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LIQUIDITY GENERATED BY HETEROGENEOUS BELIEFS AND COSTLY ESTIMATIONS

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Abstract. We study the liquidity, defined as the size of the trading volume, in a situation where an infinite number of agents with heterogeneous beliefs reach a trade-off between the cost of a precise estimation (variable depending on the agent) and the expected wealth from trading. The "true" asset price is not known and the market price is set at a level that clears the market. We show that, under some technical assumptions, the model has natural properties such as monotony of supply and demand functions with respect to the price, existence of an equilibrium and monotony with respect to the marginal cost of information. We also situate our approach within the Mean Field Games (MFG) framework of Lions and Lasry which allows to obtain an interpretation as a limit of Nash equilibrium for an infinite number of agents.

1. Introduction.

Liquidity risk has been well illustrated by the worldwide financial crisis that started in 2007 (initially centered around "subprime" lending and then extended to the financial sphere). The models used to price financial products did not take this risk into account and many well known institutions faced substantial loss (some leading to default).

More specifically, when one wants to measure the asset liquidity, several concepts have been discussed:

- the bid-ask spread, which takes into account the difference between the price at which a security can be bought and sold based on real quotes available on the market. This notion is useful for operational purposes but is sometimes too short-sighted;
- market depth: Hachmeister [9] defines the market depth as the amount of a security that can be bought and sold at various bid-ask spreads.
- immediacy: it indicates the time needed to successfully trade a certain amount of an asset at a prescribed cost.
- resilience: Hachmeister refers to this as the speed at which prices return to former levels after a shock (e.g. a large transaction, etc.); this measure requires a time window.

Several modeling approaches have been proposed, e.g., in [3] where the authors study the optimal submission strategies of bid and ask orders in a limit order book.

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They consider an agent optimizing his utility with a finite and infinite horizon and obtain results such as optimal bid/ask spread etc.

Other authors consider not one but several (types of) agents that hold non-identical estimations (also called heterogeneous beliefs) on the future price of the asset: Gallmeyer and Hollifield ([5]) study the effects of a market-wide short-sale constraint in a dynamic economy with heterogeneous beliefs and analyze the impact on the stock price as generated by the optimistic investors' intertemporal elasticity of substitution. In [19] Emilio Osambela presents a dynamic general-equilibrium economy in which one population of optimistic investors is subject to endogenous liquidity constraints. On the other hand the importance of heterogeneous beliefs on asset pricing has been recognized widely in works by e.g., Jouini et al. [11, 10].

In all these situations the typologies of the agents are intrinsically finite as the authors are not interested in what happens when an infinite number of different agents are present. Here we will suppose that an infinite number of agents are acting on the market, each having his own procedure to obtain an estimation of the "true" price of some security. We take the paradigm of heterogeneous beliefs i.e. we suppose that all the agents receive the same (costly, see latter) information but they differ in the way to interpret it, more precisely, in the way to obtain an estimation out of it. The estimation is obtained in the form of a random variable with a known mean and variance; the agent cannot change the result obtained by his procedure; the particularity of our approach is that he can diminish the variance by paying a price. Each agent optimizes a utility functional. Also, contrary to some previous works, we are not interested in the dynamics of the price itself (that we will suppose constant to simplify); instead, our focus is on the trading volume (i.e., how many units are traded at the market price) that we will consider as a proxy for the liquidity. Such a substitute for liquidity is relevant to our setting which is a one period game with no dynamics.

Considering an infinite number of optimizing agents is not technically trivial and we resort to the "Mean Field Games" approach pioneered by Lasry and Lions [16, 14, 15, 17] where a Nash equilibrium with an infinite number of agents is analyzed. Mathematical properties for special cases of functionals (e.g., quadratic) and examples of applications and numerical approaches are the object of several works: in [2] the authors present a finite difference discretization in a finite and infinite time horizon and prove approximation properties, existence and uniqueness, bounds on the solutions; they also introduce a Newton method for the coupled direct-adjoint critical point equations for the finite horizon problem in a convex setting. In [7] the author studies a prototypical case and its stability properties. In [12] the authors present a numerical method and apply it to a technological transition; another MFG model is given in [13]. In [6], MFG are stated in a finite state space. Finally, the so-called "planning problem" i.e., where the final density of agents is prescribed, is treated in [1].

Our analysis here has to take into account a dimension which is particular to this setting: the "mean field" that couples the actions of all agents appears as an equilibrium constraint. Although we only treat a particular situation in this paper, we expect that the MFG approach can be coupled with constraints on the density of agents and refer to future work for technical details.

