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NEW NUMERICAL METHODS FOR MEAN FIELD GAMES WITH QUADRATIC COSTS

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Abstract. Mean field games have been introduced by J.-M. Lasry and P.-L. Lions in [13, 14, 15] as the limit case of stochastic differential games when the number of players goes to $+\infty$. In the case of quadratic costs, we present two changes of variables that allow to transform the mean field games (MFG) equations into two simpler systems of equations. The first change of variables, introduced in [11], leads to two heat equations with nonlinear source terms. The second change of variables, which is introduced for the first time in this paper, leads to two Hamilton-Jacobi-Bellman equations. Numerical methods based on these equations are presented and numerical experiments are carried out.

1. Introduction.

Mean field games equations have been introduced by J.-M. Lasry and P.-L. Lions [13, 14, 15] to describe the dynamic equilibrium of stochastic differential games involving a continuum of players. In the time-dependent case, these equations write:

$$\partial_t u + \frac{\sigma^2}{2}\Delta u + H(x, m, \nabla u) = 0$$
$$\partial_t m + \nabla \cdot \left(m \frac{\partial H}{\partial p}(x, m, \nabla u) \right) = \frac{\sigma^2}{2}\Delta m$$

with prescribed initial condition $m(0, \cdot) = m_0(\cdot) \geq 0$ and terminal/final condition $u(T, \cdot) = u_T(\cdot)$, where u and m are scalar functions defined on $[0, T] \times \Omega$ – Ω being a smooth bounded domain – satisfying Neumann boundary conditions $\frac{\partial u}{\partial \vec{n}} = \frac{\partial m}{\partial \vec{n}} = 0$ on $(0, T) \times \partial\Omega$.

In this paper, we focus on the particular case of a quadratic hamiltonian of the form $H(x, \xi, p) = \frac{p^2}{2} + f(x, \xi)$, leading to the following mean field games equations:

$$\partial_t u + \frac{\sigma^2}{2}\Delta u + \frac{1}{2}|\nabla u|^2 = -f(x, m)$$
$$\partial_t m + \nabla \cdot (m \nabla u) = \frac{\sigma^2}{2}\Delta m$$

In this special case, a change of variables have been introduced in [11] to write the mean field games equations as two coupled heat equations with similar source terms. This change of variables consists in introducing $\phi = \exp\left(\frac{u}{\sigma^2}\right)$ and $\psi = m \exp\left(-\frac{u}{\sigma^2}\right)$ and the associated system of equations $(\mathcal{S}_1)$ is:

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$$\begin{aligned}\partial_t\phi+\frac{\sigma^2}{2}\Delta\phi &= -\frac{1}{\sigma^2}f(x,\phi\psi)\phi\\ \partial_t\psi-\frac{\sigma^2}{2}\Delta\psi &= \frac{1}{\sigma^2}f(x,\phi\psi)\psi\end{aligned}$$

with:

- Boundary conditions: $\frac{\partial\phi}{\partial\vec{n}}=\frac{\partial\psi}{\partial\vec{n}}=0$ on $(0,T)\times\partial\Omega$.
- Terminal condition: $\phi(T,\cdot)=\exp\left(\frac{u_T(\cdot)}{\sigma^2}\right)$.
- Initial condition: $\psi(0,\cdot)=\frac{m_0(\cdot)}{\phi(0,\cdot)}$.

It is noteworthy that the equation for ψ has the same structure as the equation for ϕ and we can consider, when $m_0>0$, another change of variables that consists in going back to u and introducing[1] $v=-\sigma^2\log(\psi)=u-\sigma^2\log(m)$. Then, the system of equations $(\mathcal{S}_2)$ associated to (u,v) is:

$$\partial_t u+\frac{\sigma^2}{2}\Delta u+\frac{1}{2}|\nabla u|^2=-f\left(x,e^{\frac{u-v}{\sigma^2}}\right)$$
$$\partial_t v-\frac{\sigma^2}{2}\Delta v+\frac{1}{2}|\nabla v|^2=-f\left(x,e^{\frac{u-v}{\sigma^2}}\right)$$

with:

- Boundary conditions: $\frac{\partial u}{\partial\vec{n}}=\frac{\partial v}{\partial\vec{n}}=0$ on $(0,T)\times\partial\Omega$.
- Terminal condition: $u(T,\cdot)=u_T(\cdot)$.
- Initial condition: $v(0,\cdot)=u(0,\cdot)-\sigma^2\log(m_0(\cdot))$.

Focusing on these two systems of equations, we present numerical methods to solve mean field games. These methods are completely different from the methods proposed in [1], [2] or [12] and exhibit a monotonic behavior in the approximation of (ϕ,ψ) or (u,v).

In section 2, we recall the useful results on the couple (ϕ,ψ), in particular a constructive scheme for (ϕ,ψ) introduced by J.-M. Lasry and P.-L. Lions in [16] and then studied in [7, 8]. Similar results are then deduced for the couple (u,v) in the case $m_0>0$. Section 3 introduces a numerical scheme to approximate the couple (ϕ,ψ) and presents the theoretical results: convergence of the method and monotonic behavior of the approximations for both ϕ and ψ. Then, we introduce a numerical scheme to approximate the couple (u,v) and prove similarly the associated theoretical results. Finally, section 4 presents the numerical experiments we carried out and compares the two approaches.

2. **Constructive schemes.** Throughout the paper, we suppose that m_0 and u_T are two bounded functions on Ω with $m_0\geq 0$. We suppose that $(x,\xi)\mapsto f(x,\xi)$ is a decreasing[2] function of its second variable, continuous in that variable and uniformly bounded. Moreover, to simplify the exposition[3] we suppose that $f\leq 0$.

[1] We will see that $m>0$ as soon as $m_0>0$.

[2] It is noteworthy that this monotonicity hypothesis is the key ingredient in the usual proof of uniqueness for the mean field games equations (see [15]).

[3] In terms of the initial mean field game problem, the optimal control ∇u and the subsequent distribution m are not changed if we subtract $\|f\|_\infty$ from f.

We start with a straightforward generalization[4] of a theorem established in [8]:

Theorem 2.1. *Let us introduce*

$$\mathcal{P} = \left\{ g \in L^2(0,T,H^1(\Omega))/\partial_t g \in L^2(0,T,H^{-1}(\Omega)) \right\}.$$

Let us consider $\psi^0 \in \mathcal{P}$.

There exists a unique couple of sequences $((\phi^{n+\frac{1}{2}})_n, (\psi^n)_n)$ *of* $\mathcal{P}$ *satisfying (in the weak sense):*

$$\begin{aligned} \partial_t \phi^{n+\frac{1}{2}} + \frac{\sigma^2}{2}\Delta\phi^{n+\frac{1}{2}} &= -\frac{1}{\sigma^2} f(x, \phi^{n+\frac{1}{2}}\psi^n)\phi^{n+\frac{1}{2}} \\ \partial_t \psi^{n+1} - \frac{\sigma^2}{2}\Delta\psi^{n+1} &= \frac{1}{\sigma^2} f(x, \phi^{n+\frac{1}{2}}\psi^{n+1})\psi^{n+1} \end{aligned}$$

with $\phi^{n+\frac{1}{2}}(T,\cdot) = \exp\left(\frac{u_T(\cdot)}{\sigma^2}\right)$, $\psi^{n+1}(0,\cdot) = \frac{m_0(\cdot)}{\phi^{n+\frac{1}{2}}(0,\cdot)}$ *and the Neumann boundary conditions* $\frac{\partial \phi^{n+\frac{1}{2}}}{\partial \vec{n}} = \frac{\partial \psi^{n+1}}{\partial \vec{n}} = 0$ *on* $(0,T) \times \partial\Omega$.

If $\psi^0 = 0$, *these sequences verify:*

- $(\psi^n)_n$ *and* $(\phi^{n+\frac{1}{2}})_n$ *are uniformly bounded in* $L^\infty((0,T)\times\Omega)$.
- $(\phi^{n+\frac{1}{2}})_n$ *is a decreasing sequence of functions, uniformly bounded from below by a constant*[5] $\epsilon > 0$.
- $(\psi^n)_n$ *is an increasing sequence of nonnegative functions.*
- $(\phi^{n+\frac{1}{2}}, \psi^n)_n$ *converges almost everywhere on* $(0,T)\times\Omega$, *and in* $L^2((0,T)\times\Omega)$, *towards a couple* (ϕ,ψ) *weak solution of* $(\mathcal{S}_1)$.

In the special case $m_0 > 0$, this theorem can be completed by the following proposition that provides a uniform positive lower bound for the sequence $(\psi^n)_{n\geq 1}$.

Proposition 2.2. *Let us suppose that* $m_0 > 0$.

Let us introduce $\epsilon' = \inf m_0 \times \exp\left(-\frac{1}{\sigma^2}\left(\|u_T\|_\infty + \|f\|_\infty T\right)\right) > 0$.

Then the sequence $(\psi^n)_{n\geq 1}$ *consists of functions uniformly bounded from below by* ϵ', *independently of the choice of* ψ^0.

Proof. We easily see that the function $(t,x) \mapsto \frac{\inf m_0}{\|\phi^{n+\frac{1}{2}}\|_\infty} \exp\left(-\frac{1}{\sigma^2}\|f\|_\infty t\right)$ is a subsolution of the equation:

$$\partial_t \psi^{n+1} - \frac{\sigma^2}{2}\Delta\psi^{n+1} = \frac{1}{\sigma^2} f(x, \phi^{n+\frac{1}{2}}\psi^{n+1})\psi^{n+1}$$

with $\psi^{n+1}(0,\cdot) = \frac{m_0(\cdot)}{\phi^{n+\frac{1}{2}}(0,\cdot)}$ and $\frac{\partial \psi^{n+1}}{\partial \vec{n}} = 0$ on $(0,T)\times\partial\Omega$.

Using a comparison principle on the equation for $\phi^{n+\frac{1}{2}}$, because $y \geq 0 \mapsto f(x, \phi^{n+\frac{1}{2}} y) y$ is decreasing, we see that $\|\phi^{n+\frac{1}{2}}\|_\infty \leq e^{\frac{\|u_T\|_\infty}{\sigma^2}}$.

Now, using Theorem 2.1 and a comparison principle on the equation for ψ^{n+1}, we have that:

$$\psi^{n+1} \geq \frac{\inf m_0}{\|\phi^{n+\frac{1}{2}}\|_\infty} \exp\left(-\frac{1}{\sigma^2}\|f\|_\infty T\right) \geq \frac{\inf m_0}{e^{\frac{\|u_T\|_\infty}{\sigma^2}}} \exp\left(-\frac{1}{\sigma^2}\|f\|_\infty T\right).$$

□

[4]The generalization is due to the hypothesis $m_0 \in L^\infty(\Omega)$ that provides new bounds compared to the initial case $m_0 \in L^2(\Omega)$.

