Then there exist positive constants M and ε^* such that the solution $(u^\varepsilon, v^\varepsilon)$ of Problem (P^ε) satisfies for all $\varepsilon \in (0, \varepsilon^*]$ and for all $t \in [t_0^\varepsilon, \overline{T}]$ that*

$$|u^\varepsilon(x, t) - u(x, t)| \le M\varepsilon|\ln\varepsilon|, \tag{1.8}$$

for all $x \in \{x \in \Omega,\ |dist(x, \Gamma_{t-t_0^\varepsilon})| \ge M\sqrt{\varepsilon}|\ln\varepsilon|\}$ and

$$|v^\varepsilon(x, t) - v(x, t)| \le M\sqrt{\varepsilon}|\ln\varepsilon|, \tag{1.9}$$

for all $x \in \Omega$.

Clearly, Theorem 1.1 is a direct consequence of Theorem 1.3.
This paper is organized as follows : In section 2, we state some properties of the solution of the free boundary Problem (P). In particular, we prove that the derivatives in time and in space of the function v are bounded in the whole domain $Q_{\overline{T}} = \Omega \times [0, \overline{T}]$. In section 3, we follow the ideas of X. Chen in [4] to get a precise result on the generation of interface in a short time of order $O(\varepsilon^2|\ln\varepsilon|)$, which will imply Theorem 1.2. In section 4 we first study the solution, k^ε, of an elliptic equation which approximates the equation satisfied by v. Then we use this function k^ε to built sub-supersolution of v^ε. Moreover we also construct sub-supersolution

of u^ε which have the form of the travelling wave. The last part of the section 4 is devoted to the proof of Theorem 1.3. Further in section 5 we consider the Green function associated to the operator $(-\Delta+\gamma)$ with Neumann Condition and an arbitrary hypersurface $\mathcal{H}_\lambda$ which depends continuously on $\lambda \in \Lambda$ with Λ a compact set of $\mathbf{R}^N$. We then justify rigorously that the integral of the Green function on $\mathcal{H}_\lambda$, is uniformly bounded on $\Omega \times \Lambda$. As a consequence we establish a useful estimate namely, the integral of the Green function on Γ_t is uniformly bounded on $Q_{\overline{T}}$.

2. **Properties of the solution of the free boundary Problem** (P).

We first recall the result of Existence and Uniqueness of the solution of Problem (P) obtained in [2].

Theorem 2.1. *Let $\alpha \in (0,1)$ and assume that Γ_0 is a $C^{2+\alpha}$ hypersurface which is the boundary of a domain $\Omega_+(0) \subset\subset \Omega$ then there exists a positive constant $\overline{T}$ such that the limit free boundary problem (P) has a unique smooth solution (v,Γ) on $[0,\overline{T}]$, with $\Gamma = \Big(\Gamma_t \times \{t\}\Big)_{t\in[0,\overline{T}]} \in C^{2+\alpha,\frac{2+\alpha}{2}}$ and $v|_\Gamma \in C^{2+\alpha,\frac{2+\alpha}{2}}$.*

Next we prove that the derivatives of v are bounded in the whole domain $\Omega \times (0,\overline{T})$.

Theorem 2.2. *There exists a positive constant $\overline{\mathcal{V}} > 0$ such that*

$$|v(x,t)| + |v_t(x,t)| + |\nabla v(x,t)| + |\Delta v(x,t)| \le \overline{\mathcal{V}}, \tag{2.1}$$

for all $(x,t) \in \Omega \times [0,\overline{T}]$.

Proof. Using a Maximum principle for elliptic equations, see for instance Theorem 20 in [10], we have that v is bounded on the whole domain $Q_{\overline{T}}$. Moreover since Δv is bounded we note that $v(.,t) \in C^{1+\nu}(\overline{\Omega})$ for $\nu \in (0,1)$. To prove that v_t is bounded in $Q_{\overline{T}}$ we adapt the idea of E. Logak given in [8]. Denoting by G the Green function associated to the operator $(-\Delta+\gamma)$ with Neumann Condition, we have

$$\begin{aligned} v(x,t) &= \int_\Omega G(x,x')u(x',t)dx' \\ &= -\int_{\Omega_-(t)} G(x,x')dx' + \int_{\Omega_+(t)} G(x,x')dx' \\ &= \int_\Omega G(x,x')dx' - 2\int_{\Omega_-(t)} G(x,x')dx', \end{aligned}$$

so that

$$v_t(x,t) = -2\int_{\Gamma_t} G(x,x')V_n(x',t)dx', \tag{2.2}$$

where $V_n(x',t)$ is the normal velocity of a point x' on Γ_t. Using the smoothness of Γ_t we deduce that there exists a positive constant C such that

$$\Big|v_t(x,t)\Big| \le C\int_{\Gamma_t} |G(x,x')|dx'. \tag{2.3}$$

This gives in view of (5.15) that $\|v_t\|_{L^\infty(Q_{\overline{T}})} \le C$ and concludes the proof of Theorem 2.2. □

3. **Generation of interface.** We state below the existence and uniqueness of the solution $(u^\varepsilon, v^\varepsilon)$ of Problem (P^ε).

Lemma 3.1. *For all $T \in (0,\infty)$, Problem (P^ε) admits a unique solution $(u^\varepsilon, v^\varepsilon)$. Moreover there exist positive constants C_0 and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ we have*

$$|u^\varepsilon(x,t)| + |v^\varepsilon(x,t)| \le \frac{C_0}{2}, \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,T] \tag{3.1}$$

and

$$|v^\varepsilon(x,t) - v_0(x)| \le \frac{C_0}{\varepsilon} t, \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,T]. \tag{3.2}$$

Proof. The existence of a unique solution of (P^ε) follows from standard theory of parabolic system. From the theory of the invariant region (see for instance [11]), we obtain that u^ε and v^ε are bounded, so that

$$|u^\varepsilon(x,t)| + |v^\varepsilon(x,t)| \le C, \quad \text{for all } (x,t) \in \overline{\Omega} \times [0,T].$$

Next we prove (3.2). Let $\mathcal{L}_2^\varepsilon$ be the parabolic operator associated to (1.2), namely

$$\mathcal{L}_2^\varepsilon(u,v) := \varepsilon v_t - \Delta v + \gamma v - u \tag{3.3}$$

we have for $B \ge \sup_{x\in\overline{\Omega}} |\Delta v_0| + \gamma \sup_{x\in\overline{\Omega}} |v_0| + \sup_{x\in\overline{\Omega}} |u^\varepsilon|$ that

$$\mathcal{L}_2^\varepsilon(u^\varepsilon, v_0 + \frac{B}{\varepsilon} t) = B - \Delta v_0 + \gamma v_0 + \gamma \frac{B}{\varepsilon} t - u^\varepsilon \ge 0,$$

for all $(x,t) \in \Omega \times [0,T]$. By comparison principle applied to (1.2) we deduce

$$v^\varepsilon(x,t) \le v_0(x) + \frac{B}{\varepsilon} t, \text{ for all } (x,t) \in \Omega \times [0,T].$$

Similarly, one can prove that $v^\varepsilon(x,t) \ge v_0(x) - \frac{B}{\varepsilon} t$ in $\Omega \times [0,T]$. Thus taking $C_0 \ge max(B, 2C)$, we obtain (3.1) and (3.2), which concludes the proof of lemma 3.1. □

We now state preliminary definitions, which will be useful in the sequel. For $w \in \mathcal{B} := (-2\frac{\sqrt{3}}{9}, 2\frac{\sqrt{3}}{9})$, the equation

$$f(u,w) := u(1-u^2) - w = 0,$$

has three solutions, which we denote by $h_-(w) < h_0(w) < h_+(w)$. Note that $h_\pm(0) = \pm 1$, $h_0(0) = 0$ and we list below the main properties of $h_\pm$ and h_0.

Lemma 3.2. *For $w \in \mathcal{B} := (-2\frac{\sqrt{3}}{9}, 2\frac{\sqrt{3}}{9})$ the functions $h_\pm(w)$ and $h_0(w)$ are smooth functions such that*

$$w \mapsto h_\pm(w) \text{ are strictly decreasing and } w \mapsto h_0(w) \text{ is strictly increasing.} \tag{3.4}$$

Further, let $\sigma \in (0, 1/4)$, then there exist positive constants H, H_1, H_2, H_3 and H_4 such that

$$|h_\pm(w) - h_\pm(v)| + |h_0(w) - h_0(v)| \le H|v - w|, \tag{3.5}$$

for all $(v,w) \in (-2\frac{\sqrt{3}}{9} + \frac{\sigma}{2}, 2\frac{\sqrt{3}}{9} - \frac{\sigma}{2})^2$ and

$$h_\pm(w) + H_1\delta \le h_\pm(w - \delta) \le h_\pm(w) + H_2\delta, \tag{3.6}$$

$$h_\pm(w) - H_3\delta \le h_\pm(w + \delta) \le h_\pm(w) - H_4\delta \tag{3.7}$$

for all $w \in (-2\frac{\sqrt{3}}{9} + \frac{\sigma}{2}, 2\frac{\sqrt{3}}{9} - \frac{\sigma}{2})$ and $\delta \in (0, \frac{\sigma}{4}]$.

Proof. Differentiating the equality $f(h_\pm(w), w) = 0$ with respect to w, we obtain

$$\frac{\partial h_\pm(w)}{\partial w} = \frac{1}{1 - 3(h_\pm(w))^2}. \tag{3.8}$$

Since by construction $|h_\pm(w)| > \frac{1}{\sqrt{3}}$, we deduce from (3.8) that

$$\frac{\partial h_\pm(w)}{\partial w} < 0, \text{ for all } w \in \mathcal{B} \tag{3.9}$$

and similarly since $-\frac{1}{\sqrt{3}} < h_0(w) < \frac{1}{\sqrt{3}}$ we have $\frac{\partial h_0(w)}{\partial w} = \frac{1}{1 - 3(h_0(w))^2} > 0$, which implies (3.4). (3.5) follows directly from the smoothness of $h_\pm$ and h_0 on $\mathcal{B}$. Further we have for $w \in (-2\frac{\sqrt{3}}{9} + \frac{\sigma}{2}, 2\frac{\sqrt{3}}{9} - \frac{\sigma}{2})$ and $\delta \in [0, \frac{\sigma}{4}]$ that

$$h_\pm(w - \delta) = h_\pm(w) - \frac{\partial h_\pm(\theta)}{\partial w}\delta, \tag{3.10}$$

where $\theta \in (w - \delta, w) \subset [-2\frac{\sqrt{3}}{9} + \frac{\sigma}{4}, 2\frac{\sqrt{3}}{9} - \frac{\sigma}{2}]$. Thus (3.9) and the continuity of $\frac{\partial h_\pm(w)}{\partial w}$ gives (3.6). In the same way one can check (3.7) and conclude the proof of lemma 3.2. □

In order to prove the generation of interface we now introduce some useful notations. Let $s \mapsto \rho(s) \in C^\infty(\mathbf{R})$ be the cut-off function defined by

$$\begin{cases} \rho(s) = 1, & \text{if } |s| \leq 1, \\ \rho(s) = 0, & \text{if } |s| \geq 2, \\ 0 < \rho(s) < 1, & \text{if } 1 < |s| < 2, \\ -2 < s\rho'(s) \leq 0, & \text{if } s \in \mathbf{R}, \\ |\rho''(s)| \leq 4, & \text{if } s \in \mathbf{R}, \end{cases}$$

then we set $\rho_0 = \rho\left(\frac{u - h_0(v)}{\varepsilon^{\frac{3}{2}}|\ln\varepsilon|}\right), \rho_+ = \rho\left(\frac{u - h_+(v)}{\varepsilon^{\frac{3}{2}}|\ln\varepsilon|}\right)$ and $\rho_- = \rho\left(\frac{u - h_-(v)}{\varepsilon^{\frac{3}{2}}|\ln\varepsilon|}\right)$ and

$$\tilde{f}(u,v) := \rho_0\frac{u - h_0(v)}{|\ln\varepsilon|} + \rho_+\frac{h_+(v) - u}{|\ln\varepsilon|} + \rho_-\frac{h_-(v) - u}{|\ln\varepsilon|} + (1 - \rho_0 - \rho_- - \rho_+)f(u,v). \tag{3.11}$$

As it is done in [4], we first prove some properties of the function $\tilde{f}$.

