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FROM DISCRETE TO CONTINUOUS WARDROP EQUILIBRIA

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Abstract. The notion of Wardrop equilibrium in congested networks has been very popular in congested traffic modelling since its introduction in the early 50's, it is also well-known that Wardrop equilibria may be obtained by some convex minimization problem. In this paper, in the framework of Γ-convergence theory, we analyze what happens when a cartesian network becomes very dense. The continuous model we obtain this way is very similar to the continuous model of optimal transport with congestion of Carlier, Jimenez and Santambrogio [6] except that it keeps track of the anisotropy of the network.

1. **Introduction.** Congested traffic equilibrium models on finite networks have received a lot of attention since the early 50's because of applications to road traffic and more recently to communication networks. In this line of research, the notion of Wardrop equilibrium plays a central role. Roughly speaking, Wardrop equilibrium requires that users behave rationally by choosing the shortest available paths, taking congestion into account i.e. the fact that travel times increase with the flow. Finding Wardrop equilibria is a fixed-point problem in nature that presents some analogies with mean-field games theory eventhough it is purely stationary. Soon after the work of Wardrop, it was observed by Beckmann, McGuire and Winsten [2] that the Wardrop condition actually is the first-order condition for some some convex minimization problem. This is a key property both from a theoretical point and for numerical computations. Unfortunately, the minimization problem has one (flow) variable per admissible path on the network, it may therefore quickly become intractable for realistic road or communication networks. An alternative consists in studying the dual problem, this dual formulation has one (time) variable per arc but it involves the corresponding shortest travelling times between the nodes, it is therefore nonsmooth and nonlocal. Both primal and dual formulations of Wardrop equilibria are difficult to solve for large scale networks and it becomes natural to investigate whether the problem somehow simplifies passing to the continuous limit in some sense.

The aim of this paper is to study rigorously what happens to Wardrop equilibria as the network becomes very dense. More precisely, we will consider the case of a two-dimensional cartesian network with small arc length ε and will study the

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Γ-convergence of the functionals in the dual problem as ε goes to 0. We will then obtain an optimization problem posed over certain continuous metrics variables. This limit problem is the dual of a continuous problem posed over a set of probability measures over paths which is similar to the continuous model of optimal transport with congestion of Carlier, Jimenez and Santambrogio [6] except that it keeps track of the anisotropy of the network. The optimality conditions for the continuous model of optimal transport with congestion can naturally be viewed as the continuous counterpart of Wardrop equilibria. We will first address the short-term problem in which the transport plan i.e. the amount of mass that has to be sent from each source to each destination is prescribed. We will also consider the long-term problem in which only the marginals (i.e. the distributions of supply and demand) are fixed and the transport plan is part of the unknown and has to be determined by some additional optimality requirement. In the isotropic continuous long-term case, as shown in Brasco, Carlier and Santambrogio [5], the Wardrop equilibrium problem reduces to solving some nonlinear elliptic PDE, a similar approach is possible for the anisotropic case as well but will not be developed here, we just mention that our Γ-convergence results somehow motivate the model studied in [6], [5] as a rigorous continuous limit of the discrete Wardrop problem.

The paper is organized as follows. In section 2, we set some notations, recall the definition of Wardrop equilibria and its variational characterization. The limit functional for the dual problem is identified in section 3 and a precise Γ-convergence result is stated, its proof is detailed in section 4. In section 5, we establish optimality conditions for the limit problem, these conditions may naturally be interpreted as a continuous Wardrop equilibrium. Finally, in section 6, we extend the previous analysis to the long term variant.

2. The discrete model.

2.1. Setting and definition of Wardrop equilibria.

We now describe the Wardrop equilibrium problem on a network modeled over a two-dimensional cartesian grid. In addition to the network itself and the corresponding set of admissible paths, the data of the problem are the congestion functions (which relate traveling time to the flow on each arc in a nondecreasing way) as well as the transport plan which prescribes the amount of commuting mass between nodes. The unknown is the flow configuration that will be determined through some equilibrium conditions due to Wardrop [9].