The summary of the paper is the following: in Section 2 we explain the basic properties of the model and focus on the specific investigation of this work which is the relationship between estimation cost and the trading volume. In Section 3 we situate our approach within the MFG model. Finally in Section 4 we prove the main properties of the model (the monotonicity of the supply and demand with respect to the price, existence of an equilibrium, anti-monotony with respect to precision cost, etc.) and give some illustrative examples.

2. **The liquidity model.** Let us consider a traded security of "true" value V. The true value is unknown to the market participants and will never be revealed. Instead, each agent x constructs his own estimation for V in the form of $V\tilde{A}^x$ where $\tilde{A}^x$ is a random variable; we will consider, for simplicity, that $V\tilde{A}^x$ is normal, that $\tilde{A}^x$ and $\tilde{A}^y$ are independent as soon as $x \neq y$ and that the mean of $V\tilde{A}^x$ is VA^x and variance of $V\tilde{A}^x$ is $V^2(\sigma^x)^2$ and are known to the agent x. It turns out that for technical reasons it is better to work with "precision" instead of the variance i.e. we introduce $B^x = 1/(\sigma^x)^2$.

We do not explain how the agents construct their estimation $V\tilde{A}^x$ but will suppose that each agent has his own (deterministic) procedure that is specific to himself and fixed in advance; the agent cannot influence in any way the average A^x during the process (but the mean can depend on time); in particular two different agents may (and will in practice) have different estimations (and average estimations A^x). This is not a collateral property of the model but the mere reason for which the agents trade: they trade because they have heterogeneous expectations about the final value of the security.

The only thing that the agent can do is to extract as much precision as possible from his procedure i.e., he can change B^x. However, improving the precision comes at a cost i.e., the agent has to pay $f(b)$ to attain precision b. The precision cost function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is defined on positive numbers.

There are many arguments to support such a model involving a cost for a given precision; in order to construct his estimation the agent may have to pay fees corresponding to his information sources (newspapers, databases, data streams ...), to pay the research staff, invest in computing infrastructures etc.

Based on his estimations, the agent decides to trade θ^x units; thus the size of the position of the agent on the market is $Vp \cdot \theta^x$ (where Vp is the market price); note that $\theta^x > 0$ means that the agent is long (buys) and $\theta^x < 0$ means that the agent is short (sells) the asset.

Thus, each agent is characterized by three quantities: his mean estimate VA^x, the precision B^x of the estimate and the number θ^x of units traded ; denote $X = (A, \theta, B)^T$ (here ^{T} denotes the vectorial transposition); we set the investment horizon of all the agents to be the final time $T = 1$.

Remark 1. The "time" here can be physical "wall-clock" time or "eductive time" (cf. [8] i.e., a the mental time required by the agents to reach a decision).

We denote by $m(t, X)$ the distribution of the agents (a probability measure) at time t with $m(0, X) = m_0(X)$. We will also denote by $\mathbb{E}^t$ the average with respect to the measure $m(t, X)$.

Let us denote by $\rho(t, A)$ the marginal of $m(t, X)$ with respect to the variables θ and B at time t and $\rho_0(A) = \rho(0, A)$. Note that θ, B can (and will) depend on time. However the evolution of A^x is autonomous i.e. not related to B and θ but imposed by the estimation procedure chosen by the agent once for all at the beginning. It is not subject to any choice or control between the initial and final time. Thus, even when the average estimation of each agent may depend on time, it is natural to consider an "ergodic" setting where the distribution $\rho(t, A)$, depending only on the autonomous evolution of A^x for each x, is stationary i.e., for all $t \in [0, T]$: $\rho(t, A) = \rho_0(A)$. In particular this is true when A^x does not depend on time. We introduce the average with respect to ρ_0 which will be denoted $\mathbb{E}^A$.

From a theoretical point of view it is interesting to consider the situation when the mean $\mathbb{E}^A(A) = 1$ which means that the average estimate is V i.e., in average, the agents are neither overpricing nor underpricing the security with respect to its (unknown) true value. We will see, however, that this is not necessarily indicating that the market price will be V.

In order to explain how the market price is set, we introduce the basic notions of total supply (and demand) for a price $Vp \geq 0$. The total demand, denoted $D(p)$, and total supply (also called "total offer"), denoted $O(p)$, are defined as:

$$D(p) = \mathbb{E}^T(\theta_+), \quad O(p) = \mathbb{E}^T(\theta_-). \tag{1}$$

A price p^* such that $O(p^*) = D(p^*)$ will be said to clear the market. Indeed, from the definitions of $D(\cdot)$ and $O(\cdot)$, this is equivalent to $\mathbb{E}^T(\theta) = 0$ i.e., at the price p^* the overall (signed) demand is null. Note that such a price may not exist or may not be unique, cf. Remarks 2, 3 and Figures 1,2 below.

