A-PRIORI ESTIMATES FOR STATIONARY
MEAN-FIELD GAMES

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Abstract. In this paper we establish a new class of a-priori estimates for stationary mean-field games which have a quasi-variational structure. In particular we prove $W^{1,2}$ estimates for the value function u and that the players distribution m satisfies $\sqrt{m} \in W^{1,2}$. We discuss further results for power-like nonlinearities and prove higher regularity if the space dimension is 2. In particular we also obtain in this last case $W^{2,p}$ estimates for u.

1. **Introduction.** Mean field games is a recent area of research started in the engineering community by Peter Caines and his co-workers [21], [20], and, independently, in the context of partial differential equations and viscosity solutions by Pierre Louis Lions and Jean Michel Lasry [22, 23, 24, 25].

Literature on mean field games and its applications is growing fast, for a recent survey see [27] and reference therein. Applications of mean field games arise in growth theory in economics [26] or environmental policy [2], for instance. We also believe that in the future, mean field games will play an important rôle in economics and population models. This is due to the fact that in many economics applications or population models there is a very large number of indistinguishable agents which behave in a rational but non-cooperative way. Understanding the behaviour of such systems as the number of agents tends to infinity is one of the most fundamental questions in these problems.

There is also a growing interest in numerical methods for mean-field problems [2], [1], [3]. One author and his collaborators [14] have also considered the discrete time, finite state problem, and the continuous time finite state problem [15]. Such models are relevant in many problems where a large number of agents have a choice among a finite number of states. Several problems have been worked out in detail in [18], [19], including applications to growth theory and the quadratic case.

Mean field games are frequently formulated as Hamilton-Jacobi type equation coupled with a transport equation, the adjoint of the linearization of the Hamilton-Jacobi equation. This class of problems, in the case the Hamilton-Jacobi equation does not depend on the solution of the transport equation was introduced in [9]. These methods were applied to study the vanishing viscosity problem [9], the differentiability of solutions of the infinity laplacian [6], Aubry-Mather theory in the non-convex setting [5] and systems of Hamilton-Jacobi equations and obstacle type problems [4], just to mention a few.

Given $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, the Evans-Aronsson problem [7], [8], consists in minimizing

$$\inf_{\phi} \int_{\mathbb{T}^d} e^{H(x, D\phi)} dx.$$

The stochastic Evans-Aronsson problem, which is a generalization of the previous problem and was introduced in [30], consists in minimizing

$$\inf_{\phi} \int_{\mathbb{T}^d} e^{\epsilon \Delta \phi + H(x, D\phi)} dx. \tag{1}$$

While studying this last problem two of the authors observed in [17] a new connection between mean-field games and a class of calculus of variations problems. In fact, the Euler-Lagrange equation for (1) can be written as

$$\begin{cases} \epsilon \Delta u + H(x, Du) = \ln m + \bar{H} \\ \epsilon \Delta m - \operatorname{div}(D_p H m) = 0, \end{cases}$$

which is the reference example of a stationary mean field game. Here the constant $\bar{H}$ is due to an additional normalization requirement for m, i.e. $\int m = 1$. In particular in [17] we obtained several a-priori estimates as well as the existence of smooth solutions when $H(p, x)$ is quadratic in p or in dimension 2.

In this paper we establish new a-priori estimates for a class of stationary mean-field games. We consider mean-field games of the form

$$\begin{cases} \Delta u + H(x, Du, m) = \bar{H} \\ \Delta m - \operatorname{div}(D_p H m) = 0, \end{cases} \tag{2}$$

where $H : \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ (here $\mathcal{P}(\mathbb{T}^d)$ is the set of Borel probability measures on $\mathbb{T}^d$) which satisfies suitable hypothesis. We require m to be a probability measure absolutely continuous with respect to Lebesgue measure. The constant $\bar{H}$ is chosen so that this normalization condition holds (we do not assume, however, uniqueness of $\bar{H}$ or solutions). In particular, whenever we say that $(u, m, \bar{H})$ is a solution to (2) we assume that $\int_{\mathbb{T}^d} m = 1$.

Our setting is the following: we assume that H is quasi-variational, this means that there exists $H_0(x, p) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, with g concave increasing such that

$$|H(x, p, m) - H_0(x, p) + g(m(x))| \le C. \tag{3}$$

This means that H has a local component $g(m(x))$ but does not imply that H is a local operator, that is, H could depend on m through convolution operators, difference operators or other non-local dependence. Note that if g is concave increasing then g^{-1} is convex increasing.