Network of characteristic length ε: Given Ω a bounded domain of $\mathbb{R}^2$ with a smooth boundary and $\varepsilon > 0$, we consider as network whose characteristic length is ε:

$$\Omega_\varepsilon := \varepsilon\mathbb{Z}^2 \cap \overline{\Omega}.$$

We shall denote by $(v_1, v_2, v_3, v_4) := ((1,0),(0,1),(-1,0),(0,-1))$ the directions of the network (i.e. the vectors of the canonical basis as well as their opposite) enumerated counterclockwise. In this setting, every arc of the network is of the form $[x, x+\varepsilon v_i]$ for some $x \in \Omega_\varepsilon$ and some $i \in \{1, ..., 4\}$. Arcs will therefore simply identified to pairs (x, v_i). One should think of the network as being oriented so that $[x, x+\varepsilon v_i]$ and $[x+\varepsilon v_i, x]$ really represent two distinct arcs.

Traveling times and congestion: The mass commuting on arc (x, v_i) will be denoted by $m_i^\varepsilon(x)$ and the traveling time of arc (x, v_i) will be denoted by $t_i^\varepsilon(x)$.

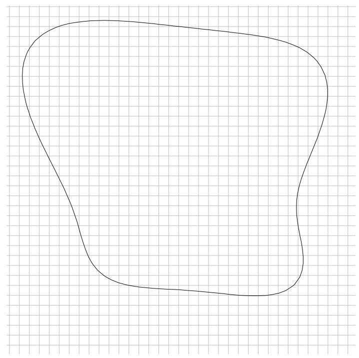

FIGURE 1. Network with characteristic length ε.

Due to congestion, traveling time and mass are related for every arc (x, v_i) by the relation:

$$t_i^\varepsilon(x) = g_i^\varepsilon(x, m_i^\varepsilon(x)) \tag{2.1}$$

where the g_i^ε are some given nonnegative functions that depend on the arc itself but also in a nondecreasing way (this is congestion) on the mass $m_i^\varepsilon(x)$ that commutes on (x, v_i). The collection of all arc-masses $m_i^\varepsilon(x)$ will be denoted $\mathbf{m}^\varepsilon$.

Remark 2.1. In this model, we do not consider the case where some time is also spent at the nodes x. This extension could be treated as well simply by considering some extra arcs. We might also allow the functions g_i^ε to take the value $+\infty$ modelling forbidden arcs or saturation effects but for the sake of simplicity we will only study the case where the travelling times are finite.

Consider two neighboring nodes x and x' with $x' = x + \varepsilon v_i$ and $v_j = -v_i$, the time to go from x to x' only depends only on the mass $m_i^\varepsilon(x)$ that uses the arc (x, v_i) whereas the time to go from x' to x depends only on the mass $m_j^\varepsilon(x')$.

Transport plan: A transport plan is also given as a collection of nonnegative masses $\gamma^\varepsilon(x, y)$, $(x, y) \in \Omega_\varepsilon \times \Omega_\varepsilon$. For each pair $(x, y) \in \Omega_\varepsilon \times \Omega_\varepsilon$ (viewed as a source/destination pair), $\gamma^\varepsilon(x, y)$ represents the amount of mass that has to be sent from the source x to the target y. Of course, if all the masses $\gamma^\varepsilon(x, y)$ are zero, no mass at all will travel along the network.

Paths: A path is a finite collection of successive nodes. A generic path σ is therefore of the form $(x_0, x_1, ..., x_L) \in \Omega_\varepsilon^{L+1}$ where $\sigma(0) := x_0 \in \Omega_\varepsilon$ and $\sigma(k+1) - \sigma(k) := x_{k+1} - x_k \in \varepsilon\{v_1, ..., v_4\}$ for $k = 0, ..., L-1$. For such a path $\sigma(0)$ is the origin of σ, εL is the (flat) length of σ and $\sigma(L)$ is the terminal point of σ. We shall use the notation $(x, v_i) \subset \sigma$ if there is a k between 0 and $L-1$ such that $\sigma(k) = x$ and $\sigma(k+1) - \sigma(k) = \varepsilon v_i$. Since commuting time on each arc is nonnegative, we shall restrict ourselves to the C^ε set of loop-free paths, this set is finite and may be partitioned as