The transaction volume at some price p is defined as the number of units that can be exchanged at that price:

$$TV(p) = \min\{O(p), D(p)\}. \tag{2}$$

A price p^* where $TV(\cdot)$ attains its maximum is of interest because it will maximize the total number of units exchanged. Note that such a price may not exist, cf. Remark 3 and Fig. 2 below. Even when it exists it may be not unique.

An elementary but important result gives information on the market price and its properties:

Theorem 2.1. *If*
- $O(p), D(p)$ are continuous,
- $O(p)$ is strictly increasing, $O(0) = 0$, $\lim_{p\to\infty} O(p) > 0$,
- $D(p)$ is strictly decreasing, $D(0) > 0$, $\lim_{p\to\infty} D(p) = 0$,
then
1/ a unique p_1^* exists such that $O(p_1^*) = D(p_1^*)$;
2/ a unique p_2^* exists such that $TV(p_2^*) \geq TV(p)$ for all $p \geq 0$;
3/ $p_1^* = p_2^*$.

Proof. For 1/ let us note that the continuous, strictly monotone function $D - O$ is such that in zero its value is strictly positive and at infinity is strictly negative. Thus there exists a unique p_1^* where the function vanishes; this gives the conclusion. We note that $(D - O)(p)$ is strictly positive for $p < p_1^*$ and strictly negative for $p > p_1^*$.

For 2/ note that

$$TV(p) = \begin{cases} O(p) \text{ for } p < p_1^* \\ D(p_1^*) = O(p_1^*) \text{ for } p = p_1^* \\ D(p) \text{ for } p > p_1^* \end{cases} \tag{3}$$

Then $TV(p_1^*) - TV(p) = O(p_1^*) - O(p)$ for $p \leq p_1^*$ and $D(p_1^*) - D(p)$ for $p \geq p_1^*$. In all situations $TV(p_1^*) - TV(p)$ is positive hence 2/ and 3/. □

Remark 2. Condition $D(0) = 0$ and similar are technical. But monotonicity is important for the equivalence between the two interpretations of the market price: the market price is the price that maximizes the transaction volume and the market price is also the price matching supply and demand. Consider for instance functions $p + \sin(\pi p)$ and $1/p$ (cf. Fig. 1): there are three points that clear the market and none maximizes the trading volume. Such a situation is ambiguous and we want to avoid it.

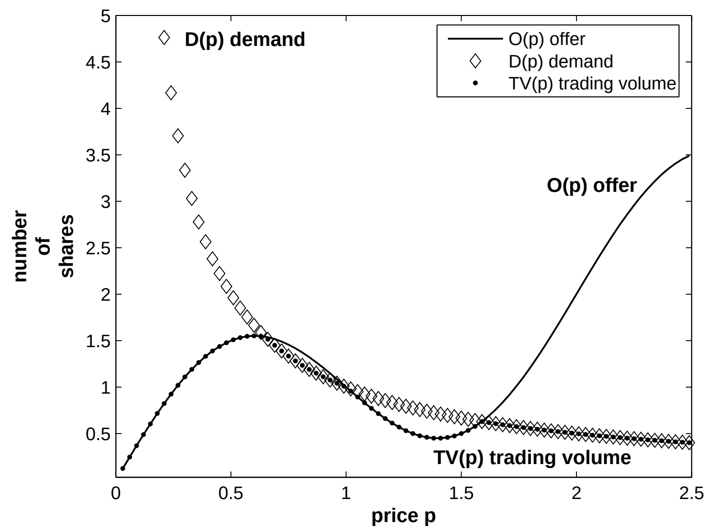

FIGURE 1. An illustration of Remark 2. Here $V = 1$, $O(p) = p + \sin(\pi p)$ and $D(p) = 1/p$ but $O(p)$ is not monotonic. Several prices exist that clear the market. The first price, situated at about $p = 0.684$, maximizes the trading volume among the points that clear the market (with value around 1.541) but it does not maximize the trading volume $TV(p)$ whose maximum value is around 1.551.

Remark 3. Continuity is also a crucial ingredient to the equality $p_1^* = p_2^*$; when the supply and demand functions are not continuous a price that maximizes trading volume may not exist, nor a price that clears the market. To illustrate this, take for instance $V = 1, O(p) = 2p^2$, $D(p) = \begin{cases} 4 - p \\ 1/p \text{ for } p \geq 1 \end{cases}$ (cf. Fig. 2). In this situation the supremum of transaction volumes is 2 but is not attained by any price; also, no price clears the market i.e., there does not exist any p such that $O(p) = D(p)$. We enter in this situation the topic of market microstructure; a market maker is necessary on such a market to smooth out supply and demand through a "pricing rule" or a "market making function", cf. [18] for details.

