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LONG TIME AVERAGE OF MEAN FIELD GAMES

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Abstract. We consider a model of mean field games system defined on a time interval $[0, T]$ and investigate its asymptotic behavior as the horizon T tends to infinity. We show that the system, rescaled in a suitable way, converges to a stationary ergodic mean field game. The convergence holds with exponential rate and relies on energy estimates and the Hamiltonian structure of the system.

1. **Introduction.** The aim of this paper is to study the asymptotic behavior, as T goes to infinity, of the solution (u^T, m^T) to the mean field game system in $[0, T]$:

$$\left\{\begin{array}{ll} (i) & -\partial_t u^T - \Delta u^T + \frac{1}{2}|Du^T|^2 = F(x, m^T) \\ (ii) & \partial_t m^T - \Delta m^T - \mathrm{div}(m^T Du^T) = 0 \\ (iii) & m^T(0) = m_0,\ u^T(x, T) = G(x) \end{array}\right. \tag{1}$$

Let us recall that the above system has been introduced by Lasry and Lions to describe differential games with a infinite number of indistinguishable players [11, 12, 13]. Its heuristic interpretation is the following. An average agent controls the stochastic differential equation

$$dX_t = \alpha_t dt + \sqrt{2} dB_t\,, \quad X_0 = x$$

where (B_t) is a standard Brownian motion. He aims at minimizing the quantity

$$\mathbb{E}\left[\int_0^T \frac{1}{2}|\alpha_s|^2 + F(X_s, m_s^T)\ ds + G(X_T)\right]\ .$$

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His value function u^T then satisfies (1)-(i) while his optimal control is -again heuristically- given in feedback form by $\alpha^*(x,t) = -Du(x,t)$. Now, if all agents argue in this way, their density $m(x,t)$ at time t will evolve with a velocity which is due, on the one hand, to the diffusion, and, on the other hand, to the drift term $-Du(x,t)$. This leads to the Kolmogorov equation (1)-(ii).

It is a natural question to investigate the behavior of the solution (u^T, m^T) of (1) when the horizon becomes larger and larger. For this study, we assume that the data are periodic in space, e.g. $\mathbb{Z}^N-$periodic to fix the ideas, and we set $Q_1 = [0,1]^N$. Then we introduce the rescaled maps

$$\upsilon^T(t,x) := u^T(tT,x) \qquad ; \qquad \mu^T(t,x) := m^T(tT,x) \qquad (t,x) \in [0,1]\times\mathbb{R}^N\ . \quad (2)$$

and the ergodic problem, with unknowns $(\bar\lambda, \bar u, \bar m)$, introduced by Lasry and Lions in [13]:

$$\begin{cases} (i) \quad \bar\lambda - \Delta \bar u + \frac{1}{2}|D\bar u|^2 = F(x,\bar m) \\ (ii) \quad -\Delta \bar m - \operatorname{div}(\bar m D\bar u) = 0 \\ \qquad \int_{Q_1} \bar u\, dx = 0\,, \quad \int_{Q_1} \bar m\, dx = 1 \end{cases} \qquad (3)$$

Our main result states that, as $T \to +\infty$, υ^T/T converges to $(1-t)\bar\lambda$ while μ^T converges to $\bar m$ in L^p for some $p \geq 1$ provided $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is increasing with respect to the last variable (Theorem 2.1). If, moreover, F is uniformly increasing with respect to m, these convergences hold with a (locally uniform in time) exponential rate (Theorem 3.1).

In order to compare our result to the existing ones and explain our method of proof, let us briefly recall what happens when the system is decoupled (i.e., $F = F(x)$). In this case, the first equation becomes a standard viscous Hamilton-Jacobi equation, for which the analysis of the long time average has been the aim of several works in the recent years: see in particular Evans [5], Arisawa-Lions [3] and Barles-Souganidis [4]. The main difficulty in the proof of the convergence is concentrated in the construction of a solution $(\bar\lambda, \bar u)$ of the ergodic problem (3-(i)). Indeed, given $(\bar\lambda, \bar u)$, one easily checks that the maps $\bar u + \bar\lambda(T-t) \pm C$ are super/subsolution of (1-(i)), and the comparison principle implies that $u^T - \bar u - \bar\lambda(T-t)$ is bounded. Hence $u^T(0,\cdot)/T$ uniformly converges to $\bar\lambda$. As for the convergence of m^T, it is just the convergence of a distribution towards the invariant measure $\bar m$, which can be deduced from the convergence of Du^T.

It turns out that the coupling in the system (1) radically changes the nature of the problem: indeed, to prove the convergence of u^T, it is mandatory to know that the term $F(x, m^T)$ has a limit, while the convergence of m^T cannot be expected unless Du^T converges. Note also that initial and terminal conditions are fixed so that we cannot expect a convergence at time $t = 0$ or at $t = T$. To the best of our knowledge, this kind of problem has been considered in the literature only in [6], for discrete mean field games (where time and space are discrete): in that paper the convergence, with an exponential rate, of the discrete equivalent of (1) to the discrete equivalent of (3) is proved. To handle the difficulties due to continuous time and continuous space, we first introduce the scaling (2) and study the behavior of (υ^T, μ^T) into compact subsets of $(0,1) \times Q_1$. The key ingredient of the proofs is the following energy equality, established in [11, 12, 13]:

$$\int_0^1 \int_{Q_1} \frac{(\mu^T + \bar{m})}{2} |D\upsilon^T - D\bar{u}|^2 + (F(x,\mu^T) - F(x,\bar{m}))(\mu^T - \bar{m})\, dxdt$$
$$= -\frac{1}{T}\left[\int_{Q_1} (\upsilon^T - \bar{u})(\mu^T - \bar{m}) dx\right]_0^1 \tag{4}$$

The (heuristic) interest of this equality is the following: by (3-(ii)), $\bar{m}$ is bounded below by a positive constant, while m is always nonnegative; on another hand, F is increasing with respect to the last variable. Therefore the above inequality implies that the quantity $\|D(\upsilon^T - \bar{u})\|_2$ is controlled by the right-hand side. It turns out that this right-hand side vanishes as $T \to +\infty$. Indeed, the mean field game system (1) has an Hamiltonian structure which entails that the quantity

$$\frac{1}{2}\int_{Q_1} m^T(t)\,|Du^T(t)|^2\, dx + \int_{Q_1} \langle Du^T(t), Dm^T(t)\rangle\, dx - \int_{Q_1} \Phi(x, m^T(t))\, dx$$

—where $\Phi(x,\cdot)$ is a primitive of $F(x,\cdot)$—does not depend on $t \in [0,T]$ and is bounded with respect to T. For $t = 0$ this gives a bound on $\|D\upsilon^T(0)\|_2$, which in turns implies that the right-hand side of (4) tends to 0. Once we know that $\|D(\upsilon^T - \bar{u})\|_2$ vanishes, one gets the convergence of μ^T to $\bar{m}$, and then the convergence of υ^T/T.

Although, for simplicity, we consider here only the case of quadratic Hamiltonians, we expect that the results actually hold for much more general systems of the form

$$\left\{\begin{array}{rl} (i) & -\partial_t u - \Delta u^T + H(x, Du^T) = F(x, m^T) \\ (ii) & \partial_t m^T - \Delta m^T - \mathrm{div}(m^T D_p H(x, Du^T)) = 0 \\ (iii) & m(0) = m_0,\ u(T) = G(x, m(T)) \end{array}\right.$$

where F and G can be local, or nonlocal, functions of m. We intend to come back to this more general framework in a future work.

The paper is organized as follows: In the first section, we assume that F is increasing and prove the convergence of υ^T/T and μ^T. In the second section we show that the convergence is actually exponential when F is uniformly increasing. We complete the paper by a proof of the existence of a classical solution to (1) without any growth restriction imposed on F.

2. **The convergence result.**

Throughout the paper, we assume that $F : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^1$ function which is $\mathbb{Z}^N-$periodic in x and increasing in m. We also assume that

$$m_0 : \mathbb{R}^N \to \mathbb{R} \quad \text{is Lipschitz continuous, } \mathbb{Z}^N\text{−periodic, } m_0 > 0 \text{ and } \textstyle\int_{Q_1} m_0 = 1.$$

and that

$$G : \mathbb{R}^N \to \mathbb{R} \text{ is } \mathbb{Z}^N\text{−periodic and } \mathcal{C}^2.$$

Under the above conditions, the Cauchy problem and the ergodic problem are well-posed:

Proposition 1. *Under the above assumptions, there is a unique classical solution* (u^T, m^T) *to (1). In the same way, equation (3) has a unique classical solution* $(\bar{\lambda}, \bar{u}, \bar{m})$. *Moreover* $\bar{m} = e^{-\bar{u}} / \left(\int_{Q_1} e^{-\bar{u}}\right) > 0$.

The proof of Proposition 1 is given in appendix. Let us stress that no condition on the growth of $F(x,\cdot)$ is assumed, moreover this result actually holds under the weaker assumption that F is nondecreasing with respect to m.