$$C^\varepsilon = \bigcup_{(x,y)\in\Omega_\varepsilon\times\Omega_\varepsilon} C^\varepsilon_{x,y}$$

where $C^\varepsilon_{x,y}$ is the set of loop-free paths having x as origin and y as terminal point. The mass traveling on the path $\sigma \in C^\varepsilon$ (therefore starting from the origin of σ and stopping at the terminal point of σ) will be denoted $w^\varepsilon(\sigma)$. The collection of all path-masses $w^\varepsilon(\sigma)$ will be denoted $\mathbf{w}^\varepsilon$. Given arc-masses $\mathbf{m}^\varepsilon$, the travel time of a path $\sigma \in C^\varepsilon$ is given by

$$\tau^\varepsilon_{\mathbf{m}^\varepsilon}(\sigma) := \sum_{(x,v_i)\in\sigma} g_i^\varepsilon(x, m_i^\varepsilon(x)).$$

Equilibria: To sum up, the data of the model are thus the masses $\gamma^\varepsilon(x,y)$ and the congestion functions g_i^ε. The unknowns are the arc-masses $m_i^\varepsilon(x)$ and path-masses $w^\varepsilon(\sigma)$ that should be determined by some equilibrium requirements. First of all, arc-masses and path-masses, should be nonnegative. In addition, arc-masses, path-masses and the data γ^ε are related by the following conditions which both express mass conversation:

$$\gamma^\varepsilon(x,y) := \sum_{\sigma\in C^\varepsilon_{x,y}} w^\varepsilon(\sigma), \ \forall (x,y) \in \Omega_\varepsilon \times \Omega_\varepsilon \tag{2.2}$$

and

$$m_i^\varepsilon(x) = \sum_{\sigma\in C^\varepsilon \,:\, (x,v_i)\subset\sigma} w^\varepsilon(\sigma). \tag{2.3}$$

The last ingredient to define equilibria is the requirement that only shortest paths (given the congestion pattern created by arc and path-masses) should actually be used. This leads to the concept of Wardrop equilibirum ([9]) that is defined precisely as follows:

Definition 2.2. *A Wardrop equilibrium is a configuration of nonnegative arc-masses* $\mathbf{m}^\varepsilon : (x,i) \mapsto (m_i^\varepsilon(x))$ *and of nonnegative path-masses* $\mathbf{w}^\varepsilon : \sigma \mapsto w^\varepsilon(\sigma)$, *that satisfy the mass conservation conditions (2.2) and (2.3) and such that for every* $(x,y) \in \Omega_\varepsilon \times \Omega_\varepsilon$ *and every* $\sigma \in C^\varepsilon_{x,y}$, *if* $w^\varepsilon(\sigma) > 0$ *then*

$$\tau^\varepsilon_{\mathbf{m}^\varepsilon}(\sigma) \le \tau^\varepsilon_{\mathbf{m}^\varepsilon}(\sigma'), \ \forall \sigma' \in C^\varepsilon_{x,y}.$$

2.2. **Variational characterizations of equilibria.** A few years after Wardrop introduced his equilibrium concept for congested networks, it was realized by Beckmann, McGuire and Winsten [2] that Wardrop equilbiria have a variational characterization. More precisely, a flow configuration $(\mathbf{w}^\varepsilon, \mathbf{m}^\varepsilon)$ is an equilibrium if and only if it minimizes

$$\sum_{x\in\Omega_\varepsilon}\sum_{i=1}^4 G_i^\varepsilon(x, m_i^\varepsilon(x)) \text{ where } G_i^\varepsilon(x,m) := \int_0^m g_i^\varepsilon(x,\alpha)d\alpha \tag{2.4}$$