[5]One can take $\epsilon = \exp\left(-\frac{1}{\sigma^2}\left(\|u_T\|_\infty + \|f\|_\infty T\right)\right)$.

The direct consequence of Proposition 2.2 is that $m_0 > 0 \implies m > 0$. Another natural corollary is that we can replace the initial function $\psi^0 = 0$ by any $\psi^0 \leq \epsilon'$ without any change in the qualitative properties of the sequences stated in Theorem 2.1[6]. In particular, we now consider $(\psi^n)_n$ and $(\phi^{n+\frac{1}{2}})_n$ for $\psi^0 = \exp\left(-\|\log(m_0)\|_\infty - \frac{1}{\sigma^2}\left(\|u_T\|_\infty + 2\|f\|_\infty T\right)\right) \leq \epsilon'$. Then, the uniform lower bound we obtained for the sequence $(\psi^n)_{n\geq 1}$ legitimates the introduction of two sequences $(u^{n+\frac{1}{2}})_n$ and $(v^n)_n$ defined by:

$$v^0 = -\sigma^2 \log(\psi^0) = \|u_T\|_\infty + \sigma^2\|\log(m_0)\|_\infty + 2\|f\|_\infty T$$

$$\forall n \geq 0, \quad u^{n+\frac{1}{2}} = \sigma^2 \log(\phi^{n+\frac{1}{2}}), \qquad v^{n+1} = -\sigma^2 \log(\psi^{n+1})$$

We have the following Theorem that is a consequence of Theorem 2.1:

Theorem 2.3. *Let suppose that u_T, $m_0 > 0$ and f are such that $\forall n \in \mathbb{N}, \phi^{n+\frac{1}{2}}, \psi^n \in C^{1,2}([0,T] \times \overline{\Omega})$ and $\phi, \psi \in C^{1,2}([0,T] \times \overline{\Omega})$. Then $(u^{n+\frac{1}{2}})_n$ and $(v^n)_n$ are sequences of functions in $C^{1,2}([0,T] \times \overline{\Omega})$ satisfying:*

$$\partial_t u^{n+\frac{1}{2}} + \frac{\sigma^2}{2}\Delta u^{n+\frac{1}{2}} + \frac{1}{2}|\nabla u^{n+\frac{1}{2}}|^2 = -f\left(x, e^{\frac{u^{n+\frac{1}{2}} - v^n}{\sigma^2}}\right)$$

$$\partial_t v^{n+1} - \frac{\sigma^2}{2}\Delta v^{n+1} + \frac{1}{2}|\nabla v^{n+1}|^2 = -f\left(x, e^{\frac{u^{n+\frac{1}{2}} - v^{n+1}}{\sigma^2}}\right)$$

with:

- *Boundary conditions: $\frac{\partial u^{n+\frac{1}{2}}}{\partial \vec{n}} = \frac{\partial v^{n+1}}{\partial \vec{n}} = 0$ on $(0,T) \times \partial\Omega$.*
- *Terminal condition: $u^{n+\frac{1}{2}}(T, \cdot) = u_T(\cdot)$.*
- *Initial condition: $v^{n+1}(0, \cdot) = u^{n+\frac{1}{2}}(0, \cdot) - \sigma^2 \log(m_0(\cdot))$.*

Moreover:

- *$(v^n)_n$ and $(u^{n+\frac{1}{2}})_n$ are uniformly bounded in $C([0,T] \times \overline{\Omega})$.*
- *$(u^{n+\frac{1}{2}})_n$ is a decreasing sequence.*
- *$(v^n)_n$ is a decreasing sequence.*
- *$(u^{n+\frac{1}{2}}, v^n)_n$ converges uniformly towards a couple (u, v) classical solution of $(\mathcal{S}_2)$.*

Proof. Theorem 2.3 is a simple consequence of Theorem 2.1 and of the remark we made after Proposition 2.2. Uniform convergence is a consequence of Dini's theorem since all the functions involved are continuous. □

3. **Numerical schemes.**

Now, our goal is to provide a way to approximate (ϕ, ψ) and (u, v) with recursive finite difference schemes that have the same properties as the constructive schemes presented in the preceding section. Such a scheme has been introduced for (ϕ, ψ) in [7] and we review the results in the first subsection[7]. A new scheme is then introduced for (u, v). What is important to notice is that the monotonicity properties obtained in Theorem 2.1 or Theorem 2.3 are rooted to comparison principles for the partial differential equations. Hence, one of the main requirement for a scheme to have the same properties as the corresponding

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[6]One can indeed reiterate the proof of the above Theorem as in [8] and see that the only important point is that $\psi^1 \geq \psi^0$.

[7]For the proofs, the interested reader may look at [7].

constructive scheme is to verify similar comparison principles. As a consequence, many schemes can be proposed in addition to those proposed in this article. Our goal is in fact to illustrate the process with one specific scheme for each change of variables.

In what follows we consider $\Omega = (0,1)$ and we consider a uniform subdivision $(x_0, \dots, x_J)$ of the interval $[0,1]$ where $x_j = j\Delta x$ for $j \in \mathcal{J} = \{0, \dots, J\}$. Also we consider a uniform subdivision $(t_0, \dots, t_I)$ of $[0,T]$ where $t_i = i\Delta t$ for $i \in \mathcal{I} = \{0, \dots, I\}$.

3.1. **Numerical scheme for** (ϕ, ψ).

3.1.1. *Definition and monotonicity properties of the scheme.* The numerical scheme defines recursively approximations $(\widehat{\psi}^n)_n$ of ψ and approximations $(\widehat{\phi}^{n+\frac{1}{2}})_n$ of ϕ on the grid introduced above, in the following way:

$\forall (i,j) \in \mathcal{I} \times \mathcal{J}, \quad \widehat{\psi}^0_{i,j} = 0$, and for $n \geq 0$:

$\forall j \in \mathcal{J}, \quad \widehat{\phi}^{n+\frac{1}{2}}_{I,j} = \exp\left(\frac{u_T(x_j)}{\sigma^2}\right)$ and $\forall i \in \mathcal{I} - \{I\}, \forall j \in \mathcal{J}$,

$$\frac{\widehat{\phi}^{n+\frac{1}{2}}_{i+1,j} - \widehat{\phi}^{n+\frac{1}{2}}_{i,j}}{\Delta t} + \frac{\sigma^2}{2} \frac{\widehat{\phi}^{n+\frac{1}{2}}_{i,j+1} - 2\widehat{\phi}^{n+\frac{1}{2}}_{i,j} + \widehat{\phi}^{n+\frac{1}{2}}_{i,j-1}}{(\Delta x)^2} = -\frac{1}{\sigma^2} f(x_j, \widehat{\phi}^{n+\frac{1}{2}}_{i,j} \widehat{\psi}^n_{i,j}) \widehat{\phi}^{n+\frac{1}{2}}_{i,j}$$

$\forall j \in \mathcal{J}, \quad \widehat{\psi}^{n+1}_{0,j} = \frac{m_0(x_j)}{\widehat{\phi}^{n+\frac{1}{2}}_{0,j}}$ and $\forall i \in \mathcal{I} - \{I\}, \forall j \in \mathcal{J}$,

$$\frac{\widehat{\psi}^{n+1}_{i+1,j} - \widehat{\psi}^{n+1}_{i,j}}{\Delta t} - \frac{\sigma^2}{2} \frac{\widehat{\psi}^{n+1}_{i+1,j+1} - 2\widehat{\psi}^{n+1}_{i+1,j} + \widehat{\psi}^{n+1}_{i+1,j-1}}{(\Delta x)^2} = \frac{1}{\sigma^2} f(x_j, \widehat{\phi}^{n+\frac{1}{2}}_{i+1,j} \widehat{\psi}^{n+1}_{i+1,j}) \widehat{\psi}^{n+1}_{i+1,j}$$

where we defined $\widehat{\psi}^n_{i,-1} = \widehat{\psi}^n_{i,0}$, $\widehat{\psi}^n_{i,J+1} = \widehat{\psi}^n_{i,J}$, $\widehat{\phi}^{n+\frac{1}{2}}_{i,-1} = \widehat{\phi}^{n+\frac{1}{2}}_{i,0}$, $\widehat{\phi}^{n+\frac{1}{2}}_{i,J+1} = \widehat{\phi}^{n+\frac{1}{2}}_{i,J}$ to take account of Neumann conditions on the boundary[8].

The above scheme is implicit and we need to prove that it is well-posed. To that purpose, let us introduce $\mathcal{M} = M_{I+1,J+1}(\mathbb{R})$ the space of matrices with $I+1$ lines and $J+1$ columns and

$$\mathcal{M}_\epsilon = \{(m_{i,j})_{i \in \mathcal{I}, j \in \mathcal{J}} \in \mathcal{M}, \quad \forall (i,j) \in \mathcal{I} \times \mathcal{J}, m_{i,j} \geq \epsilon\}$$

Let us start with the well-posedness of $\widehat{\phi}^{n+\frac{1}{2}}$ for a given $\widehat{\psi}^n$.

Proposition 3.1. *$\forall \widehat{\psi} \in \mathcal{M}_0$, there is a unique solution $\widehat{\phi} \in \mathcal{M}$ to the following equation:*

$\forall i \in \mathcal{I} - \{I\}, \forall j \in \mathcal{J}$,

$$\frac{\widehat{\phi}_{i+1,j} - \widehat{\phi}_{i,j}}{\Delta t} + \frac{\sigma^2}{2} \frac{\widehat{\phi}_{i,j+1} - 2\widehat{\phi}_{i,j} + \widehat{\phi}_{i,j-1}}{(\Delta x)^2} = -\frac{1}{\sigma^2} f(x_j, \widehat{\phi}_{i,j} \widehat{\psi}_{i,j}) \widehat{\phi}_{i,j}$$

with $\widehat{\phi}_{I,j} = \exp\left(\frac{u_T(x_j)}{\sigma^2}\right) \quad \forall j \in \mathcal{J}$, and the conventions $\widehat{\phi}_{i,-1} = \widehat{\phi}_{i,0}$, $\widehat{\phi}_{i,J+1} = \widehat{\phi}_{i,J}$.

Hence $\widehat{\Phi} : \widehat{\psi} \in \mathcal{M}_0 \mapsto \widehat{\phi} \in \mathcal{M}$ is well defined.

Moreover, there exists a constant $\epsilon > 0$ such that $\forall \widehat{\psi} \in \mathcal{M}_0, \widehat{\phi} = \widehat{\Phi}(\widehat{\psi}) \in \mathcal{M}_\epsilon$.

Similarly for the equation that defines $\widehat{\psi}^{n+1}$ from $\widehat{\phi}^{n+\frac{1}{2}}$, we have:

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[8] The reason why we did not consider centered differences is linked to the proof of convergence (see [7]).

Proposition 3.2. *Let us fix* $\epsilon > 0$.