Let us recall the notation for the scaled functions

$$\upsilon^T(x,t) := u^T(x,tT) \qquad ; \qquad \mu^T(x,t) := m^T(x,tT)$$

Theorem 2.1. *As $T \to +\infty$,*

- *the map $\upsilon^T(t,\cdot)/T$ converges to $(1-t)\bar\lambda$ in $L^1(Q_1)$ uniformly with respect to $t \in [0,1]$,*
- *the maps υ^T/T and $\upsilon^T - \int_{Q_1} \upsilon^T(t)dy$ converge to the maps $(1-t)\bar\lambda$ and $\bar u$ respectively in $L^2((0,1)\times Q_1)$,*
- *the map μ^T converges to $\bar m$ in $L^p((0,1)\times Q_1)$, for any $p < \frac{N+2}{N}$.*
- *the maps $F(\cdot,\mu^T)$ and $F(\cdot,\mu^T)\mu^T$ converge respectively to $F(\cdot,\bar m)$ and to $F(\cdot,\bar m)\bar m$ in $L^1((0,1)\times Q_1)$.*

Remark 1. If we only assume that F is nondecreasing, we show the same convergence for the maps υ^T/T and $\upsilon^T - \int_{Q_1}\upsilon^T(t)dy$. However, in this case we only prove weak convergences of the maps μ^T and $F(\cdot,\mu^T)$.

Remark 2. The assumptions on m_0 and $G(x)$ could be weakened, obtaining the same convergence result. In particular, the C^2 character of $G(x)$ is only assumed here to work with classical solutions. Up to a slight refinement of the preliminary a priori estimates below, one can actually take a final cost $u(x,T) = G(x,m(T))$ where G is nondecreasing with respect to m.

We need some preliminary estimates. In order to simplify some computations, we assume, without loss of generality, that F is non negative (otherwise, one can replace F by $F - \min_{x\in Q_1} F(x,0)$, which is nonnegative since $F(x,\cdot)$ is nondecreasing).

Lemma 2.2. *For any $0 \le t_1 < t_2 \le T$ we have*

$$\left[\int_{Q_1} (u^T - \bar u)(m^T - \bar m)dx\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{Q_1} \frac{(m^T+\bar m)}{2}|Du^T - D\bar u|^2 + (F(x,m^T) - F(x,\bar m))(m^T - \bar m)\ dxdt = 0$$

Proof. This computation comes directly from [13]. Since T is fixed, we simply write m and u instead of m^T and u^T. We first integrate over $(t_1,t_2)\times Q_1$ the equation satisfied by $(u-\bar u)$ multiplied by $(m-\bar m)$. Since $\int_{Q_1}(m(x,t) - \bar m(x))dx = 0$, we get, after integration by parts:

$$\int_{t_1}^{t_2}\int_{Q_1} -\partial_t u(m-\bar m) + \langle D(m-\bar m), D(u-\bar u)\rangle + \frac{1}{2}(m-\bar m)(|Du|^2 - |D\bar u|^2) = \int_{t_1}^{t_2}\int_{Q_1} (F(x,m) - F(x,\bar m))(m-\bar m)\ .$$

In the same way we integrate over $(t_1,t_2)\times Q_1$ the equation satisfied by $(m-\bar m)$ multiplied by $(u-\bar u)$:

$$\int_{t_1}^{t_2}\int_{Q_1} (u-\bar u)\partial_t m + \langle D(m-\bar m), D(u-\bar u)\rangle + \langle mDu - \bar m D\bar u, D(u-\bar u)\rangle = 0\ .$$

We now compute the difference between the second equation and the first one:

$$\int_{t_1}^{t_2}\int_{Q_1} \partial_t[(u-\bar u)(m-\bar m)] + \langle mDu-\bar m D\bar u, D(u-\bar u)\rangle$$
$$-\frac{1}{2}(m-\bar m)(|Du|^2-|D\bar u|^2) + (F(x,m)-F(x,\bar m))(m-\bar m) = 0\,.$$

To complete the proof we just note that

$$\langle mDu-\bar m D\bar u, D(u-\bar u)\rangle - \frac{1}{2}(m-\bar m)(|Du|^2-|D\bar u|^2) = \frac{(m+\bar m)}{2}|Du-D\bar u|^2\,.$$

□

Another crucial point is given by the following lemma, which exploits the fact that system (1) has an Hamiltonian structure. Note that this is directly related to the optimal control interpretation of the Mean Field Games system ([12]).

Lemma 2.3. *There exists a constant M^T such that*

$$\frac{1}{2}\int_{Q_1} m^T(t)\,|Du^T(t)|^2\,dx + \int_{Q_1}\langle Du^T(t), Dm^T(t)\rangle\,dx$$
$$-\int_{Q_1}\Phi(x,m^T(t))\,dx = M^T \qquad \forall t\in[0,T]$$

where $\Phi(x,m)=\int_0^m F(x,\rho)\,d\rho$.

Proof. We multiply (1)-(i) by $\partial_t m^T(t)$ and (1)-(ii) by $\partial_t u^T(t)$. Summing the two equations we get, at (t,x),

$$-\Delta u^T\partial_t m^T + \frac{1}{2}|Du^T|^2\,\partial_t m^T - F(x,m^T)\partial_t m^T = \Delta m^T\partial_t u^T + \mathrm{div}(m^T Du^T)\partial_t u^T$$

Integrating with respect to x gives:

$$\int_{Q_1}\left(\langle Du^T,\partial_t Dm^T\rangle + \langle Dm^T,\partial_t Du^T\rangle\right)\,dx$$
$$+\int_{Q_1}\left[\frac{1}{2}|Du^T|^2\,\partial_t m^T + m^T\,\langle Du^T,\partial_t Du^T\rangle\right]\,dx - \int_{Q_1}F(x,m^T)\partial_t m^T\,dx = 0$$

This means that

$$\frac{d}{dt}\left\{\int_{Q_1}\langle Du^T, Dm^T\rangle\,dx + \frac{1}{2}\int_{Q_1} m^T|Du^T|^2\,dx - \int_{Q_1}\Phi(x,m^T)\,dx\right\} = 0\,,$$

hence the conclusion. □

We deduce the following

Corollary 1. *We have*

(i) M^T is bounded with respect to T.
(ii) $|Du^T(0)|$ is bounded in $L^2(Q_1)$.

Proof. On one hand we have (since $F\geq 0$ and $u(T)=G(x)$)

$$\begin{aligned} M^T &= \int_{Q_1}\langle Du(T), Dm(T)\rangle\,dx + \frac{1}{2}\int_{Q_1} m(T)|Du(T)|^2\,dx - \int_{Q_1}\Phi(x,m(T))\,dx \\ &\leq -\int_{Q_1}\Delta u(T)\,m(T)\,dx + \frac{1}{2}\int_{Q_1} m(T)|Du(T)|^2\,dx \leq C\|m(T)\|_{L^1(Q_1)} = C\,. \end{aligned}$$

On the other hand we have

$$M^T = \int_{Q_1} \langle Du(0), Dm_0\rangle \, dx + \frac{1}{2}\int_{Q_1} m_0 |Du(0)|^2 \, dx - \int_{Q_1} \Phi(x, m_0)\, dx$$

where, since $m_0 > 0$,

$$\left|\int_{Q_1} \langle Du(0), Dm_0\rangle dx\right| \le \frac{1}{4}\int_{Q_1} m_0 |Du(0)|^2 \, dx + \int_{Q_1} \frac{|Dm_0|^2}{m_0}\, dx\,.$$

Hence

$$M^T \ge \frac{1}{4}\int_{Q_1} m_0 |Du(0)|^2 \, dx - C\,.$$

In particular M^T is bounded both from above and from below. We also deduce from our last inequality that

$$\int_{Q_1} |Du(0)|^2 \, dx \le C\,,$$

so that $|Du(0)|$ is bounded in $L^2(Q_1)$. □

Combining Corollary 1 with Lemma 2.2 we get:

Lemma 2.4.

$$\int_0^T \int_{Q_1} \frac{(m^T + \bar m)}{2} |Du^T - D\bar u|^2 + (F(x, m^T) - F(x, \bar m))(m^T - \bar m) \; dxdt \le C \quad (5)$$

In particular

$$\frac{1}{T}\int_0^T \int_{Q_1} m^T \, |Du^T|^2 \, dxdt \le C \quad (6)$$

and

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \int_{Q_1} |Du^T - D\bar u|^2 \; dxdt = 0\,. \quad (7)$$

Proof. Using Lemma 2.2, we have

$$\int_0^T \int_{Q_1} \frac{(m^T + \bar m)}{2} |Du^T - D\bar u|^2 + (F(x, m^T) - F(x, \bar m))(m^T - \bar m) \; dxdt$$
$$= \int_{Q_1} (u^T(0) - \bar u)(m_0 - \bar m) dx - \int_{Q_1} (u^T(T) - \bar u)(m^T(T) - \bar m) dx$$

Recalling that $u^T(T) = G$ and the bounds assumed on G, the last term is bounded. If we set $\tilde u^T = \int_{Q_1} u^T \, dx$, we have

$$\int_{Q_1} u^T(0)(m_0 - \bar m) dx = \int_{Q_1} (u^T(0) - \tilde u^T(0))(m_0 - \bar m) dx$$
$$\le C\left(\|m_0\|_\infty + \|\bar m\|_\infty\right) \|Du^T(0)\|_{L^2(Q_1)}$$

and using Corollary 1 we conclude that this is bounded. Therefore, we obtain that (5) holds. □

Next we estimate the space derivative of m^T.

Lemma 2.5.

$$\int_0^T\int_{Q_1}\left|D\sqrt{m^T(x,t)}\right|^2 dxdt \le C\int_0^T\int_{Q_1} m^T(x,t)|Du^T(x,t)|^2 dxdt + \int_{Q_1} L(m_0(x))dx$$

where $L(s)=s\log(s)-s+1$.

Remark 3. According to Lemma 2.4, the right-hand-side of the above inequality is bounded by CT.