subject to nonnegativity constraints and the mass conservation conditions (2.2)-(2.3). Note that this is a convex program (since the functions g_i^ε are nondecreasing with respect to mass) so that existence results and numerical schemes can easily be derived from this variational formulation. Note however that this problem uses the whole path flow configuration $\mathbf{w}^\varepsilon$ and enumerating all such paths flows becomes extremely costly as soon as the network becomes dense, that prevents in practice the use of this formulation for realistic congested networks. This explains why one may often prefer to work with the dual formulation which reads as:

$$\inf_{\mathbf{t}^\varepsilon\in\mathbb{R}_+^{4\#\Omega_\varepsilon}} \sum_{x\in\Omega_\varepsilon}\sum_{i=1}^4 H_i^\varepsilon(x, t_i^\varepsilon(x)) - \sum_{(x,y)\in{\Omega_\varepsilon}^2} \gamma^\varepsilon(x,y) T^\varepsilon_{\mathbf{t}^\varepsilon}(x,y) \tag{2.5}$$

where $\mathbf{t}^\varepsilon \in \mathbb{R}_+^{4\#\Omega_\varepsilon}$ should be understood as $\mathbf{t}^\varepsilon = (t_i^\varepsilon(x))_{(x,i)\in\Omega_\varepsilon\times\{1,\cdots,4\}}$, $H_i^\varepsilon(x,.) := (G_i^\varepsilon(x,.))^*$ is the Legendre Transform of $G_i^\varepsilon(x,.)$ that is

$$H_i^\varepsilon(x,t) := \sup_{m\geq 0}\{mt - G_i^\varepsilon(x,m)\},\ \forall t \in \mathbb{R}_+ \tag{2.6}$$

and $T_{\mathbf{t}^\varepsilon}^\varepsilon$ is the minimal length functional:

$$T_{\mathbf{t}^\varepsilon}^\varepsilon(x,y) := \min_{\sigma\in C_{x,y}^\varepsilon} \sum_{(x,v_i)\subset\sigma} t_i^\varepsilon(x). \tag{2.7}$$

In the dual formulation (2.5) (of course again a convex program), we only have $4\#\Omega_\varepsilon = O(\varepsilon^{-2})$ variables which is much better than having one variable per path but is still huge for small ε. Note however that the price to pay in working with (2.5) is the term that depends on $T_{\mathbf{t}^\varepsilon}^\varepsilon$ since it is nonsmooth, nonlocal and might be complicated to optimize, it is not unrealistic however to expect that passing to the continuous limit will actually simplify the structure because one will then be allowed to use the theory of Hamilton-Jacobi equations. The connection between this dual formulation and Wardrop equilibria (i.e. the minimization of (2.4) under the constraints (2.2)-(2.3)) is that whenever $(\mathbf{m}^\varepsilon, \mathbf{w}^\varepsilon)$ is a Wardrop equilbrium then $\mathbf{t}^\varepsilon := (g_i^\varepsilon(x, m_i^\varepsilon(x))$ solves (2.5), in fact, solving (2.5) amounts to find the equilibrium travelling times (and thus also the corresponding arc-masses $m_i^\varepsilon(x)$ by inverting the relation $t_i^\varepsilon(x) = g_i^\varepsilon(x, m_i^\varepsilon(x)))$. We refer to the recent paper of Baillon and Cominetti [1] for more details, references and an extension of the model to a Markovian setting.

3. **The Γ-convergence result.**

3.1. **Scaling and assumptions.** Of course, if one wants to be able to pass to the continuous limit, $\varepsilon \to 0^+$ in the Wardrop equilibrium problem, some structural assumptions have to be made on the ε-dependence of the data. The first assumption is the convergence of the transport plans γ^ε, namely we assume that there exists a finite nonnegative measure γ on $\overline{\Omega}\times\overline{\Omega}$ to which γ^ε weakly star converges in the sense that the family of discrete measures $\sum_{(x,y)\in\Omega_\varepsilon{}^2}\gamma^\varepsilon(x,y)\delta_{(x,y)}$ weakly star converges to γ:

$$\lim_{\varepsilon\to 0^+} \sum_{(x,y)\in\Omega_\varepsilon{}^2} \gamma^\varepsilon(x,y)\varphi(x,y) = \int_{\overline{\Omega}\times\overline{\Omega}} \varphi d\gamma,\ \forall\varphi \in C(\overline{\Omega}\times\overline{\Omega}). \tag{3.1}$$