The market price at time T, denoted $V\overline{\mathcal{P}}$, balances total supply and demand i.e., the overall demand / supply balance is null; this gives an implicit equation for $\overline{\mathcal{P}}$:

$$\int \theta dm(T, A, \theta, B) = 0 \text{ or equivalently } \mathbb{E}^T(\theta) = 0. \tag{4}$$

In order to model the choices of the agents, we will consider the classical situation of an agent that is maximizing a utility function. Since the uncertainty appears as

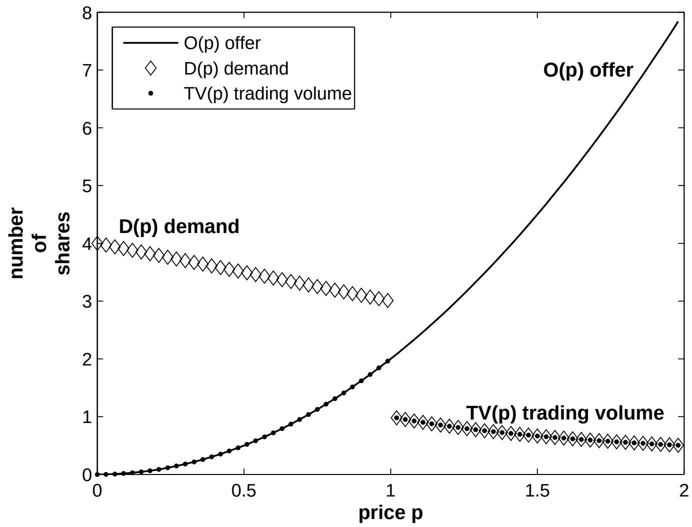

FIGURE 2. An illustration of the Remark 3: supply and demand functions are discontinuous: no price exists that clears the market; the maximum trading volume is not attained by any price.

a normal variable, we have two alternatives which coincide: either consider that the utility is a function of the mean and variance of the wealth or, equivalently, take an expected utility framework. To keep intuitive understanding, we will keep the simplest situation of a utility function $U(u,v) = u - \frac{\lambda}{2}v$ where u is the expected wealth and v its variance; the parameter $\lambda \in \mathbb{R}_+$ is called the risk aversion coefficient. Of course, this basic utility function has several known drawbacks (not a coherent risk measure etc.), but this does not play an important role here and this choice considerably simplifies the results.

Note that all agents have the same utility function.

Of course, the wealth itself is a function of θ^x, B^x; the wealth is computed under the assumption that the agent enters the transaction (buys or sells) at the market price and exits the transaction (sells or buys) at a price equal to his estimation. Thus, for a given price Vp (not necessarily the market equilibrium price $\overline{\mathcal{P}}$) the average expected wealth for agent x, denoted u^x is $u^x = V\theta^x(A^x - p) - f(B^x)$; the variance of the wealth, denoted v^x is $v^x = \frac{(\theta^x)^2 V^2}{(B^x)_+}$ (here $(B^x)_+$ is the positive part of B^x with convention that division by zero equals $+\infty$).

Thus, for a given price Vp (not necessarily the market equilibrium price $V\overline{\mathcal{P}}$) the utility of the agent x to be optimized is:

$$J(X^x) = V\theta^x(A^x - p) - f(B^x) - \frac{\lambda}{2}\frac{(\theta^x)^2 V^2}{(B^x)_+}. \tag{5}$$

Let us discuss some technicalities concerning the precision cost function $f(b)$ (the "research cost" to reach the precision b). Conditions for f, that seem very natural, are $f(0) = 0$, $f'(0) = 0$ (this is to fix the marginal cost at start; this is a non-trivial choice but its implications are not important for the technical results of the paper).

We will also consider that f is increasing, strictly convex (this will be seen later to ensure well-posedness), C^2 and $\lim_{x\to\infty} f(x)/x = \infty$.

Now that the model has been set, several important questions are to be addressed in order to prove that the model corresponds to the intuitive picture one may have and also to prove the mathematical well-posedness of the overall problem:

- is the solution unique i.e., does there exist a unique $\overline{\mathcal{P}}$ that solves the equilibrium equation (4);

- are the total demand $D(p)$ and total supply $O(p)$ monotonic functions of p (in order to be within the framework of Thm. 2.1)?

Note that $\overline{\mathcal{P}}$ (given by Thm. 2.1) is not necessarily equal to 1 even if $\mathbb{E}^0(A) = 1$.

3. **Comparison and interpretation as Mean Field Games (MFG).** The Mean Field Games framework (MFG) is a mathematical model describing the interaction among a large number of agents / players. An agent can control his situation, based on a set of preferences and by acting on some parameters. MFG can show the emergence of a collective behavior (fashion trends, financial crises, real estates valuation, etc.) out of individual optimizations performed by each agent: while an agent by himself cannot influence the collective behavior (his decisions have negligible impact on the collective parameters and as such he only optimizes his own situation given the environmental situation) the collective choices of all agents create an overall environment (the "mean field") that affects in return the individual decisions.