Lemma 3.3. *Let*

$$\overline{\sigma} := -\frac{\sqrt{3}}{3} + h_+\Big(\frac{2\sqrt{3}}{9} - \frac{\sigma}{4}\Big) = \frac{\sqrt{3}}{3} - h_-\Big(-\frac{2\sqrt{3}}{9} + \frac{\sigma}{4}\Big),$$

then there exist positive constant $\overline{\varepsilon}$ *and* C_f *such that for all* $\varepsilon \in (0, \overline{\varepsilon}]$ *and* $v \in [-\frac{2\sqrt{3}}{9} + \frac{\sigma}{2}, \frac{2\sqrt{3}}{9} - \frac{\sigma}{2}]$ *the function* $\tilde{f}$ *defined by (3.11) satisfies*

$$|\tilde{f}(u,v) - f(u,v)| \leq C_f \varepsilon^{\frac{3}{2}} |\ln \varepsilon|, \text{ for all } u \in [-C_0, C_0] \tag{3.12}$$

$$|\tilde{f}_u(u,v)| \leq C_f \text{ and } |\tilde{f}_v(u,v)| \leq C_f, \text{ for all } u \in [-C_0, C_0] \tag{3.13}$$

$$\tilde{f}_u(u,v) \geq \frac{1}{|\ln \varepsilon|}, \text{ for all } u \in [-\frac{\sqrt{3}}{3} + \overline{\sigma}, \frac{\sqrt{3}}{3} - \overline{\sigma}] \tag{3.14}$$

$$\tilde{f}_u(u,v) \leq -\frac{1}{|\ln \varepsilon|}, \text{ for all } u \in [-C_0, -\frac{\sqrt{3}}{3} - \overline{\sigma}] \cup [\frac{\sqrt{3}}{3} + \overline{\sigma}, C_0] \tag{3.15}$$

$$|\tilde{f}_{vv}(u,v)| \leq \frac{C_f}{\varepsilon^{\frac{3}{2}} |\ln \varepsilon|}, \text{ for all } u \in [-C_0, C_0] \tag{3.16}$$

$$|\tilde{f}_v(u,v)| \leq C_f |\tilde{f}_u(u,v)|, \text{ for all } u \in [-C_0, C_0], \ |u \pm \frac{\sqrt{3}}{3}| \geq \overline{\sigma}, \tag{3.17}$$

where C_0 *is defined in lemma 3.1.*

Proof. Since the proof is very similar to its of lemma 3.1 in [4], we only give the sketch of the proof. We set

$$A(v):$$
$$=\{\eta, \ |\eta - h_0(v)| \leq 2\varepsilon^{\frac{3}{2}} |\ln \varepsilon| \text{ or } |\eta - h_-(v)| \leq 2\varepsilon^{\frac{3}{2}} |\ln \varepsilon| \text{ or } |\eta - h_+(v)| \leq 2\varepsilon^{\frac{3}{2}} |\ln \varepsilon|\}$$

and we note that $\tilde{f} = f$ for $u \notin A(v)$. Thus the inequalities of lemma 3.3 are obvious for $u \notin A(v)$.

1. We now consider the case $u \in A(v)$. Indeed we first assume that

$$|u - h_0(v)| \leq 2\varepsilon^{\frac{3}{2}} |\ln \varepsilon|. \tag{3.18}$$

If ε is small enough then $\rho_- = \rho_+ = 0$ and

$$\tilde{f}(u,v) := \rho_0 \frac{u - h_0(v)}{|\ln \varepsilon|} + (1 - \rho_0) f(u,v). \tag{3.19}$$

Using (3.18) and the fact that $f(h_0(v), v) = 0$ we deduce that

$$\left| \frac{u - h_0(v)}{|\ln \varepsilon|} - f(u,v) \right| = |u - h_0(v)| \left| \frac{1}{|\ln \varepsilon|} - \frac{f(u,v) - f(h_0(v),v)}{u - h_0(v)} \right| \leq C\varepsilon^{\frac{3}{2}} |\ln \varepsilon|. \tag{3.20}$$

So by (3.19) we have

$$\left| f(u,v) - \tilde{f}(u,v) \right| \leq C\varepsilon^{\frac{3}{2}} |\ln \varepsilon|,$$

which coincides with (3.12). Differentiating (3.19) with respect to u one has

$$\tilde{f}_u(u,v) = \frac{\rho'}{\varepsilon^{\frac{3}{2}} |\ln \varepsilon|} \left(\frac{u - h_0(v)}{|\ln \varepsilon|} - f(u,v) \right) + \frac{\rho_0}{|\ln \varepsilon|} + (1 - \rho_0) f_u(u,v). \tag{3.21}$$

As in [4] one can check that the first term of (3.21) is positive and thus

$$\tilde{f}_u(u,v) \geq \frac{\rho_0}{|\ln \varepsilon|} + (1 - \rho_0) f_u(u,v) \geq \frac{1}{|\ln \varepsilon|},$$

which implies (3.14). Using (3.19), (3.20) and the fact that ρ' and f_u are bounded we obtain the first part of (3.13). Differentiating (3.19) with respect to v one gets

$$\tilde{f}_v(u,v) = -\frac{\rho' h_0'}{\varepsilon^{\frac{3}{2}} |\ln \varepsilon|} \left(\frac{u - h_0(v)}{|\ln \varepsilon|} - f(u,v) \right) - \frac{\rho_0 h_0'}{|\ln \varepsilon|} + (1 - \rho_0) f_v(u,v), \tag{3.22}$$

so that by (3.20) the last part of (3.13) is obtained. Comparing (3.21) with (3.22) and using the same arguments as in [4] one can check (3.17). Differentiating (3.22) with respect to v we have

$$\tilde{f}_{vv}(u,v) = \frac{\rho''(h_0')^2}{(\varepsilon^{\frac{3}{2}}|\ln\varepsilon|)^2}\left(\frac{u-h_0(v)}{|\ln\varepsilon|} - f(u,v)\right) + O\left(\frac{1}{\varepsilon^{\frac{3}{2}}|\ln\varepsilon|}\right), \tag{3.23}$$

which together with (3.20) implies (3.16). Similarly one can obtain (3.12)-(3.17) in the cases $|\eta - h_-(v)| \le 2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|$ and $|\eta - h_+(v)| \le 2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|$. This concludes the proof of lemma 3.3. □

In what follows, we prove that in a short time of order $O(\varepsilon^2)$ the solution u^ε can be approximated by the solution of the following ordinary differential equation

$$(ODE)\begin{cases} \omega_\tau(\zeta,\tau,w) = \tilde{f}(\omega,w), \text{ for all } \tau > 0, \\ \omega(\zeta,0,w) = \zeta, \end{cases}$$

where $\zeta \in [-C_0, C_0]$ and $w \in \mathcal{B}$. We next recall some qualitative properties of ω and we refer to lemma 3.2 in [4] for the proof.

Lemma 3.4. *Assume that* $\zeta \in [-C_0, C_0]$ *and* $w \in (-2\frac{\sqrt{3}}{9} + \frac{\sigma}{2}, 2\frac{\sqrt{3}}{9} - \frac{\sigma}{2})$ *and let* $\omega(\zeta,\tau,w)$ *be the solution of (ODE). Then* $\omega \in C^2([-C_0,C_0]\times\mathbf{R}^+\times(-2\frac{\sqrt{3}}{9} + \frac{\sigma}{2}, 2\frac{\sqrt{3}}{9} - \frac{\sigma}{2}))$ *and*

$$\omega_\zeta(\zeta,\tau,w) > 0. \tag{3.24}$$

Moreover there exist positive constants τ_0 *and* ε_0 *such that for all* $\varepsilon \in (0,\varepsilon_0]$ *and* $\tau \ge \tau_0|\ln\varepsilon|$, *we have*

$$\omega(\zeta,\tau,w) \ge h_+(w) - 2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|, \forall\zeta \in [h_0(w) + 2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|, \infty), \tag{3.25}$$

$$\omega(\zeta,\tau,w) \le h_-(w) + 2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|, \forall\zeta \in (-\infty, h_0(w) - 2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|], \tag{3.26}$$

and

$$h_-(w) - 2\varepsilon^{\frac{3}{2}}|\ln\varepsilon| \le \omega(\zeta,\tau,w) \le h_+(w) + 2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|, \forall\zeta \in [-C_0,C_0]. \tag{3.27}$$

Further there exists a positive constant C_1 *such that for all* $\varepsilon \in (0,\varepsilon_0]$ *and* $\tau \in [0,\tau_0|\ln\varepsilon|]$, *we have*

$$|\omega_{\zeta\zeta}| \le C_1\frac{\omega_\zeta}{\varepsilon^{\frac{3}{2}}}, \tag{3.28}$$

$$|\omega_v| \le C_1(1+\omega_\zeta), \tag{3.29}$$

$$|\omega_{\zeta v}| + |\omega_{vv}| \le C_1\frac{(1+\omega_\zeta)}{\varepsilon^{\frac{3}{2}}}. \tag{3.30}$$

We are now in a position to prove the generation interface result, namely

Lemma 3.5. *Assume (H1)-(H4) then there exist positive constant* ε_0, M_0 *and* τ_0 *such that for all* $\varepsilon \in (0,\varepsilon_0]$ *the solution* $(u^\varepsilon, v^\varepsilon)$ *of Problem* (P^ε) *satisfies*

$$h_-(\varepsilon v_0) - M_0\varepsilon^{\frac{3}{2}}|\ln\varepsilon| \le u^\varepsilon(x,t_0^\varepsilon) \le h_+(\varepsilon v_0) + M_0\varepsilon^{\frac{3}{2}}|\ln\varepsilon|, \forall x \in \Omega \tag{3.31}$$

with $t_0^\varepsilon := \tau_0\varepsilon^2|\ln\varepsilon|$ *and*

$$|u^\varepsilon(x,t_0^\varepsilon) - h_+(\varepsilon v_0)| \le M_0\varepsilon^{\frac{3}{2}}|\ln\varepsilon|, \forall x \in \tilde{\Omega}_+^\varepsilon, \tag{3.32}$$

$$|u^\varepsilon(x,t_0^\varepsilon) - h_-(\varepsilon v_0)| \le M_0\varepsilon^{\frac{3}{2}}|\ln\varepsilon|, \forall x \in \tilde{\Omega}_-^\varepsilon, \tag{3.33}$$

where

$$\tilde{\Omega}_+^\varepsilon := \{x \in \Omega,\ u_0(x) \geq h_0(\varepsilon v_0) + M_0\sqrt{\varepsilon}|\ln\varepsilon|\},$$
$$\tilde{\Omega}_-^\varepsilon := \{x \in \Omega,\ u_0(x) \leq h_0(\varepsilon v_0) - M_0\sqrt{\varepsilon}|\ln\varepsilon|\}.$$