Our second assumption concerns the form of the congestion functions t_i^ε that we assume to be of the form

$$g_i^\varepsilon(x,m) = \varepsilon g_i\Big(x, \frac{m}{\varepsilon}\Big),\ \forall\varepsilon > 0,\ (x,i) \in \Omega_\varepsilon \times \{1,...,4\} \tag{3.2}$$

where g_i is a given continuous, nonnegative function on $\overline{\Omega}\times\mathbb{R}_+$, that is nondecreasing in its second argument. This assumption is natural in terms of scaling, it means that the travelling time on an arc of length ε is of order ε and depends on the flow per unit of length i.e. m/ε.

Under assumption (3.2), the functions G_i^ε and H_i^ε that appear in the primal and dual variational characterizations of Wardrop equilibria are thus given by

$$G_i^\varepsilon(x,m) = \varepsilon^2 G_i\Big(x, \frac{m}{\varepsilon}\Big) \text{ where } G_i(x,m) := \int_0^m g_i(x,\alpha)d\alpha \tag{3.3}$$

and

$$H_i^\varepsilon(x,t) = \varepsilon^2 H_i\Big(x, \frac{t}{\varepsilon}\Big) \text{ where } H_i(x,.) = (G_i(x,.))^* \tag{3.4}$$

i.e. for every $\xi \in \mathbb{R}_+$:

$$H_i(x,\xi) = \sup_{m\in\mathbb{R}_+} \{m\xi - G_i(x,m)\}.$$

Note also that the previous assumptions imply that $H_i(x,.)$ is actually strictly convex. In view of (2.5) and (3.4) it is natural to rescale the arc-times $\mathbf{t}^\varepsilon$ by defining the time per unit of length or *metric* variables

$$\xi^\varepsilon := \frac{\mathbf{t}^\varepsilon}{\varepsilon}, \text{ i.e. } \xi_i^\varepsilon(x) = \frac{t_i^\varepsilon(x)}{\varepsilon}, \ \forall x \in \Omega_\varepsilon, \ \forall i \in \{1,...,4\} \tag{3.5}$$

and then to rewrite (2.5) in terms of ξ^ε as:

$$\inf_{\xi^\varepsilon \in \mathbb{R}_+^{4\#\Omega_\varepsilon}} J^\varepsilon(\xi^\varepsilon) := I_0^\varepsilon(\xi^\varepsilon) - I_1^\varepsilon(\xi^\varepsilon) \tag{3.6}$$

where

$$I_0^\varepsilon(\xi^\varepsilon) := \varepsilon^2 \sum_{x\in\Omega_\varepsilon} \sum_{i=1}^4 H_i(x, \xi_i^\varepsilon(x)) \tag{3.7}$$

and

$$I_1^\varepsilon(\xi^\varepsilon) := \varepsilon \sum_{(x,y)\in{\Omega_\varepsilon}^2} \gamma^\varepsilon(x,y) T^\varepsilon_{\xi^\varepsilon}(x,y). \tag{3.8}$$

Our last assumption is that H_i is continuous in its first argument and there exists $p > 2$ and two constants $0 < \lambda \le \Lambda$ such that for every $(x,\xi,i) \in \overline{\Omega} \times \mathbb{R}_+ \times \{1,...,4\}$ one has

$$\lambda(\xi^p - 1) \le H_i(x,\xi) \le \Lambda(\xi^p + 1). \tag{3.9}$$

The growth condition (3.9) is natural if one wants to work in L^p in the continuous limit and thus to obtain a simple convex integral term as the limit of I_0^ε (recall that by construction H_i is convex in its second argument), the requirement $p > 2$ is technical and less natural, it will however turn out to be crucial to pass to the limit in the more involved nonlocal term I_1^ε in (3.6) which will make use of Morrey's inequality as explained below. From now on, we will always assume that assumptions (3.1), (3.2) and (3.9) are satisfied.