We refer to [16, 14, 15, 17] for further information. The MFG theory shows that a Nash equilibrium for a game of N players will tend, in some specified sense, when $N \to \infty$, to the so-called MFG equations.

Let X_t^x be the characteristics at time t of a agent/ player starting in x at time 0. It evolves with SDE:

$$dX_t^x = \alpha(t, X_t^x)dt + \sigma dW_t^x,\ X_0^x = x \tag{6}$$

where $\alpha(t, X_t^x)$ is the control that can be chosen by the agent/ player.

Note that each agent has his own randomness modeled with an independent Brownian. Denote by $m(t,x)$ the density of players at time t and position $x \in E$ with E being the state space. The optimization problem of the agent is: for a (fixed) finite horizon T, optimize:

$$\inf_\alpha \mathbb{E}\left\{\int_0^T L(X_t^x, \alpha(t, X_t^x)) + V(X_t^x; m(t,\cdot))dt + V_0(X_T^x; m(T,\cdot))\right\}. \tag{7}$$

The operator L encodes constraints or costs on the control while V and V_0 encode the goal. Define $H(x,\xi) = \sup_\alpha \langle \xi, \alpha\rangle - L(x,\alpha)$; $\nu = \sigma^2/2$.

For a finite number of agents (i.e., when $m(t,x)$ is a sum of N Dirac masses) critical point equations can be written that describe a Nash equilibrium; these equations converge (up to sub-sequences) to solutions of the following MFG system for $N \to \infty$:

$$\partial_t m + \operatorname{div}(\alpha m) - \nu \Delta m = 0, \tag{8}$$

$$m(0,x) = m_0(x), \int m = 1, \; m \geq 0 \tag{9}$$

$$\alpha = -\frac{\partial}{\partial p} H(x, \nabla u) \tag{10}$$

$$\partial_t u + \nu \Delta u - H(x, \nabla u) + V(x,m) = 0, \tag{11}$$

$$u(T,x) = V_0(x, m(T,\cdot)), \int u = 0. \tag{12}$$

To model the situation in Section 2 the evolution equations and the initial probability distribution will be:

$$dX_t^x = d \begin{pmatrix} A_t^x \\ \theta_t^x \\ B_t^x \end{pmatrix} = \begin{pmatrix} \alpha(t, A_t^x) \\ \alpha_\theta(t, X_t^x) \\ \alpha_B(t, X_t^x) \end{pmatrix} dt + \begin{pmatrix} \sigma_A(t, A_t^x) dW_t^A \\ \sigma_\theta(t, X_t^x) dW_t^\theta \\ \sigma_B(t, X_t^x) dW_t^B \end{pmatrix} \tag{13}$$

$$m(t,X)\Big|_{t=0} = m_0(X). \tag{14}$$

We will take operators L and V to be null. Recall that we supposed that autonomous evolution of A^x is defining a stationary distribution ρ_0. To simplify even more the setting we can take A^x to be constant and θ^x and B^x to have a deterministic evolution.

$$d \begin{pmatrix} A_t^x \\ \theta_t^x \\ B_t^x \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_\theta(t, X_t^x) \\ \alpha_B(t, X_t^x) \end{pmatrix} dt \tag{15}$$

$$m(t,X)\Big|_{t=0} = m_0(X). \tag{16}$$

To this we add the equilibrium condition (4). This framework allows to expect an interpretation of our setting: a Nash equilibrium for an infinite number of players. Note that we do not explicitly show the relationship between the Nash equilibrium of N agents and the results in the next section corresponding to an infinite number of agents.

4. Theoretical results.

4.1. Existence of an equilibrium.

Theorem 4.1. *Suppose that the precision cost function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $f(0) = 0$, $f'(0) = 0$. Suppose also that f is increasing, strictly convex, of C^2 class and $\lim_{x\to\infty} f(x)/x = \infty$. Then:*

- *the optimal precision cost B^x and trading size θ^x are*

$$B^x = (f')^{-1}\left(\frac{(A^x - p)^2}{2\lambda}\right); \tag{17}$$

$$\theta^x = \frac{(A^x - p)B^x}{\lambda V} = \frac{(A^x - p)}{\lambda V}(f')^{-1}\left(\frac{(A^x - p)^2}{2\lambda}\right). \tag{18}$$

  In particular both are explicit functions of A^x.
- *supply $O(p)$ and demand $D(p)$ are strictly monotone with respect to p.*