Proof. By (3.2) we have

$$|v^\varepsilon(x,t) - v_0(x)| \leq C_0\tau_0\varepsilon|\ln\varepsilon|, \text{ for all } (x,t) \in \Omega \times [0, t_0^\varepsilon]. \tag{3.34}$$

We set

$$u^\pm(x,t) := \omega\left(u_0 \pm l_1 \frac{t}{\varepsilon^{\frac{3}{2}}}, \frac{t}{\varepsilon^2}, \varepsilon v_0(x) \mp l_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|\right), \text{ for all } (x,t) \in \Omega \times [0,T], \tag{3.35}$$

where l_1 and l_2 are two constants to be chosen later. We now prove that u^- and u^+ are respectively sub- supersolution to the parabolic equation (1.1). To that purpose, we compute the derivatives of u^+, namely

$$u_t^+ = \frac{l_1}{\varepsilon^{\frac{3}{2}}}\omega_\zeta + \frac{1}{\varepsilon^2}\omega_\tau, \tag{3.36}$$

and by (3.28)-(3.30)

$$\begin{aligned} |\Delta u^+| &= |\Delta u_0 \omega_\zeta + |\nabla u_0|^2\omega_{\zeta\zeta} + 2\varepsilon\nabla u_0.\nabla v_0\omega_{\zeta w} + \varepsilon\Delta v_0\omega_w + \varepsilon^2|\nabla v_0|^2\omega_{ww}| \\ &\leq \tilde{A}_0\left(\frac{1}{\sqrt{\varepsilon}} + \frac{\omega_\zeta}{\varepsilon^{\frac{3}{2}}}\right), \end{aligned} \tag{3.37}$$

where $\tilde{A}_0$ is a positive constant. Denoting by $\mathcal{L}_1^\varepsilon$ the parabolic operators associated to (1.1) we have using (3.36) and (3.37) that

$$\begin{aligned} \mathcal{L}_1^\varepsilon(u^+, v^\varepsilon) \geq\ & \frac{l_1}{\varepsilon^{\frac{3}{2}}}\omega_\zeta - \tilde{A}_0\left(\frac{1}{\sqrt{\varepsilon}} + \frac{\omega_\zeta}{\varepsilon^{\frac{3}{2}}}\right) + \frac{1}{\varepsilon^2}\Big(\tilde{f}(\omega, \varepsilon v_0 - l_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|) \\ & - f(\omega, \varepsilon v_0 - l_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|) + \varepsilon(v^\varepsilon - v_0) + l_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|\Big). \end{aligned}$$

This in view of (3.34) and (3.12) gives

$$\mathcal{L}_1^\varepsilon(u^+, v^\varepsilon) \geq \frac{|\ln\varepsilon|}{\sqrt{\varepsilon}}\left(l_2 - C_f - C_0\tau_0\sqrt{\varepsilon} - \frac{\tilde{A}_0}{|\ln\varepsilon|}\right) + \frac{\omega_\zeta}{\varepsilon^{\frac{3}{2}}}\left(l_1 - \tilde{A}_0\right),$$

for $t \in [0, \tau_0\varepsilon^2|\ln\varepsilon|]$. Choosing $l_1 \geq \tilde{A}_0$ and $l_2 \geq C_f + C_0\tau_0\sqrt{\varepsilon} + \frac{\tilde{A}_0}{|\ln\varepsilon|}$, we deduce that $\mathcal{L}_1^\varepsilon(u^+, v^\varepsilon) \geq 0$. Similarly, one can check that $\mathcal{L}_1^\varepsilon(u^-, v^\varepsilon) \leq 0$. By comparison principle, this gives since

$$u^\pm(x,0) = \omega(u_0, 0, \varepsilon v_0 \mp l_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|) = u_0(x)$$

and

$$\frac{\partial u^\pm}{\partial n} = \omega_\zeta\frac{\partial u_0}{\partial n} + \varepsilon\omega_w\frac{\partial v_0}{\partial n} = 0 = \frac{\partial u^\varepsilon}{\partial n}$$

that

$$u^-(x,t) \leq u^\varepsilon(x,t) \leq u^+(x,t), \text{ for all } (x,t) \in \Omega \times [0, \tau_0\varepsilon^2|\ln\varepsilon|]. \tag{3.38}$$

Let us apply (3.38) at $t = t_0^\varepsilon = \tau_0\varepsilon^2|\ln\varepsilon|$; then since $u_0(x) + l_1\tau_0\frac{\varepsilon^2}{\varepsilon^{\frac{3}{2}}}|\ln\varepsilon| = u_0(x) + l_1\tau_0\sqrt{\varepsilon}|\ln\varepsilon| \in [-C_0, C_0]$ for ε small enough, we deduce from (3.5) and (3.27) that

$$u^\varepsilon(x, t_0^\varepsilon) \leq h_+(\varepsilon v_0 - l_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|) + 2\varepsilon^{\frac{3}{2}}|\ln\varepsilon| \leq h_+(\varepsilon v_0) + (Hl_2 + 2)\varepsilon^{\frac{3}{2}}|\ln\varepsilon|$$

and

$$u^\varepsilon(x, t_0^\varepsilon) \geq h_-(\varepsilon v_0 + l_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|) - 2\varepsilon^{\frac{3}{2}}|\ln\varepsilon| \geq h_-(\varepsilon v_0) - (Hl_2 + 2)\varepsilon^{\frac{3}{2}}|\ln\varepsilon|, \tag{3.39}$$

which gives (3.31). Similarly, by (3.5), (3.26) and (3.38) we have

$$\begin{aligned} u^\varepsilon(x,t_0^\varepsilon) &\le u^+(x,t_0^\varepsilon) \le h_-(\varepsilon v_0 - l_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|) + 2\varepsilon^{\frac{3}{2}}|\ln\varepsilon| \\ &\le h_-(\varepsilon v_0) + (Hl_2+2)\varepsilon^{\frac{3}{2}}|\ln\varepsilon|, \end{aligned}$$

if $u_0(x) + l_1\tau_0\sqrt{\varepsilon}|\ln\varepsilon| \in (-\infty, h_0(\varepsilon v_0 - l_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|) - 2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|]$, which is fulfilled if

$$u_0(x) \le h_0(\varepsilon v_0) - (l_1\tau_0 + 4)\sqrt{\varepsilon}|\ln\varepsilon|.$$

This together with (3.39) implies (3.33). In the same way one can prove (3.32) and achieves the proof of lemma 3.5. □

Proof of Theorem 1.2. Finally, Theorem 1.2 follows directly from (3.5) and lemma 3.5.

4. **Propagation of interface.**

4.1. **Preliminary results.**

We now state preliminary definitions, which will be useful in the sequel. Let $(U(z,w),C(w))$ be the solution of the system

$$(TW)\begin{cases} U_{zz}(z,w) + C(w)U_z(z,w) + f(U,w) = 0, \forall z\in\mathbf{R}, \\ \lim_{z\to+\infty} U(z,w) = h_+(w),\ \lim_{z\to-\infty} U(z,w) = h_-(w), \\ U(0,w) = h_0(w). \end{cases}$$

The velocity $C(w)$ is a smooth function, which satisfies

$$C(w) = \frac{1}{\sqrt{2}}\Big(2h_0(w) - h_-(w) - h_+(w)\Big) = \frac{3}{\sqrt{2}}h_0(w), \tag{4.1}$$

so that $C(0)=0$. Next we describe some qualitative properties of the travelling wave solution $(U(z,w),C(w))$.

Lemma 4.1. *There exist positive constants A and β such that for all $w\in\mathcal{B}$*

$$|U|\le A,\ 0<U_z\le A \text{ and } |U_w|\le A, \text{ for all } z\in\mathbf{R} \tag{4.2}$$

$$|U_z(z,w)| + |U_{zz}(z,w)| \le Ae^{-\beta|z|} \text{ for all } z\in\mathbf{R} \tag{4.3}$$

and

$$|U(z,w) - h_+(w)| \le Ae^{-\beta z} \text{ if } z\ge 0, \tag{4.4}$$

$$|U(z,w) - h_-(w)| \le Ae^{\beta z} \text{ if } z\le 0. \tag{4.5}$$

Moreover the velocity $C(w)$ satisfies

$$C(w) = \frac{3}{\sqrt{2}}w + O(|w|^2). \tag{4.6}$$

Proof. We refer to [6], [4] or [5] for the proof of (4.2)-(4.5). Further since $h_0'(0) = \dfrac{1}{1-3h_0^2(0)} = 1$ we have

$$C(w) = \frac{3}{\sqrt{2}}h_0(w) = \frac{3}{\sqrt{2}}w + O(w^2),$$

which coincides with (4.6) and concludes the proof of lemma 4.1. □

We now introduce a smooth truncated approximation of the signed distance function, namely

$$d(x,t) = \begin{cases} dist(x,\Gamma_t) & \text{if } x \in \Omega_+(t)\cup\Omega_-(t) \text{ and } |dist(x,\Gamma_t)| \le \frac{L}{2} \\ L & \text{if } x \in \Omega_+(t) \text{ and } |dist(x,\Gamma_t)| \ge L \\ -L & \text{if } x \in \overline{\Omega}\setminus\Omega_+(t) \text{ and } |dist(x,\Gamma_t)| \ge L \\ \frac{L}{2} \le |d| \le L & \text{if } \frac{L}{2} \le |dist(x,\Gamma_t)| \le L. \end{cases}$$

Taking L small enough we may assume $0 < L < \delta$ where δ is defined by (5.1) and also $-dist(x,\Gamma_t) > L$ for all $(x,t) \in \partial\Omega \times [0,\overline{T}]$, so that

$$\frac{\partial d}{\partial n}(x,t) = 0 \text{ for all } (x,t) \in \partial\Omega\times[0,\overline{T}]. \tag{4.7}$$

We next state an auxiliary result, which will be useful to obtain sub-supersolution of (P^ε).