Proof. Since T is fixed in the estimate, we simply set $(u,m)=(u^T,m^T)$. We multiply by $\log(m)$ the equation satisfied by m and integrate over $(0,T)\times Q_1$ to get

$$\int_{Q_1} L(m(T)) + 4\int_0^T\int_{Q_1}\left|D\sqrt{m}\right|^2 dxdt \le \int_0^T\int_{Q_1}|Dm||Du| + \int_{Q_1} L(m_0)$$

(Note that $L'(s)=\log(s)$ and $L(s)\ge 0$.) Since, by Young's inequality, we have

$$\begin{aligned}\int_0^T\int_{Q_1}|Dm||Du| &\le \frac12\int_0^T\int_{Q_1}\frac{|Dm|^2}{m}+\frac12\int_0^T\int_{Q_1} m|Du|^2\\ &= 2\int_0^T\int_{Q_1}|D\sqrt{m}|^2+\frac12\int_0^T\int_{Q_1} m|Du|^2\ ,\end{aligned}$$

the desired inequality is proved. □

Corollary 2.

$$\int_0^T \|m^T\|_{L^{N_*/(N_*-2)}(Q_1)} \le CT \tag{8}$$

where $N_*=N$ *if* $N>2$ *while, if* $N=2$, N_* *is any value such that* $N_*>2$. *Moreover, if* $N>2$ *and* $\alpha=\frac{N+2}{N}$, *then we have*

$$\frac1T\int_0^T\int_{Q_1}(m^T(x,t))^\alpha \le C\ . \tag{9}$$

If $N=2$, *inequality (9) holds true for every* $\alpha<2$.

Proof. As usual we set $(u,m)=(u^T,m^T)$. Since

$$\int_{Q_1}(\sqrt{m(t)})^2 = 1 \qquad \forall t\in[0,T]\ ,$$

using Sobolev inequality combined with Lemma 2.5 we obtain

$$\int_0^T \|\sqrt{m(t)}\|^2_{L^{\frac{2N_*}{N_*-2}}(Q_1)} \le C\int_0^T (1+\|D\sqrt{m(t)}\|^2_{L^2(Q_1)}) \le CT\ , \tag{10}$$

where $N_*=N$ if $N>2$, while, if $N=2$, $N_*>2$. In particular, we get (8). Now, for $\theta\in(0,1)$ and $\alpha=1-\theta+\theta\frac{N_*}{N_*-2}\in(1,\frac{N_*}{N_*-2})$, we use the interpolation inequality

$$\|m(t)\|^\alpha_{L^\alpha(Q_1)} \le \|m(t)\|^{1-\theta}_{L^1(Q_1)}\|m(t)\|^{\theta\frac{N_*}{N_*-2}}_{L^{N_*/(N_*-2)}(Q_1)}\ ,$$

to get

$$\frac1T\int_0^T\int_{Q_1}(m(x,t))^\alpha dxdt \le \frac1T\int_0^T \|\sqrt{m}\|^{\theta\frac{2N_*}{N_*-2}}_{L^{\frac{2N_*}{N_*-2}}(Q_1)} dt\ .$$

If $N > 2$, we have $N_* = N$ and choosing $\theta = \frac{N-2}{N}$ (i.e. $\alpha = \frac{N+2}{N}$) we conclude using (10). If $N = 2$, for any $\alpha < 2$ we can choose $N_* = \frac{2}{\alpha-1}$ so that $\theta\frac{N_*}{N_*-2} = 1$, and again we conclude. $\square$

Proof of Theorem 2.1. Since (μ^T) is bounded in $L^\alpha((0,1)\times Q_1)$ for $N > 2$ (or in L^p for any $p < 2$ if $N = 2$), a subsequence converges weakly to some μ. Note that μ is $\mathbb{Z}^N-$periodic, $\mu \geq 0$ a.e. and $\int_{Q_1}\mu(x,t)dx = 1$ for a.e. $t \in (0,1)$. Our aim is to show that $\mu = \bar m$. Let $\phi = \phi(x)$ be a smooth test function. For any $0 < t_1 < t_2 < T$ we integrate over $(t_1,t_2)\times Q_1$ the equation satisfied by μ^T multiplied by ϕ: we get

$$\frac{1}{T}\left[\int_{Q_1}\mu^T\phi\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{Q_1} -\mu^T\Delta\phi + \mu^T\langle Dv^T, D\phi\rangle = 0\,. \tag{11}$$

Since Lemma 2.4 implies that

$$\lim_{T\to+\infty}\int_{t_1}^{t_2}\int_{Q_1}\mu^T|Dv^T - D\bar u|^2\,dxdt = 0$$

using Schwartz inequality and since $\|\mu^T\|_{L^1(Q_1)} = 1$ we deduce that

$$\lim_{T\to+\infty}\int_{t_1}^{t_2}\int_{Q_1}\mu^T\langle(Dv^T - D\bar u), D\phi\rangle = 0\,.$$

Thus, the weak convergence of μ^T implies, being $\bar u$ and ϕ smooth, that

$$\int_{t_1}^{t_2}\int_{Q_1}\mu^T\langle Dv^T, D\phi\rangle \to \int_{t_1}^{t_2}\int_{Q_1}\mu\langle D\bar u, D\phi\rangle\,.$$

We let then $T\to+\infty$ in (11) to get:

$$\int_{t_1}^{t_2}\int_{Q_1} -\mu\Delta\phi + \mu\langle D\bar u, D\phi\rangle = 0$$

for any smooth test function $\phi = \phi(x)$. In particular, $z := (1/(t_2 - t_1))\int_{t_1}^{t_2}\mu$ is a weak solution to

$$-\Delta z - \mathrm{div}\,(zD\bar u) = 0, \qquad \int_{Q_1} z = 1\,.$$

The next Lemma 2.6 implies that $(1/(t_2 - t_1))\int_{t_1}^{t_2}\mu = \bar m$, and then, since t_1, t_2 are arbitrary, $\mu = \bar m$, which proves the weak convergence of (μ^T) to $\bar m$. Note that this weak convergence is established without using the increasing character of F with respect to m, see Remark 1.

Lemma 2.6. *Let $\bar u : \mathbb{R}^N \to \mathbb{R}$ be a $\mathcal{C}^2$ periodic function. Then there is at most one signed measure μ on Q_1, which is periodic and a weak solution to*

$$-\,\Delta\mu - \mathrm{div}\,(\mu D\bar u) = 0, \qquad \int_{Q_1}\mu = 1 \tag{12}$$

We postpone the proof of the Lemma and proceed with the analysis of the behavior of μ^T. By Lemma 2.4, we have

$$\lim_{T\to+\infty}\int_0^1\int_Q (F(x,\mu^T(t,x)) - F(x,\bar m(x))(\mu^T(t,x) - \bar m(x))\,dxdt = 0\,. \tag{13}$$

Using now that $F(x,\cdot)$ is increasing, the above limit implies that (μ^T) converges to $\bar m$ a.e. and, therefore, strongly in L^p for any $p < \alpha$.

We now claim that $F(\cdot,\mu^T)$ converges to $F(\cdot,\bar m)$ in $L^1((0,1)\times Q_1)$. Indeed, let L be a Lipschitz constant of F on $Q_1\times[0,\|\bar m\|_\infty+1]$. We have

$$\begin{aligned}\int_0^1\int_{Q_1}|F(x,\mu^T)-F(x,\bar m)|\,dxdt&\\ \leq\int\int_{\{\mu^T>\bar m+1\}}&(F(x,\mu^T)-F(x,\bar m))(\mu^T-\bar m)\,dxdt\\ &+\int\int_{\{\mu^T\leq\bar m+1\}}|F(x,\mu^T(x,t))-F(x,\bar m(x))|\,dxdt\\ \leq\frac{C}{T}+L&\int\int_{\{\mu^T\leq\bar m+1\}}|\mu^T(x,t))-\bar m(x)|\,dxdt\end{aligned}$$

where the last inequality comes from Lemma 2.4. Since the right-hand side vanishes as $T\to+\infty$, our claim is proved. Similarly, using (13) and since μ^T converges to $\bar\mu$ a.e., we conclude that $F(\cdot,\mu^T)\mu^T$ converges to $F(\cdot,\bar m)\bar m$ in $L^1((0,1)\times Q_1)$.

Let us now check the convergence of υ^T. We know from Lemma 2.4 that $D\upsilon^T$ converges to $D\bar u$ in $L^2((0,1)\times Q_1)$. Let us integrate the equation satisfied by υ^T on $Q_1\times(0,1)$:

$$\begin{aligned}\frac{1}{T}\left(\int_{Q_1}\upsilon^T(x,t)dx-\int_{Q_1}G(x)dx\right)+\frac12\int_t^1\int_{Q_1}|D\upsilon^T|^2dxds&\\ =\int_t^1\int_{Q_1}F(x,\mu^T(s))dxds&\end{aligned}$$

where $D\upsilon^T\to D\bar u$ in L^2 and $F(\cdot,\mu^T(\cdot))\to F(\cdot,\bar m)$ in $L^1((0,1\times Q_1)$. So

$$\lim_{T\to+\infty}\frac1T\int_{Q_1}\upsilon^T(x,t)dx=(1-t)\int_{Q_1}\left[-\frac12|D\bar u|^2+F(x,\bar m)\right]dx=(1-t)\bar\lambda\ .$$

Note that the convergence is actually uniform with respect to $t\in[0,1]$. Using Poincaré-Wirtinger inequality, we get, setting $\langle\upsilon^T\rangle=\int_{Q_1}\upsilon^Tdx$ and $\tilde\upsilon^T=\upsilon^T-\langle\upsilon^T\rangle$,

$$\int_0^1\int_{Q_1}\left|\tilde\upsilon^T-\bar u\right|^2\leq C\int_0^1\int_{Q_1}|D(\upsilon^T-\bar u)|^2\to0$$

This shows the convergence in L^2 of $\tilde\upsilon^T$ to $\bar u$ and, since $\frac{\langle\upsilon^T\rangle}{T}\to(1-t)\bar\lambda$, the convergence in L^2 of $\frac1T\upsilon^T$ to $(1-t)\bar\lambda$.