- *an equilibrium price $\overline{\mathcal{P}}$ that clears the market (eqn.(4)) exists and is unique:*

$$\overline{\mathcal{P}} = \frac{\mathbb{E}^A(AB)}{\mathbb{E}^A(B)}. \tag{19}$$

Proof. An agent only sees the others through the market price $V\overline{\mathcal{P}}$. If we consider now a (possibly non-equilibrium) price Vp as given, then the agent optimizes the functional $J(X^x(T))$ which only depends on the final state $X^x(T)$ and not on the controls. Since the controls allow to obtain each possible configuration for $X^x(T)$ (A^x is given and fixed), then the values $B^x(T)$ and $\theta^x(T)$ will correspond to an optimum of the function:

$$\mathcal{J} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} : \ \mathcal{J}(y,z) = Vy(A^x - p) - f(z) - \frac{\lambda}{2}\frac{y^2V^2}{z}. \tag{20}$$

Let us denote y^*, z^* an optimum candidate. Asking that $\frac{\partial \mathcal{J}}{\partial y} = 0$ one obtains $y^* = \frac{(A^x-p)z^*}{\lambda V}$; then z^* optimizes the function $\frac{(A^x-p)^2}{2\lambda}z - f(z)$. It is straightforward to see that under hypothesis taken on f optimal points indeed exist and satisfy (17)-(18).

In order to prove the strict monotonicity of the supply and demand functions with respect to p, it is enough to prove that e.g. $(\theta^x)_+$ is monotone with respect to p (similar arguments hold for $(\theta^x)_-$). This is a consequence of the fact that $(A^x - p)_+$ is (strictly) monotone with respect to $p < A^x$ and in the same domain $(f')^{-1}\left(\frac{(A^x-p)^2}{2\lambda}\right)$ is also monotone because of the assumptions on f, namely convexity, regularity and $f(0) = 0 = f'(0)$.

The monotonicity, by Thm 2.1, implies that a unique price that clears the market exists and this price also maximizes the trading volume.

Note that θ^x is a function of A^x, that can be written $\theta^x = \theta(A^x)$, same for $B^x = B(A^x)$. Thus, equation (4) can be written $\mathbb{E}^T(\theta) = 0$ and also $\mathbb{E}^A\left(\frac{(A-p)B(A)}{\lambda V}\right) = 0$ which gives the conclusion. □

Remark 4. Assumptions on f can be weakened (cf. [4]).

In general, the price $V\overline{\mathcal{P}}$ depends on the cost function $f(\cdot)$. But for the particular case where ρ_0 is symmetric the following result proves independence:

Corollary 1. *Under assumptions in Thm. 4.1 on function f if there exists $p^1 > 0$ such that*

$$\forall \alpha \in \mathbb{R} : \ \rho_0(p^1 - \alpha) = \rho_0(p^1 + \alpha) \tag{21}$$

(with the convention that ρ_0 is null on $\mathbb{R}_-$) then $\overline{\mathcal{P}} = p^1$ and in particular $\overline{\mathcal{P}}$ is independent of f.

Proof. Of course (21) is equivalent to say that ρ_0 is symmetric around p^1. As a side remark note that its support will be contained in $[0, 2p^1]$. The proof builds on the remark that the function $B(A)$ is symmetric around p^1 thus $\theta(A)$ is anti-symmetric. Since the distribution ρ_0 is symmetric then for $p = p^1$ we have $\mathbb{E}^A(\theta(A)) = 0$; this implies, by uniqueness, that $\overline{\mathcal{P}} = p^1$. □

Remark 5. Analog results hold for more general utility functions U (cf. [20]).

The relative market price $\overline{\mathcal{P}}$ is solution to the equation:

$$\mathbb{E}^A\left[(A - \overline{\mathcal{P}})(f')^{-1}\left(\frac{(A-\overline{\mathcal{P}})^2}{2\lambda}\right)\right] = 0. \tag{22}$$

We denote by TV_f the equilibrium trading volume for precision cost function f; it satisfies the relation:

$$TV_f = \frac{1}{\lambda V}\mathbb{E}^A\left[(A-\overline{\mathcal{P}})_+ (f')^{-1}\left(\frac{(A-\overline{\mathcal{P}})^2}{2\lambda}\right)\right]. \tag{23}$$