$\forall \widehat{\phi} \in \mathcal{M}_\epsilon$, *there is a unique solution* $\widehat{\psi} \in \mathcal{M}$ *to the following equation:*

$\forall i \in \mathcal{I} - \{I\}, \forall j \in \mathcal{J}$,

$$\frac{\widehat{\psi}_{i+1,j} - \widehat{\psi}_{i,j}}{\Delta t} - \frac{\sigma^2}{2} \frac{\widehat{\psi}_{i+1,j+1} - 2\widehat{\psi}_{i+1,j} + \widehat{\psi}_{i+1,j-1}}{(\Delta x)^2} = \frac{1}{\sigma^2} f(x_j, \widehat{\phi}_{i+1,j}\widehat{\psi}_{i+1,j})\widehat{\psi}_{i+1,j}$$

with $\widehat{\psi}_{0,j} = \frac{m_0(x_j)}{\widehat{\phi}_{0,j}}$ $\quad \forall j \in \mathcal{J}$, *and the conventions* $\widehat{\psi}_{i,-1} = \widehat{\psi}_{i,0}$, $\widehat{\psi}_{i,J+1} = \widehat{\psi}_{i,J}$.

Hence $\widehat{\Psi} : \widehat{\phi} \in \mathcal{M}_\epsilon \mapsto \widehat{\psi} \in \mathcal{M}$ *is well defined.*

Moreover, $\forall \widehat{\phi} \in \mathcal{M}_\epsilon, \widehat{\psi} = \widehat{\Psi}(\widehat{\phi}) \in \mathcal{M}_0$.

Let us now turn to a monotonicity property of the functions $\widehat{\Phi}$ and $\widehat{\Psi}$ that is based on a discrete comparison principle for the scheme we considered.

Proposition 3.3.

$$\forall \widehat{\psi}_1 \le \widehat{\psi}_2 \in \mathcal{M}_0, \widehat{\Phi}(\widehat{\psi}_1) \ge \widehat{\Phi}(\widehat{\psi}_2)$$
$$\forall \widehat{\phi}_1 \le \widehat{\phi}_2 \in \mathcal{M}_\epsilon, \widehat{\Psi}(\widehat{\phi}_1) \ge \widehat{\Psi}(\widehat{\phi}_2)$$

Combining the above monotonicity inequalities, we get that the sequences $\left(\widehat{\psi}^n\right)_n$ and $\left(\widehat{\phi}^{n+\frac{1}{2}}\right)_n$ converge monotonically. More precisely we have that:

Proposition 3.4. *The numerical scheme verifies the following properties:*

- $\left(\widehat{\phi}^{n+\frac{1}{2}}\right)_n$ *is a decreasing sequence of* $\mathcal{M}_\epsilon$ *where* ϵ *is as in Proposition 3.1.*
- $\left(\widehat{\psi}^n\right)_n$ *is an increasing sequence of* $\mathcal{M}_0$, *bounded from above by* $\frac{\|m_0\|_\infty}{\epsilon}$.
- $\left(\widehat{\phi}^{n+\frac{1}{2}}, \widehat{\psi}^n\right)_n$ *converges towards a couple* $(\widehat{\phi}, \widehat{\psi}) \in \mathcal{M}_\epsilon \times \mathcal{M}_0$ *that verifies:*

$\forall j \in \mathcal{J}, \quad \widehat{\phi}_{I,j} = \exp\left(\frac{u_T(x_j)}{\sigma^2}\right)$ *and* $\forall i \in \mathcal{I} - \{I\}, \forall j \in \mathcal{J}$,

$$\frac{\widehat{\phi}_{i+1,j} - \widehat{\phi}_{i,j}}{\Delta t} + \frac{\sigma^2}{2} \frac{\widehat{\phi}_{i,j+1} - 2\widehat{\phi}_{i,j} + \widehat{\phi}_{i,j-1}}{(\Delta x)^2} = -\frac{1}{\sigma^2} f(x_j, \widehat{\phi}_{i,j}\widehat{\psi}_{i,j})\widehat{\phi}_{i,j}$$

$\forall j \in \mathcal{J}, \quad \widehat{\psi}_{0,j} = \frac{m_0(x_j)}{\widehat{\phi}_{0,j}}$ *and* $\forall i \in \mathcal{I} - \{I\}, \forall j \in \mathcal{J}$,

$$\frac{\widehat{\psi}_{i+1,j} - \widehat{\psi}_{i,j}}{\Delta t} - \frac{\sigma^2}{2} \frac{\widehat{\psi}_{i+1,j+1} - 2\widehat{\psi}_{i+1,j} + \widehat{\psi}_{i+1,j-1}}{(\Delta x)^2} = \frac{1}{\sigma^2} f(x_j, \widehat{\phi}_{i+1,j}\widehat{\psi}_{i+1,j})\widehat{\psi}_{i+1,j}$$

$$\widehat{\phi}_{i,-1} = \widehat{\phi}_{i,0}, \quad \widehat{\phi}_{i,J+1} = \widehat{\phi}_{i,J}, \quad \widehat{\psi}_{i,-1} = \widehat{\psi}_{i,0}, \quad \widehat{\psi}_{i,J+1} = \widehat{\psi}_{i,J}$$

3.1.2. *Convergence of the scheme.* To deal with the convergence of the scheme, we introduce the $L^\infty(L^2)$ norm $\|\cdot\|$ on the set $\mathcal{M}$:

$$\forall m = (m_{i,j})_{i,j} \in \mathcal{M}, \|m\|^2 = \sup_{0 \le i \le I} \frac{1}{J+1} \sum_{j=0}^{J} m_{i,j}^2$$

The convergence results we provide are based on the following recursive stability bounds for the scheme[9]:

[9] The result is slightly more general than the result stated in [7] but the proof is the same noticing that the sequence of approximations are uniformly bounded.

Theorem 3.5. *Let us suppose, in addition to the hypotheses made above, that f is uniformly locally Lipschitz with respect to its second variable, i.e.*

$$\forall M > 0, \exists K, \forall x \in [0,1], \forall \xi_1, \xi_2 \in [0, M], |f(x,\xi_2) - f(x,\xi_1)| \leq K|\xi_2 - \xi_1|$$

Then: $\forall \nu > 0$, $\exists C > 0$, $\forall I, J \in \mathbb{N}^*$ such that $\frac{1}{\Delta t} > 1 + \frac{K}{\sigma^2} \max\left(\left\| e^{\frac{u_T}{\sigma^2}} \right\|_\infty^2, \frac{\|m_0\|_\infty^2}{\epsilon^2}\right) + \nu$, we have $\forall n \in \mathbb{N}$:

$$\|\widehat{\phi}^{n+\frac{1}{2}} - \tilde{\phi}^{n+\frac{1}{2}}\|^2 \leq C\|\widehat{\psi}^n - \tilde{\psi}^n\|^2 + C\|\tilde{\eta}^{n+\frac{1}{2}}\|^2$$

$$\|\widehat{\psi}^{n+1} - \tilde{\psi}^{n+1}\|^2 \leq C\|\widehat{\phi}^{n+\frac{1}{2}} - \tilde{\phi}^{n+\frac{1}{2}}\|^2 + C\|\tilde{\eta}^{n+1}\|^2$$

where $\tilde{\phi}_{i,j}^{n+\frac{1}{2}} = \phi^{n+\frac{1}{2}}(t_i, x_j)$ and $\tilde{\psi}_{i,j}^n = \psi^n(t_i, x_j)$ are the values on the grid of the solutions[10] of the equations of Theorem 2.1 (for $\psi^0 = 0$) and where $\tilde{\eta}^{n+\frac{1}{2}}$ and $\tilde{\eta}^{n+1}$ are the consistency errors associated to the equations that define respectively $\phi^{n+\frac{1}{2}}$ and ψ^{n+1}.

Using these bounds recursively, we obtain for fixed n the convergence of the schemes defining respectively $\widehat{\phi}^{n+\frac{1}{2}}$ and $\widehat{\psi}^n$.

Theorem 3.6. *Let us suppose, in addition to the hypotheses of Theorem 3.5 that u_T, m_0 and f are so that $\forall n \in \mathbb{N}, \phi^{n+\frac{1}{2}}, \psi^n \in C^{1,2}([0,T] \times [0,1])$.*
Then $\forall n \in \mathbb{N}$:

$$\lim_{\Delta t, \Delta x \to 0} \|\widehat{\phi}^{n+\frac{1}{2}} - \tilde{\phi}^{n+\frac{1}{2}}\| = 0$$

$$\lim_{\Delta t, \Delta x \to 0} \|\widehat{\psi}^{n+1} - \tilde{\psi}^{n+1}\| = 0$$

Then, a weak form of convergence for the entire scheme can be obtained:

Theorem 3.7. *Let us suppose, in addition to the hypotheses made in Theorem 3.5 that u_T, m_0 and f are so that $\forall n \in \mathbb{N}, \phi^{n+\frac{1}{2}}, \psi^n \in C^{1,2}([0,T] \times [0,1])$ and $\phi, \psi \in C^{1,2}([0,T] \times [0,1])$*
Then:

$$\lim_{n \to \infty} \limsup_{\Delta t, \Delta x \to 0} \|\widehat{\phi}^{n+\frac{1}{2}} - \tilde{\phi}\| = 0$$

$$\lim_{n \to \infty} \limsup_{\Delta t, \Delta x \to 0} \|\widehat{\psi}^{n+1} - \tilde{\psi}\| = 0$$

3.2. **Numerical scheme for** (u, v)**.**

3.2.1. *Definition, bounds and monotonicity properties of the scheme.* We now consider a numerical scheme that defines recursively approximations $(\widehat{v}^n)_n$ of v and approximations $(\widehat{u}^{n+\frac{1}{2}})_n$ of u on the grid introduced above:

$$\widehat{v}_{i,j}^0 = \|u_T\|_\infty + \sigma^2 \|\log(m_0)\|_\infty + 2\|f\|_\infty T, \quad (i,j) \in \mathcal{I} \times \mathcal{J}$$

and for $n \geq 0$:

$$\frac{\widehat{u}_{i+1,j}^{n+\frac{1}{2}} - \widehat{u}_{i,j}^{n+\frac{1}{2}}}{\Delta t} + \frac{\sigma^2}{2} \frac{\widehat{u}_{i+1,j+1}^{n+\frac{1}{2}} - 2\widehat{u}_{i+1,j}^{n+\frac{1}{2}} + \widehat{u}_{i+1,j-1}^{n+\frac{1}{2}}}{(\Delta x)^2} + f\left(x_j, e^{\frac{\widehat{u}_{i+1,j}^{n+\frac{1}{2}} - \widehat{v}_{i+1,j}^n}{\sigma^2}}\right)$$