3.2. **The limit functional.** In view of the previous paragraph, it is natural to introduce

$$L_+^p := \{\xi = (\xi_1, ..., \xi_4), \ \xi_i \in L^p(\Omega), \ \xi_i \ge 0, \ i = 1, ..., 4\}$$

as well as the integral functional

$$I_0(\xi) := \sum_{i=1}^4 \int_\Omega H_i(x, \xi_i(x))dx, \ \forall \xi \in L_+^p \tag{3.10}$$

which naturally arises as the continuous limit of I_0^ε. The construction of the term that plays the same role as I_1^ε is more involved, to understand this term let us define for every $u = (u_1, u_2) \in \mathbb{R}^2$:

$$\Phi(u) := ((u \cdot v_i)_+)_{i=1,...,4} = ((u_1)_+, (u_2)_+, (u_1)_-, (u_2)_-) \tag{3.11}$$

Now, let $\xi^\varepsilon \in \mathbb{R}_+^{4\#\Omega_\varepsilon}$, $(x,y) \in \Omega_\varepsilon \times \Omega_\varepsilon$, let $\sigma \in C^\varepsilon_{x,y}$, let $\varepsilon L(\sigma)$ be the euclidean length of σ and slightly abusing notations let us extend ξ^ε on each arc by letting ξ_i^ε be constant with value $\xi_i^\varepsilon(x)$ on the arc $[x, x + \varepsilon v_i]$, let us also identify σ with the piecewise affine curve $t \in [0, L(\sigma)] \mapsto \sigma(t)$ defined by $\sigma(t) = \sigma_k + (t-k)(\sigma_{k+1} - \sigma_k)$ for $t \in [k, k+1]$ with $k = 0, ..., L(\sigma) - 1$, we then have

$$\varepsilon \sum_{(x,v_i)\subset\sigma} \xi_i^\varepsilon(x) = \sum_{k=0}^{L(\sigma)-1} \Phi(\sigma_{k+1} - \sigma_k) \cdot \xi^\varepsilon(\sigma_k) = \int_0^{L(\sigma)} \Phi(\dot\sigma(t)) \cdot \xi^\varepsilon(\sigma(t))dt$$

so that

$$\begin{aligned} \varepsilon T_{\xi^\varepsilon}^\varepsilon(x,y) &= \inf_{\sigma\in C_{x,y}^\varepsilon} \int_0^{L(\sigma)} \Phi(\dot\sigma(t)) \cdot \xi^\varepsilon(\sigma(t))dt \\ &= \inf_{\sigma\in C_{x,y}^\varepsilon} \int_0^1 \Phi(\dot{\widetilde\sigma}(t)) \cdot \xi^\varepsilon(\widetilde\sigma(t))dt \end{aligned}$$

where $\widetilde\sigma : [0,1] \to \overline\Omega$ simply is the reparameterization of $\widetilde\sigma(t) = \sigma(L(\sigma)t)$, $t \in [0,1]$. This strongly suggests to define for $\xi = (\xi_1, ..., \xi_4) \in C(\overline\Omega, \mathbb{R}_+^4)$

$$c_\xi(x,y) := \inf\left\{ \int_0^1 \Phi(\dot\sigma(t)) \cdot \xi(\sigma(t))dt \right\} \tag{3.12}$$

where the infimum is over the set of absolutely continuous curves σ with values in $\overline\Omega$ and such that $\sigma(0) = x$ and $\sigma(1) = y$. Note c_ξ is a sort of Finsler distance which keeps track of the anisotropy of the network but it is actually not a distance : it is indeed not separating if ξ vanishes somewhere and it is not symmetric since Φ is not even. Now, our aim is to extend the definition of c_ξ to the case where ξ is only L_+^p, to do so we proceed as in Carlier, Jimenez and Santambrogio [6] by remarking thanks to an easy dynamic programming argument that if ξ is continuous then c_ξ is actually Lipschitz and for all x and a.e. y one has

