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GROW UP AND SLOW DECAY IN THE CRITICAL SOBOLEV CASE

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Dedicated to Hiroshi Matano on the occasion of his 60th birthday

Abstract. We present conjectures on asymptotic behaviour of threshold solutions of the Cauchy problem for a semilinear heat equation with Sobolev critical nonlinearity. The conjectures say that, depending on the decay rate of initial data and the space dimension, the threshold solutions may grow up, stabilize, or decay to zero as $t \to \infty$. The rates of grow up or decay are computed formally using matched asymptotics.

1. **Introduction.** In this paper we study the large-time behaviour of global classical solutions of the Cauchy problem

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where we assume that u_0 is continuous, and

$$p = p_S := \frac{N+2}{N-2}, \qquad N > 2.$$

Problem (1.1) with $p > 1$ has been studied as an archetypal superlinear problem and as a canonical problem of more general superlinear equations after taking a scaling limit. In spite of its simple appearance, (1.1) is known to have a very rich mathematical structure, and has been studied extensively by many authors. The exponent p_S is a well-known critical exponent for the dynamics of (1.1), see the recent monograph [28] and references therein.

For $1 < p < p_S$ there are no bounded positive radial steady states while for $p \geq p_S$ such steady states exist. For $p = p_S$ they are of the form

$$\Phi(|x|;k) := \left(k + \frac{|x|^2}{kN(N-2)}\right)^{-\frac{N-2}{2}}, \qquad k > 0. \tag{1.2}$$

It was shown in [33] (see also [12]) that radial steady states are threshold solutions for all $p \geq p_S$. A solution $u = u(x,t;u_0)$ is called a threshold solution if its initial

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function u_0 is such that $u(\cdot,t;\lambda u_0)$ exists for all $t>0$ if $0<\lambda<1$ while $u(\cdot,t;\lambda u_0)$ blows up in finite time in the L^∞-norm if $\lambda>1$.

For $p>1$ and $\Omega\subset\mathbb{R}^N$ bounded, threshold solutions of the Dirichlet problem

$$\begin{cases} u_t=\Delta u+u^p, & x\in\Omega,\ t>0,\\ u=0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.3}$$

were studied in [23] and later in [11] and other papers, see [28] and references therein. If Ω is a ball then we have a complete picture which can be summarized as follows. Every threshold solution converges to the unique positive steady state if $p<p_S$. For $p>p_S$, every radial threshold solution of (1.3) blows up in finite time but exists globally as a weak solution (incomplete blow up). In fact, it can develop a singularity at the centre of the ball Ω only finitely many times, see [4], [19], [20]. If $p=p_S$ then all radial threshold solutions of (1.3) grow up, which means that they exist globally and become unbounded as $t\to\infty$. Their rate of grow up was studied in [10].

For the Cauchy problem (1.1), the situation is much more complicated. Global positive solutions exist if and only if $p>p_F:=1+2/N$, see [9], [28]. For $p_F<p<p_S$, several sufficient conditions for global existence of threshold solutions and their decay to zero have been established, see [17, 18, 27, 24, 28, 29]. If $p>p_S$ then threshold solutions of (1.1) can exhibit incomplete blow up (cf. [21, 22, 28]), they can grow up (cf. [1, 2, 3, 5, 6, 25, 26]), converge to steady states (cf. [7, 14, 30, 31]) or decay to zero at slow rates (cf. [13, 8, 30]). Not many results have been established for threshold solutions of (1.1) when $p=p_S$. Among some other things, sufficient conditions for global existence of threshold solutions for $p=p_S$ were given in [24, 27, 28]; for some other results see for example [15, 16].

The aim of this paper is to present evidence that radial threshold solutions of (1.1) with $p=p_S$ can exhibit grow up and slow decay at various rates.

In what follows we consider solutions of

$$\begin{cases} u_t=u_{rr}+\frac{N-1}{r}u_r+u^{p_S}, & r>0,\ t>0,\\ u_r(0,t)=0, & t>0,\\ u(r,0)=u_0(r)>0, & r\ge 0, \end{cases} \tag{1.4}$$

where u_0 is continuous and $N>2$, $N\in\mathbb{R}$.