[10]We implicitly suppose here that the functions are defined pointwise and for instance that $\forall n \in \mathbb{N}, \phi^{n+\frac{1}{2}}, \psi^n \in C([0,T] \times [0,1])$

$$+\frac{1}{2}\left[\left(\frac{\widehat{u}^{n+\frac{1}{2}}_{i+1,j+1}-\widehat{u}^{n+\frac{1}{2}}_{i+1,j}}{\Delta x}\right)_+^2+\left(\frac{\widehat{u}^{n+\frac{1}{2}}_{i+1,j}-\widehat{u}^{n+\frac{1}{2}}_{i+1,j-1}}{\Delta x}\right)_-^2\right]=0,\quad i\in\mathcal{I}-\{I\},j\in\mathcal{J}$$

$$\widehat{u}^{n+\frac{1}{2}}_{I,j}=u_T(x_j),\quad j\in\mathcal{J}$$

$$\frac{\widehat{v}^{n+1}_{i+1,j}-\widehat{v}^{n+1}_{i,j}}{\Delta t}-\frac{\sigma^2}{2}\frac{\widehat{v}^{n+1}_{i,j+1}-2\widehat{v}^{n+1}_{i,j}+\widehat{v}^{n+1}_{i,j-1}}{(\Delta x)^2}+f\left(x_j,e^{\frac{\widehat{u}^{n+1}_{i,j}-\widehat{v}^{n}_{i,j}}{\sigma^2}}\right)$$

$$+\frac{1}{2}\left[\left(\frac{\widehat{v}^{n+1}_{i,j+1}-\widehat{v}^{n+1}_{i,j}}{\Delta x}\right)_-^2+\left(\frac{\widehat{v}^{n+1}_{i,j}-\widehat{v}^{n+1}_{i,j-1}}{\Delta x}\right)_+^2\right]=0,\quad i\in\mathcal{I}-\{I\},j\in\mathcal{J}$$

$$\widehat{v}^{n+1}_{0,j}=\widehat{u}^{n+\frac{1}{2}}_{0,j}-\sigma^2\log(m_0(x_j)),\quad j\in\mathcal{J}$$

where we defined $\widehat{v}^n_{i,-1}=\widehat{v}^n_{i,1}$, $\widehat{v}^n_{i,J+1}=\widehat{v}^n_{i,J-1}$, $\widehat{u}^{n+\frac{1}{2}}_{i,-1}=\widehat{u}^{n+\frac{1}{2}}_{i,1}$ and $\widehat{u}^{n+\frac{1}{2}}_{i,J+1}=\widehat{u}^{n+\frac{1}{2}}_{i,J-1}$ to take account of Neumann conditions on the boundary.

To study this scheme and as in Theorem 3.5 above, we suppose that f is uniformly locally Lipschitz with respect to its second variable. We shall denote $K(R)$ the Lipschitz constant (with respect to its second argument) of the restriction of f to $[0,1]\times[0,e^{\frac{2R}{\sigma^2}}]$.

The above scheme is explicit and hence well defined. We will start with a key lemma in the proof of the monotonicity properties.

Lemma 3.8. *Let us consider $R>0$ and $B_R=[-R,R]^{J+1}$.*

Let us consider the function $G_u:(a,b)\in B_R^2\mapsto c\in\mathbb{R}^{J+1}$ defined by:

$$c_j=a_j+\Delta t\left[\frac{\sigma^2}{2}\frac{a_{j+1}-2a_j+a_{j-1}}{(\Delta x)^2}\right.$$
$$\left.+f\left(x_j,e^{\frac{a_j-b_j}{\sigma^2}}\right)+\frac{1}{2}\left[\left(\frac{a_{j+1}-a_j}{\Delta x}\right)_+^2+\left(\frac{a_j-a_{j-1}}{\Delta x}\right)_-^2\right]\right]$$

where $a_{-1}=a_1$ and $a_{J+1}=a_{J-1}$.

Then:

$$\forall a\in B_R, G_u(a,\cdot)\textit{ is an increasing function of each coordinate.}$$

and

$$(\sigma^2+4R)\frac{\Delta t}{(\Delta x)^2}+\frac{\Delta t}{\sigma^2}K(R)e^{\frac{2R}{\sigma^2}}\le 1$$

$$\implies\forall b\in B_R, G_u(\cdot,b)\textit{ is an increasing function of each coordinate.}$$

Proof. The first assertion follows from the fact that f is a decreasing function of its second variable.

Now, because of the form of the discrete Hamiltonian, $\frac{dc_j}{da_{j+1}}\ge 0$ and $\frac{dc_j}{da_{j-1}}\ge 0$.

Turning to $\frac{dc_j}{da_j}$ we have:

$$\begin{aligned}\frac{dc_j}{da_j} =&1+\Delta t\Bigg[-\sigma^2\frac{1}{(\Delta x)^2}+\frac{1}{\sigma^2}e^{\frac{a_j-b_j}{\sigma^2}}f'\left(x_j, e^{\frac{a_j-b_j}{\sigma^2}}\right)\\&-\frac{1}{(\Delta x)^2}\left[(a_{j+1}-a_j)_+ + (a_j-a_{j-1})_-\right]\Bigg]\\&\geq 1-(\sigma^2+4R)\frac{\Delta t}{(\Delta x)^2}-\frac{\Delta t}{\sigma^2}K(R)e^{\frac{2R}{\sigma^2}}\\&\geq 0\end{aligned}$$

□

Similarly, we have:

Lemma 3.9. *Let us consider the function* $G_v : (a,b) \in B_R^2 \mapsto c \in \mathbb{R}^{J+1}$ *defined by:*

$$\begin{aligned}c_j = a_j + \Delta t\Bigg[&\frac{\sigma^2}{2}\frac{a_{j+1}-2a_j+a_{j-1}}{(\Delta x)^2}\\&-f\left(x_j, e^{\frac{b_j-a_j}{\sigma^2}}\right)-\frac{1}{2}\left[\left(\frac{a_{j+1}-a_j}{\Delta x}\right)_-^2+\left(\frac{a_j-a_{j-1}}{\Delta x}\right)_+^2\right]\Bigg]\end{aligned}$$

where $a_{-1} = a_1$ *and* $a_{J+1} = a_{J-1}$.

Then:

$$\forall a \in B_R, G_v(a,\cdot) \text{ is an increasing function of each coordinate.}$$

and

$$(\sigma^2+4R)\frac{\Delta t}{(\Delta x)^2}+\frac{\Delta t}{\sigma^2}K(R)e^{\frac{2R}{\sigma^2}} \leq 1$$

$$\implies \forall b \in B_R, G_v(\cdot,b) \text{ is an increasing function of each coordinate.}$$

Proof. The proof is *mutatis mutandis* the same as the proof of Lemma 3.8. □

From these two lemmas, we can deduce uniform bounds for $\widehat{u}_{i,j}^{n+\frac{1}{2}}$ and $\widehat{v}_{i,j}^{n}$:

Proposition 3.10. *Let us introduce* $R = \|u_T\|_\infty + \sigma^2\|\log(m_0)\|_\infty + 2\|f\|_\infty T$. *Suppose that the following condition is satisfied:*

$$(\sigma^2+4R)\frac{\Delta t}{(\Delta x)^2}+\frac{\Delta t}{\sigma^2}K(R)e^{\frac{2R}{\sigma^2}} \leq 1 \qquad (*)$$

Then, $\forall n \in \mathbb{N}$*:*

$$\|\widehat{u}^{n+\frac{1}{2}}\|_\infty \leq \|u_T\|_\infty + \|f\|_\infty T \leq R, \quad \|\widehat{v}^n\|_\infty \leq R$$

Proof. The result is true for $\|\widehat{v}^0\|_\infty$ by definition. Let us suppose that $\|\widehat{v}^n\|_\infty \leq R$, for a given n.

Since we have that $\|\widehat{u}_I^{n+\frac{1}{2}}\|_\infty = \|u_T\|_\infty$, using Lemma 3.8 we obtain:

$$\begin{aligned}\widehat{u}_I^{n+\frac{1}{2}} \leq \|u_T\|_\infty \implies \widehat{u}_{I-1}^{n+\frac{1}{2}} &= G_u\left(\widehat{u}_I^{n+\frac{1}{2}}, \widehat{v}_I^n\right)\\&\leq G_u\left(\|u_T\|_\infty, \widehat{v}_I^n\right) \leq \|u_T\|_\infty + \|f\|_\infty \Delta t\end{aligned}$$

Similarly,

$$\widehat{u}_I^{n+\frac{1}{2}} \geq -\|u_T\|_\infty \implies \widehat{u}_{I-1}^{n+\frac{1}{2}} = G_u\left(\widehat{u}_I^{n+\frac{1}{2}}, \widehat{v}_I^n\right)$$
$$\geq G_u\left(-\|u_T\|_\infty, \widehat{v}_I^n\right) \geq -\|u_T\|_\infty - \|f\|_\infty \Delta t$$

Hence, $\|\widehat{u}_{I-1}^{n+\frac{1}{2}}\|_\infty \leq \|u_T\|_\infty + \|f\|_\infty \Delta t$ and by immediate induction we obtain:

$$\forall i \in \mathcal{I}, \quad \|\widehat{u}_{I-i}^{n+\frac{1}{2}}\|_\infty \leq \|u_T\|_\infty + \|f\|_\infty i\Delta t$$

Consequently, $\|\widehat{u}^{n+\frac{1}{2}}\|_\infty \leq \|u_T\|_\infty + \|f\|_\infty T \leq R$.