Lemma 4.2. *Let D_1, D_2, S_1 and m_1 be strictly positive constants and let $\mathcal{G}_2$ be the constant defined in Corollary 1, then the solution k^ε of the following elliptic problem*

$$(K)\begin{cases} -\Delta k^\varepsilon + \gamma k^\varepsilon = D_1\sqrt{\varepsilon}|\ln\varepsilon| + D_2\chi_{\{-2S_1\sqrt{\varepsilon}|\ln\varepsilon|e^{m_1t}\le d(x,t)\le 0\}} & \text{in } \Omega \\ \dfrac{\partial k^\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies for all $(x,t)\in\Omega\times[0,\overline{T}]$

$$0 < \mathcal{K}_m\sqrt{\varepsilon}|\ln\varepsilon| \le k^\varepsilon(x,t) \le \mathcal{K}_M(t)\sqrt{\varepsilon}|\ln\varepsilon|, \tag{4.8}$$

where $\mathcal{K}_m = \frac{D_1}{2\gamma}$, $\mathcal{K}_M(t) = \frac{D_1}{\gamma} + D_2C\mathcal{G}_2 2S_1e^{m_1t}$. Further there exists a constant, $\overline{\mathcal{K}}$, independent of D_1, S_1 and m_1 such that

$$|\nabla k^\varepsilon(x,t)| + |\Delta k^\varepsilon(x,t)| \quad \le \overline{\mathcal{K}}, \text{ for all } (x,t)\in\Omega\times[0,\overline{T}] \tag{4.9}$$

and

$$|k^\varepsilon_t(x,t)| \le \overline{\mathcal{K}}, \text{ for all } (x,t)\in\Omega\times[0,\overline{T}]. \tag{4.10}$$

Proof. We first note that (K) admits a unique solution, k^ε, and that $\frac{D_1}{2\gamma}\sqrt{\varepsilon}|\ln\varepsilon|$ is a subsolution of (K), so

$$k^\varepsilon(x,t) \ge \frac{D_1}{2\gamma}\sqrt{\varepsilon}|\ln\varepsilon| > 0, \text{ for all } (x,t)\in\Omega\times[0,\overline{T}]. \tag{4.11}$$

Further k^ε is determined by

$$k^\varepsilon(x,t) = \int_\Omega G(x,x')\Big(D_1\sqrt{\varepsilon}|\ln\varepsilon| + D_2\chi_{\{-2S_1\sqrt{\varepsilon}|\ln\varepsilon|e^{m_1t}\le d(x',t)\le 0\}}\Big)dx', \tag{4.12}$$

where G is the Green function associated to the operator $(-\Delta+\gamma)$ with Neumann Condition. We remark that

$$\int_\Omega G(x,x')dx' = \frac{1}{\gamma}, \text{ for all } x\in\Omega$$

and thus

$$0 < k^\varepsilon(x,t) \le \frac{D_1}{\gamma}\sqrt{\varepsilon}|\ln\varepsilon| + D_2E(x,t), \tag{4.13}$$

where

$$E(x,t) := \int_\Omega |G(x,x')|\chi_{\{-2S_1\sqrt{\varepsilon}|\ln\varepsilon|e^{m_1t}\le d(x',t)\le 0\}}dx'. \tag{4.14}$$

Next we estimate E. Taking L small enough we assume that

$$\Gamma_t(L) := \{x' \in \mathbf{R}^N,\ |d(x',t)| \le L\} \subset\subset \Omega.$$

One can remark that the existence of such L is ensured by lemma 5.1. Next we introduce for any fixed $t \in [0, \overline{T}]$ the change of variables

$$\begin{cases} \Gamma_t(L) & \to & \Gamma_t \times [-L, L] \\ x' & \mapsto & (p, r), \end{cases}$$

where $p = p(x', t)$ is the projection of x' on Γ_t. We write this change of variable as $x' = X(p, r) = p + rn$ where n denotes the normal vector to Γ_t pointing from $\Omega_+(t)$ to $\Omega_-(t)$. Thus

$$E(x,t) = \int_{-2S_1\sqrt{\varepsilon}|\ln\varepsilon|e^{m_1 t}}^{0} \int_{\Gamma_t} |G(x, X(p,r))| Jac(X(p,r)) dp dr, \tag{4.15}$$

where $Jac(X(p, r))$ is the Jacobian of the change of variable, X. Since X is smooth, one gets

$$|Jac(X)| \le C \text{ for all } (p, r) \in \Gamma_t \times [-L, L] \text{ and all } t \in [0, \overline{T}],$$

so that

$$E(x,t) \le C \int_{-2S_1\sqrt{\varepsilon}|\ln\varepsilon|e^{m_1 t}}^{0} \int_{\Gamma_t} |G(x, p + rn)| dp dr.$$

Thus setting $q(p) = p + rn$ we have

$$E(x,t) \le C \int_{-2S_1\sqrt{\varepsilon}|\ln\varepsilon|e^{m_1 t}}^{0} \int_{\Gamma_t + rn} |G(x, q)| dq dr.$$

Therefore by (5.17) we deduce that

$$E(x,t) \le C\mathcal{G}_2 2S_1\sqrt{\varepsilon}|\ln\varepsilon|e^{m_1 t},$$

which with (4.13) gives (4.8). Furthermore by the elliptic equation satisfied by k^ε we have

$$\|\Delta k^\varepsilon\|_{L^\infty(Q_{\overline{T}})} \le C. \tag{4.16}$$

As it is done in Theorem 2.2 to prove that $\|\nabla v\|_{L^\infty(Q_{\overline{T}})} \le \overline{\mathcal{V}}$, one can deduce from (4.16) that $\|\nabla k^\varepsilon\|_{L^\infty(Q_{\overline{T}})}$ is also bounded. This together with (4.16) implies (4.9). To achieve the proof of lemma 4.2, it remains to check (4.10). By (4.12) we have

$$k^\varepsilon(x,t) = D_1\sqrt{\varepsilon}|\ln\varepsilon| \int_\Omega G(x,x')dx' + D_2\left(\int_{\Omega_-(t)} G(x,x')dx' - \int_{\Omega_-(t,\varepsilon)} G(x,x')dx'\right), \tag{4.17}$$

where

$$\Omega_-(t,\varepsilon) := \{x \in \Omega,\ d^\varepsilon(x,t) < 0\}, \text{ with } d^\varepsilon(x,t) = d(x,t) + 2S_1\sqrt{\varepsilon}|\ln\varepsilon|e^{m_1 t}.$$

Let $\Omega_+(t,\varepsilon) := \{x \in \Omega,\ d^\varepsilon(x,t) > 0\}$ and $\Gamma_{t,\varepsilon} := \partial\Omega_+(t,\varepsilon)$ then differentiating (4.17) with respect to t one has

$$\begin{aligned} k_t^\varepsilon(x,t) &= D_2\Big(\int_{\Gamma_t} G(x,x')V_n(x',t)dx' \\ &\quad - \int_{\Gamma_{t,\varepsilon}} G(x,x')(V_n(x',t) + 2S_1 m_1\sqrt{\varepsilon}|\ln\varepsilon|e^{m_1 t})dx'\Big). \end{aligned} \tag{4.18}$$

Using (2.1) and (2.2) we obtain

$$\left|\int_{\Gamma_t} G(x,x')V_n(x',t)dx'\right| \le \frac{1}{2}\overline{\mathcal{V}}. \tag{4.19}$$

Moreover by (5.16) and the smoothness of Γ_t we have

$$\left|\int_{\Gamma_{t,\varepsilon}} G(x,x')\Big(V_n(x',t)+2S_1m_1\sqrt{\varepsilon}|\ln\varepsilon|e^{m_1t}\Big)dx'\right| \le C\mathcal{G}_1,$$

where C is a positive constant. This with (4.18) and (4.19) gives (4.10) and concludes the proof of lemma 4.2. □

4.2. **Construction of sub-supersolution.** Let S_1, m_1, S_2, be some positive constants to be chosen later, we set

$$S_1^\varepsilon(t) \quad := S_1\sqrt{\varepsilon}|\ln\varepsilon|e^{m_1t}(1+t) \tag{4.20}$$

$$S_2^\varepsilon \quad := S_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|. \tag{4.21}$$

Moreover we define for all $t\in[0,\overline{T}]$

$$U^\pm(x,t) = U\bigg(\frac{d(x,t)\pm S_1^\varepsilon(t)}{\varepsilon}, \varepsilon(v\mp 2k^\varepsilon)(x,t)\mp S_2^\varepsilon\bigg) \tag{4.22}$$

and

$$V^\pm(x,t) = v(x,t)\pm k^\varepsilon(x,t), \tag{4.23}$$

where k^ε satisfied the system (K).

Lemma 4.3. *Let $\mathcal{L}_1^\varepsilon$ and $\mathcal{L}_2^\varepsilon$ be the parabolic operators associated to (1.1) and (1.2) respectively, then we have*

$$\mathcal{L}_1^\varepsilon(U^+,V^-) \ge 0, \tag{4.24}$$

$$\mathcal{L}_1^\varepsilon(U^-,V^+) \le 0, \tag{4.25}$$

$$\mathcal{L}_2^\varepsilon(U^+,V^+) \ge 0, \tag{4.26}$$

$$\mathcal{L}_2^\varepsilon(U^-,V^-) \le 0, \tag{4.27}$$

in $\Omega\times[0,\overline{T}]$.

Proof. We first prove (4.24). By a standard computation we have

$$\mathcal{L}_1^\varepsilon(U^+,V^-) = I_1+I_2+I_3+\frac{1}{\varepsilon}k^\varepsilon+\frac{1}{\varepsilon^2}S_2^\varepsilon, \tag{4.28}$$

with

$$I_1 := \frac{U_{zz}}{\varepsilon^2}\Big(1-|\nabla d|^2\Big) \tag{4.29}$$

$$I_2 := \frac{U_z}{\varepsilon}\bigg(d_t-\Delta d+\frac{C(\varepsilon(v-2k^\varepsilon)-S_2^\varepsilon)}{\varepsilon}+(S_1^\varepsilon)_t\bigg) \tag{4.30}$$

$$\begin{aligned} I_3 := \quad & \varepsilon(v_t-2k_t^\varepsilon)U_\nu - 2(\nabla d.(\nabla v-2\nabla k^\varepsilon))U_{z\nu} \\ & -\varepsilon(\Delta v-2\Delta k^\varepsilon)U_\nu-\varepsilon^2(\nabla v-2\nabla k^\varepsilon)^2U_{\nu\nu} \end{aligned} \tag{4.31}$$

and where the derivatives of U are evaluated at the point

$$\bigg(\frac{d(x,t)+S_1^\varepsilon(t)}{\varepsilon}, \varepsilon(v-2k^\varepsilon)(x,t)-S_2^\varepsilon\bigg).$$

Using the definition of d and the property (4.3) of U we have

$$\begin{aligned}\left|U_{zz}(1-|\nabla d|^2)\right| &\leq \sup_{|d|\geq\frac{L}{2}}\left|U_{zz}\left(\frac{d(x,t)+S_1^\varepsilon(t)}{\varepsilon},\varepsilon(v-2k^\varepsilon)(x,t)-S_2^\varepsilon\right)\right| \\ &\leq A\sup_{|d|\geq\frac{L}{2}} e^{-\beta\left|\frac{d(x,t)+S_1^\varepsilon(t)}{\varepsilon}\right|} \leq Ae^{-\beta\frac{L/2-S_1^\varepsilon(t)}{\varepsilon}}.\end{aligned}$$

This gives since $\lim_{\varepsilon\downarrow 0} S_1^\varepsilon(\overline{T})=0$ that

$$I_1 \geq -C_1. \tag{4.32}$$

Next we estimate the term I_2. Since $V_n=d_t$ and $\Delta d=-(N-1)K$, we first note that the motion equation

$$d_t=\Delta d-\frac{3}{\sqrt{2}}v,\ \forall x\in\Gamma_t,$$

together with the mean value theorem and the smoothness of the function d implies

$$\left|d_t-\Delta d+\frac{3}{\sqrt{2}}v\right|\leq \tilde{D}|d|,\quad \text{in } \overline{\Omega}\times[0,\overline{T}],$$

where $\tilde{D}$ is a positive constant. This yields in view of (4.30) and the fact that $U_z\geq 0$ that

$$I_2\geq\frac{U_z}{\varepsilon}\left[-\tilde{D}|d+S_1^\varepsilon|\right]+\frac{U_z}{\varepsilon}\tilde{I}_2, \tag{4.33}$$

with

$$\tilde{I}_2:=\left[(S_1^\varepsilon)_t+\frac{C(\varepsilon(v-2k^\varepsilon)-S_2^\varepsilon)}{\varepsilon}-\frac{3}{\sqrt{2}}v-\tilde{D}S_1^\varepsilon\right]. \tag{4.34}$$