It was shown in [27] that if

$$\lim_{r\to\infty} r^{\frac{N-2}{2}}u_0(r)=0,$$

and u is a threshold solution then it exists for all $t>0$ and

$$\limsup_{t\to\infty} t^{\frac{N-2}{4}}\|u(\cdot,t)\|_\infty=\infty.$$

Here $t^{-(N-2)/4}$ is the selfsimilar decay rate.

We shall assume that the initial function u_0 satisfies

$$\lim_{r\to\infty} r^\gamma u_0(r)=A \quad \text{for some} \quad A>0 \quad \text{and} \quad \gamma>\frac{N-2}{2}. \tag{1.5}$$

The conjecture below says that decay rates which are slower than the selfsimilar one occur for threshold solutions of (1.4) if $2<N<4$, $(N-2)/2<\gamma<N-2$ or $4\le N<6$, $(N-2)/2<\gamma<2$. On the other hand, grow up occurs if $2<N<4$, $\gamma>N-2$ or $N=4$, $\gamma>2$. Notice that $\gamma=N-2$, which separates decay from grow up when $2 < N < 4$, corresponds to the decay rate of the steady states given by (1.2).

Conjecture 1.1. *Let u be a threshold solution of (1.4) with initial function u_0 satisfying (1.5). Then there is a constant $C = C(N, u_0) > 0$ such that*

$$\lim_{t\to\infty} \frac{\|u(\cdot,t)\|_\infty}{\varphi(t;N,\gamma)} = C,$$

where the function $\varphi(\cdot;N,\gamma)$ is given in the table below for $2 < N < 6$.

	$\frac{N-2}{2} < \gamma < 2$	$\gamma = 2$	$\gamma > 2$
$2 < N < 4$	$t^{\frac{\gamma+2-N}{2}}$	$t^{\frac{4-N}{2}}(\ln t)^{-1}$	$t^{\frac{4-N}{2}}$
$N = 4$	$t^{-\frac{2-\gamma}{2}}\ln t$	1	$\ln t$
$4 < N < 6$	$t^{-\frac{(2-\gamma)(N-2)}{2(6-N)}}$	$(\ln t)^{-\frac{N-2}{6-N}}$	1

If $N \geq 6$ and $\gamma > (N-2)/2$ then $\varphi(\cdot;N,\gamma) \equiv 1$.

In the subsequent sections we compute the decay and grow up rates formally using matched asymptotics. In Section 2 we give an inner expansion and Sections 3-5 are devoted to outer solutions and matching for $N < 4$, $N > 4$ and $N = 4$, respectively.

2. **Inner solution.**

The inner variables are

$$\xi := \frac{r}{\rho(t)}, \qquad \phi(\xi,t) := \rho^{\frac{N-2}{2}}(t)u(r,t),$$

where

$$\rho(t) := u^{-\frac{2}{N-2}}(0,t) \tag{2.1}$$

is to be determined (cf. Section 4 in [10]). Thus

$$\rho^2\phi_t - \rho\rho'\left(\xi\phi_\xi + \frac{N-2}{2}\phi\right) = \phi_{\xi\xi} + \frac{N-1}{\xi}\phi_\xi + \phi^p.$$

The first two terms in the expansion for the relevant (singly-unstable – separating blowing-up solutions from fast (linear-diffusively) decaying ones) solution take the form

$$\phi \sim \Phi_0(\xi) + \rho\rho'\Phi_1(\xi), \tag{2.2}$$

implying for consistency that

$$\rho \ll \sqrt{t} \quad \text{as} \quad t \to \infty. \tag{2.3}$$

Thus

$$(\Phi_0)_{\xi\xi} + \frac{N-1}{\xi}(\Phi_0)_\xi + \Phi_0^{p_S} = 0, \tag{2.4}$$