Now, $\widehat{v}_0^{n+1} \leq \|\widehat{u}_0^{n+\frac{1}{2}}\|_\infty + \sigma^2\|\log(m_0)\|_\infty \leq \|u_T\|_\infty + \sigma^2\|\log(m_0)\|_\infty + \|f\|_\infty T$. Hence, using Lemma 3.9, we have:

$$\widehat{v}_1^{n+1} = G_v\left(\widehat{v}_0^{n+1}, \widehat{u}_0^{n+\frac{1}{2}}\right) \leq G_v\left(\|u_T\|_\infty + \sigma^2\|\log(m_0)\|_\infty + \|f\|_\infty T, \widehat{u}_0^{n+\frac{1}{2}}\right)$$
$$\leq \|u_T\|_\infty + \sigma^2\|\log(m_0)\|_\infty + \|f\|_\infty T + \|f\|_\infty \Delta t$$

By the same symmetry argument as above we obtain:

$$\|\widehat{v}_1^{n+1}\|_\infty \leq \|u_T\|_\infty + \sigma^2\|\log(m_0)\|_\infty + \|f\|_\infty T + \|f\|_\infty \Delta t$$

By immediate induction we have:

$$\forall i \in \mathcal{I}, \quad \|\widehat{v}_i^{n+1}\|_\infty \leq \|u_T\|_\infty + \sigma^2\|\log(m_0)\|_\infty + \|f\|_\infty T + \|f\|_\infty i\Delta t$$

Consequently, $\|\widehat{v}^{n+1}\|_\infty \leq R$, and the result is proved by induction on n. □

Using the condition $(*)$ introduced above, we can prove that the two sequences have a monotonic behavior:

Proposition 3.11. *If $(\Delta t, \Delta x)$ verifies the condition $(*)$ of Proposition 3.10, then the numerical scheme verifies the following properties:*

- $\left(\widehat{u}^{n+\frac{1}{2}}\right)_n$ *is a decreasing sequence of $\mathcal{M}$.*
- $\left(\widehat{v}^{n}\right)_n$ *is a decreasing sequence of $\mathcal{M}$.*
- $\left(\widehat{u}^{n+\frac{1}{2}}, \widehat{v}^n\right)_n$ *converges towards a couple $(\widehat{u}, \widehat{v}) \in \mathcal{M} \times \mathcal{M}$ that verifies:*

$$\frac{\widehat{u}_{i+1,j} - \widehat{u}_{i,j}}{\Delta t} + \frac{\sigma^2}{2}\frac{\widehat{u}_{i+1,j+1} - 2\widehat{u}_{i+1,j} + \widehat{u}_{i+1,j-1}}{(\Delta x)^2} + f\left(x_j, e^{\frac{\widehat{u}_{i+1,j} - \widehat{v}_{i+1,j}}{\sigma^2}}\right)$$
$$+\frac{1}{2}\left[\left(\frac{\widehat{u}_{i+1,j+1} - \widehat{u}_{i+1,j}}{\Delta x}\right)_+^2 + \left(\frac{\widehat{u}_{i+1,j} - \widehat{u}_{i+1,j-1}}{\Delta x}\right)_-^2\right] = 0, \quad i \in \mathcal{I} - \{I\}, j \in \mathcal{J}$$
$$\widehat{u}_{I,j} = u_T(x_j), \quad j \in \mathcal{J}$$

$$\frac{\widehat{v}_{i+1,j} - \widehat{v}_{i,j}}{\Delta t} - \frac{\sigma^2}{2}\frac{\widehat{v}_{i,j+1} - 2\widehat{v}_{i,j} + \widehat{v}_{i,j-1}}{(\Delta x)^2} + f\left(x_j, e^{\frac{\widehat{u}_{i,j} - \widehat{v}_{i,j}}{\sigma^2}}\right)$$
$$+\frac{1}{2}\left[\left(\frac{\widehat{v}_{i,j+1} - \widehat{v}_{i,j}}{\Delta x}\right)_-^2 + \left(\frac{\widehat{v}_{i,j} - \widehat{v}_{i,j-1}}{\Delta x}\right)_+^2\right] = 0, \quad i \in \mathcal{I} - \{I\}, j \in \mathcal{J}$$
$$\widehat{v}_{0,j} = \widehat{u}_{0,j} - \sigma^2\log(m_0(x_j)), \quad j \in \mathcal{J}$$

with $\widehat{u}_{i,-1} = \widehat{u}_{i,1}$, $\widehat{u}_{i,J+1} = \widehat{u}_{i,J-1}$, $\widehat{v}_{i,-1} = \widehat{v}_{i,1}$ and $\widehat{v}_{i,J+1} = \widehat{v}_{i,J-1}$.

Proof. Because of Proposition 3.10, we know that $\|\widehat{v}^1\|_\infty \leq R = \widehat{v}^0$. Hence, we know that $\widehat{v}^1 \leq \widehat{v}^0$.

To compare $\widehat{u}^{\frac{1}{2}}$ and $\widehat{u}^{\frac{3}{2}}$, we start with $\widehat{u}_I^{\frac{3}{2}} = \widehat{u}_I^{\frac{1}{2}} \leq \widehat{u}_I^{\frac{1}{2}}$. Now, if we have $\widehat{u}_{i+1}^{\frac{3}{2}} \leq \widehat{u}_{i+1}^{\frac{1}{2}}$ for some $i \in \mathcal{I} - \{I\}$, then we have $\widehat{u}_i^{\frac{3}{2}} = G_u(\widehat{u}_{i+1}^{\frac{3}{2}}, \widehat{v}_{i+1}^1) \leq G_u(\widehat{u}_{i+1}^{\frac{1}{2}}, \widehat{v}_{i+1}^1)$. Then, using the first part of Lemma 3.8, we get $G_u(\widehat{u}_{i+1}^{\frac{1}{2}}, \widehat{v}_{i+1}^1) \leq G_u(\widehat{u}_{i+1}^{\frac{1}{2}}, \widehat{v}_{i+1}^0) = \widehat{u}_i^{\frac{1}{2}}$.

By induction, $\forall i \in \mathcal{I}, \widehat{u}_i^{\frac{3}{2}} \leq \widehat{u}_i^{\frac{1}{2}}$, *i.e.* $\widehat{u}^{\frac{3}{2}} \leq \widehat{u}^{\frac{1}{2}}$.

In particular we have $\widehat{u}_0^{\frac{3}{2}} \leq \widehat{u}_0^{\frac{1}{2}}$ and hence by definition $\widehat{v}_0^2 \leq \widehat{v}_0^1$.

Now, if we have $\widehat{v}_i^2 \leq \widehat{v}_i^1$ for some $i \in \mathcal{I} - \{I\}$, we use the same argument as above to get:

$$\widehat{v}_{i+1}^2 = G_v(\widehat{v}_i^2, \widehat{u}_i^{\frac{3}{2}}) \leq G_v(\widehat{v}_i^1, \widehat{u}_i^{\frac{3}{2}}) \leq G_v(\widehat{v}_i^1, \widehat{u}_i^{\frac{1}{2}}) = \widehat{v}_{i+1}^1$$

Hence, by induction, we get $\forall i \in \mathcal{I}, \widehat{v}_i^2 \leq \widehat{v}_i^1$, *i.e.* $\widehat{v}^2 \leq \widehat{v}^1$.

Reiterating the process, we see that $\forall n \in \mathbb{N}, \widehat{u}^{n+\frac{3}{2}} \leq \widehat{u}^{n+\frac{1}{2}}$ and $\widehat{v}^{n+1} \leq \widehat{v}^n$.

Because of the bounds obtained in Proposition 3.10, we then know that $\left(\widehat{u}^{n+\frac{1}{2}}, \widehat{v}^n\right)_n$ converges towards a couple $(\widehat{u}, \widehat{v})$ and this couple satisfies the equation of Proposition 3.11 straightforwardly. □

3.2.2. *Convergence of the scheme.* In contrast with the first scheme we considered, we are now going to deal with convergence in L^∞ norm and we equip $\mathcal{M}$ with the norm $\|\cdot\|_\infty$.

The convergence results we provide are based on the following recursive stability bounds for the scheme:

Theorem 3.12. *Let us introduce $M = 3R$.*

Let us suppose that $(\Delta t, \Delta x)$ verifies the following condition:

$$(\sigma^2 + 4M)\frac{\Delta t}{(\Delta x)^2} + \frac{\Delta t}{\sigma^2} K(M) e^{\frac{2M}{\sigma^2}} \leq 1 \qquad (**)$$

Then, there exists a constant C, independent of the grid, such that $\forall n \in \mathbb{N}$:

$$\|\widehat{u}^{n+\frac{1}{2}} - \tilde{u}^{n+\frac{1}{2}}\|_\infty \leq C\|\widehat{v}^n - \tilde{v}^n\|_\infty + C\|\tilde{\eta}^{n+\frac{1}{2}}\|_\infty$$

$$\|\widehat{v}^{n+1} - \tilde{v}^{n+1}\|_\infty \leq C\|\widehat{u}^{n+\frac{1}{2}} - \tilde{u}^{n+\frac{1}{2}}\|_\infty + C\|\tilde{\eta}^{n+1}\|_\infty$$

where $\tilde{u}_{i,j}^{n+\frac{1}{2}} = u^{n+\frac{1}{2}}(t_i, x_j)$ and $\tilde{v}_{i,j}^{n+1} = v^{n+1}(t_i, x_j)$ are the values on the grid of the functions[11] *defined in section 1 and where $\tilde{\eta}^{n+\frac{1}{2}}$ and $\tilde{\eta}^{n+1}$ are the consistency errors associated to the equations that characterize respectively $u^{n+\frac{1}{2}}$ and v^{n+1}.*

Proof. Let us first notice that the bound we obtained in Proposition 3.10 for $\widehat{u}^{n+\frac{1}{2}}$ (resp. $\widehat{v}^n$) also applies to $\tilde{u}^{n+\frac{1}{2}}$ (resp. $\tilde{v}^n$) because of a straightforward application of the comparison principle to $u^{n+\frac{1}{2}}$ (resp. v^n).

Let us consider $i \in \mathcal{I} - \{I\}$. By definition of the scheme and the consistency errors, we have:

$$\widehat{u}_i^{n+\frac{1}{2}} = G_u(\widehat{u}_{i+1}^{n+\frac{1}{2}}, \widehat{v}_{i+1}^n), \qquad \tilde{u}_i^{n+\frac{1}{2}} = G_u(\tilde{u}_{i+1}^{n+\frac{1}{2}}, \tilde{v}_{i+1}^n) - \Delta t \tilde{\eta}_i^{n+\frac{1}{2}}$$

Hence:

[11] We implicitly suppose here that the functions are defined pointwise and for instance that $\forall n \in \mathbb{N}, u^{n+\frac{1}{2}}, v^n \in C([0,T] \times [0,1])$

$$\begin{aligned}\|\widehat{u}_i^{n+\frac{1}{2}} - \tilde{u}_i^{n+\frac{1}{2}}\|_\infty &\leq \|G_u(\widehat{u}_{i+1}^{n+\frac{1}{2}}, \widehat{v}_{i+1}^n) - G_u(\tilde{u}_{i+1}^{n+\frac{1}{2}}, \tilde{v}_{i+1}^n)\|_\infty + \Delta t \|\tilde{\eta}_i^{n+\frac{1}{2}}\|_\infty \\ &\leq \|G_u(\widehat{u}_{i+1}^{n+\frac{1}{2}}, \widehat{v}_{i+1}^n) - G_u(\tilde{u}_{i+1}^{n+\frac{1}{2}}, \widehat{v}_{i+1}^n)\|_\infty \\ &+ \|G_u(\tilde{u}_{i+1}^{n+\frac{1}{2}}, \widehat{v}_{i+1}^n) - G_u(\tilde{u}_{i+1}^{n+\frac{1}{2}}, \tilde{v}_{i+1}^n)\|_\infty + \Delta t \|\tilde{\eta}^{n+\frac{1}{2}}\|_\infty\end{aligned}$$