We are finally left with the pointwise convergence of $\frac{\upsilon^T(t,\cdot)}{T}$ to $\bar\lambda(1-t)$ for any $t\in[0,1]$. For this we note that

$$\partial_t\left(\frac{\upsilon^T}{T}\right)=-\Delta\upsilon^T+\frac12|D\upsilon^T|^2-F(x,\mu^T)$$

where the right-hand side is converging in $L^2((0,1);H^{-1})+L^1((0,1)\times Q_1))$. We recall from [16], Theorem 1.1 that the space

$$\left\{v\in L^2((a,b);H_0^1(Q_1))\text{ with }v_t\in L^2((a,b);H^{-1})+L^1((a,b)\times Q_1))\right\}$$

is continuously embedded into $C^0([a,b];L^1(Q_1))$. A straightforward adaptation of this result is possible to our case, i.e. $u\in L^2((0,1);H^1(Q_1))$ with periodicity conditions. Therefore, we actually conclude that $\frac{\upsilon^T(t,\cdot)}{T}$ converges strongly in $L^1(Q_1)$ to $\bar\lambda(1-t)$ uniformly with respect to $t\in[0,1]$. □

Proof of Lemma 2.6. Let ν_1 and ν_2 be two solutions. Let us set $\nu = \nu_1 - \nu_2$. For any smooth periodic map $f : \mathbb{R}^N \to \mathbb{R}$ there is a unique ergodic constant $\bar\delta$ for which the following equation

$$\bar\delta - \Delta w + \langle Dw, D\bar u\rangle = f$$

has a smooth, periodic solution w. Then

$$\int_{Q_1} f d\nu = \int_{Q_1} (-\Delta w + \langle Dw, D\bar u\rangle) d\nu = 0$$

because $\nu(Q_1) = 0$ and ν is a solution to (12). Since f is arbitrary, this implies that $\nu = 0$. □

3. **The convergence rate.** In order to estimate the rate of convergence in Theorem 2.1, we now assume that $F = F(x, m)$ has a growth rate which is bounded from below: namely we assume that there is some $\gamma > 0$ such that

$$F(x,s) - F(x,t) \geq \gamma(s-t) \qquad \forall s \geq t,\ \forall x \in \mathbb{R}^N\ . \tag{14}$$

Then we have

Theorem 3.1. *Let us set* $\tilde u^T = u^T - \int_{Q_1} u^T(x,t)\, dx$*. Under assumption (14) there is some* $\kappa > 0$ *(independent of* T*) such that*

$$\|\tilde u^T(t) - \bar u\|_{L^1(Q_1)} \leq \frac{C}{T-t}\left(e^{-\kappa(T-t)} + e^{-\kappa t}\right) \qquad \forall t \in (0,T)\ , \tag{15}$$

$$\|m^T(t) - \bar m\|_{L^1(Q_1)} \leq \frac{C}{t}\left(e^{-\kappa(T-t)} + e^{-\kappa t}\right) \qquad \forall t \in (0,T) \tag{16}$$

and

$$\left\| \frac{u^T(t)}{T} - \bar\lambda\left(1 - \frac{t}{T}\right)\right\|_{L^1(Q_1)} \leq \frac{C}{T} \qquad \forall t \in (0, T-1)\ . \tag{17}$$

Remark 4. The estimates (15) and (16) can be rephrased in terms of the scaled functions υ^T, μ^T, giving

$$\|\upsilon^T(t) - \int_{Q_1} \upsilon^T(t)\, dy - \bar u\|_{L^1(Q_1)} \leq \frac{C}{T}\ \left(\frac{e^{-\kappa T(1-t)} + e^{-\kappa t T}}{1-t}\right) \qquad \forall t \in (0,1)$$

and

$$\|\mu^T(t) - \bar m\|_{L^1(Q_1)} \leq \frac{C}{T}\ \left(\frac{e^{-\kappa T(1-t)} + e^{-\kappa t T}}{t}\right) \qquad \forall t \in (0,1)\ .$$

With respect to Theorem 2.1, we obtain here that the convergences are pointwise in time, and actually hold uniformly on compact subsets of $(0,1)$ at an exponential rate. Finally, (17) implies

$$\left\| \frac{\upsilon^T(t)}{T} - \bar\lambda\,(1-t))\right\|_{L^1(Q_1)} \leq \frac{C}{T}$$

providing a rate for the L^1- convergence of υ^T/T which was established in Theorem 2.1.

The main ingredient of the proof of Theorem 3.1 is the following energy estimate:

Lemma 3.2. *Under the assumptions of Theorem 3.1 there exists $\sigma > 0$ (independent of T) such that*

$$\int_{\delta T}^{(1-\delta)T} \int_{Q_1} |Du^T(x,t) - D\bar{u}(x)|^2 \, dxdt + \int_{\delta T}^{(1-\delta)T} \int_{Q_1} |m^T(x,t) - \bar{m}(x)|^2 \, dxdt \le C e^{-\sigma\,\delta T} \tag{18}$$

for every $\delta > 0$.

Proof of Lemma 3.2. As usual we drop the dependence on T of (u^T, m^T). Let us set

$$\varphi(t) := \int_{Q_1} (\tilde{u} - \bar{u})(m - \bar{m}) \, dx \, .$$

It is a consequence of Lemma 2.2 that

$$\frac{d}{dt}\varphi(t) = -\int_{Q_1} \left\{ \frac{m + \bar{m}}{2} |D(u - \bar{u})|^2 + \left(F(x,m) - F(x,\bar{m})\right)(m - \bar{m}) \right\} dx \, . \tag{19}$$

By assumption (14) we have

$$\left(F(x,m) - F(x,\bar{m})\right)(m - \bar{m}) \ge \gamma (m - \bar{m})^2$$

while using Poincaré–Wirtinger inequality and the fact that $\min_{Q_1} \bar{m} > 0$ we have, for some $c_0 > 0$,

$$\int_{Q_1} \frac{\bar{m}}{2} |D(u - \bar{u})|^2 \ge c_0 \int_{Q_1} |\tilde{u} - \bar{u}|^2 \, dx \, .$$

Therefore

$$\begin{aligned} \int_{Q_1} \left\{ \frac{m + \bar{m}}{2} |D(u - \bar{u})|^2 + \left(F(x,m) - F(x,\bar{m})\right)(m - \bar{m}) \right\} dx \\ \ge c_0 \int_{Q_1} |\tilde{u} - \bar{u}|^2 \, dx + \gamma \int_{Q_1} (m - \bar{m})^2 \, dx \\ \ge \sigma \left| \int_{Q_1} (\tilde{u} - \bar{u})(m - \bar{m}) \, dx \right| \end{aligned}$$

for some $\sigma > 0$. We have obtained so far that, for some $\sigma > 0$,

$$\frac{d}{dt}\varphi \le -\sigma\varphi \quad \text{and} \quad \frac{d}{dt}\varphi \le \sigma\varphi \quad \forall t \in (0,T) \, .$$

Integrating the first inequality in $(0,t)$ and the second one in (t,T) we get

$$e^{-\sigma(T-t)}\varphi(T) \le \varphi(t) \le \varphi(0) e^{-\sigma t} \quad \forall t \in (0,T) \, .$$

Since $u(T)$ is bounded and $\int_{Q_1} m \, dx = 1$ we have

$$\varphi(T) \ge -C \, ,$$

while in view of Corollary 1 combined with the Poincaré-Wirtinger inequality, we have

$$\varphi(0) \le (\|m_0\|_\infty + \|\bar{m}\|_\infty) \int_{Q_1} |\tilde{u}(0)| + \|\bar{u}\|_\infty \le C \, .$$

Then we conclude that

$$-K e^{-\sigma(T-t)} \le \varphi(t) \le K e^{-\sigma t}$$

for some constant $K > 0$. Coming back to (19) we deduce that

$$\int_{t_1}^{t_2}\int_{Q_1}\left\{\frac{m+\bar m}{2}|D(u-\bar u)|^2+(F(x,m)-F(x,\bar m))\,(m-\bar m)\right\}dx\,dt \\ =\varphi(t_1)-\varphi(t_2)\le K\left(e^{-\sigma t_1}+e^{-\sigma(T-t_2)}\right)\,. \tag{20}$$

Choosing $t_1 = \delta T$, $t_2 = (1-\delta)T$, and recalling that $\bar m > 0$, we get

$$\int_{\delta T}^{(1-\delta)T}\int_{Q_1}\left\{|D(u-\bar u)|^2+(F(x,m)-F(x,\bar m))\,(m-\bar m)\right\}dxdt\le C\,Ke^{-\sigma\,\delta T}$$

for every $\delta > 0$. Using assumption (14) we also deduce (18). □

Proof of Theorem 3.1. Let us consider the equation of $\tilde u - \bar u$. Setting $v = \tilde u - \bar u$, we have

$$-v_t-\Delta v+\frac12|Dv|^2+\langle Dv,D\bar u\rangle=F(x,m)-F(x,\bar m)+\bar\lambda+\frac{d}{dt}\int_{Q_1}u(y,t)dy\,. \tag{21}$$