whereby in view of (2.1) and (1.2) we have

$$\Phi_0(\xi) = \Phi(\xi;1). \tag{2.5}$$

At next order we have the initial value problem

$$(\Phi_1)_{\xi\xi} + \frac{N-1}{\xi}(\Phi_1)_\xi + \frac{N+2}{N-2}\Phi_0^{\frac{4}{N-2}}\Phi_1 = -\xi(\Phi_0)_\xi - \frac{N-2}{2}\Phi_0, \tag{2.6}$$

subject to

$$\Phi_1 = (\Phi_1)_\xi = 0 \quad \text{at } \xi = 0,$$

the complementary function of which satisfies

$$\Phi_1(\xi) \sim \beta \quad \text{as } \xi \to \infty \tag{2.7}$$

and, because

$$\Phi_0(\xi) \sim (N(N-2))^{-\frac{N-2}{2}} \xi^{-(N-2)} \quad \text{as } \xi \to \infty,$$

a particular integral has asymptotic behaviour

$$\Phi_1(\xi) \sim \alpha \xi^{4-N} \quad \text{as } \xi \to \infty, \qquad N \neq 4. \tag{2.8}$$

In (2.8), α can be evaluated by direct substitution; in (2.7), β can be determined only by integration of (2.6), which can be achieved by reduction of order using the fact that $\partial\Phi/\partial k(\xi;1)$ satisfies the homogeneous version of (2.6). The exception to (2.8) is when $N = 4$, when we instead have

$$\Phi_1(\xi) \sim 4 \ln \xi \quad \text{as } \xi \to \infty, \qquad N = 4. \tag{2.9}$$

3. **Outer solution and matching:** $2 < N < 4$**.** In this regime (2.8) dominates (2.7) and the inner expansion (2.2) thus breaks down at

$$\xi = O\left((\rho\rho')^{-1/2}\right). \tag{3.1}$$

The outer problem reads

$$u_t \sim u_{rr} + \frac{N-1}{r} u_r \tag{3.2}$$

and thus has $r = O(\sqrt{t})$, so that (3.1) implies that $\rho'/\rho = O(1/t)$. Moreover, since (2.7) is negligible, the boundary condition on (3.2) is that no r^0 term be present at $r = 0$ i.e. that the solution to (3.2) satisfy

$$u \sim (N(N-2)\rho)^{\frac{N-2}{2}} r^{-(N-2)} + O(r^{4-N}) \quad \text{as } r \to 0, \tag{3.3}$$

where we have matched with (2.5) and where $\rho(t)$ will be determined asymptotically by this outer problem.

As a side remark we mention that in view of (1.5), the relevant contributions to the far field expansion read

$$u \sim Ar^{-\gamma} + \left(A\gamma(\gamma + 2 - N) r^{-\gamma-2} + A^{p_S} r^{-\gamma p_S}\right) t, \tag{3.4}$$

this requiring for consistency with $r = O(\sqrt{t})$ that

$$t^{\frac{2\gamma}{N-2}} \gg t \quad \text{as } t \to \infty$$

i.e. that $\gamma > (N-2)/2$. This condition will prove important later; if $\gamma < (N-2)/2$ then finite-time blow up occurs.

From (3.2), (3.3) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty ru \, \mathrm{d}r = [ru_r + (N-2)u]_0^\infty = 0 \tag{3.5}$$

when the integral exists i.e. when $\gamma > 2$; an alternative statement of (3.3) is that $ru_r + (N-2)u = 0$ at $r = 0$. In the case $\gamma > 2$

$$u \sim t^{-1} g(r/\sqrt{t}), \qquad g(\eta) := B\eta^{-(N-2)} e^{-\eta^2/4} \tag{3.6}$$

where B depends on the initial data (and cannot be determined explicitly because the conservation law (3.5) of course holds only asymptotically as $t \to \infty$). Matching (3.6) with (3.3) yields

$$\begin{aligned} \rho &\sim \tfrac{1}{N(N-2)} B^{\frac{2}{N-2}} t^{-\frac{4-N}{N-2}}, \\ u_{max} &\sim (N(N-2))^{-\frac{2}{N-2}} B^{-1} t^{\frac{4-N}{2}} \end{aligned} \tag{3.7}$$

for $N < 4$, $\gamma > 2$, representing grow up.