To explain the name quasi-variational, note that if we consider the variational problem

$$\inf_{\phi} \int_{\mathbb{T}^d} F(\Delta\phi + H_0(x, D\phi) - \bar{H})dx,$$

where F is convex increasing, the corresponding Euler-Lagrange equation can be written as

$$\begin{cases} \Delta u + H_0(x, Du) - \bar{H} = (F')^{-1}(m) \\ \Delta m - \operatorname{div}(D_p H_0(x, Du)m) = 0. \end{cases}$$

By defining $g = (F')^{-1}$, we get

$$H(x, p, m) = H_0(x, p) - g(m). \tag{4}$$

Thus we can think of quasi-variational mean-field games as perturbations of mean-field games with a variational structure.

If H is as in (4) then the monotonicity technique by Lions and Lasry [22] can be used to establish uniqueness. Note that, in this paper we do not require monotonicity in H. We should also observe that the setting we work here is different from the one in [22] as we do not require continuity in m with respect to weak convergence. In fact this fails even for Hamiltonians of the form (4).

In this paper, to simplify the exposition we consider the periodic setting. In this way the discussion of boundary conditions is avoided. We believe, however, that the same ideas can be applied in a variety of problems which include boundary conditions of various types.

The hypothesis under which our results hold are discussed in §2. Then in the next section we prove various a-priori bounds, and is structured as follows:

1. a-priori bounds in $\bar{H}$, Proposition 2;
2. $W^{1,2}$ integrability for u, Corollary 1;
3. regularity for m, $\sqrt{m} \in W^{1,2}$, Proposition 2;
4. higher integrability for m, if $g(m) = m^\gamma$, Corollary 3;
5. $D(\ln m) \in L^2$, Proposition 5;
6. $W^{2,2}$ and $W^{2,p}$ estimates for u, in dimension 2, Propositions 6 and 7.

The first two bounds, namely a-priori bounds for $\bar{H}$ and $W^{1,2}$ estimates for u are immediate for variational problems in the form (4) as one can use the variational formulation to extract bounds on any possible minimizer. However for quasi-variational problems we cannot use this technique and these are not obvious at all. These a-priori bounds, combined with additional regularity hypothesis on H can be used to prove a-priori bounds for higher norms of u and m, at least if $d = 2$. These in turn can be combined with continuation methods to establish existence of smooth solutions. This will be the subject of a future publication.

2. **Main assumptions.** We now describe the main assumptions, in addition to quasi-variationality, that we will be working with. We suppose that H_0 satisfies

$$|p|^2 \le C + C H_0(x, p). \tag{5}$$

This is a natural coercivity condition.

Given $(x, p, m) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{T}^d)$ we define the Lagrangian as

$$L(x, p, m) = -H(x, p, m) + p \cdot D_p H(x, p, m). \tag{6}$$

This is the natural definition of Lagrangian if one recalls that in classical mechanics the Lagrangian corresponding to a Hamiltonian $H(x, p)$ is given by

$$L(x, v) = \sup_p v \cdot p - H(p, x),$$

and the supremum is achieved for

$$v = D_p H(p, x).$$

Thus (6) corresponds to the Lagrangian function written in terms of the position-momentum (x, p) coordinates rather than position-velocity (x, v) as it is customary.

We suppose that

$$L(x, p, m) \geq cH_0(x, p) - C. \tag{7}$$

In view of (5) this is a coercivity hypothesis on L and implies

$$L(x, p, m) \geq c|p|^2 - C.$$

From (5) and (7) it follows that there exists C such that for every function $\varphi : \mathbb{T}^d \to \mathbb{R}^d$, we have

$$\int_{\mathbb{T}^d} L(x, \varphi(x), m) dm \geq -C, \tag{8}$$

note that this last fact can hold independently of (7), for instance if

$$L(x, v, m) = \frac{v^2}{2} + \ln m$$

then, because

$$m \ln m \geq -1$$

we obtain (8).

We assume further that H is uniformly convex in p,

$$D_{pp}^2 H(x, p, m) \geq \gamma > 0. \tag{9}$$

We assume the following bound

$$|D_p H(x, p, m)|^2 \leq C + CH_0(x, p), \tag{10}$$

which can can be relaxed to the more general bound

$$|D_p H(x, p, m)|^2 \leq C + CH_0(x, p, m) + Cm^\delta, \tag{11}$$

for $0 < \delta < 1 - \frac{2}{2^*}$, where 2^* is the Sobolev conjugated exponent $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{d}$.