Let $T_1(s) = \max(-1, \min(s, 1))$ and $\Theta_1(s) = \int_0^s T_1(y)dy$. Observe that $DT_1(v) = Dv\chi_{\{|v|<1\}}$ a.e. We multiply (21) by $T_1(v)(t_0 - t)$ and integrate over $(\tau, t_0) \times Q_1$. Since

$$\int_\tau^{t_0}\int_{Q_1}-v_tT_1(v)(t_0-t)\;dxdt=(t_0-\tau)\int_{Q_1}\Theta_1(v(\tau))\;dx-\int_\tau^{t_0}\int_{Q_1}\Theta_1(v)\;dxdt\,,$$

and

$$\int_\tau^{t_0}\int_{Q_1}-\Delta v\;T_1(v)(t_0-t)\;dxdt=\int_\tau^{t_0}\int_{Q_1}|DT_1(v)|^2(t_0-t)\;dxdt\ge0\,,$$

we have

$$(t_0-\tau)\int_{Q_1}\Theta_1(v(\tau))\;dx\;\le\;\int_\tau^{t_0}\int_{Q_1}\Theta_1(v)\;dxdt \\ +\int_\tau^{t_0}\int_{Q_1}\Big(-\frac12|Dv|^2-\langle Dv,D\bar u\rangle \\ +F(x,m)-F(x,\bar m)+\bar\lambda+\frac{d}{dt}\int_{Q_1}u(t,y)dy\Big)\,T_1(v)(t_0-t)\;dxdt \tag{22}$$

We now estimate each term in the right-hand side. Since $\Theta_1(s) \le s^2/2$ we have by Poincaré–Wirtinger inequality,

$$\int_{Q_1}\Theta_1(v)\;dx\le\int_{Q_1}\frac{v^2}{2}\;dx\le C\int_{Q_1}|Dv^2|\;dx\,.$$

Using that $|T_1(v)| \le \min(1, |v|)$ and Poincaré–Wirtinger inequality again, we get

$$\left|\int_{Q_1}\left(\frac12|Dv|^2+\langle Dv,D\bar u\rangle\right)T_1(v)\,dx\right| \\ \le\frac12(1+\|D\bar u\|_\infty)\int_{Q_1}|Dv|^2\,dx+\frac12\|D\bar u\|_\infty\int_{Q_1}|v|^2\,dx \\ \le C\int_{Q_1}|Dv|^2\,dx$$

Moreover we have

$$\int_{Q_1} |F(x,m) - F(x,\bar m)|\, |T_1(v)| dx = \int\limits_{\{|m-\bar m|\le 1\}} |F(x,m) - F(x,\bar m)|\, |T_1(v)| dx + \int\limits_{\{|m-\bar m|> 1\}} |F(x,m) - F(x,\bar m)|\, |T_1(v)| dx$$

If $|m-\bar m|\le 1$ we have $|m|\le 1+|\bar m|$, so that m, $\bar m$ are in a bounded set; since F is locally Lipschitz continuous, we have

$$|F(x,m) - F(x,\bar m)| \le \bar\gamma |m-\bar m| \quad \forall m, \bar m \, : |m-\bar m|\le 1\,, |\bar m| \le \|\bar m\|_\infty$$

for some constant $\bar\gamma$. Therefore, using Young's inequality we have that

$$\int\limits_{\{|m-\bar m|\le 1\}} |F(x,m) - F(x,\bar m)|\, |T_1(v)| dx \le \frac12 \int_{Q_1} |v|^2\, dx + C\int_{Q_1} (m-\bar m)^2\, dx\,.$$

Using once more Poincaré inequality and (14), we obtain therefore

$$\int\limits_{\{|m-\bar m|\le 1\}} |F(x,m) - F(x,\bar m)|\, |T_1(v)| dx \le C\int_{Q_1} |Dv|^2\, dx + C\int_{Q_1} \left(F(x,m) - F(x,\bar m)\right)(m-\bar m) dx\,.$$

Since obviously

$$\int\limits_{\{|m-\bar m|> 1\}} |F(x,m) - F(x,\bar m)|\, |T_1(v)| dx \le \int_{Q_1} \left(F(x,m) - F(x,\bar m)\right)(m-\bar m) dx\,,$$

we conclude that

$$\int_{Q_1} |F(x,m) - F(x,\bar m)|\, |T_1(v)| dx \le C\int_{Q_1} |Dv|^2\, dx + C\int_{Q_1} \left(F(x,m) - F(x,\bar m)\right)(m-\bar m) dx\,.$$

In order to estimate the remaining terms in (22), let us recall that

$$\begin{aligned} \bar\lambda &= \int_{Q_1} \left[F(y,\bar m) - \frac12 |D\bar u|^2\right] dy\,, \\ \frac{d}{dt}\int_{Q_1} u(y,t) dy &= \int_{Q_1} \left[\frac12 |Du|^2 - F(y,m)\right] dy\,. \end{aligned} \tag{23}$$

So we can argue as above to get

$$\begin{aligned} &\left|\int_{Q_1} \left(\bar\lambda + \frac{d}{dt}\int_{Q_1} u(t,y) dy\right) T_1(v)\right| \le \\ &\le \int_{Q_1}\int_{Q_1} |F(y,m) - F(y,\bar m)||T_1(v)| + \frac12 |Dv|^2 + |\langle D\bar u, Dv\rangle||T_1(v)|\; dxdy \\ &\le C\int_{Q_1} |Dv|^2\, dx + C\int_{Q_1} \left(F(x,m) - F(x,\bar m)\right)(m-\bar m)\; dx\,. \end{aligned}$$

Putting all these estimates together we obtain finally from (22) that

$$(t_0-\tau)\int_{Q_1}\Theta_1(v(\tau))\,dx \le C\int_\tau^{t_0}\int_{Q_1}|Dv|^2\,dxdt + C\int_\tau^{t_0}\int_{Q_1}|Dv|^2\,(t_0-t)\,dxdt$$
$$+C\int_\tau^{t_0}\int_{Q_1}\left(F(x,m)-F(x,\bar m)\right)(m-\bar m)(t_0-t)\;dxdt\,,$$

which yields

$$\int_{Q_1}\Theta_1(v(\tau))\,dx \le C\left(1+\frac{1}{t_0-\tau}\right)\int_\tau^{t_0}\int_{Q_1}|Dv|^2\,dxdt$$
$$+\,C\int_\tau^{t_0}\int_{Q_1}\left(F(x,m)-F(x,\bar m)\right)(m-\bar m)dxdt\,.$$

Using (20) we deduce

$$\int_{Q_1}\Theta_1(v(\tau))\,dx \le C\left(1+\frac{1}{t_0-\tau}\right)\left(e^{-\sigma\tau}+e^{-\sigma(T-t_0)}\right)\quad \forall t_0\in(\tau,T)$$

Choosing $t_0=\tau+\frac{1}{2}(T-\tau)$ and since $|s|\le C\max(\Theta_1(s),\sqrt{\Theta_1(s)})$ we deduce (15) for some $\kappa>0$.

We use a similar argument in the equation of $m-\bar m$. Let us denote by $\varphi=m-\bar m$. Note that φ satisfies

$$\varphi_t-\Delta\varphi-\mathrm{div}(\varphi Du+\bar m D(u-\bar u))=0\,.$$

We multiply by $T_1(\varphi)(t-t_0)$, where $t_0\in(0,\tau)$, and we integrate in (t_0,τ). We get

$$(\tau-t_0)\int_{Q_1}\Theta_1(\varphi(\tau))\,dx + \int_{t_0}^{\tau}\int_{Q_1}|DT_1(\varphi)|^2\,(t-t_0)dxdt$$
$$=\int_{t_0}^{\tau}\int_{Q_1}\Theta_1(\varphi)\,dxdt + \int_{t_0}^{\tau}\int_{Q_1}\left(D\bar u + D(u-\bar u)\right)\varphi\, DT_1(\varphi)\,(t-t_0)\,dxdt$$
$$+\int_{t_0}^{\tau}\int_{Q_1}\bar m D(u-\bar u)DT_1(\varphi)\,(t-t_0)dxdt\,. \tag{24}$$

We have

$$\int_{Q_1}\left(D\bar u+D(u-\bar u)\right)\varphi\,DT_1(\varphi)\,dx \le \frac12\int_{Q_1}|DT_1(\varphi)|^2dx$$
$$+\frac12\|D\bar u\|_\infty^2\int_{Q_1}|\varphi|^2\,dx + \frac12\int_{Q_1}|D(u-\bar u)|^2\,dx\,.$$

Similarly, we estimate

$$\int_{Q_1}\bar m D(u-\bar u)DT_1(\varphi)\,dx \le \frac12\int_{Q_1}|DT_1(\varphi)|^2\,dx + \frac12\|\bar m\|_\infty^2\int_{Q_1}|D(u-\bar u)|^2\,dx\,.$$

Then we deduce from (24) that

$$(\tau-t_0)\int_{Q_1}\Theta_1(\varphi(\tau))\,dx \le \frac12\int_{t_0}^{\tau}\int_{Q_1}\varphi^2\,dxdt$$
$$+C\int_{t_0}^{\tau}\int_{Q_1}|\varphi|^2\,(t-t_0)dxdt + C\int_{t_0}^{\tau}\int_{Q_1}|D(u-\bar u)|^2\,(t-t_0)dxdt$$

which yields

$$\int_{Q_1} \Theta_1(\varphi(\tau))\,dx \le C\left(1+\tfrac{1}{\tau-t_0}\right)\int_{t_0}^{\tau}\int_{Q_1}\varphi^2\,dxdt + C\int_{t_0}^{\tau}\int_{Q_1}|D(u-\bar u)|^2\,dxdt\,.$$

From our assumption (14) on F we have

$$|\varphi|^2 \le \frac{1}{\gamma}(F(x,m)-F(x,\bar m))(m-\bar m)\,.$$

Therefore, using (20), we can conclude that

$$\int_{Q_1}\Theta_1(\varphi)(\tau)\,dx \le C\left(1+\frac{1}{\tau-t_0}\right)\left(e^{-\sigma t_0}+e^{-\sigma(T-\tau)}\right)\,.$$

We choose $t_0=\frac{\tau}{2}$ and we conclude as before that (16) holds true.