For $(N-2)/2 < \gamma < 2$ we instead have

$$u \sim t^{-\frac{\gamma}{2}} g(r/\sqrt{t}), \qquad g(\eta) \sim \sigma A \eta^{-(N-2)} \quad \text{as } \eta \to 0$$

where $g(\eta)$ is specified via $\eta g_\eta + (N-2)g = 0$ at $\eta = 0$,

$$g \sim A\eta^{-\gamma} \quad \text{as } \eta \to \infty,$$

where the constant $\sigma(\gamma)$ can be determined from the boundary value problem for $g(\eta)$. In this case

$$\begin{aligned} \rho &\sim \tfrac{1}{N(N-2)} (\sigma A)^{\frac{2}{N-2}} t^{-\frac{\gamma+2-N}{N-2}}, \\ u_{max} &\sim (N(N-2))^{-\frac{2}{N-2}} (\sigma A)^{-1} t^{\frac{\gamma+2-N}{2}} \end{aligned} \tag{3.8}$$

for $N < 4$, $\gamma < 2$. This gives grow up for $N-2 < \gamma < 2$ and decay for $(N-2)/2 < \gamma < N-2$; for $\gamma = N-2$ we have $g(\eta) = A\eta^{-(N-2)}$, and u neither grows nor decays; this is of course the case in which the decay in (1.5) coincides with that of (2.5). The constraint (2.3) requires $\gamma > (N-2)/2$, an inequality already deduced from other considerations.

Finally here, for $\gamma = 2$ we have

$$u \sim t^{-1} \ln t \; g(r/\sqrt{t}) + t^{-1} h(r/\sqrt{t})$$

whereby

$$g(\eta) := \sigma A \eta^{-(N-2)} e^{-\eta^2/4}$$

(where σ will again be determined by matching) and

$$g - h - \frac{\eta}{2} h_\eta = h_{\eta\eta} + \frac{N-1}{\eta} h_\eta$$

so that (since no η^0 term can be present at $\eta = 0$)

$$\sigma A \int_0^\eta \theta^{-(N-3)} e^{-\theta^2/4} \, d\theta - \frac{1}{2}\eta^2 h = \eta h_\eta + (N-2)h,$$

from which it follows (on imposing the required far-field behaviour) that

$$\sigma = \left(2 \int_0^\infty \theta^{-(N-3)} e^{-\theta^2/4} \, d\theta \right)^{-1}.$$

Thus

$$\begin{aligned} \rho &\sim \tfrac{1}{N(N-2)} (\sigma A)^{\frac{2}{N-2}} t^{-\frac{4-N}{N-2}} \ln^{\frac{2}{N-2}} t, \\ u_{max} &\sim (N(N-2))^{-\frac{2}{N-2}} (\sigma A)^{-1} t^{\frac{4-N}{2}} \ln^{-1} t, \end{aligned} \tag{3.9}$$

for $N < 4$, $\gamma = 2$; the latter grows slightly more slowly than in (3.7).

4. Outer solution and matching: $N > 4$.

4.1. Scenario 1. In this regime it is plausible that there is an intermediate region where

$$\xi = O\left((\rho\rho')^{-1/(N-2)}\right) \tag{4.1}$$

and whereby at leading order

$$0 = u_{rr} + \frac{N-1}{r}u_r, \tag{4.2}$$

so that

$$u \sim \rho^{-\frac{N-2}{2}}\left((N(N-2))^{\frac{N-2}{2}}\xi^{-(N-2)} + \beta\rho\rho'\right); \tag{4.3}$$

over this region the two contributions from (4.2) thus swap dominance. Since (4.1) implies

$$r = O\left(\rho^{\frac{N-3}{N-2}}(\rho')^{-\frac{1}{N-2}}\right) \tag{4.4}$$

the requirement that $r \ll \sqrt{t}$ in some circumstances simply leads again to (2.3); however, as we shall see later the situation can be more complicated and we shall subsequently (in Subsection 4.2) need to revisit the self-consistency of the scenario that we now present.