Moreover we have by (4.6) and (4.8) that

$$|C(\varepsilon(v-2k^\varepsilon)-S_2^\varepsilon)-\frac{3}{\sqrt{2}}(\varepsilon v-2\varepsilon k^\varepsilon-S_2^\varepsilon)|\leq C_1\varepsilon^2. \tag{4.35}$$

Substituting (4.35) into (4.34) we obtain

$$\tilde{I}_2\geq S_1\sqrt{\varepsilon}|\ln\varepsilon|e^{m_1t}(t+1)\left[m_1-\tilde{D}\right]+\sqrt{\varepsilon}|\ln\varepsilon|\left[S_1-C_1\frac{\sqrt{\varepsilon}}{|\ln\varepsilon|}-\frac{3}{\sqrt{2}}S_2\right]-3\sqrt{2}k^\varepsilon.$$

Thus choosing $m_1>2\tilde{D}$ and $S_1>1+\frac{3}{\sqrt{2}}S_2$ and also using (4.8) we deduce

$$\tilde{I}_2\geq\sqrt{\varepsilon}|\ln\varepsilon|\left(S_1e^{m_1t}\frac{m_1}{4}-3\sqrt{2}D_1\mathcal{G}\right)+\sqrt{\varepsilon}|\ln\varepsilon|S_1e^{m_1t}\left(\frac{m_1}{4}-6\sqrt{2}C\mathcal{G}_2\right),$$

for ε small enough. So for $m_1>24\sqrt{2}D_2C\mathcal{G}_2$ and $m_1\geq 12\sqrt{2}D_1\mathcal{G}$ we obtain $\tilde{I}_2>0$. This together with (4.33) gives $I_2\geq-\tilde{D}U_z\frac{|d+S_1^\varepsilon|}{\varepsilon}$ and then by (4.3)

$$I_2\geq -A\tilde{D}\sup_{\mathbf{R}}(|z|e^{-\beta|z|})\geq -C_2. \tag{4.36}$$

Moreover since U, v, k^ε and their derivatives are bounded we have

$$I_3\geq -C_3. \tag{4.37}$$

Substituting (4.32), (4.36), (4.37) into (4.28) we obtain

$$\mathcal{L}_1^\varepsilon(U^+,V^-)\geq -C_1-C_2-C_3+\frac{1}{\varepsilon}k^\varepsilon+\frac{S_2|\ln\varepsilon|}{\sqrt{\varepsilon}}.$$

Since k^ε is a positive function, we conclude that

$$\mathcal{L}_1^\varepsilon(U^+,V^-)\geq 0,\ \text{for } \varepsilon \text{ small enough}. \tag{4.38}$$

Next we compute $\mathcal{L}_2^\varepsilon(U^+, V^+)$. We have that

$$\mathcal{L}_2^\varepsilon(U^+, V^+) = \varepsilon(v + k^\varepsilon)_t - \Delta(v + k^\varepsilon) - U^+ + \gamma(v + k^\varepsilon).$$

Since v satisfies the limit equation $-\Delta v = u - \gamma v$ one has

$$\mathcal{L}_2^\varepsilon(U^+, V^+) = \varepsilon k_t^\varepsilon - \Delta k^\varepsilon + \gamma k^\varepsilon + \varepsilon v_t + (1 - U^+)\chi_{\{d>0\}} + (-1 - U^+)\chi_{\{d\leq 0\}}. \quad (4.39)$$

Using (3.5), (2.1), (4.8) we note that

$$|h_\pm(0) - h_\pm(\varepsilon(v - 2k^\varepsilon) - S_2^\varepsilon)| \leq H(\varepsilon|v - 2k^\varepsilon| + S_2^\varepsilon) \leq (H\overline{\mathcal{V}} + 1)\varepsilon. \quad (4.40)$$

Moreover we have

$$|1 - U^+|\chi_{\{d>0\}} \leq |1 - h_+(\varepsilon(v - 2k^\varepsilon) - S_2^\varepsilon)| + |h_+(\varepsilon(v - 2k^\varepsilon) - S_2^\varepsilon) - U|\chi_{\{d>0\}}, \quad (4.41)$$

where U is evaluated at the point $\left(\dfrac{d(x,t) + S_1^\varepsilon(t)}{\varepsilon}, \varepsilon(v - 2k^\varepsilon)(x,t) - S_2^\varepsilon\right)$. For $d > 0$ we have $S_1\sqrt{\varepsilon}|\ln\varepsilon| \leq d + S_1^\varepsilon(t)$, which together with (4.4) gives

$$|h_+(\varepsilon(v - 2k^\varepsilon) - S_2^\varepsilon) - U|\chi_{\{d>0\}} \leq Ae^{-\beta\frac{d + S_1^\varepsilon(t)}{\varepsilon}} \leq Ae^{-\beta S_1\frac{|\ln\varepsilon|}{\sqrt{\varepsilon}}} \leq \varepsilon, \quad (4.42)$$

for ε small enough. By (4.40), (4.41) and (4.42) we obtain

$$|1 - U^+|\chi_{\{d>0\}} \leq (H\overline{\mathcal{V}} + 2)\varepsilon, \quad (4.43)$$

for ε small enough. Similarly, we have

$$\begin{aligned} |-1 - U|\chi_{\{d\leq 0\}} &\leq |-1 - h_-(\varepsilon(v - 2k^\varepsilon) - S_2^\varepsilon)|\chi_{\{d<-2S_1^\varepsilon(t)\}} \\ &+ |h_-(\varepsilon(v - 2k^\varepsilon) - S_2^\varepsilon) - U|\chi_{\{d<-2S_1^\varepsilon(t)\}} + |-1 - U|\chi_{\{-2S_1^\varepsilon(t)\leq d\leq 0\}}. \end{aligned} \quad (4.44)$$

Further for $d < -2S_1^\varepsilon(t)$ we have $\dfrac{d + S_1^\varepsilon(t)}{\varepsilon} < -\dfrac{S_1^\varepsilon(t)}{\varepsilon} < -S_1\frac{|\ln\varepsilon|}{\sqrt{\varepsilon}} < 0$ and then in view of (4.5)

$$|h_-(\varepsilon(v - 2k^\varepsilon) - S_2^\varepsilon) - U|\chi_{\{d<-2S_1^\varepsilon(t)\}} \leq Ae^{\beta\frac{d + S_1^\varepsilon(t)}{\varepsilon}} \leq Ae^{-\beta S_1\frac{|\ln\varepsilon|}{\sqrt{\varepsilon}}} \leq \varepsilon, \quad (4.45)$$

for ε small enough. By (4.2), (4.40), (4.44) and (4.45) we obtain

$$|-1 - U|\chi_{\{d\leq 0\}} \quad \leq \quad (H\overline{\mathcal{V}} + 2)\varepsilon + (1 + A)\chi_{\{-2S_1^\varepsilon(t)\leq d\leq 0\}}. \quad (4.46)$$

Substituting (4.46) and (4.43) into (4.39) and also using (2.1) we deduce that

$$\mathcal{L}_2^\varepsilon(U^+, V^+) \geq \varepsilon k_t^\varepsilon - \Delta k^\varepsilon + \gamma k^\varepsilon - (\overline{\mathcal{V}} + 2H\overline{\mathcal{V}} + 4)\varepsilon - (1 + A)\chi_{\{-2S_1^\varepsilon(t)\leq d\leq 0\}}.$$

Let us apply lemma 4.2 with $D_1 = \overline{\mathcal{V}} + 2H\overline{\mathcal{V}} + 4$ and $D_2 = 1 + A$, then we deduce from the elliptic equation satisfied by k^ε that

$$\mathcal{L}_2^\varepsilon(U^+, V^+) \geq \varepsilon k_t^\varepsilon + D_1\sqrt{\varepsilon}|\ln\varepsilon| - D_1\varepsilon. \quad (4.47)$$

This with (4.10) gives $\mathcal{L}_2^\varepsilon(U^+, V^+) \geq 0$, for ε small enough.

Thus choosing $D_1 = \overline{\mathcal{V}} + 2H\overline{\mathcal{V}} + 4$, $D_2 = 1 + A$, $S_1 > 1 + \dfrac{3}{\sqrt{2}}S_2$ and $m_1 > \max(2\tilde{D}, 24\sqrt{2}D_2C\mathcal{G}_2, 12\sqrt{2}D_1)$ we have shown that the estimates (4.24) and (4.26) are satisfied. Similarly one can check (4.25) and (4.27) and conclude the proof of lemma 4.3. □

Furthermore using (4.7) and the fact that $\dfrac{\partial k^\varepsilon}{\partial n} = \dfrac{\partial v}{\partial n} = 0$ we obtain

$$\frac{\partial U^+}{\partial n} = \frac{\partial U^-}{\partial n} = \frac{\partial V^+}{\partial n} = \frac{\partial V^-}{\partial n} = 0. \quad (4.48)$$

We now check the initial conditions.

Lemma 4.4.

$$V^-(x,0) \le v^\varepsilon(x,t_0^\varepsilon) \le V^+(x,0) \tag{4.49}$$

and

$$U^-(x,0) \le u^\varepsilon(x,t_0^\varepsilon) \le U^+(x,0). \tag{4.50}$$

Proof. It follows from (3.2) that

$$v_0(x) - C_0\tau_0\varepsilon|\ln\varepsilon| \le v^\varepsilon(x,\tau_0\varepsilon^2|\ln\varepsilon|) \le v_0(x) + C_0\tau_0\varepsilon|\ln\varepsilon|,$$

thus by (4.8) and the definition of $V^\pm$, (4.23), we have

$$V^-(x,0) \le v^\varepsilon(x,t_0^\varepsilon) \le V^+(x,0), \text{ for } \varepsilon \text{ small enough.} \tag{4.51}$$

To obtain the initial condition for u^ε we consider two cases, namely $d(x,0) \le -2\frac{M_0}{\eta_0}\sqrt{\varepsilon}|\ln\varepsilon|$ and $d(x,0) > -2\frac{M_0}{\eta_0}\sqrt{\varepsilon}|\ln\varepsilon|$, where M_0 and η_0 are respectively defined in lemma 3.5 and in asumption (H5).