We now prove (17). To this purpose, we first claim that

$$\left|\int_{Q_1} u(y,t)dy - \bar\lambda\,(T-t)\right| \le C \qquad \forall t\in(0,T-1)\,. \tag{25}$$

Proof of (25): the computation are very close from the ones we did previously. Recalling the notation $v=\tilde u-\bar u$ and (23), we have

$$\bar\lambda + \frac{d}{dt}\int_{Q_1} u(y,t)dy = \int_{Q_1}(F(x,\bar m)-F(x,m)) + \frac12|Dv|^2 + \langle D\bar u, Dv\rangle\;dx\,.$$

Hence we get

$$\begin{aligned}(T-t)\bar\lambda - \int_{Q_1} u(y,t)dy \;&= -\int_{Q_1} u(y,T)dy - \int_t^T\int_{Q_1}(F(x,m)-F(x,\bar m))\\ &\quad + \int_t^T\int_{Q_1}\frac12|Dv|^2 + \langle D\bar u, Dv\rangle\;dxd\tau\,.\end{aligned} \tag{26}$$

As before we estimate

$$\begin{aligned}\left|\int_t^T\int_{Q_1}(F(x,m)-F(x,\bar m))\right| &\le \int_t^T\int_{Q_1}(F(x,m)-F(x,\bar m))(m-\bar m)dxd\tau\\ &\quad + \bar\gamma\int_{\{|m-\bar m|\le 1\}}|m-\bar m|dxd\tau\end{aligned}$$

which yields, due to (20),

$$\left|\int_t^T\int_{Q_1}(F(x,m)-F(x,\bar m))\right| \le K(1+e^{-\sigma t}) + \bar\gamma\int_{\{|m-\bar m|\le 1\}}|m-\bar m|dxd\tau\,.$$

We split last integral in the subsets $(t,t+1)$ and $(t+1,T)$, and in this last term we use (16) obtaining

$$\begin{aligned}\int_{\{|m-\bar m|\le 1\}}|m-\bar m|dxd\tau &\le 1 + \int_{t+1}^T\int_{Q_1}|m-\bar m|dxd\tau\\ &\le 1 + C\int_{t+1}^T\frac{1}{\tau}(e^{-\kappa(T-\tau)}+e^{-\kappa\tau})d\tau \le C(1+e^{-\kappa t})\,.\end{aligned}$$

Therefore, we conclude that

$$\left| \int_t^T \int_{Q_1} (F(x,m) - F(x,\bar m)) \right| \leq C\,. \tag{27}$$

Similarly we estimate the energy terms. Splitting the integral between $(t, T-1)$ and $(T-1, T)$ we have

$$\begin{aligned} &\left| \int_t^T \int_{Q_1} \langle D\bar u, Dv\rangle \; dx d\tau \right| \\ &\quad \leq \left| \int_t^{T-1} \int_{Q_1} \langle D\bar u, Dv\rangle \; dx d\tau \right| + \|D\bar u\|_{L^2(Q_1)} \left(\int_{T-1}^T \int_{Q_1} |Dv|^2 \, dx d\tau \right)^{\frac12} \\ &\quad \leq \left| \int_t^{T-1} \int_{Q_1} \langle D\bar u, Dv\rangle \; dx d\tau \right| + C(1+e^{-\sigma T})^{\frac12} \end{aligned}$$

due to (20). On the other hand, since

$$\int_t^{T-1} \int_{Q_1} \langle D\bar u, Dv\rangle \; dx d\tau = -\int_t^{T-1} \int_{Q_1} v \Delta \bar u \, dx d\tau\,,$$

using the C^2 character of $\bar u$ and (15) we have

$$\left| \int_t^{T-1} \int_{Q_1} \langle D\bar u, Dv\rangle \; dx d\tau \right| \leq C \int_t^{T-1} \frac{1}{T-\tau} (e^{-\kappa(T-\tau)} + e^{-\kappa\tau}) d\tau \leq C(1+e^{-\kappa t})\,.$$

On account of (20), overall we deduce that

$$\left| \int_t^T \int_{Q_1} \frac12 |Dv|^2 + 2\langle D\bar u, Dv\rangle \; dx d\tau \right| \leq C\,. \tag{28}$$

Since u_T is bounded, we conclude from (26), (27), (28) that (25) holds true.
Finally, to show (17), we just note that, for $t \in (0, T-1)$, using (15) and (25) we get

$$\begin{aligned} &\int_{Q_1} \left| \frac{u(t)}{T} - \bar\lambda \left(1 - \frac{t}{T}\right) \right| \; dx \\ &\quad \leq \frac1T \int_{Q_1} |\tilde u(t) - \bar u| \; dx + \frac1T \int_{Q_1} |\bar u| \; dx + \frac1T \left| \int_{Q_1} u(t) \; dx - \bar\lambda \,(T-t) \right| \\ &\quad \leq \frac{C}{T(T-t)} \left(e^{-\kappa(T-t)} + e^{-\kappa t} \right) + \frac{C}{T} \leq \frac{C}{T}\,. \end{aligned}$$

□

Remark 5. As addressed in [12], the system (1) can be interpreted as the optimality system of an optimal control problem, more precisely a bilinear control problem where the state equation is the Fokker–Planck equation

$$\partial_t m - \Delta m - \mathrm{div}(m\psi) = 0 \quad \text{in } (0,T)\times Q_1, \quad m(0) = m_0$$

and the control problem given by

$$\inf_{\psi} \int_0^T \int_{Q_1} \left\{ \frac12 m\,|\psi|^2 \, + \Phi(x,m) \right\} dx dt$$

where $\Phi(x,m) = \int_0^m F(x,\rho)\,d\rho$. On this viewpoint, our analysis proves the convergences of the optimal control $\psi^T = \nabla u^T$ and the optimal state m^T towards the optimal control and state $\nabla \bar{u}$, $\bar{m}$ of the associated stationary control problem. In particular, Lemma 3.2 shows the exponential L^2 convergence (in the transient time) when Φ has at least quadratic growth.

4. Appendix.

We prove here the existence of classical solutions to (1). Uniqueness, under the assumption that F is nondecreasing with respect to m, is established in [12, 13]. Existence and uniqueness of the solution of the ergodic problem can be obtained in a similar (and even simpler) way, so we omit the proof.

Theorem 4.1. *Assume that the map $F : \mathbb{R}^N \times [0,+\infty) \to \mathbb{R}$ is of class $\mathcal{C}^1$, $\mathbb{Z}^N-$periodic with respect to the first variable and bounded below. Then for any $\mathcal{C}^2$, periodic maps $G(x)$, m_0, there exists a classical solution (u,m) of (1).*

Weak solutions for system (1) are proved to exist in [12, 13] for a general class of Hamiltonians. In the special case of a quadratic Hamiltonian, considered here, a Hopf-Cole transform can also be used to derive existence of weak solution through a monotone scheme (see [9], for bounded maps F). As for smooth solutions, the approach described in [9] can actually be used to obtain regularity results similar to the one presented here (see [15]). Related results for the ergodic problem and special class of F's are also established in [7], [8], with a variational approach.

The proof of Theorem 4.1 requires several steps. First we show the existence of a classical solution when F is bounded (Lemma 4.2) and derive an integral estimate for this solution. The key step is Lemma 4.4 which explains that the solution is bounded in L^∞ by a constant depending mainly on the quantity

$$\|F\|_{\infty,m_0} := \sup_{Q_1\times[0,2\|m_0\|_\infty]} |F(x,r)|\,.$$

The existence of a classical solution in the general case can then be derived by standard approximation arguments.

Since, from our assumption, F is bounded from below, we can assume from now on and without loss of generality that $F \geq 0$. Moreover, we denote henceforth $u(T)$ by u_T.

Lemma 4.2. *Assume that the conditions of Theorem 4.1 holds and that F is bounded. Then there is at least one classical, periodic solution (u,m) of (1).*

Proof. We construct a fixed point in $L^1((0,T)\times Q_1)$: given $m \in L^1((0,T)\times Q_1)$, we solve

$$-\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F(x,m(t,x))\,, \qquad u(T) = u_T$$

and we define $\varphi = \varphi(m)$ as the solution of

$$\partial_t \varphi - \Delta \varphi - \mathrm{div}(\varphi Du) = 0\,, \quad \varphi(0) = m_0\,.$$

Using the change of variables $w = e^{-u/2}$, w satisfies the linear equation

$$-\partial_t w - \Delta w + \frac{1}{2} w F(x,m(t,x)) = 0\,, \qquad w(T) = e^{-u_T/2}$$

Since F is bounded and u_T is $\mathcal{C}^2$, there is a solution w to the above equation—and therefore a solution u of the first equation—which is bounded (uniformly with respect to m) in $L^p(0,T;W^{2,p}(Q_1)) \cap L^\infty((0,T)\times Q_1)$ for every $p < \infty$ (Theorem

IV.9.1 of [10]). Using the bound on Du, we get a solution φ of the second equation which is bounded (again uniformly with respect to m) in $L^2((0,T),H^1(Q_1))\cap L^\infty((0,T)\times Q_1)$ while $\partial_t\varphi\in L^2(0,T;H^{-1}(Q_1))$ and, in addition, φ is uniformly Hölder continuous (Theorems III.4.1, III.7.1 and III.10.1 of [10]). Then the map $\varphi : m\to\varphi(m)$ is compact in L^1. Since it is also continuous, there exists a solution (u,m) of (1) with $u\in L^p(0,T;W^{2,p}(Q_1))\cap L^\infty((0,T)\times Q_1)$ and $m\in L^2((0,T),H^1(Q_1))\cap L^\infty((0,T)\times Q_1)$. We obtain that the solution (u,m) is classical by a bootstrapping argument. □