Set

$$\hat{H}_{xx}(x, p, m) = D_{xx}(H(x, p, m) + g(m)),$$

and

$$\hat{H}_{px} = D_x(D_p H(x, p, m)),$$

then we suppose the following estimates hold:

$$|\hat{H}_{xx}| \leq C + CH_0, \quad \text{and} \quad |\hat{H}_{px}|^2 \leq C + CH_0; \tag{12}$$

As we will see in section 3, remark 3, this bound can sometimes be relaxed to

$$|\hat{H}_{xx}|, |\hat{H}_{px}|^2 \leq C + CH_0(x, p) + Cm^{2^*/2}.$$

Finally we suppose

$$H_0(x, p) \leq C + C|p|^2. \tag{13}$$

As a model example we could consider

$$H(x, p, m) = \frac{|p|^2}{2} + V(x) + W(\eta * m) - m^\gamma,$$

where V and W are smooth functions, η is a smoothing kernel and $0 < \gamma < 1$. In this case

$$L(x, p, m) = \frac{|p|^2}{2} - V(x) - W(\eta * m) + m^\gamma.$$

All the previous hypothesis can easily be checked. This example represents a situation in which there is a local interaction term of the form $m^\gamma(x)$, which is in fact repulsive. This means that increasing local concentrations at a point x increase the cost for a reference player located at x. The non-local term $W(\eta * m)$ takes into account concentration effects in a neighborhood of the position of each player, and the potential $V(x)$ encodes the different desirability of the various positions.

3. **A-priori estimates.** In this section we study several a-priori estimates for the solutions of the mean-field game equation (2). Whereas for variational mean-field games one has, from the variational principle, estimates for $\bar{H}$ and $\|u\|_{W^{1,2}}$, see [17], these are not obvious, for general mean-field games, even with the quasi-variational structure. As the key objective of this section is to establish a-priori estimates we assume all solutions to be classical smooth solutions.

Proposition 1. Let $(u, m, \bar{H})$ solve (2), with $\int_{\mathbb{T}^d} m = 1$, then

$$\int_{\mathbb{T}^d} L(x, Du, m)dm = -\bar{H}. \tag{14}$$

Proof.

$$
\begin{aligned}
\bar{H} &= \int_{\mathbb{T}^d} \Delta u + H(x, Du, m)dm \\
&= \int_{\mathbb{T}^d} H(x, Du, m) - D_p H(x, Du, m)Du\, dm \\
&= -\int_{\mathbb{T}^d} L(x, Du, m)dm
\end{aligned}
$$

$\square$

In Aubry-Mather theory (see [10, 11, 12, 13, 28, 29], for instance) one considers this first order case where H does not depend on m. If one looks for the unique value $\bar{H}$ for which the cell problem

$$H(x, Du) = \bar{H}$$

has a viscosity solution, then one has

$$\int_{\mathbb{T}^d} L(x, Du)dm = -\bar{H}.$$

A similar result also holds for the stochastic Mather problem [16].

Proposition 2. Let $(u, m, \bar{H})$ solve (2). Assume (3), (5), and (8). Then there exists a constant C, independent of (u, m) such that

$$|\bar{H}| \leq C. \tag{15}$$

Proof. By Proposition 1 and (8)

$$\bar{H} = -\int_{\mathbb{T}^d} L(x, Du, m)dm \leq C.$$

To prove the opposite inequality, observe that by the quasi-variationality hypothesis (3) we have

$$\bar{H} \geq H_0(x, Du) - g(m) - C + \Delta u.$$

Then, using (5),

$$g(m) \geq -C - \bar{H} + \Delta u$$

Because g is increasing, we have

$$m \geq g^{-1} \left(-C - \bar{H} + \Delta u\right).$$

Since g^{-1} is a convex function, by Jensen's inequality we have

$$\int_{\mathbb{T}^d} g^{-1} \left(\Delta u - C - \bar{H}\right) dx \geq g^{-1} \left(\int_{\mathbb{T}^d} \Delta u - C - \bar{H}\right) dx = g^{-1} \left(-C - \bar{H}\right).$$

Since m is a probability measure, it follows that

$$1 \geq g^{-1} \left(-C - \bar{H}\right)$$

and then

$$\bar{H} \geq -C.$$

$\square$

Corollary 1. *Let $(u, m, \bar{H})$ solve* (2). *Assume* (3), (5), *and* (8). *Then*

$$\int_{\mathbb{T}^d} H_0(x, Du) dx \leq C,$$

and so, $Du \in L^2$.