1/ We first suppose that $d(x,0) \le -2\frac{M_0}{\eta_0}\varepsilon|\ln\varepsilon|$. Then else $d(x,0) = dist(x,\Gamma_0) < 0$ or $dist(x,\Gamma_0) \le -L/2$; thus $x \in \Omega_-(0)$ and

$$dist(x,\Gamma_0) \le -2\frac{M_0}{\eta_0}\sqrt{\varepsilon}|\ln\varepsilon|.$$

This gives in view of $(H5)$ that

$$u_0(x) \le -2M_0\sqrt{\varepsilon}|\ln\varepsilon|.$$

Then using (3.1), (3.5) and the fact that $h_0(0) = 0$ we deduce

$$u_0(x) \le -2M_0\sqrt{\varepsilon}|\ln\varepsilon| \le -M_0\sqrt{\varepsilon}|\ln\varepsilon| + h_0(\varepsilon v_0), \text{ for } \varepsilon \text{ small enough,}$$

so that $x \in \tilde{\Omega}_-^\varepsilon$. Thus by (3.33) we obtain

$$u^\varepsilon(x,\tau_0\varepsilon^2|\ln\varepsilon|) \le h_-(\varepsilon v_0) + M_0\varepsilon^{\frac{3}{2}}|\ln\varepsilon|. \tag{4.52}$$

Furthermore since

$$U^+(x,0) = U\left(\frac{d(x,0) + S_1\sqrt{\varepsilon}|\ln\varepsilon|}{\varepsilon}, \varepsilon(v_0(x) - 2k^\varepsilon(x,0)) - S_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|\right)$$

we deduce from (3.4) and (3.6) that

$$\begin{aligned} U^+(x,0) \quad &\ge h_-(\varepsilon(v_0(x) - 2k^\varepsilon(x,0)) - S_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|) \ge h_-(\varepsilon v_0(x) - S_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|) \\ &\ge h_-(\varepsilon v_0(x)) + H_1 S_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|. \end{aligned}$$

Thus by (4.52) we deduce

$$U^+(x,0) \ge u^\varepsilon(x,\tau_0\varepsilon^2|\ln\varepsilon|), \tag{4.53}$$

for $S_2 \ge \dfrac{M_0}{H_1}$.

2/ In the case $d(x,0) > -2\frac{M_0}{\eta_0}\sqrt{\varepsilon}|\ln\varepsilon|$, we have for $S_1 > 2\frac{M_0}{\eta_0} + \frac{1}{\beta}$, where β is defined in lemma 4.1, that

$$\frac{d(x,0) + S_1\sqrt{\varepsilon}|\ln\varepsilon|}{\varepsilon} \ge -2\frac{M_0}{\eta_0}\frac{|\ln\varepsilon|}{\sqrt{\varepsilon}} + \left(2\frac{M_0}{\eta_0} + \frac{1}{\beta}\right)\frac{|\ln\varepsilon|}{\sqrt{\varepsilon}} = \frac{1}{\beta}\frac{|\ln\varepsilon|}{\sqrt{\varepsilon}}.$$

This implies since $U_z > 0$ that

$$U^+(x,0) \ge U\left(\frac{1}{\beta}\frac{|\ln\varepsilon|}{\sqrt{\varepsilon}}, \varepsilon(v_0(x) - 2k^\varepsilon(x,0)) - S_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|\right).$$

Thus using (4.4) and (3.4) one gets

$$\begin{aligned} U^+(x,0) &\geq h_+(\varepsilon(v_0(x) - 2k^\varepsilon(x,0)) - S_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|) - Ae^{-\frac{|\ln\varepsilon|}{\sqrt{\varepsilon}}} \\ &\geq h_+(\varepsilon v_0(x) - S_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|) - A\varepsilon^2, \end{aligned}$$

which in view of (3.6) gives

$$U^+(x,0) \geq h_+(\varepsilon v_0(x)) + H_1 S_2 \varepsilon^{\frac{3}{2}}|\ln\varepsilon| - A\varepsilon^2. \tag{4.54}$$

Furthermore by the estimate (3.31) we have

$$u^\varepsilon(x, \tau_0\varepsilon^2|\ln\varepsilon|) \leq h_+(\varepsilon v_0) + M_0\varepsilon^{\frac{3}{2}}|\ln\varepsilon|,$$

which together with (4.54) implies

$$U^+(x,0) \geq u^\varepsilon(x, \tau_0\varepsilon^2|\ln\varepsilon|),$$

for $S_2 \geq \dfrac{M_0 + A}{H_1}$ and ε small enough. This with (4.53) proves that $U^+(x,0) \geq u^\varepsilon(x, \tau_0\varepsilon^2|\ln\varepsilon|)$ for all $x \in \Omega$. Similarly one can check that $U^-(x,0) \leq u^\varepsilon(x, \tau_0\varepsilon^2|\ln\varepsilon|)$ for all $x \in \Omega$ and concludes the proof of lemma 4.4. □

4.3. **Proof of Theorem 1.3.** We are now in a position to prove Theorem 1.3. Using (4.51) and the lemmas 4.3, 4.4 and A.1 with $\tilde{t} = t_0^\varepsilon = \tau_0\varepsilon^2|\ln\varepsilon|$, we have

$$U^-(x,t) \leq u^\varepsilon(x, t_0^\varepsilon + t) \leq U^+(x,t), \tag{4.55}$$

$$V^-(x,t) \leq v^\varepsilon(x, t_0^\varepsilon + t) \leq V^+(x,t), \tag{4.56}$$

for all $(x,t) \in \Omega \times [0, \overline{T}]$. From the definition of $V^\pm$, (4.23), and the property (4.8) of k^ε we obtain

$$v(x,t) - \mathcal{K}_M(\overline{T})\sqrt{\varepsilon}|\ln\varepsilon| \leq v^\varepsilon(x, t_0^\varepsilon + t) \leq v(x,t) + \mathcal{K}_M(\overline{T})\sqrt{\varepsilon}|\ln\varepsilon|, \tag{4.57}$$

for all $(x,t) \in \Omega \times [0, \overline{T}]$. Thus using (2.1) and noting that

$$v(x,t) - v(x, t - \tau_0\varepsilon^2|\ln\varepsilon|) = \int_{t-\tau_0\varepsilon^2|\ln\varepsilon|}^{t} v_t(x,s)ds$$

we obtain

$$\begin{aligned} v(x,t) - \mathcal{K}_M(\overline{T})\sqrt{\varepsilon}|\ln\varepsilon| - \overline{\mathcal{V}}\tau_0\varepsilon^2|\ln\varepsilon| &\leq v^\varepsilon(x,t) \\ &\leq v(x,t) + \overline{\mathcal{V}}\tau_0\varepsilon^2|\ln\varepsilon| + \mathcal{K}_M(\overline{T})\sqrt{\varepsilon}|\ln\varepsilon|, \end{aligned}$$

for all $(x,t) \in \Omega \times [\tau_0\varepsilon^2|\ln\varepsilon|, \overline{T}]$, which implies (1.9). Next we show (1.8). We have by (4.55) and the definition of $U^\pm$, (4.22), that

$$U\left(\frac{d(x,t) - S_1^\varepsilon(t)}{\varepsilon}, \varepsilon(v(x,t) + 2k^\varepsilon(x,t)) + S_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|\right) \leq u^\varepsilon(x, \tau_0\varepsilon^2|\ln\varepsilon| + t) \tag{4.58}$$

and

$$u^\varepsilon(x, \tau_0\varepsilon^2|\ln\varepsilon| + t) \leq U\left(\frac{d(x,t) + S_1^\varepsilon(t)}{\varepsilon}, \varepsilon(v(x,t) - 2k^\varepsilon(x,t)) - S_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|\right), \tag{4.59}$$

for all $(x,t) \in \Omega \times [0, \overline{T}]$.

Let $(x,t) \in \Omega \times [0, \overline{T}]$ such that $dist(x, \Gamma_t) \geq 2S_1^\varepsilon(\overline{T})$ then we have

$$d(x,t) - S_1^\varepsilon(t) \geq S_1\sqrt{\varepsilon}|\ln\varepsilon| > 0. \tag{4.60}$$

Thus using (4.58), (4.59), (4.4) and the fact that $U_z > 0$ we deduce that

$$h_+\Big(\varepsilon(v(x,t)+2k^\varepsilon(x,t))+S_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|\Big)-Ae^{-\beta\dfrac{d(x,t)-S_1^\varepsilon(t)}{\varepsilon}}$$

$$\leq u^\varepsilon(x,\tau_0\varepsilon^2|\ln\varepsilon|+t)\leq h_+\Big(\varepsilon(v(x,t)-2k^\varepsilon(x,t))-S_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|\Big).$$

In view of (3.5) we deduce that

$$h_+(0)-H(\varepsilon\overline{\mathcal{V}}+2\mathcal{K}_M(\overline{T})\varepsilon\sqrt{\varepsilon}|\ln\varepsilon|+S_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|)-Ae^{-\beta\dfrac{d(x,t)-S_1^\varepsilon(t)}{\varepsilon}}$$

$$\leq u^\varepsilon(x,\tau_0\varepsilon^2|\ln\varepsilon|+t)\leq h_+(0)+H(\varepsilon\overline{\mathcal{V}}+2\mathcal{K}_M(\overline{T})\varepsilon\sqrt{\varepsilon}|\ln\varepsilon|+S_2\varepsilon^{\frac{3}{2}}|\ln\varepsilon|).$$

Further by (4.60)

$$e^{-\beta\frac{d(x,t)-S_1^\varepsilon(t)}{\sqrt{\varepsilon}}}\leq e^{-\beta S_1\frac{|\ln\varepsilon|}{\sqrt{\varepsilon}}}\leq\varepsilon,\text{ for }\varepsilon\text{ small enough}$$

and thus for all $(x,t)\in\Omega\times[0,\overline{T}]$ satisfying $dist(x,\Gamma_t)\geq 2S_1^\varepsilon(\overline{T})$ we conclude that

$$1-\varepsilon M_1\leq u^\varepsilon(x,\tau_0\varepsilon^2|\ln\varepsilon|+t)\leq 1+\varepsilon M_1,$$

where M_1 is a positive constant. In the same way, one can show the existence of a positive constant M_2 such that

$$-1-\varepsilon M_2\leq u^\varepsilon(x,\tau_0\varepsilon^2|\ln\varepsilon|+t)\leq -1+\varepsilon M_2,$$

for all $(x,t)\in\Omega\times[0,\overline{T}]$ satisfying $dist(x,\Gamma_t)\leq -2S_1^\varepsilon(\overline{T})$. This implies (1.8) and concludes the proof of Theorem 1.3.

5. Integral of the Green function on a family of hypersurface $(\mathcal{H}_\lambda)_{\lambda\in\Lambda}$.

5.1. **Preliminary notations.** Let $\mathcal{H}_0$ be a compact hypersurface of $\mathbf{R}^N$ of class C^k, with $k\geq 2$ and $\nu_0:\mathcal{H}_0\to\mathbf{R}^N$ a unit normal vector on $\mathcal{H}_0$ of class C^{k-1}. Let $\delta>0$ be such that the function ψ_0 defined by

$$\psi_0:\begin{cases}\mathcal{H}_0\times(-\delta,\delta) & \to\mathbf{R}^N\\ (x,s) & \mapsto\psi_0(x,s)=x+s\nu_0(x)\end{cases}$$

is a diffeomorphism from $\mathcal{H}_0\times(-\delta,\delta)$ to a tubular neighborhood $\mathcal{H}_0(\delta)$ of $\mathcal{H}_0$. (5.1)

Let Λ be a compact set and $\phi:\Lambda\times\mathcal{H}_0\to\mathbf{R}^N$ a continuous function satisfying the two following assumptions :

($\mathcal{A}1$) For all $\lambda\in\Lambda$, $\phi_\lambda : x\mapsto\phi(\lambda,x)$ is a C^k-diffeomorphism from $\mathcal{H}_0$ to a hypersurface $\mathcal{H}_\lambda$,

($\mathcal{A}2$) The differential of ϕ_λ depends continuously on $(\lambda,x)\in\Lambda\times\mathcal{H}_0$.