The next step consists in deriving a general estimate for the solution of (1):

Lemma 4.3. *Under the conditions of Lemma 4.2, let (u,m) be a classical, periodic solution of (1). Then there is a constant c, depending only on T, $\|F\|_{\infty,m_0}$, $\|u_T\|_\infty$, $\|u_0\|_\infty$, such that*

$$\int_0^T\int_{Q_1}F(x,m)m\,dxdt+\int_0^T\int_{Q_1}(1+m)|Du|^2\,dxdt+\sup_{t\in[0,T]}\int_{Q_1}|u(t)|dx\le c \quad (29)$$

Proof. Since $F\ge 0$, we have $u\ge -\|u_T\|_\infty$. Let us subtract inequality (1-(i)) multiplied by m to inequality (1-(ii)) multiplied by u and integrate over $(\tau,T)\times Q_1$: we get

$$\left[\int_{Q_1}um\;dx\right]_\tau^T+\int_\tau^T\int_{Q_1}\frac{m}{2}|Du|^2+mF(x,m)\;dxdt=0\,. \quad (30)$$

On another hand, integrating (1-(i)) over $(\tau,T)\times Q_1$ gives

$$\left[-\int_{Q_1}u\;dx\right]_\tau^T+\int_\tau^T\int_{Q_1}\frac{1}{2}|Du|^2-F(x,m)\;dxdt=0\,. \quad (31)$$

From (31), we have

$$\begin{aligned}\int_{Q_1}u(0)\;dx &\le \int_0^T\int_{Q_1}F(x,m)\;dxdt+\|u_T\|_\infty\\ &\le \int\int_{m\ge 2\|m_0\|_\infty}F(x,m)\,dxdt+\int\int_{m<2\|m_0\|_\infty}F(x,m)\,dxdt+\|u_T\|_\infty\\ &\le \frac{1}{2\|m_0\|_\infty}\int_0^T\int_{Q_1}mF(x,m)\;dxdt+C\end{aligned}$$

Using inequality (30) we then derive that

$$\begin{aligned}\int_0^T\int_{Q_1}\frac{m}{2}|Du|^2+mF(x,m)\;dxdt &\le \|m_0\|_\infty\int_{Q_1}u(0)\;dx+C\\ &\le \frac{1}{2}\int_0^T\int_{Q_1}mF(x,m)\;dxdt+C\,,\end{aligned}$$

hence $\int_0^T\int_{Q_1}\frac{m}{2}|Du|^2+mF(x,m)\;dxdt$ and $\int_0^T\int_{Q_1}F(x,m)\;dxdt$ are bounded. Coming back to (31) gives the bound on $\int_{Q_1}u(t)\;dx$ for any $t\in[0,T]$ as well as the bound on $|Du|^2$. □

Lemma 4.4. *Under the assumptions of Lemma 4.2, there is a constant C, depending only on T, N, $\|u_T\|_\infty$ and $\|m_0\|_\infty$ and $\|F\|_{\infty,m_0}$, such that any classical solution (u,m) to (1) satisfies*

$$\|u\|_\infty+\|m\|_\infty\le C\,.$$

Proof. Throughout the proof, we denote by c a generic constant, which might change from line to line, and which depends only on T, N, $\|u_T\|_\infty$, $\|m_0\|_\infty$ and $\|F\|_{\infty,m_0}$. We will use the classical idea of Moser iterations to achieve the L^∞ estimate as limit of L^p estimates. We will need however several steps.
Step 1: some basic estimates. Following [14], let us set

$$\psi := m\, e^{\frac{1}{2}u}\,.$$

Since $\psi_t = m_t e^{\frac{1}{2}u} + \frac{1}{2} m e^{\frac{1}{2}u} u_t$, a straightforward computation shows that ψ satisfies

$$\psi_t - \Delta\psi + \frac{1}{2}F(x,m)\psi = 0\,.$$

Fix $\gamma > 1$. Multiplying the above equation by ψ^γ we obtain, for $\tau \in [0,T]$,

$$\begin{aligned}\frac{1}{\gamma+1}\int_{Q_1}\psi(\tau)^{\gamma+1}dx + \gamma\int_0^\tau\int_{Q_1}|D\psi|^2\psi^{\gamma-1}dxdt \\ +\frac{1}{2}\int_0^\tau\int_{Q_1}F(x,m)\psi^{\gamma+1}\,dxdt = \frac{1}{\gamma+1}\int_{Q_1} m_0^{\gamma+1}e^{\frac{1}{2}u(\gamma+1)}(0)\,dx\,.\end{aligned} \tag{32}$$

On the other hand the equation of u multiplied by $e^{\frac{1}{2}u(1+\gamma)}$ implies, for any $\tau \in [0,T]$,

$$\begin{aligned}\frac{2}{\gamma+1}\int_{Q_1}e^{\frac{1}{2}u(\gamma+1)}(\tau)\,dx + (\frac{\gamma}{2}+1)\int_\tau^T\int_{Q_1}|Du|^2e^{\frac{1}{2}u(\gamma+1)}\,dxdt \\ \le \frac{2}{\gamma+1}\int_{Q_1}e^{\frac{1}{2}u(\gamma+1)}(T)dx + \int_\tau^T\int_{Q_1}F(x,m)e^{\frac{1}{2}u(\gamma+1)}\,dxdt\end{aligned} \tag{33}$$

Step 2: We are going to play with the two estimates (32) and (33) in order to bound the last term in (33). Namely we claim that:

$$\int_0^T\int_{Q_1}F(x,m)e^{\frac{1}{2}u(\gamma+1)}\,dxdt \le \frac{2}{\gamma+1}\int_{Q_1}e^{\frac{1}{2}u(\gamma+1)}(T)dx + c\int_0^T\int_{Q_1}e^{\frac{1}{2}u(\gamma+1)}\,dxdt \tag{34}$$

Proof of (34): Thanks to (33) with $\tau = 0$, we can estimate the right-hand side of (32) as follows:

$$\begin{aligned}\int_{Q_1}m_0^{\gamma+1}e^{\frac{1}{2}u(\gamma+1)}(0)\,dx &\le \|m_0\|_\infty^{\gamma+1}\int_{Q_1}e^{\frac{1}{2}u(\gamma+1)}(0)\,dx \\ &\le \|m_0\|_\infty^{\gamma+1}\int_{Q_1}e^{\frac{1}{2}u(\gamma+1)}(T)dx \\ &\quad + \|m_0\|_\infty^{\gamma+1}\frac{\gamma+1}{2}\int_0^T\int_{Q_1}F(x,m)e^{\frac{1}{2}u(\gamma+1)}\,dxdt\,.\end{aligned}$$

Plugging this inequality into (32) with $\tau = T$ gives

$$\begin{aligned}&\frac{1}{2}\int_0^T\int_{Q_1}F(x,m)m^{\gamma+1}e^{\frac{1}{2}u(\gamma+1)}(t)\,dxdt \\ &\quad\le \tfrac{\|m_0\|_\infty^{\gamma+1}}{\gamma+1}\int_{Q_1}e^{\frac{1}{2}u(\gamma+1)}(T)dx + \tfrac{\|m_0\|_\infty^{\gamma+1}}{2}\int_0^T\int_{Q_1}F(x,m)e^{\frac{1}{2}u(\gamma+1)}\,dxdt\,.\end{aligned}$$

So we have

$$2^{\gamma+1}\|m_0\|_\infty^{\gamma+1}\int\int_{\{m\geq 2\|m_0\|_\infty\}} F(x,m)e^{\frac12 u(\gamma+1)}(t)\,dxdt$$
$$\leq \frac{2\|m_0\|_\infty^{\gamma+1}}{\gamma+1}\int_{Q_1} e^{\frac12 u(\gamma+1)}(T)dx + \|m_0\|_\infty^{\gamma+1}\int_0^T\int_{Q_1} F(x,m)e^{\frac12 u(\gamma+1)}\,dxdt\,.$$

Therefore

$$(2^{\gamma+1}-1)\int\int_{\{m\geq 2\|m_0\|_\infty\}} F(x,m)e^{\frac12 u(\gamma+1)}(t)\,dxdt$$
$$\leq \frac{2}{\gamma+1}\int_{Q_1} e^{\frac12 u(\gamma+1)}(T)dx + \int\int_{\{m<2\|m_0\|_\infty\}} F(x,m)e^{\frac12 u(\gamma+1)}\,dxdt\ .$$

Hence, since $2^{\gamma+1}-1\geq 1$,

$$\int_0^T\int_{Q_1} F(x,m)e^{\frac12 u(\gamma+1)}\,dxdt$$
$$\leq \frac{2}{(\gamma+1)}\int_{Q_1} e^{\frac12 u(\gamma+1)}(T)dx + 2\int\int_{\{m<2\|m_0\|_\infty\}} F(x,m)e^{\frac12 u(\gamma+1)}\,dxdt$$
$$\leq \frac{2}{(\gamma+1)}\int_{Q_1} e^{\frac12 u(\gamma+1)}(T)dx + c\int_0^T\int_{Q_1} e^{\frac12 u(\gamma+1)}\,dxdt$$

which is (34).