4.2. **Application for a power function.** Let us take a particular case $f(b) = \mu\frac{b^\alpha}{\alpha}$ with $\alpha > 1$, $\mu > 0$. Then we have

$$B(A) = \left(\frac{(A-\overline{\mathcal{P}})^2}{2\lambda\mu}\right)^{\frac{1}{\alpha-1}}, \tag{24}$$

and $\overline{\mathcal{P}}$ satisfies:

$$\mathbb{E}^A(A-\overline{\mathcal{P}})|A-\overline{\mathcal{P}}|^{\frac{2}{\alpha-1}} = 0, \tag{25}$$

and

$$TV_f = \frac{1}{\lambda(2\mu\lambda)^{\frac{1}{\alpha-1}}}\mathbb{E}^A(A-\overline{\mathcal{P}})_+|A-\overline{\mathcal{P}}|^{\frac{2}{\alpha-1}}. \tag{26}$$

Remark 6. We note that the trading volume is inversely correlated with the risk aversion coefficient λ which means more risk averse agents are, less they trade. The same holds for the "cost of precision" μ: more expensive the information is, less transactions the market has; this behavior is consistent with a liquidity crisis where a sudden increase in the cost of precision can limit the market liquidity.

It is also interesting to compute the (optimal) expected wealth for an agent having average estimation A; this wealth is:

$$\left(\frac{\alpha-1}{\lambda\alpha(2\mu\lambda)^{\frac{1}{\alpha-1}}}\right)|A-\overline{\mathcal{P}}|^{\frac{2\alpha}{\alpha-1}}. \tag{27}$$

Remark 7. Note that for $\alpha > 1$ the wealth is finite and strictly positive. For $\alpha = 1$ the formula is not valid and the wealth is infinity.

If A tends to infinity, then the expected wealth also tends to infinity which means that larger A is more the agent expects to win. In order for the distribution to have finite moments the density of A has to decrease when A becomes large; then the probability to be in this situation is small which means that large wealths are only expected by a negligible amount of agents involved. Of course the real wealth of each agent is zero because the price does not change in our model.

The total expected wealth of the entire market is finite as soon as the distribution $\rho_0(A)$ has moments of order $\frac{2\alpha}{\alpha-1}$ i.e.

$$\mathbb{E}^A\left(\frac{\alpha-1}{\lambda\alpha(2\mu\lambda)^{\frac{1}{\alpha-1}}}\right)|A-\overline{\mathcal{P}}|^{\frac{2\alpha}{\alpha-1}} < \infty. \tag{28}$$

For the particular case $\alpha = 2$, we obtain (after simplifications) the equation for the relative market price:

$$\mathbb{E}^A(A-\overline{\mathcal{P}})^3 = 0, \tag{29}$$

which tells us that if the third central moment of the distribution $\rho_0(A)$ is null then $\overline{\mathcal{P}} = 1$ and thus the price is exactly the true price V. The formula is interesting in itself and also because it shows that the mere condition $\mathbb{E}^A(A) = 1$ is not enough to insure that the market will trade at the "true" price V.

Other information cost functions can be proposed e.g., exponential function $f(b) = \mu\left(e^{\xi b} - 1 - b\xi\right)$, $\xi \in \mathbb{R}$.

4.3. **Dependence of the trading volume on the precision cost function.** A result that addresses the properties of the trading volume in relation to f is the following:

Theorem 4.2 (anti-monotony of the trading volume)**.** *Let f, g be two precision cost functions fulfilling the hypothesis of the Thm. 4.1. Also suppose that $g'(b) \geq f'(b)$ for any $b \in \mathbb{R}_+$. Denote by TV_f and TV_g the equilibrium trading volumes for precision cost functions f and g respectively. Then $TV_f \geq TV_g$.*

Proof. Let us recall that if a function is monotone its inverse (when it exists) is also monotone and of the same type of monotonicity. Since f and g are convex, it follows that f' and g' are monotone increasing.

Denote $F = (f')^{-1}$ and $G = (g')^{-1}$; using the hypothesis we obtain from $g' > f'$ that $F \geq G$ with both F and G being increasing functions. For any precision cost function h the demand at price p denoted $D(h,p)$ is given by the formula

$$D(h,p) = \frac{1}{V\lambda}\mathbb{E}^A\left[(A-p)_+(h')^{-1}\left(\frac{(A-p)^2}{2\lambda}\right)\right], \tag{30}$$

and symmetrically the supply at price p

$$O(h,p) = \frac{1}{V\lambda}\mathbb{E}^A\left[(A-p)_-(h')^{-1}\left(\frac{(A-p)^2}{2\lambda}\right)\right]. \tag{31}$$

Note that $D(h,p)$ is a decreasing function of p and $O(h,p)$ is increasing. Recall that the equilibrium price $\overline{\mathcal{P}}_f$ balances supply and demand i.e., satisfies:

$$D(f,\overline{\mathcal{P}}_f) = O(f,\overline{\mathcal{P}}_f). \tag{32}$$

We also have a similar equation for g. Since $F \geq G$, one obtains that for any price p: $O(g,p) \leq O(f,p)$ and also $D(g,p) \leq D(f,p)$. In particular $O(g,\overline{\mathcal{P}}_g) = D(g,\overline{\mathcal{P}}_g) \leq D(f,\overline{\mathcal{P}}_g)$. Define P_1 as the solution of the equation : $O(g,P_1) = D(f,P_1)$ (such a solution exists because $O(g,\cdot) - D(f,\cdot)$ is continuous, the value in zero is strictly negative and the value at infinity is strictly positive). Then $P_1 \geq \overline{\mathcal{P}}_g$ because $O(g,p)$ is increasing and $D(f,p)$ is decreasing. In a symmetric way one can prove that $P_1 \geq \overline{\mathcal{P}}_f$.