Let ν_λ be the unit normal vector on $\mathcal{H}_\lambda$ of class C^{k-1}. We denote by $Jac(\phi_\lambda(x))$ the jacobian of ϕ_λ. Since by ($\mathcal{A}2$), $Jac(\phi_\lambda(x))$ is a continuous function on the compact set $\Lambda\times\mathcal{H}_0$ we deduce that there exists a positive constant C_ϕ such that

$$|Jac(\phi_\lambda(x))|\leq C_\phi,\text{ for all }(\lambda,x)\in\Lambda\times\mathcal{H}_0. \tag{5.2}$$

Finally, we define ϕ^* by

$$\phi^*:\begin{cases}\Lambda\times\mathcal{H}_0\times\mathbf{R} & \to\mathbf{R}^N\\ (\lambda,x,s) & \mapsto\phi^*(\lambda,x,s)=\phi_\lambda(x)+s\nu_\lambda(\phi_\lambda(x)).\end{cases}$$

Further in what follows we denote by $B(x,R)$ the open ball of $\mathbf{R}^N$ of center x and radius R.

5.2. **Uniform tubular neighborhood of $\mathcal{H}_\lambda$.**

Lemma 5.1. *1/ There exists $s_0 > 0$ such that for all $\lambda \in \Lambda$ the function*

$$\phi^*_\lambda : \begin{cases} \mathcal{H}_0 \times (-s_0, s_0) & \to \mathbf{R}^N \\ (x,s) & \mapsto \phi^*_\lambda(x,s) = \phi^*(\lambda, x, s) \end{cases}$$

is a C^{k-1} diffeomorphism from $\mathcal{H}_0 \times (-s_0, s_0)$ to the tubular neighborhood of $\mathcal{H}_\lambda$ denoted by $\mathcal{H}_\lambda(s_0)$.
2/ Setting ψ_λ the diffeomorphism defined from $\mathcal{H}_0(s_0)$ to $\mathcal{H}_\lambda(s_0)$ by $\psi_\lambda := \phi^*_\lambda \circ (\psi_0)^{-1}$, then there exists a strictly positive constant $c > 0$ such that

$$|\psi_\lambda(z') - \psi_\lambda(z)| \geq c|z' - z|, \text{ for all } \lambda \in \Lambda \text{ and } z, z' \in \mathcal{H}_0(s_0). \tag{5.3}$$

Proof. We first claim that
there exists $s_1 > 0$ such that for all $\lambda \in \Lambda$ and all $(x,s) \in \mathcal{H}_0 \times (-s_1, s_1)$, $D\phi^*_\lambda(x,s)$ is an isomorphism from $T_x\mathcal{H}_0 \times \mathbf{R}$ to $\mathbf{R}^N$, where $T_x\mathcal{H}_0$ is the tangent hyperplane to $\mathcal{H}_0$ at x.

Proof of the claim : Let $\Delta(\lambda, x, s) := \left| det\Big(D\phi^*_\lambda(x,s)\Big)\right|$, then Δ is continuous on $\Lambda \times \mathcal{H}_0 \times \mathbf{R}$. Moreover we deduce from ($\mathcal{A}1$) that $\Delta(\lambda, x, 0) > 0$. Further from ($\mathcal{A}2$) we deduce the continuity of Δ on the compact set $\Lambda \times \mathcal{H}_0$. Thus there exists a constant $m > 0$ such that $\Delta(\lambda, x, 0) \geq m$, for all $(\lambda, x) \in \Lambda \times \mathcal{H}_0$. Since Δ is uniformly continuous on the compact set $\Lambda \times \mathcal{H}_0 \times [-1, 1]$ there exists $s_1 \in (0,1)$ such that

$$|\Delta(\lambda, x, s) - \Delta(\lambda, x, 0)| < m, \text{ for all } \lambda \in \Lambda \text{ and all } (x,s) \in \mathcal{H}_0 \times (-s_1, s_1),$$

which implies that

$$\Delta(\lambda, x, s) > 0, \text{ for all } \lambda \in \Lambda \text{ and all } (x,s) \in \mathcal{H}_0 \times (-s_1, s_1). \tag{5.4}$$

Thus the preliminary claim is obtained.

Further since $\psi_\lambda \in C^1(\mathcal{H}_0 \times (-s_1, s_1))$ and since $(\lambda, x, s) \to D\psi_\lambda(x,s)$ is continuous on $\Lambda \times \mathcal{H}_0 \times (-s_1, s_1)$ we deduce from the previous claim and the local inversion theorem with parameter that there exist $\eta > 0$ and $c > 0$ such that $\mathcal{H}_0(\eta) \subset \mathcal{H}_0(s_1)$ and

$$|\psi_\lambda(z) - \psi_l(z')| \geq c|z - z'|, \ \forall \lambda \in \Lambda \text{ and all } (z, z') \in \mathcal{H}_0(\eta).$$

Choosing $s_0 \in (0, \eta]$, we obtain (5.3), which in particular implies that ψ_λ is injective for all $\lambda \in \Lambda$. Finally by (5.4) we conclude that ψ_λ is a diffeomorphism from $\mathcal{H}_0(s_0)$ to $\mathcal{H}_\lambda(s_0)$, which achieves the proof of lemma 5.1. □

5.3. **Volume of the balls of the hypersurface $\mathcal{H}_\lambda$.**

We first estimate the volume of a ball of $\mathcal{H}_0$. More precisely setting for $x \in \mathcal{H}_0$ and $r > 0$ $\beta(x,r) := B(x,r) \cap \mathcal{H}_0 = \{y \in \mathcal{H}_0, \ |x - y| < r\}$ we prove the following lemma

Lemma 5.2. *There exists a constant $\mathcal{C}$ such that*

$$Vol(\beta(x,r)) \leq \mathcal{C} r^{N-1}, \text{ for all } (x,r) \in \mathcal{H}_0 \times (0, \infty). \tag{5.5}$$

Proof. Let $x \in \mathcal{H}_0$. There exists $i \in \{1, ..., N\}$ such that the projection, p_i from $\mathbf{R}^N$ to the hyperplan H_i of equation $X_i = 0$, induces a diffeomorphism from an open neighborhood $O_x \subset \mathcal{H}_0$ of x to an open set O'_x of H_i. We denote by φ_x the inverse of the restriction of p_i to O_x, namely

$$\varphi_x = (p_{i_{|O_x}})^{-1} : \; O'_x \to O_x.$$

Let $R > 0$ be such that the closed ball of center $p_i(x)$ and radius R, $\overline{B(p_i(x), R)}$, is contained in O'_x, then there exists a constant C_x such that

$$|Jac(\varphi_x(y))| \leq C_x \;\; \text{for all } y \in \overline{B(p_i(x), R)}, \tag{5.6}$$

where $Jac(\varphi_x)$ is the Jacobian of φ_x. Further we set $U_x := \varphi_x(B(p_i(x), R))$. Since $\mathcal{H}_0$ is compact there exists an open finite subcover of $\mathcal{H}_0$, $(U_{x_j})_{j\in\{1,...,\mathcal{J}\}}$. Let $\rho > 0$ be the Lebesgue number of the cover $(U_{x_j})_{j\in\{1,...,\mathcal{J}\}}$, then by definition of ρ every subset of $\mathcal{H}_0$ having diameter less than ρ is contained in some U_{x_j}.
Let $x \in \mathcal{H}_0$, there exists $j \in \{1, ..., \mathcal{J}\}$ such that $\beta(x, r) \subset U_{x_j}$ for all $r \leq \rho/2$. By the previous remark, there exists a projection p_{i_0} which is a diffeomorphism from U_{x_j} to $p_{i_0}(U_{x_j}) = B(p_{i_0}(x_j), R)$, so that for all $r \leq \rho/2$

$$\begin{aligned} Vol(\beta(x, r)) = \int_{\beta(x,r)} d\sigma(x) &= \int_{p_{i_0}(\beta(x,r))} |Jac(\varphi(y))| dy \\ &\leq \int_{B(p_{i_0}(x_j),r))} |Jac(\varphi(y))| dy. \end{aligned}$$

Setting $C := \max(C_{x_1}, ..., C_{x_\mathcal{J}})$ this gives in view of (5.6) that

$$Vol(\beta(x, r)) \leq C \int_{B(p_{i_0}(x),r)} dy = Ca_{N-1} r^{N-1} \;\; \text{for all } r \leq \rho/2, \tag{5.7}$$

where a_{N-1} denotes the volume of the unit ball of R^{N-1}. Moreover since $\lambda \mapsto Vol(\mathcal{H}_\lambda)$ is continuous on the compact set Λ one has

$$\frac{Vol(\beta(x, r))}{r^{N-1}} \leq \frac{2^{N-1} Vol(\mathcal{H}_\lambda)}{\rho^{N-1}} \leq \frac{2^{N-1}}{\rho^{N-1}} \tilde{C}, \;\text{ for all } r \geq \rho/2.$$

This together with (5.7) implies (5.5) and concludes the proof of lemma 5.2. □

We now consider a ball of $\mathcal{H}_\lambda$ and prove a similar estimate to (5.5).

Lemma 5.3. *Setting $\beta(\lambda, x, r) = B(x, r) \cap \mathcal{H}_\lambda$ for $x \in \mathcal{H}_\lambda$ and $r > 0$, then there exists a constant $\tilde{\mathcal{C}}$ such that for all $\lambda \in \Lambda$ we have*

$$Vol(\beta(\lambda, x, r)) \leq \tilde{\mathcal{C}} r^{N-1}, \;\text{ for all } (x, r) \in \mathcal{H}_\lambda \times (0, \infty). \tag{5.8}$$

Proof. Using the change of variables ϕ_λ on gets

$$\begin{aligned} Vol(\beta(\lambda, x, r)) &= \int_{\beta(x,r)\cap\mathcal{H}_\lambda} d\sigma_\lambda(y) = \int_{\phi_\lambda^{-1}(\beta(\lambda,x,r))} |Jac(\phi_\lambda(z))| d\sigma_0(z) \\ &\leq C_\phi Vol(\phi_\lambda^{-1}(\beta(\lambda, x, r))). \end{aligned} \tag{5.9}$$

Further let M, $M' \in \phi_\lambda^{-1}(\beta(\lambda, x, r)) \subset \mathcal{H}_0$ and let $N, N' \in \beta(\lambda, x, r)$ such that $N := \phi_\lambda(M)$ and $N' := \phi_\lambda(M')$, then applying lemma 5.1, there exists $s_0 > 0$ such that for all $\lambda \in \Lambda$ the diffeomorphism ψ_λ from $\mathcal{H}_0(s_0)$ to $\mathcal{H}_\lambda(s_0)$ satisfies (5.3), so that in particular

$$|M - M'| \leq \frac{1}{c} |\psi_\lambda(M) - \psi_\lambda(M')|. \tag{5.10}$$

Noting that $M, M' \in \mathcal{H}_0$ and then $\psi_\lambda(M) = \phi_\lambda(M)$, $\psi_\lambda(M') = \phi_\lambda(M')$, we obtain from (5.10) that $|M - M'| \leq \frac{1}{c}|N - N'| \leq \frac{2r}{c}$. Thus $diam(\phi_\lambda^{-1}(\beta(\lambda, x, r)) \leq \frac{2r}{c}$ and

$\phi_\lambda^{-1}(\beta(x,r)\cap\mathcal{H}_\lambda)$ is contained in a ball, B', of radius $r'=\frac{2r}{c}$. Finally, by lemma 5.2 we obtain

$$Vol(\phi_\lambda^{-1}(\beta(\lambda,x,r)))\leq\int_{\mathcal{H}_0\cap B'}d\sigma_0(y)\leq\mathcal{C}\left(\frac{2r}{c}\right)^{N-1},$$

which together with (5.9) implies (5.8) and achieves the proof of lemma 5.3. □

5.4. **Integral of the Green function on $\mathcal{H}_\lambda$.** We set for $N>2$ and $p\in\mathbf{N}^*$

$$J(x,y):=\frac{1}{|x-y|^{N-2}}\text{ and }J_p(x,y)=\begin{cases}J(x,y) & \text{if } |x-y|>\frac{1}{p}\\ p^{N-2} & \text{otherwise,}\end{cases}$$

and for $N=2$ and $p\in\mathbf{N}^*$

$$J(x,y):=\Big|\ln|x-y|\Big|\text{ and }J_p(x,y)=\begin{cases}J(x,y) & \text{if } |x-y|>\frac{1}{p}\\ \ln(p) & \text{otherwise.}\end{cases}$$

Let

$$P(\lambda,x):=\int_{\mathcal{H}_\lambda}J(x,y)d\sigma_\lambda(y)\text{ and }P_p(\lambda,x):=\int_{\mathcal{H}_\lambda}J_p(x,y)d\sigma_\lambda(y)\tag{5.11}$$

we prove below that P and P_p are bounded.