Step 3: Let us set $v_\gamma = e^{\frac14 u(\gamma+1)}$ and $v=v_1=e^{\frac12 u}$. We now claim that

$$\|v_\gamma\|^2_{L^{2+\frac4N}((0,T)\times Q_1)} \leq c\int_{Q_1} |v_\gamma(T)|^2\,dx + c(\gamma+1)\int_0^T\int_{Q_1}|v_\gamma|^2\,dxdt \tag{35}$$

Proof of (35): In view of (34), (33) implies

$$\sup_{\tau\in[0,T]}\int_{Q_1} e^{\frac12 u(\gamma+1)}(\tau)\,dx + (\gamma+1)^2\int_0^T\int_{Q_1}|Du|^2 e^{\frac12 u(\gamma+1)}\,dxdt$$
$$\leq c\int_{Q_1} e^{\frac12 u(\gamma+1)}(T)dx + c(\gamma+1)\int_0^T\int_{Q_1} e^{\frac12 u(\gamma+1)}\,dxdt\,.$$

The above inequality can be rewritten as

$$\begin{aligned}&\sup_{\tau\in[0,T]}\int_{Q_1}|v_\gamma(\tau)|^2\,dx + \int_0^T\int_{Q_1}|Dv_\gamma|^2\,dxdt\\ &\quad\leq c\int_{Q_1}|v_\gamma(T)|^2\,dx + c(\gamma+1)\int_0^T\int_{Q_1}|v_\gamma|^2\,dxdt\,.\end{aligned} \tag{36}$$

Let us denote by RHS the right-hand side of (36). Using first Hölder inequality, interpolating the exponent $2+\frac4N$ between 2 and $\frac{2N}{N-2}$, and then Sobolev inequality together with (36), we have

$$\begin{aligned}\int_0^T\int_{Q_1}|v_\gamma|^{2+\frac4N} &\leq \int_0^T\left(\int_{Q_1}|v_\gamma|^2\,dx\right)^{\frac2N}\left(\int_{Q_1}|v_\gamma|^{\frac{2N}{N-2}}\,dx\right)^{\frac{N-2}{N}}\,dt\\ &\leq \sup_{\tau\in[0,T]}\left(\int_{Q_1}|v_\gamma(\tau)|^2\right)^{\frac2N}\int_0^T\int_{Q_1}\left(|v_\gamma|^2+|Dv_\gamma|^2\right)\,dxdt\\ &\leq c(RHS)^{\frac2N+1}\end{aligned}$$

hence we deduce (35) holds. Note that we used here $N > 2$ in Sobolev inequality, but a standard adaptation (replacing $\frac{2N}{N-2}$ with any exponent $q < \infty$) allows to deal with the case $N \le 2$ in a similar way.

Step 5 (Bounds on $\|v\|_p$). Since, from (29), $\sup_{\tau\in[0,T]} \int_{Q_1} |u(\tau)|dx \le c$, we have, for any $\tau \in [0,T]$ and any L large,

$$|\{v_\gamma \ge L\} \cap (\ (0,T) \times Q_1)| = \left| \{u \ge \frac{4\ln(L)}{(\gamma+1)}\} \cap ((0,T)\times Q_1) \right| \le \frac{c(\gamma+1)}{\ln(L)} \,.$$

Then (35) implies, thanks to Hölder inequality, that

$$\begin{aligned}
\|v_\gamma\|^2_{L^{2+\frac{4}{N}}((0,T)\times Q_1)} &\le c\, e^{\|u_T\|_\infty(\gamma+1)} + c(\gamma+1) \int\int_{\{v_\gamma \ge L\}} |v_\gamma|^2\, dxdt + c(\gamma+1)L^2 \\
&\le c\, e^{\|u_T\|_\infty(\gamma+1)} + c(\gamma+1)\ \|v_\gamma\|^2_{L^{2+\frac{4}{N}}((0,T)\times Q_1)} |\{v_\gamma \ge L\}|^{\frac{2}{N+2}} + c(\gamma+1)L^2 \\
&\le c\, e^{\|u_T\|_\infty(\gamma+1)} + \frac{c(\gamma+1)^{1+\frac{2}{N+2}}}{(\ln(L))^{\frac{2}{N+2}}}\ \|v_\gamma\|^2_{L^{2+\frac{4}{N}}((0,T)\times Q_1)} + c(\gamma+1)L^2
\end{aligned}$$

Therefore, if we choose L sufficiently large (depending on γ), we obtain a bound on $\|v_\gamma\|_{L^{2+\frac{4}{N}}((0,T)\times Q_1)}$ by a constant $c = c(\gamma)$, which implies a bound on $\|v\|_p$ by a constant $c = c(p)$ depending only on p, T, N, $\|u_T\|_\infty$ and $\|m_0\|_\infty$ and $\|F\|_{\infty,m_0}$.

Step 6 (Bound on $\|v\|_\infty$): Let us set $\alpha = 1 + \frac{2}{N}$, and recall that $v = e^{\frac{1}{2}u}$. Then inequality (35) can be rewritten as

$$\|v\|^{\gamma+1}_{L^{\alpha(\gamma+1)}} \le c\|v(T)\|^{\gamma+1}_\infty + c(\gamma+1)\|v\|^{\gamma+1}_{L^{\gamma+1}} \,.$$

Hence

$$\int_0^T \int_{Q_1} |v|^{\alpha(\gamma+1)}\ dxdt \le c\|v(T)\|^{\alpha(\gamma+1)}_\infty + c(\gamma+1)^\alpha \left(\int_0^T \int_{Q_1} |v|^{\gamma+1}\ dxdt \right)^\alpha \,.$$

Let us set $I_k = \int_0^T \int_{Q_1} |v|^{\alpha^k}\ dxdt$ (for $k \in \mathbb{N}$). Then choosing $\gamma+1 = \alpha^k$ (for $k \ge k_0$, where k_0 is the smallest integer for which $\alpha\, k_0 > 2$) we get

$$I_{k+1} \le c\|v_T\|^{\alpha^{k+1}}_\infty + c\,\alpha^{\alpha k}(I_k)^\alpha$$

So, if we set $J_k = \max\{I_k, \|v_T\|^{\alpha^k}_\infty\}$, we obtain

$$J_{k+1} \le c\,\alpha^{\alpha k}(J_k)^\alpha$$

By induction this implies that

$$J_k \le c^{\frac{\alpha^{k-k_0-1}}{\alpha-1}} \alpha^{\sum_{l=0}^{k-k_0}(k-1-l)\alpha^{l+1}} (J_{k_0})^{\alpha^{k-k_0}}$$

In particular

$$\|v\|_{L^{\alpha^k}} \le c^{\frac{\alpha^{-k_0}-\alpha^{-k}}{\alpha-1}} \alpha^{\sum_{l=0}^{k-k_0}(k-1-l)\alpha^{l+1-k}} (J_{k_0})^{\alpha^{-k_0}}$$

Letting $k \to +\infty$ gives the desired bound:

$$\|v\|_\infty \le c^{\frac{\alpha^{-k_0}}{\alpha-1}} \alpha^{\frac{\alpha}{(1-\alpha)^2}} (J_{k_0})^{\alpha^{-k_0}}\ \le\ c$$

since, from step 5, $J_{k_0} \le c$.

Step 7: conclusion. Since u is bounded from below by $-\|u_T\|_\infty$, the L^∞ bound on $v = e^{\frac{1}{2}u}$ gives the L^∞ bound on u. In particular, we have now a bound for $\psi(0) := m_0 e^{\frac{1}{2}u(0)}$, hence we immediately deduce a L^∞ bound for ψ (e.g. see (32)). Since $m = \psi e^{-\frac{1}{2}u}$, the L^∞ bound on m then follows. □

Proof of Theorem 4.1. Let (F_n) is a sequence of smooth, bounded functions which converges locally uniformly to F. Let (u_n, m_n) be a solution of

$$\begin{cases} (i) & -\partial_t u - \Delta u + \frac{1}{2}|Du|^2 = F_n(x,m) \\ (ii) & \partial_t m - \Delta m - \operatorname{div}(mDu) = 0 \\ (iii) & m(0) = m_0,\ u(x,T) = u_T \end{cases} \tag{37}$$

given by Lemma 4.2. Then we know from the previous step that the (u_n, m_n) are uniformly bounded (since $\sup_n \|F_n\|_{\infty,m_0}$ is bounded). Then, as in the proof of Lemma 4.2, classical estimates apply and (u_n, m_n) converge to some pair (u, m) which is a bounded solution of (1). A bootstrapping argument then shows that (u, m) is classical. □

Remark 6. We explicitly point out that the estimates only depend on the L^∞ bound of m_0 and u_T and on the (local in m) bound of $F(x,m)$. In particular, after the above estimate, and thanks to standard compactness results, one can prove the existence of solutions (u,m) only asking for bounded initial and terminal conditions, and requiring e.g. $F(x,m)$ to be only continuous (and bounded from below). In that case solutions are so-called strong, namely they will belong to $L^\infty((0,T)\times Q_1) \cap L^2((0,T); H^1(Q_1))$ and to $L^p((a,b); W^{2,p}(Q_1))$ for every $p < \infty$ and $[a,b] \subset (0,T)$. Such solutions are classical if in addition F is C^1, and they are regular up to $t = 0$ or $t = T$ depending whether m_0 and u_T are regular enough.

Moreover, the above L^∞ estimate can be easily extended to the case that $u(T) = G(x, m(T))$, where $G(x,m)$ satisfies the same assumptions as F, i.e. being $\mathbb{Z}^N-$periodic with respect to the first variable and bounded below. It is enough to handle the term $G(x, m(T))$ in the same way as it was done with $F(x,m)$, in this case the estimate will depend on $\|G\|_{\infty,m_0} := \sup_{Q_1\times[0,2\|m_0\|_\infty]} |G(x,r)|$.

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