Then $TV_g = O(g,\overline{\mathcal{P}}_g) \leq O(g,P_1) = D(f,P_1) \leq D(f,\overline{\mathcal{P}}_f) = TV_f$, hence the conclusion. ☐

Remark 8. Note that each function will generate **its own** equilibrium market price i.e., $\overline{\mathcal{P}}_f$ may be different from $\overline{\mathcal{P}}_g$.

4.4. **Results under weaker hypotheses on the precision cost function.** The hypotheses accepted so far on the precision cost function $f(B)$ are rather strong: strictly convex of C^2 class. We relax in this section these assumptions but refer to [4] for optimal results.

Theorem 4.3. *Suppose that the precision cost function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is convex (thus continuous for $b > 0$), $f(0) = 0$ and f is continuous in 0. Also assume f to be coercive in the sense that $\liminf_{x\to\infty} f(x)/x = \infty$. Then for each given price p each agent x attains its optimum in at least a (possibly non-unique) configuration with precision B^x and order volume θ^x; moreover θ^x is monotone (decreasing) with respect to p. Finally, the overall demand and supply functions $D(p)$ and $O(p)$ are also monotone with respect to p.*

Proof. As in the proof of Thm 4.1 we denote by $y^*(p), z^*(p)$ an optimum candidate where we explicitly mark the dependence on p. Since the functional $\mathcal{J}$ in equation (20) is differentiable with respect to z, we obtain as before from $\frac{\partial \mathcal{J}}{\partial y} = 0$ that

$$y^*(p) = \frac{(A^x - p) z^*(p)}{\lambda V}; \tag{33}$$

moreover $z^*(p)$ optimizes the function $g_p(z) = \frac{(A^x-p)^2}{2\lambda} z - f(z)$. Since $f(0) = 0$, f is continuous and coercive the optimum exists but is not necessarily unique (because f is not necessarily differentiable neither strictly convex).

Take $A^x \leq p_1 \leq p_2$ and suppose that some choice of optimums $z^*(p_1)$ and $z^*(p_2)$ exists such that $z^*(p_1) > z^*(p_2)$. Using the optimality properties for $z^*(p_1)$ and $z^*(p_2)$, we obtain:

$$g_{p_2}(z^*(p_2)) - g_{p_1}(z^*(p_2)) \geq g_{p_2}(z^*(p_1)) - g_{p_1}(z^*(p_1)),$$

thus

$$\frac{(p_2 - A^x)^2}{2\lambda} z^*(p_2) - \frac{(p_1 - A^x)^2}{2\lambda} z^*(p_2) \geq \frac{(p_2 - A^x)^2}{2\lambda} z^*(p_1) - \frac{(p_1 - A^x)^2}{2\lambda} z^*(p_1),$$

which implies

$$z^*(p_2) \geq z^*(p_1). \tag{34}$$

which contradicts $z^*(p_1) > z^*(p_2)$. We conclude that that $z^*(p_1) \leq z^*(p_2)$.

Recall now that z^* stands for the optimal value of B^x thus we have monotonicity for B^x.

Recall also that the optimal value of θ^x is given by the formula (33); we obtain thus the monotonicity of θ^x for $p \geq A^x$. An analogous argument works on the branch $p \leq A^x$ and, since the optimal θ^x for $p = A^x$ is zero, we obtain the monotony of θ^x with respect to p. The monotony of overall supply and demand functions $D(p)$ and $O(p)$ follows. □

Remark 9. We do not claim that $O(p)$ and $D(p)$ are necessarily continuous functions nor that the monotonicity is strict. This precludes the use of Thm. 2.1. But it is obvious that convexity is better than just continuity, which means that additional properties of $D(p)$ and $O(p)$ can be proved. We refer to [4] for details.

4.5. **Further comments and perspectives.** As this model is concerned only with deriving a formula for the trading volume, the dynamics of the "true" price was not considered. Of course, it would be interesting to take this into account; also a further refinement concerns the estimation process $\tilde{A}$ and its cost that may possess stochastic dynamics; we refer to future work for some follow-ups.

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