In the current scenario, the outer region has (3.2) subject on matching with (4.3) to no $r^{-(N-2)}$ term being present at $r = 0$ i.e. to $r^{N-1}u_r = 0$ and (3.5) is thus replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty r^{N-1}u \,\mathrm{d}r = 0$$

if $\gamma > N$. Thus we would have for $\gamma > N$ that

$$u \sim t^{-\frac{N}{2}}g(r/\sqrt{t}), \qquad g(\eta) := Be^{-\eta^2/4} \tag{4.5}$$

and matching with (3.3) would then yield

$$\beta\rho^{-\frac{N-4}{2}}\rho' = Bt^{-\frac{N}{2}} \tag{4.6}$$

where

$$\beta = \frac{2N(N+2)\Gamma^2(N/2)}{(N-2)(N-4)\Gamma(N)} > 0.$$

In fact this argument is incorrect, for reasons we explain in Subsection 4.2, but is instructive in that the implication

$$\rho(t) \to \rho_\infty \quad \text{as } t \to \infty \tag{4.7}$$

is indeed valid, implying that a steady state is attained: trying instead to set $\rho_\infty = 0$ leads from (4.6) to unacceptable signs for $N < 6$ and to a violation of (2.3) for $N > 6$ – similar comment applies in other cases below.

For $\gamma < N$ under the current scenario we instead have

$$u \sim t^{-\frac{\gamma}{2}}g(r/\sqrt{t}), \qquad g(\eta) \sim \sigma A \quad \text{as } \eta \to 0,$$

$g(\eta)$ being specified via $g_\eta = 0$ at $\eta = 0$ and

$$g \sim A\eta^{-\gamma} \quad \text{as } \eta \to \infty,$$

for some constant $\sigma(\gamma)$ with

$$\beta\rho^{-\frac{N-4}{2}}\rho' = \sigma At^{-\frac{\gamma}{2}}. \tag{4.8}$$

For $\gamma > 2$ we thus recover (4.7) (this also encompasses the case $\gamma = N$ when a $\ln t$ term appears on the right-hand side of (4.8), cf. the discussion of $\gamma = 2$ in the previous section). Since we need $\gamma > (N-2)/2$, it follows that (4.7) in fact applies for all relevant γ when $N \geq 6$. For $4 < N < 6$, $(N-2)/2 < \gamma < 2$, however, we have from (4.8) that

$$\begin{aligned} \rho &\sim \left(\frac{(6-N)\sigma A}{(2-\gamma)\beta}\right)^{\frac{2}{6-N}} t^{\frac{2-\gamma}{6-N}}, \\ u_{max} &\sim \left(\frac{(6-N)\sigma A}{(2-\gamma)\beta}\right)^{-\frac{N-2}{6-N}} t^{-\frac{(N-2)(2-\gamma)}{2(6-N)}}, \end{aligned} \tag{4.9}$$

as $t \to \infty$; as $\gamma \to (N-2)/2^+$ we approach the usual self-similar behaviour. For $4 < N < 6$, $\gamma = 2$ we have

$$\begin{aligned} \rho &\sim \left(\frac{(6-N)\sigma A}{\beta}\right)^{\frac{2}{6-N}} (\ln t)^{\frac{2}{6-N}}, \\ u_{max} &\sim \left(\frac{(6-N)\sigma A}{\beta}\right)^{-\frac{N-2}{6-N}} (\ln t)^{-\frac{N-2}{6-N}}. \end{aligned} \tag{4.10}$$

Both (4.9) and (4.10) imply slow decay.