Now, $\widehat{u}_{i+1}^{n+\frac{1}{2}} \leq \tilde{u}_{i+1}^{n+\frac{1}{2}} + \|\widehat{u}_{i+1}^{n+\frac{1}{2}} - \tilde{u}_{i+1}^{n+\frac{1}{2}}\|_\infty \leq M$ and we can apply Lemma 3.8 because the condition $(**)$ is satisfied:

$$\begin{aligned}G_u(\widehat{u}_{i+1}^{n+\frac{1}{2}}, \widehat{v}_{i+1}^n) &\leq G_u(\tilde{u}_{i+1}^{n+\frac{1}{2}} + \|\widehat{u}_{i+1}^{n+\frac{1}{2}} - \tilde{u}_{i+1}^{n+\frac{1}{2}}\|_\infty, \widehat{v}_{i+1}^n) \\ &\leq G_u(\tilde{u}_{i+1}^{n+\frac{1}{2}}, \widehat{v}_{i+1}^n) + \|\widehat{u}_{i+1}^{n+\frac{1}{2}} - \tilde{u}_{i+1}^{n+\frac{1}{2}}\|_\infty \\ + \sup_j &\left| f\left(x_j, e^{\frac{\tilde{u}_{i+1,j}^{n+\frac{1}{2}} + \|\widehat{u}_{i+1}^{n+\frac{1}{2}} - \tilde{u}_{i+1}^{n+\frac{1}{2}}\|_\infty - \widehat{v}_{i+1,j}^n}{\sigma^2}}\right) - f\left(x_j, e^{\frac{\tilde{u}_{i+1,j}^{n+\frac{1}{2}} - \widehat{v}_{i+1,j}^n}{\sigma^2}}\right)\right| \Delta t\end{aligned}$$

Using the bounds we obtained, and the Lipschitz property of both the function f and the function $y \mapsto e^y$ on compact sets, there exists a constant α such that:

$$G_u(\widehat{u}_{i+1}^{n+\frac{1}{2}}, \widehat{v}_{i+1}^n) \leq G_u(\tilde{u}_{i+1}^{n+\frac{1}{2}}, \widehat{v}_{i+1}^n) + (1+\alpha\Delta t)\|\widehat{u}_{i+1}^{n+\frac{1}{2}} - \tilde{u}_{i+1}^{n+\frac{1}{2}}\|_\infty$$

Reversing the roles played by $\widehat{u}_{i+1}^{n+\frac{1}{2}}$ and $\tilde{u}_{i+1}^{n+\frac{1}{2}}$, we eventually obtain:

$$\|G_u(\widehat{u}_{i+1}^{n+\frac{1}{2}}, \widehat{v}_{i+1}^n) - G_u(\tilde{u}_{i+1}^{n+\frac{1}{2}}, \widehat{v}_{i+1}^n)\|_\infty \leq (1+\alpha\Delta t)\|\widehat{u}_{i+1}^{n+\frac{1}{2}} - \tilde{u}_{i+1}^{n+\frac{1}{2}}\|_\infty$$

Now,

$$\begin{aligned}&\|G_u(\tilde{u}_{i+1}^{n+\frac{1}{2}}, \widehat{v}_{i+1}^n) - G_u(\tilde{u}_{i+1}^{n+\frac{1}{2}}, \tilde{v}_{i+1}^n)\|_\infty \\ &= \sup_j \left| f\left(x_j, e^{\frac{\tilde{u}_{i+1,j}^{n+\frac{1}{2}} - \widehat{v}_{i+1,j}^n}{\sigma^2}}\right) - f\left(x_j, e^{\frac{\tilde{u}_{i+1,j}^{n+\frac{1}{2}} - \tilde{v}_{i+1,j}^n}{\sigma^2}}\right)\right| \Delta t\end{aligned}$$

Using the same argument as above, there exists a constant β such that:

$$\|G_u(\tilde{u}_{i+1}^{n+\frac{1}{2}}, \widehat{v}_{i+1}^n) - G_u(\tilde{u}_{i+1}^{n+\frac{1}{2}}, \tilde{v}_{i+1}^n)\|_\infty \leq \beta\Delta t \|\widehat{v}^n - \tilde{v}^n\|_\infty$$

Combining the inequalities, we have:

$$\|\widehat{u}_i^{n+\frac{1}{2}} - \tilde{u}_i^{n+\frac{1}{2}}\|_\infty \leq (1+\alpha\Delta t)\|\widehat{u}_{i+1}^{n+\frac{1}{2}} - \tilde{u}_{i+1}^{n+\frac{1}{2}}\|_\infty + \beta\Delta t\|\widehat{v}^n - \tilde{v}^n\|_\infty + \Delta t\|\tilde{\eta}^{n+\frac{1}{2}}\|_\infty$$

By immediate induction, we see that there exists a constant A such that:

$$\forall i \in \mathcal{I}, \|\widehat{u}_i^{n+\frac{1}{2}} - \tilde{u}_i^{n+\frac{1}{2}}\|_\infty \leq A\|\widehat{u}_I^{n+\frac{1}{2}} - \tilde{u}_I^{n+\frac{1}{2}}\|_\infty + A\|\widehat{v}^n - \tilde{v}^n\|_\infty + A\|\tilde{\eta}^{n+\frac{1}{2}}\|_\infty$$

Since $\widehat{u}_I^{n+\frac{1}{2}} = \tilde{u}_I^{n+\frac{1}{2}}$, we have:

$$\|\widehat{u}^{n+\frac{1}{2}} - \tilde{u}^{n+\frac{1}{2}}\|_\infty \leq A\|\widehat{v}^n - \tilde{v}^n\|_\infty + A\|\tilde{\eta}^{n+\frac{1}{2}}\|_\infty$$

Turning to the second inequality, we proceed in the same way and we obtain a constant B such that:

$$\forall i \in \mathcal{I}, \|\widehat{v}_i^{n+1} - \tilde{v}_i^{n+1}\|_\infty \leq B\|\widehat{v}_0^{n+1} - \tilde{v}_0^{n+1}\|_\infty + B\|\widehat{u}^{n+\frac{1}{2}} - \tilde{u}^{n+\frac{1}{2}}\|_\infty + B\|\tilde{\eta}^{n+1}\|_\infty$$

But, by definition $\|\widehat{v}_0^{n+1} - \tilde{v}_0^{n+1}\|_\infty = \|\widehat{u}_0^{n+\frac{1}{2}} - \tilde{u}_0^{n+\frac{1}{2}}\|_\infty$ and hence:

$$\|\widehat{v}^{n+1} - \tilde{v}^{n+1}\|_\infty \leq 2B\|\widehat{u}^{n+\frac{1}{2}} - \tilde{u}^{n+\frac{1}{2}}\|_\infty + B\|\tilde{\eta}^{n+1}\|_\infty$$

□

As for the preceding scheme, these recursive bounds allow to write a convergence result for fixed n:

Theorem 3.13. *Let us suppose that u_T, $m_0 > 0$ and f are so that $\forall n \in \mathbb{N}, u^{n+\frac{1}{2}}$, $v^n \in C^{1,2}([0,T] \times [0,1])$.*
Then $\forall n \in \mathbb{N}$:

$$\lim_{\substack{\Delta t, \Delta x \to 0 \\ (\Delta t, \Delta x) \text{ satisfying } (**)}} \|\widehat{u}^{n+\frac{1}{2}} - \tilde{u}^{n+\frac{1}{2}}\|_\infty = 0$$

$$\lim_{\substack{\Delta t, \Delta x \to 0 \\ (\Delta t, \Delta x) \text{ satisfying } (**)}} \|\widehat{v}^{n+1} - \tilde{v}^{n+1}\|_\infty = 0$$

Proof. By immediate application of the inequalities of Theorem 3.12, we get $\forall n \in \mathbb{N}$:

$$\|\widehat{u}^{n+\frac{1}{2}} - \tilde{u}^{n+\frac{1}{2}}\|_\infty \leq \sum_{k=0}^{n} C^{2k+1}\|\tilde{\eta}^{n+\frac{1}{2}-k}\|_\infty + C^{2k+2}\|\tilde{\eta}^{n-k}\|_\infty$$

and

$$\|\widehat{v}^{n+1} - \tilde{v}^{n+1}\|_\infty \leq \sum_{k=0}^{n} C^{2k+1}\|\tilde{\eta}^{n+1-k}\|_\infty + C^{2k+2}\|\tilde{\eta}^{n+\frac{1}{2}-k}\|_\infty$$

Hence, because our assumptions imply that the consistency errors tend to 0 as Δt and Δx tend to 0, we have the result. □

Then, a weak form of convergence for the entire scheme can be obtained:

Theorem 3.14. *Let us suppose that u_T, $m_0 > 0$ and f are so that $\forall n \in \mathbb{N}, u^{n+\frac{1}{2}}, v^n \in C^{1,2}([0,T] \times [0,1])$ and $u, v \in C^{1,2}([0,T] \times [0,1])$*
Then:

$$\lim_{n\to\infty} \limsup_{\substack{\Delta t, \Delta x \to 0 \\ (\Delta t, \Delta x) \text{ satisfying } (**)}} \|\widehat{u}^{n+\frac{1}{2}} - \tilde{u}\| = 0$$

$$\lim_{n\to\infty} \limsup_{\substack{\Delta t, \Delta x \to 0 \\ (\Delta t, \Delta x) \text{ satisfying } (**)}} \|\widehat{v}^{n+1} - \tilde{v}\| = 0$$

Proof. We have seen that $(u^{n+\frac{1}{2}})_n$ converges uniformly towards u by Theorem 2.2. Consequently:

$$\forall \varepsilon > 0, \exists n_0, \forall n \geq n_0, \forall \Delta t, \Delta x > 0, \|\tilde{u}^{n+\frac{1}{2}} - \tilde{u}\|_\infty \leq \varepsilon$$

Hence, $\forall n \geq n_0$, we know from Theorem 3.13 that:

$$\limsup_{\substack{\Delta t, \Delta x \to 0 \\ (\Delta t, \Delta x) \text{ satisfying } (**)}} \|\widehat{u}^{n+\frac{1}{2}} - \tilde{u}\| \leq \limsup_{\substack{\Delta t, \Delta x \to 0 \\ (\Delta t, \Delta x) \text{ satisfying } (**)}} \|\widehat{u}^{n+\frac{1}{2}} - \tilde{u}^{n+\frac{1}{2}}\| + \|\tilde{u}^{n+\frac{1}{2}} - \tilde{u}\| \leq \varepsilon$$

Thus, we get that:

$$\lim_{n\to\infty} \limsup_{\substack{\Delta t, \Delta x \to 0 \\ (\Delta t, \Delta x) \text{ satisfying } (**)}} \|\widehat{u}^{n+\frac{1}{2}} - \tilde{u}\| = 0$$

The result is obtained similarly for v. □

4. **Numerical examples and analysis.** Let us consider, as an example, the following problem on $\Omega = (0,1)$. Agents prefer to live far from the others and we consider the following[12] running payoff function f:

$$f(x,\xi) = -\min(1.4, \max(\xi, 0.7))$$

At the end of the period, when $t = T = 1$, we suppose that agents have the terminal payoff $u_T(x) = x^2(1-x)^2$.