Proof.

$$\int_{\mathbb{T}^d} H_0(x, Du) dx \quad \leq \quad \int_{\mathbb{T}^d} \Delta u + H(x, Du, m) + g(m) dx + C \qquad (16)$$

$$\leq \quad \bar{H} + \int_{\mathbb{T}^d} g(m) dx + C \qquad (17)$$

$$\leq \quad \tilde{C}, \qquad (18)$$

where (16) comes from hypothesis (3) and the fact that $\int_{\mathbb{T}^d} \Delta u dx = 0$ by periodicity, then (17) follows from (2), (18) from Jensen's inequality using the concavity of g, and estimate (15) for $\bar{H}$. $\square$

Proposition 3. *Let $(u, m, \bar{H})$ be a solution of* (2). *Assume* (3), (5), *and* (7). *Then*

$$\int_{\mathbb{T}^d} H_0(x, Du) dm \leq C. \qquad (19)$$

Proof. By Proposition 1 and (7)

$$\bar{H} \quad = \quad -\int_{\mathbb{T}^d} L(x, Du, m) dm$$

$$\leq \quad -c \int_{\mathbb{T}^d} H_0(x, Du) dm + C.$$

This, together with (15) implies (19). $\square$

The previous a-priori estimates are valid for deterministic mean-field games of the form

$$\begin{cases} H(x, Du, m) = \bar{H} \\ \operatorname{div}(D_p H m) = 0, \end{cases}$$

with similar proofs. However, in order to obtain further regularity the presence of the Laplacian is essential.

Corollary 2. *Let $(u, m, \bar{H})$ be a solution of* (2). *Assume* (3), (5), (7), *and either* (10) *or* (11). *Then*

$$\|\sqrt{m}\|_{W^{1,2}} \leq C. \tag{20}$$

Proof. Multiply

$$\Delta m - \operatorname{div}(D_p H m) = 0$$

by $\ln m$ and integrate by parts to obtain

$$\int_{\mathbb{T}^d} \frac{|Dm|^2}{m} dx = \int D_p H Dm \, dx$$

$$\leq \frac{1}{2} \int_{\mathbb{T}^d} |D_p H|^2 dm + \frac{1}{2} \int_{\mathbb{T}^d} \frac{|Dm|^2}{m} dx.$$

Assuming (10) we get

$$4 \int_{\mathbb{T}^d} |D\sqrt{m}|^2 dx \leq \int_{\mathbb{T}^d} |D_p H|^2 dm \leq C + C \int H_0 \, dm. \tag{21}$$

From Proposition 3 and $\int m = 1$ we get (20). $\qquad\qquad\square$

Remark 1. In the proof of the last proposition we could replace the assumption (10) by the more general hypothesis (11).

Indeed, taking into account Hölder's inequality

$$\int |fh| \leq \left(\int |f|^p\right)^{\frac{1}{p}} \left(\int |h|^q\right)^{\frac{1}{q}} ; \quad \frac{1}{p} + \frac{1}{q} = 1,$$

let's consider

$$|f|^p = |m| ; \quad |h|^q = \left|\sqrt{m}\right|^{2^*},$$

so that

$$|fh| = m^{1+\delta}.$$

This means that

$$\frac{1}{p} + \frac{2^*}{2} \frac{p-1}{p} = 1 + \delta,$$

from which we deduce

$$\frac{1}{p} = 1 - \frac{2\delta}{2^* - 2} ; \quad \frac{1}{q} = \frac{2\delta}{2^* - 2}$$

and get

$$\int m^{1+\delta} \leq \left(\int m\right)^{\frac{1}{p}} \left(\int \left|\sqrt{m}\right|^{2^*}\right)^{\frac{2\delta}{2^* - 2}}.$$

Since $\int m = 1$, we have

$$\int m^{1+\delta} \leq \left(\int \left|\sqrt{m}\right|^{2^*}\right)^{\frac{2\delta}{2^* - 2}} \leq C \left(\int \left|D\sqrt{m}\right|^2\right)^{\frac{2^*\delta}{2^* - 2}}.$$