4.2. **Scenario 2.** We must now reassess the applicability of the above asymptotic structure in the light of (4.3). For $(N-2)/2 < \gamma < N-2$ all is well, with (4.4) necessarily having $r \ll \sqrt{t}$. However, (4.6) implies for $\gamma > N$ that

$$\rho = \rho_\infty + O\left(t^{-\frac{N-2}{2}}\right) \quad \text{as } t \to \infty$$

so that (4.4) has

$$r = O\left(t^{\frac{N}{2(N-2)}}\right),$$

which is never consistent with $r \ll \sqrt{t}$, while (4.8) gives

$$\rho = \rho_\infty + O\left(t^{-\frac{\gamma-2}{2}}\right) \quad \text{as } t \to \infty$$

for $\max(2, (N-2)/2) < \gamma < N$, so (4.4) corresponds to

$$r = O\left(t^{\frac{\gamma}{2(N-2)}}\right)$$

and self-consistency then demands that $\gamma < N-2$, as already implied. Thus the asymptotic structure differs from that explored in Subsection 4.1 when $\gamma \geq N-2$: the intermediate layer is absent, while (4.3), in the form

$$u \sim (N(N-2)\rho_\infty)^{\frac{N-2}{2}} r^{-(N-2)} + \beta \rho_\infty^{-\frac{N-4}{2}} \rho', \tag{4.11}$$

is applied as a matching condition on the outer region, the first term in (4.11) implying that the leading-order outer solution takes the form

$$u \sim t^{-\frac{N-2}{2}} g(r/\sqrt{t}) \tag{4.12}$$

with

$$g(\eta) = K\eta^{-(N-2)} \int_\eta^\infty \theta^{N-3} e^{-\theta^2/4}\, d\theta + \begin{cases} 0, & N-2 < \gamma, \\ A\eta^{-(N-2)}, & \gamma = N-2, \end{cases} \tag{4.13}$$

where K depends on the initial data but can be determined in terms of ρ_∞ (which of course itself depends on the initial data) using (4.11), in the form

$$\rho_\infty^{(N-2)/2} (N(N-2))^{(N-2)/2} = K \int_0^\infty \theta^{N-3} e^{-\theta^2/4}\, d\theta + \begin{cases} 0, & N-2 < \gamma, \\ A, & \gamma = N-2, \end{cases}$$

and matching with the second term in (4.11) gives

$$\beta\rho_\infty^{-(N-4)/2}\rho' \sim -\frac{K}{N-2}t^{-\frac{N-2}{2}} \tag{4.14}$$

so that ρ decays towards ρ_∞ at a rate given by (4.14) rather than by (4.6) or (4.8) when $\gamma \geq N-2$.

5. **Outer solution and matching:** $N = 4$.

This critical case is of some interest because there is a significant change in behaviour between $N < 4$ and $N > 4$, as indicated by the analysis of the previous two sections. Here (4.3) is replaced by

$$u \sim \rho^{-1}\left(8\xi^{-2} + 4\rho\rho' \ln(r/\rho)\right) \tag{5.1}$$

in matching into $r = O(\sqrt{t})$. To describe the outer region for $\gamma > 2$ we accordingly introduce variables $\eta = r/\sqrt{t}$, $\tau = \ln t$ and seek an expansion in the form

$$u \sim t^{-1}\big(\psi(\tau)g(\eta) + \dot{\psi}(\tau)G(\eta)\big) \tag{5.2}$$

where $\dot{}$ denotes $\mathrm{d}/\mathrm{d}\tau$, $g(\eta), G(\eta) = o(\eta^{-2})$ as $\eta \to \infty$ and $\psi(\tau)$ is specified by the requirement that

$$u \sim t^{-1}\psi(\tau)\eta^{-2} \quad \text{as } \eta \to 0 \tag{5.3}$$

applies as a matching condition to all orders in τ; matching with (4.1) then immediately implies that

$$\psi = 8\rho. \tag{5.4}$$

At leading order we find

$$g(\eta) = \eta^{-2}e^{-\eta^2/4},$$

while the first correction term satisfies

$$\eta^{-2}e^{-\eta^2/4} - G - \frac{\eta}{2}G_\eta = G_{\eta\eta} + \frac{3}{\eta}G_\eta$$

and hence

$$-\frac{1}{2}\big(E_1(\eta^2/4) + \eta^2 G\big) = \eta G_\eta + 2G,$$

where

$$E_1(\zeta) := \int_\zeta^\infty \vartheta^{-1}e^{-\vartheta}\mathrm{d}\vartheta$$