We suppose that volatility is $\sigma = 0.8$.

We assume that the population distribution at time $t = 0$ is $m_0(x) = 1 - 0.2\cos(\pi x)$.

We consider the first numerical scheme[13] to approximate (ϕ, ψ) with $\Delta t = 1/250$ and $\Delta x = 1/50$. For the second numerical scheme to approximate (u, v), we used the scheme[14] with $\Delta t = 1/2000$ and $\Delta x = 1/50$. In both cases, we stopped the iterations in n when the difference between two approximations of the distribution m was below 10^{-6}, *i.e.*:

$$\left\| \widehat{\phi}^{n+\frac{1}{2}} \widehat{\psi}^{n+1} - \widehat{\phi}^{n-\frac{1}{2}} \widehat{\psi}^{n} \right\|_\infty < 10^{-6}$$

or

$$\left\| \exp\left(\frac{\widehat{u}^{n+\frac{1}{2}} - \widehat{v}^{n+1}}{\sigma^2} \right) - \exp\left(\frac{\widehat{u}^{n-\frac{1}{2}} - \widehat{v}^{n}}{\sigma^2} \right) \right\|_\infty < 10^{-6}$$

The algorithm stopped after respectively 34 iterations and 35 iterations for the first and the second scheme.

To illustrate the monotonicity of the first scheme, we plot on Figure 1 and Figure 2 the values of $\widehat{\phi}^{n+\frac{1}{2}}(0,\cdot)$ and $\widehat{\psi}^{n+1}(T,\cdot)$ as n increases. We see, as announced, that $(\widehat{\phi}^{n+\frac{1}{2}})_n$ is decreasing and $(\widehat{\psi}^{n})_n$ is increasing.

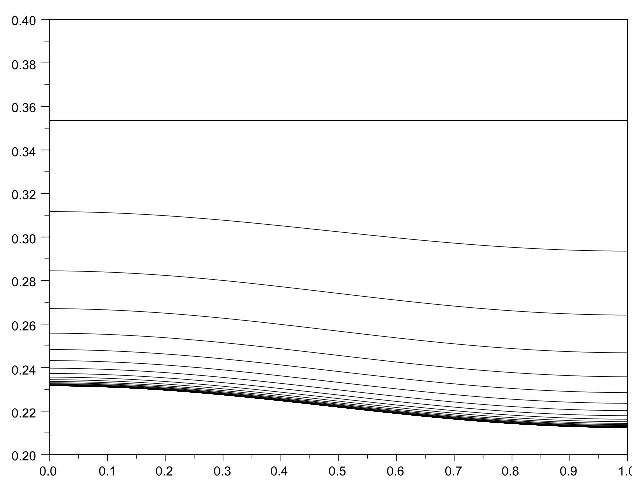

FIGURE 1. Evolution of $\widehat{\phi}^{n+\frac{1}{2}}(0,\cdot)$ as n increases.

[12] The bounds on ξ guarantee that f is bounded. In practice, these bounds are not binding when n is large.

[13] Using a Newton method to solve the implicit equations.

[14] $\widehat{v}^0$ was chosen empirically so that the monotonicity property is true.

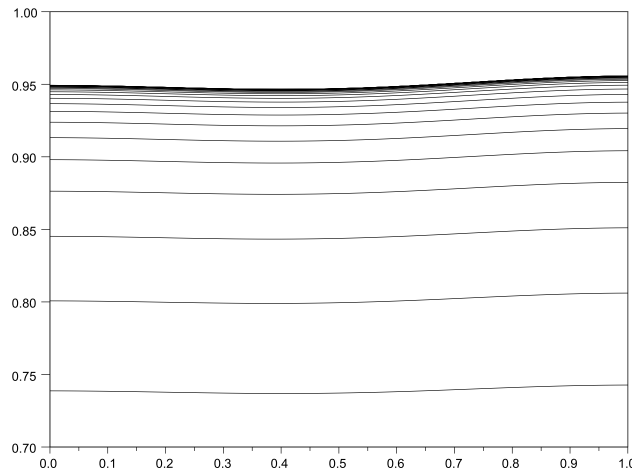

FIGURE 2. Evolution of $\widehat{\psi}^{n+1}(T,\cdot)$ as n increases.

The associated evolution of $\widehat{m}^{n+1}(T,\cdot) = \widehat{\phi}^{n+\frac{1}{2}}(T,\cdot)\widehat{\psi}^{n+1}(T,\cdot)$ as n increases (Figure 3) shows the specificity of the approach since we approximate a probability distribution function and only conserve total mass when $n \to \infty$.

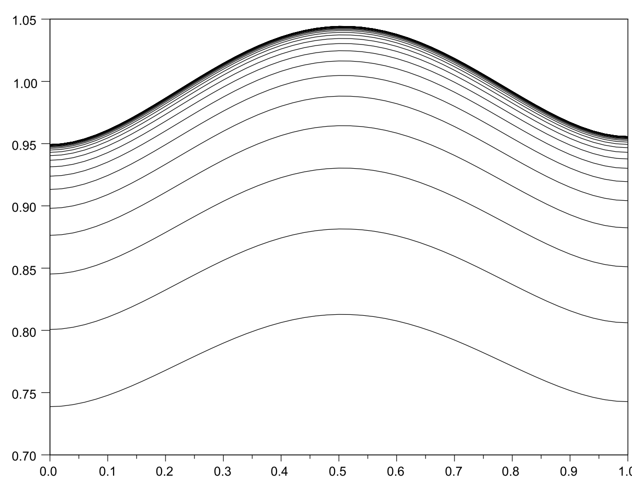

FIGURE 3. Evolution of $\widehat{m}^{n+1}(T,\cdot) = \widehat{\phi}^{n+\frac{1}{2}}(T,\cdot)\widehat{\psi}^{n+1}(T,\cdot)$ as n increases.

Coming to the second scheme, we plot on Figure 4 and Figure 5 the values of $\widehat{u}^{n+\frac{1}{2}}(0,\cdot)$ and $\widehat{v}^{n+1}(T,\cdot)$ as n increases. We see, as announced, that $(\widehat{u}^{n+\frac{1}{2}})_n$ and $(\widehat{v}^n)_n$ are both decreasing.

The associated evolution of $\widehat{m}^{n+1}(T,\cdot) = \exp\left(\frac{\widehat{u}^{n+\frac{1}{2}}(T,\cdot)-\widehat{v}^{n+1}(T,\cdot)}{\sigma^2}\right)$ as n increases (Figure 6) shows the same behavior as for the first scheme.

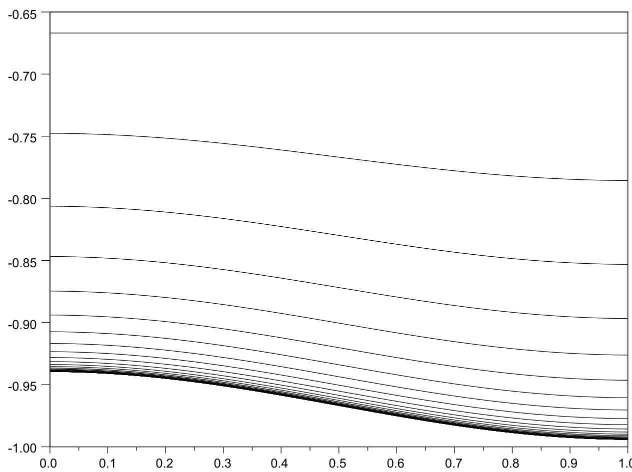

FIGURE 4. Evolution of $\widehat{u}^{n+\frac{1}{2}}(0,\cdot)$ as n increases.

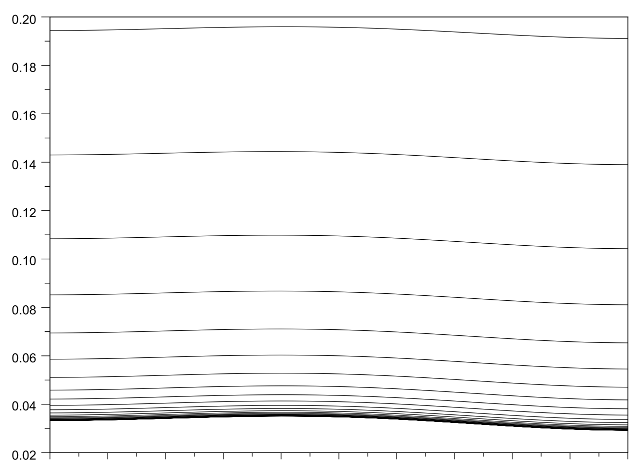

FIGURE 5. Evolution of $\widehat{v}^{n+1}(T,\cdot)$ as n increases.

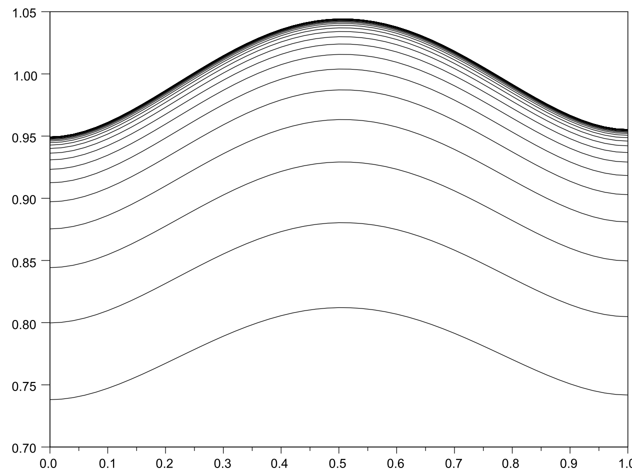

FIGURE 6. Evolution of $\widehat{m}^{n+1}(T,\cdot) = \exp\left(\frac{\widehat{u}^{n+\frac{1}{2}}(T,\cdot)-\widehat{v}^{n+1}(T,\cdot)}{\sigma^2}\right)$ as n increases.

Now, we illustrate the approximations we obtain after 34 iterations for ϕ and ψ (Figures 7 and 8) and the corresponding approximations for m and u (Figures 9 and 10) .

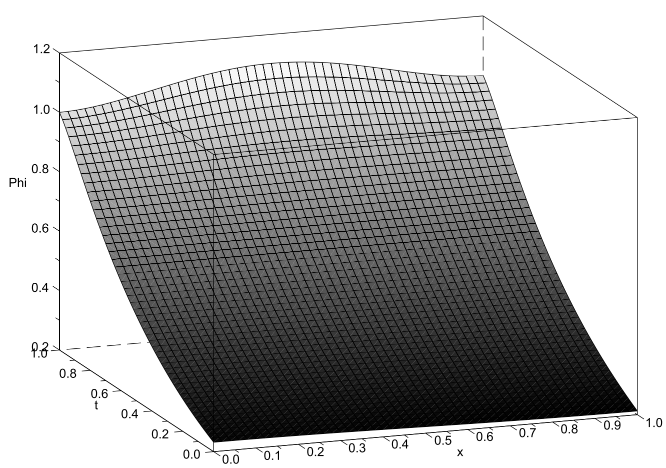

FIGURE 7. $\widehat{\phi}^{n+\frac{1}{2}}$ for $n = 34$.