Lemma 5.4. *Let the assumptions ($\mathcal{A}1$) and ($\mathcal{A}2$) be satisfied, then the function $(\lambda,x)\to P(\lambda,x)$ is continuous on $\Lambda\times\mathbf{R}^N$. Moreover there exists a positive constant $\mathcal{P}$ such that*

$$P(\lambda,x)=\int_{\mathcal{H}_\lambda}J(x,y)d\sigma_\lambda(y)\leq\mathcal{P},\ \textit{for all }\lambda\in\Lambda\textit{ and }x\in\mathbf{R}^N.\tag{5.12}$$

Proof. We first consider the case $N>2$. Using the change of variables ϕ_λ on has

$$P_p(\lambda,x)=\int_{\mathcal{H}_0}J_p(x,\phi_\lambda(z))|Jac(\phi_\lambda(z))|d\sigma_0(z).$$

Since $(\lambda,x,z)\mapsto J_p(x,\phi_\lambda(x))|Jac(\phi_\lambda(x))|$ is continuous and bounded by $C_\phi p^{N-2}$, which is integrable on $\mathcal{H}_0$, we deduce that $(\lambda,x)\mapsto P_p(\lambda,x)$ is continuous on $\Lambda\times\mathbf{R}^N$. We next prove the uniform convergence of (P_p) to P on $\Lambda\times\mathbf{R}^N$, as $p\uparrow\infty$. We have

$$\begin{aligned}0\leq P(\lambda,x)-P_p(\lambda,x) &= \int_{\mathcal{H}_\lambda\cap B(x,\frac{1}{p})}(J(x,y)-p^{N-2})d\sigma_\lambda(y)\\ &\leq \int_{\mathcal{H}_\lambda\cap B(x,\frac{1}{p})}J(x,y)d\sigma_\lambda(y).\end{aligned}\tag{5.13}$$

Let $q\in\mathbf{N}$, we set $\beta_q=\mathcal{H}_\lambda\cap B(x,\frac{1}{p2^q})$, so that $|x-y|\geq\frac{1}{p2^{q+1}}$ for $y\in\beta_q\setminus\beta_{q+1}$. We obtain from (5.13) and lemma 5.3

$$\begin{aligned}&0\leq P(\lambda,x)-P_p(\lambda,x)\leq\Sigma_{q\in\mathbf{N}}\int_{\beta_q\setminus\beta_{q+1}}J(x,y)d\sigma_\lambda(y)\\ &\leq\Sigma_{q\in\mathbf{N}}(p2^{q+1})^{N-2}Vol(\beta_q\setminus\beta_{q+1})\leq\Sigma_{q\in\mathbf{N}}(p2^{q+1})^{N-2}Vol(\beta_q),\end{aligned}$$

so that

$$0\leq P(\lambda,x)-P_p(\lambda,x)\leq\Sigma_{q\in\mathbf{N}}(p2^{q+1})^{N-2}\left(\frac{1}{p2^q}\right)^{N-1}\tilde{\mathcal{C}}=\frac{2^{N-1}}{p}\tilde{\mathcal{C}}.$$

Thus the sequence of continuous functions (P_p) converges uniformly to P on $\Lambda\times\mathbf{R}^N$, as $p\uparrow\infty$ and then P is continuous on $\Lambda\times\mathbf{R}^N$. Further let R_0 be such that $\phi(\Lambda\times\mathcal{H}_0)\subset B(0,R_0)$ then the continuity of P on $\Lambda\times\mathbf{R}^N$ implies that

$$P \text{ is bounded on the compact set } \Lambda\times\overline{B(0,R_0+1)}. \tag{5.14}$$

Moreover let $x\notin\overline{B(0,R_0+1)}$ we have $0\le J(x,y)\le 1$ for $y\in\mathcal{H}_\lambda$, so that

$$P(\lambda,x)\le\int_{\mathcal{H}_\lambda}d\sigma_\lambda(y)=\int_{\mathcal{H}_0}|Jac(\phi_\lambda(x))|d\sigma_0(z)\le C_\phi Vol(\mathcal{H}_0).$$

This together with (5.14) implies (5.12) and concludes the proof of lemma 5.4 in the case $N>2$. Similarly, one can show lemma 5.4 in the case $N=2$. □

Finally, we deduce from lemma 5.4 that the integral of the Green function on Γ_t or on continuous perturbation of Γ_t is bounded. More precisely we will apply lemma 5.4 with successively $\lambda=t$, $\Lambda=[0,T]$ and with $\lambda=(t,\varepsilon)$, $\Lambda=[0,T]\times[0,\tilde{\varepsilon}_0]$ with $\tilde{\varepsilon}_0$ to be chosen later to deduce the following Corollary.

Corollary 1. *Let (u,v,Γ), with $\Gamma=(\Gamma_t\times\{t\})_{t\in[0,\overline{T}]}$ be the solution of the free boundary Problem (P) and let G be the Green function associated with $-\Delta+\gamma$ with Neumann condition then there exists a positive constant $\mathcal{G}$ such that*

$$\int_{\Gamma_t}|G(x,x')|dx'\le\mathcal{J}\int_{\Gamma_t}|J(x,x')|dx'\le\mathcal{G},\ \textit{for all } (x,t)\in\Omega\times[0,\overline{T}]. \tag{5.15}$$

Moreover, setting

$$\Gamma_{t,\varepsilon}:=\{x\in\Omega,\ d^\varepsilon(x,t)=0\},\ \textit{where } d^\varepsilon(x,t)=d(x,t)+2S_1\sqrt{\varepsilon}|\ln\varepsilon|e^{m_1t}$$

then there exist $\tilde{\varepsilon}_0>0$ and a positive constant $\mathcal{G}_1$ independent of S_1 such that

$$\int_{\Gamma_{t,\varepsilon}}|G(x,x')|dx'\le\mathcal{G}_1,\ \textit{for all } (x,t,\varepsilon)\in\Omega\times[0,\overline{T}]\times[0,\tilde{\varepsilon}_0]. \tag{5.16}$$

Similarly there exists a positive constant $\mathcal{G}_2$ such that

$$\int_{\Gamma_t+rn}|G(x,x')|dx'\le\mathcal{G}_2,\ \textit{for all } (x,t,r)\in\Omega\times[0,\overline{T}]\times[-L,L], \tag{5.17}$$

where n is the normal to Γ_t and $L\in(0,s_0)$.

Proof. We recall (see for example [3] p.1214) that there exists a positive constant $\mathcal{J}$ such that

$$|G(x,x')|\le\mathcal{J}J(x,x'),\ \text{for all } (x,x')\in\Omega^2. \tag{5.18}$$

Since $\Gamma\in C^{2+\alpha,\frac{2+\alpha}{2}}$, the function ϕ_λ related to Γ and defined by $(\mathcal{A}1)$ is a $C^{2+\alpha}$ diffeomorphism from Γ_0 to Γ_t and $t\mapsto\Gamma_{t,\varepsilon}\in C^{\frac{2+\alpha}{2}}([0,\overline{T}])\subset C([0,\overline{T}])$. Thus applying lemma 5.4 we deduce from (5.12) and (5.18) the estimate (5.15). Further let ε small enough such that $2S_1\sqrt{\varepsilon}|\ln\varepsilon|e^{m_1\overline{T}}\le s_0$ then since $(t,\varepsilon)\mapsto\Gamma_{t,\varepsilon}\in C^{\frac{2+\alpha}{2},\infty}([0,\overline{T}]\times[0,\tilde{\varepsilon}_0])\subset C([0,\overline{T}]\times[0,\tilde{\varepsilon}_0])$ we also obtain from lemma 5.4 the estimate (5.16). In the same way, assuming that $L<s_0$ and then using (5.12) and the fact that $(t,r)\mapsto\Gamma_t+rn\in C^{\frac{2+\alpha}{2},\infty}([0,\overline{T}]\times[-L,L])\subset C([0,\overline{T}]\times[-L,L])$ we deduce (5.17). This ends the proof of Corollary 1. □

Appendix A. **Appendix : Comparison principle.** In this Section, we recall a classical comparison principle lemma and we refer to [4] for its proof.

Lemma A.1. *We assume that* $\underline{u}$, $\underline{v}$, $\overline{u}$ *and* $\overline{v}$ *are four bounded functions satisfying the following inequalities*

$$\mathcal{L}_1^\varepsilon(\overline{u},\underline{v}) \geq 0 \text{ and } \mathcal{L}_1^\varepsilon(\underline{u},\overline{v}) \leq 0 \qquad \text{in } \Omega\times(0,T) \tag{A.1}$$

$$\mathcal{L}_2^\varepsilon(\overline{u},\overline{v}) \geq 0 \text{ and } \mathcal{L}_2^\varepsilon(\underline{u},\underline{v}) \leq 0 \qquad \text{in } \Omega\times(0,T) \tag{A.2}$$

$$\frac{\partial \underline{u}}{\partial n} \leq 0 \leq \frac{\partial \overline{u}}{\partial n} \text{ and } \frac{\partial \underline{v}}{\partial n} \leq 0 \leq \frac{\partial \overline{v}}{\partial n} \qquad \text{on } \partial\Omega\times(0,T) \tag{A.3}$$

$$\underline{u}(x,0) \leq u^\varepsilon(x,\tilde{t}) \leq \overline{u}(x,0) \qquad \text{for } x\in\Omega \tag{A.4}$$

$$\underline{v}(x,0) \leq v^\varepsilon(x,\tilde{t}) \leq \overline{v}(x,0) \qquad \text{for } x\in\Omega, \tag{A.5}$$

for some $\tilde{t}\in[0,T)$. *Then we have*

$$\underline{u}(x,t) \leq u^\varepsilon(x,\tilde{t}+t) \leq \overline{u}(x,t) \qquad \text{in } \Omega\times[0,T) \tag{A.6}$$

$$\underline{v}(x,t) \leq v^\varepsilon(x,\tilde{t}+t) \leq \overline{v}(x,t) \qquad \text{in } \Omega\times[0,T). \tag{A.7}$$

Moreover $\overline{u}$ *and* $\underline{u}$ *are called respectively supersolution and subsolution of* u^ε. *In the same way,* $\overline{v}$ *and* $\underline{v}$ *are called respectively supersolution and subsolution of* v^ε.

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