is the exponential integral. Thus

$$G(\eta) \sim \frac{1}{2}\ln\eta \quad \text{as } \eta \to 0 \tag{5.5}$$

and matching to the second term in (5.1) (wherein the $\ln\rho$ contribution is negligible) implies

$$4\dot{\rho}\ln r \sim -\frac{1}{4}\psi + \frac{\dot{\psi}}{2}\ln(r/\sqrt{t}), \tag{5.6}$$

the first term on the right-hand side coming from $g(\eta)$ and the second from (5.5). In view of (5.4), the $\ln r$ terms in (5.6) automatically match and

$$\tau\dot{\rho} \sim -\rho,$$

so that

$$\rho \sim \frac{1}{K\ln t}, \qquad u_{max} \sim K\ln t$$

(representing extremely slow grow up) for some constant K that depends on the initial data. Tracking back through the above analysis, we may observe as an aside that the matching condition (5.1) is equivalent to

$$r^3 u_r \sim -16\rho, \qquad r u_r + 2u \sim 4\rho' \ln t \quad \text{at } r = 0$$

i.e. to (3.2) being subject to the mixed condition

$$r u_r + 2u \sim -\frac{1}{4}\left(r^3 u_r\right)_t \ln t \quad \text{at } r = 0,$$

the right-hand side of which is only logarithmically larger than the left for $r = O(\sqrt{t})$. We note that in the analysis above the leading-order matching condition on (3.2) can be viewed as being

$$r u_r + (N-2)u = 0 \quad \text{at } r = 0$$

for $N < 4$ and

$$\left(r^{N-1} u_r\right)_t = 0 \quad \text{at } r = 0$$

for $N > 4$, clearly illustrating the borderline role of the case $N = 4$.

For $\gamma = 2 (= (N-2))$ we can replace (5.2) by

$$u \sim t^{-1}\left(A\eta^{-2} + \psi(\tau)g(\eta) + \dot{\psi}(\tau)G(\eta)\right) \tag{5.7}$$

where g and G are as given above but (5.3) is replaced by

$$u \sim t^{-1}\left(A + \psi(\tau)\right)\eta^{-2} \quad \text{as } \eta \to 0.$$

Matching proceeds as before, the only changes following from the obvious one to (5.4), whereby

$$A + \psi = 8\rho, \qquad \tau\dot{\psi} \sim -\psi$$

so that the ψ contribution does not appear at leading order in (5.7) and

$$\rho \sim A/8, \qquad u_{max} \sim 8/A.$$

We note here that no $\ln t$ terms appear at leading order in the borderline case for $N = 4$, this being a consequence of the similarity solution $A\eta^{-2}$ having the exceptional feature that neither η^0 nor η^{N-4} terms are present as $\eta \to 0$ (the two of course coinciding for $N = 4$).

Finally, for $((N-2)/2 =) 1 < \gamma < 2$ we have

$$u \sim t^{-\frac{\gamma}{2}} g(r/\sqrt{t}), \qquad g(\eta) \sim \sigma A \quad \text{as } \eta \to 0, \qquad g(\eta) \sim A\eta^{-\gamma} \quad \text{as } \eta \to \infty,$$

and matching with (5.1) gives

$$4\ln(\sqrt{t}/\rho)\rho' \sim \sigma A t^{-\gamma/2}$$

so that, roughly speaking, $\rho \sim t^{1-\gamma/2}$ and hence

$$\rho' \sim \frac{\sigma A t^{-\gamma/2}}{2(\gamma-1)\ln t}$$

so that, more precisely,

$$\rho \sim \frac{\sigma A t^{(2-\gamma)/2}}{(\gamma-1)(2-\gamma)\ln t}, \qquad u_{max} \sim \frac{1}{\rho},$$

representing slow decay. As a final check, in (5.1) we have a balance

$$\xi^{-2} \approx t^{1-\gamma}/\ln t$$

so

$$r \approx \sqrt{t/\ln t}$$

which indeed satisfies $r \ll \sqrt{t}$, but 'only just'.

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