Thus instead of (21) we have

$$4 \int_{\mathbb{T}^d} |D\sqrt{m}|^2 dx \leq \int_{\mathbb{T}^d} |D_p H|^2 dm$$

$$\leq C + C \int H_0 dm + C \int m^{1+\delta} dx$$

$$\leq C + C \int H_0 dm + C \left(\int |D\sqrt{m}|^2 \right)^{\frac{2^*\delta}{2^* - 2}}$$

This inequality yields (20) if

$$\frac{2^*\delta}{2^* - 2} < 1,$$

that is,

$$0 < \delta < 1 - \frac{2}{2^*}.$$

Remark 2. Under either conditions of Corollary 2, we have $\sqrt{m} \in W^{1,2}$ from which it follows $\sqrt{m} \in L^{2^*}$ and so $m \in L^{2^*/2}$.

Proposition 4. *Let* $(u, m, \bar{H})$ *be a solution of* (2). *Assume* (3), (5), (7), (9), *and* (12). *Then*

$$\int_{\mathbb{T}^d} g'(m)|Dm|^2 dx \leq C, \int_{\mathbb{T}^d} |D^2 u|^2 dm \leq C \tag{22}$$

Proof. Applying the Laplacian Δ to first equation of (2) we get

$$\Delta \Delta u + \hat{H}_{x_i x_i}(x, Du, m) + 2\hat{H}_{p_k x_i}(x, Du, m)u_{x_k x_i} + \text{Tr}(D^2_{pp}H(x, Du, m)(D^2 u)^2)$$
$$+ D_p H(x, Du, m)D\Delta u - \text{div}(g'(m)Dm) = 0.$$

Integrating w.r.t. m

$$\int_{\mathbb{T}^d} g'(m)|Dm|^2 dx$$

$$+ \int_{\mathbb{T}^d} \text{Tr}(D^2_{pp}H(x, Du, m)(D^2 u)^2) + \hat{H}_{x_i x_i}(x, Du, m) + 2\hat{H}_{p_k x_i}(x, Du, m)u_{x_k x_i} dm = 0.$$

By elementary estimates

$$\int_{\mathbb{T}^d} g'(m)|Dm|^2 dx + \frac{\gamma}{2} \int_{\mathbb{T}^d} |D^2 u|^2 m \leq \int_{\mathbb{T}^d} |\hat{H}_{xx}(x, Du, m)| + C|\hat{H}_{px}(x, Du, m)|^2 dm.$$

From (12) and Proposition 3 we have

$$\int_{\mathbb{T}^d} g'(m)|Dm|^2 dx + \frac{\gamma}{2} \int_{\mathbb{T}^d} |D^2 u|^2 dm \leq C + C \int_{\mathbb{T}^d} H_0(x, Du)dm \leq C. \tag{23}$$

Hence $\int_{\mathbb{T}^d} g'(m)|Dm|^2 dx$ and $\int_{\mathbb{T}^d} |D^2 u|^2 dm$ are bounded. $\qquad \square$

Remark 3. In the previous proposition we could have replaced (12) by

$$|\hat{H}_{xx}|, |\hat{H}_{xp}|^2 \leq C + CH_0 + Cm^{2^*/2 - 1},$$

and requiring in addition (10) or (11) as in Remark 1. In this situation (23) is replaced by

$$\int_{\mathbb{T}^d} g'(m)|Dm|^2 dx + \frac{\gamma}{2} \int_{\mathbb{T}^d} |D^2 u|^2 dm \leq C + C \int_{\mathbb{T}^d} H_0(x, Du)dm + C \int_{\mathbb{T}^d} m^{2^*/2} dx.$$

This, combined with Corollary (2), would give a similar estimate, using Remark 2. Similarly, under the assumptions of Remark 1, using again Remark 2 we get the same estimate.

Corollary 3. *Let $g(m) = m^\gamma$, with $0 < \gamma < 1$, and $(u, m, \bar{H})$ be a solution of (2). Assume (3), (5), (7), (9), and (12). Then*

$$\int_{\mathbb{T}^d} m^{\frac{2^*}{2}(\gamma+1)} \leq C \tag{24}$$

where 2^* is the Sobolev conjugate exponent $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{d}$.