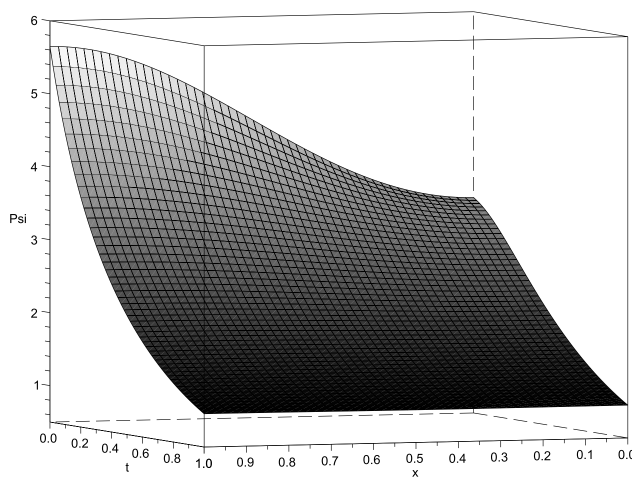

FIGURE 8. $\widehat{\psi}^{n+1}$ for $n = 34$.

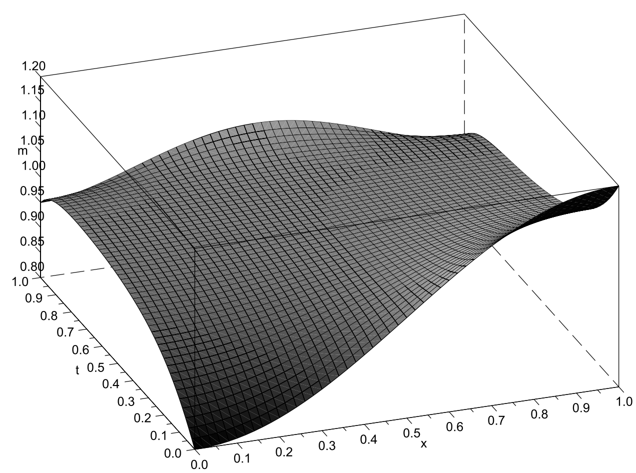

FIGURE 9. $\widehat{m}^{n+1} = \widehat{\phi}^{n+\frac{1}{2}}\widehat{\psi}^{n+1}$ for $n = 34$.

We see in particular that the population evolves in accordance with the intuition. Because people do not want to live together, agents on the right side move to the left and the distribution spreads. Then, because the final payoff is high near the center, the agents go toward the center when t is close to T.

Similarly, we illustrate the approximations we obtain after 35 iterations of the second scheme for u and v (Figures 11 and 12) and the corresponding approximations for m (Figure 13).

The difference between the approximations for m obtained with the two schemes (although they were not based on the same grid) is small and bounded by $1.2\ 10^{-3}$.

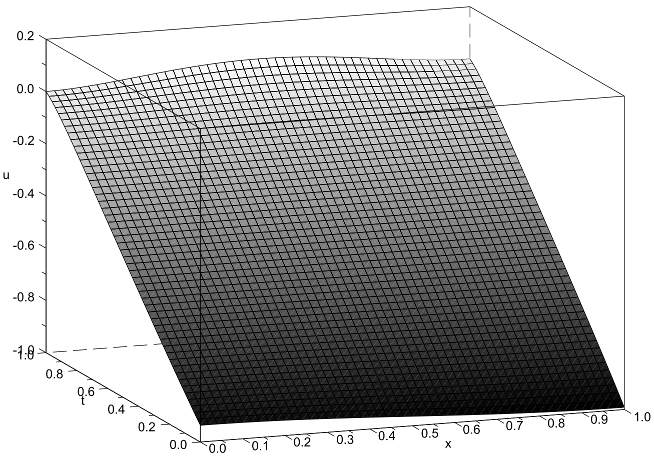

FIGURE 10. $\widehat{u}^{n+\frac{1}{2}} = \sigma^2 \log(\widehat{\phi}^{n+\frac{1}{2}})$ for $n = 34$.

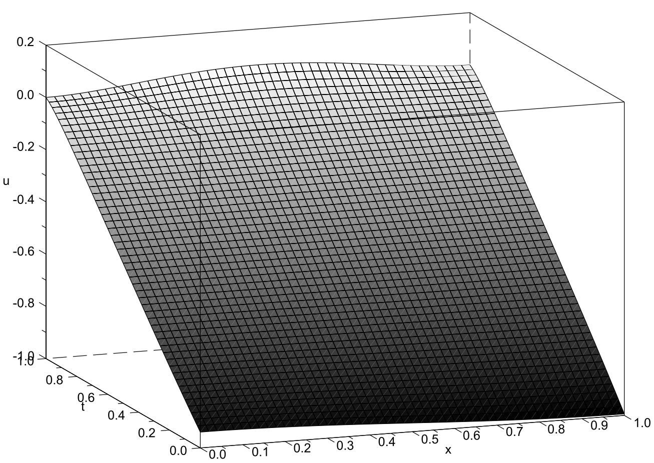

FIGURE 11. $\widehat{u}^{n+\frac{1}{2}}$ for $n = 35$.

In fact, if we analyze the behavior of the two schemes with respect to their convergence speed or as far as the evolution of computation time with Δt and Δx is concerned, we see that the respective performances are similar. However, the important point we want to focus on is not the comparison between the specific schemes we chose to illustrate the approximation of the couples (ϕ, ψ) and (u, v). One can indeed propose another scheme to approximate (ϕ, ψ) or a better scheme to approximate (u, v) – for example an implicit scheme with a better CFL condition based on bounds for the discrete gradients (see for instance the general method developed in [4] for Hamilton-Jacobi equations) – and we prefer to focus on the general

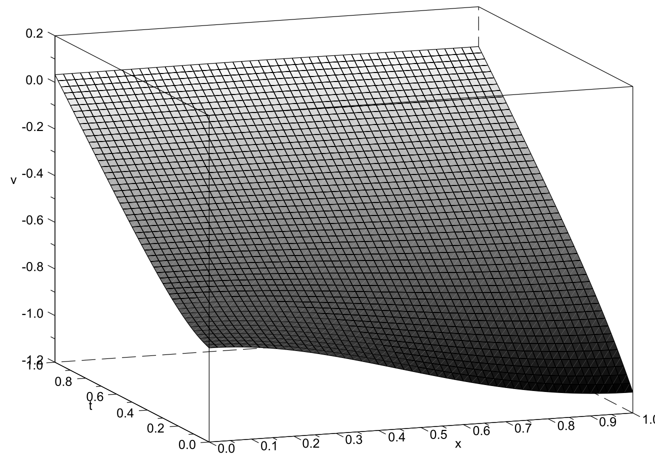

FIGURE 12. $\widehat{v}^{n+1}$ for $n = 35$.

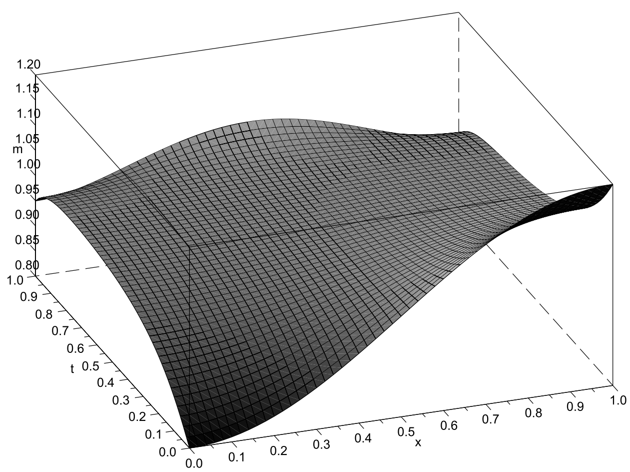

FIGURE 13. $\widehat{m}^{n+1} = \exp\left(\frac{\widehat{u}^{n+\frac{1}{2}} - \widehat{v}^{n+1}}{\sigma^2}\right)$ for $n = 35$.

differences between the approaches to approximate (ϕ, ψ) and those to approximate (u, v).

If we look at the definition of ϕ and ψ, we can assert that their value depend strongly on the value of the volatility parameter σ. Although u depends on σ, the influence of σ on $\phi = e^{\frac{u}{\sigma^2}}$ is indeed very important. Similarly, although m also depends on σ, the influence of σ on $\psi = me^{-\frac{u}{\sigma^2}}$ is often dominated by the term $\frac{1}{\sigma^2}$ when $\sigma \to 0$. As a consequence, ϕ and ψ take very large or very small values (depending on the sign of u) when σ is small and the distribution $m = \phi\psi$ will be computed multiplying two factors of different magnitudes. In addition to that, the partial differential equations $(\mathcal{S}_1)$ exhibit terms in σ^2 and terms in $\frac{1}{\sigma^2}$. A natural consequence is that a scheme based on an approximation of (ϕ, ψ) is not relevant when σ is small (see also [7]). The situation is different when one considers a scheme to approximate (u, v). Since $u - v = \sigma^2 \log(m)$, small values of σ do not appear to be as much a problem as they were when the couple (ϕ, ψ) was considered. Hence, schemes that approximate (u, v) are certainly better suited when it comes to considering vanishing viscosity.

5. **Conclusion.** In this paper we consider a change of variables already used in [7, 8, 11, 16] in order to introduce a new change of variables which transforms the mean field games equations into a system of Hamilton-Jacobi-Bellman equations when the hamiltonian is quadratic and $m_0 > 0$. A new constructive scheme is presented for the associated system of equations and we show that we can build decreasing sequences $(u^{n+\frac{1}{2}})_n$ and $(v^n)_n$ which converge respectively towards u and $u - \sigma^2 \log(m)$. As for the initial change of variables $(\phi, \psi) = (e^{\frac{u}{\sigma^2}}, me^{-\frac{u}{\sigma^2}})$ whose properties are recalled in this paper (see also [7]), we can build numerical schemes that have the same monotonic behavior as the constructive scheme. Such a scheme is proposed in this paper and we prove its conditional convergence. One of the main advantages of this new approach is that the values taken by the new variable $v = u - \sigma^2 \log(m)$ do not naturally blow up when $\sigma \to 0$.

Another potential application of this new change of variables is based on the interpretation of the system $(\mathcal{S}_2)$ as the system of equations of two coupled stochastic control problems, especially when $f(x, \xi) = f(x) - \log(\xi)$.

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