If in addition

$$2\gamma + 1 \leq \frac{2^*}{2}(\gamma+1),$$

then

$$\int_{\mathbb{T}^d} |Du|^4 dm \leq C. \tag{25}$$

Proof. Let

$$f = m^{\frac{1}{2}(\gamma+1)}.$$

Since $\gamma < 1$ we have

$$0 \leq \int f\, dx \leq 1.$$

By Poincaré inequality

$$\int_{\mathbb{T}^d} f^2 dx - \left(\int f\, dx\right)^2 = \int_{\mathbb{T}^d} \left(f - \int_{\mathbb{T}^d} f\, dx\right)^2 dx \leq C \int_{\mathbb{T}^d} |Df|^2 dx.$$

Thus, using (22)

$$\|f\|_{W^{1,2}} \leq 1 + C \int_{\mathbb{T}^d} |Df|^2 dx$$

$$\leq 1 + C \int_{\mathbb{T}^d} g'(m)|Dm|^2 dx \leq \tilde{C}.$$

By Sobolev inequality

$$\|f\|_{2^*} \leq C\|f\|_{W^{1,2}} \leq \hat{C},$$

proving (24) .

Assume now

$$p = 2\frac{2\gamma + 1}{\gamma + 1} \leq 2^*,$$

then

$$\int_{\mathbb{T}^d} g(m)^2 m = \int_{\mathbb{T}^d} f^p \leq C\|f\|_{2^*}^p \leq \bar{C}. \tag{26}$$

By (3) and Proposition 2

$$H_0(x, Du) \leq C - \Delta u + g(m).$$

Then

$$|Du|^4 \leq C + C H_0(x, Du)^2$$

$$\leq K + K g(m)^2 + K|D^2 u|^2.$$

Integrating w.r.t. m, and using (26), (22), we get estimate (25).   □

Proposition 5. *Let $(u, m, \bar{H})$ be a solution of* (2). *Assume* (3), (5), (8), *and*(10). *Then*

$$\int_{\mathbb{T}^d} \frac{|Dm|^2}{m^2} \leq C. \tag{27}$$

Proof. Multiply

$$\Delta m - \mathrm{div}(D_p H m) = 0$$

by $\frac{1}{m}$ and integrate by parts to obtain

$$\int_{\mathbb{T}^d} \frac{|Dm|^2}{m^2} \leq C \int_{\mathbb{T}^d} |D_p H(x, Du, m)|^2$$

$$\leq \tilde{C} + \tilde{C} \int H_0(x, Du).$$

Corollary 1 gives the estimate. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4. Since $D \ln m = \frac{Dm}{m}$, the previous proposition states that $D \ln m \in L^2$. In [17] using the variational structure we also have that $\ln m \in L^2$. It is not clear, however, if the same holds in the generalized setting of the present paper.

Remark 5. Note that (27) also holds under the conditions of Corollary 2 if (10) is replaced by (11).

Proposition 6. *Let $(u, m, \bar{H})$ be a solution of* (2). *Suppose* (3), (5), (7), (10) *and* (13) *hold. Assume $g(m) = m^\gamma$ with $0 < \gamma < 1$. Suppose $d = 2$. Then $u \in W^{2,2}$.*

Proof. Observe that

$$\int_{\mathbb{T}^2} (\Delta u)^2 \leq C \int_{\mathbb{T}^2} g(m)^2 + C \int_{\mathbb{T}^2} H_0(x, Du)^2 + C.$$

Because $\sqrt{m} \in W^{1,2}$ we have

$$\int_{\mathbb{T}^2} m^{2\gamma} \leq C.$$

As in [17] we have(using (13))

$$\int_{\mathbb{T}^2} H_0(x, Du)^2 \leq C,$$

which then yields the result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 7. *Let $(u, m, \bar{H})$ be a solution of* (2). *Suppose* (3), (5), (7), (10) *and* (13) *hold. Assume $g(m) = m^\gamma$ with $0 < \gamma < 1$. Suppose $d = 2$. Then $u \in W^{2,p}$, for all $1 < p < \infty$.*

Proof. We have

$$|\Delta u| \leq C + |H_0(x, Du)| + |g(m)|.$$

Since $u \in W^{2,2}$, by Proposition (6), $Du \in L^q$ for all q, by Sobolev's Theorem.

From (13) it follows $H_0(x, Du) \in L^p$, for all $1 < p < \infty$. Also, since $\sqrt{m} \in W^{1,2}$, we have $g(m) \in L^p$ for all $1 < p < \infty$. Thus $\Delta u \in L^p$ for all $1 < p < \infty$ from which it follows $u \in W^{2,p}$, by standard elliptic regularity. $\qquad\qquad\qquad\qquad\square$

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