$$|\nabla_y c_\xi(x,y)| \le \sum_{i=1}^4 \xi_i(y)$$

and a similar inequality gives a bound for $\nabla_x c_\xi(x,y)$ for every y and a.e. x. Now recall that we have assumed that $p > 2$ in assumption (3.9), so that $W^{1,p}(\Omega) \subset C^{0,\alpha}(\Omega)$ with $\alpha := 1 - 2/p$ and we deduce from Morrey's inequality that there is a constant C such that for every $(x, y_1, y_2) \in \Omega^3$ one has

$$|c_\xi(x,y_1) - c_\xi(x,y_2)| \le C\|\xi\|_{L^p}|y_1 - y_2|^\alpha$$

and similarly, for every $(x_1, x_2, y) \in \Omega^3$ one has

$$|c_\xi(x_1,y) - c_\xi(x_2,y)| \le C\|\xi\|_{L^p}|x_1 - x_2|^\alpha$$

and thus

$$|c_\xi(x_1,y_1) - c_\xi(x_2,y_2)| \le C\|\xi\|_{L^p}(|x_1 - x_2|^\alpha + |y_1 - y_2|^\alpha).$$

Since c_ξ vanishes on the diagonal, we deduce from Arzelà-Ascoli theorem that if $(\xi_n)_n$ is a sequence $C(\overline\Omega, \mathbb{R}_+^4)$ that is bounded in L^p then the sequence c_{ξ_n} admits a subsequence that converges in $C(\overline\Omega \times \overline\Omega)$. For $\xi \in L_+^p$ this enables us to define

$$\overline{c}_\xi(x,y) = \sup\left\{c(x,y) \,:\, c \in \mathcal{A}(\xi)\right\}, \forall (x,y) \in \overline\Omega \times \overline\Omega \tag{3.13}$$

where

$$\mathcal{A}(\xi) = \left\{ \lim_n c_{\xi_n} \text{ in } C(\overline\Omega \times \overline\Omega) \,:\, (\xi_n)_n \in C(\overline\Omega, \mathbb{R}_+^4),\ \xi_n \to \xi \text{ in } L^p \right\}. \tag{3.14}$$

For further use, let us state the following result which is a straightforward generalization of Lemmas 3.4 and 3.5 in [6]:

Lemma 3.1. *If $\xi \in C(\overline{\Omega}, \mathbb{R}^4_+)$ then $c_\xi = \overline{c}_\xi$. If $\xi \in L^p_+$ there exists a sequence $(\xi_n)_n$ in $C(\overline{\Omega}, \mathbb{R}^4_+)$ such that c_{ξ_n} converges to $\overline{c}_\xi$ in $C(\overline{\Omega} \times \overline{\Omega})$ as $n \to \infty$.*

Having defined $\overline{c}_\xi$ when ξ is only L^p_+, let us now define

$$I_1(\xi) := \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi d\gamma, \tag{3.15}$$

and the continuous limit (this term will be justified precisely by Γ-convergence in the next section) of J^ε by:

$$J(\xi) := I_0(\xi) - I_1(\xi) = \sum_{i=1}^{4} \int_\Omega H_i(x, \xi_i(x))dx - \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi d\gamma, \ \forall \xi \in L^p_+. \tag{3.16}$$

3.3. **The Γ-convergence result.** The theory of Γ-convergence, initially due to Ennio de Giorgi is a powerful tool to study the convergence of variational problems (convergence of values but also of minimizers) depending on a parameter. It is particularly well suited to study problems involving a scale parameter, as is the case in the present paper where ε represents the network scale and to identify discrete to continuous limits in variational problems, as we shall see in our Wardrop equilibrium problem. We refer to the books of Dal Maso [7] and Braides [4] for the general theory of Γ-convergence as well as for many applications.

First let us define weak L^p convergence of a discrete family $\xi^\varepsilon \in \mathbb{R}^{4\#\Omega_\varepsilon}_+$:

Definition 3.2. *For $\varepsilon > 0$, let $\xi^\varepsilon \in \mathbb{R}^{4\#\Omega_\varepsilon}_+$ and $\xi \in L^p_+$, then ξ^ε is said to weakly converge to ξ in L^p (which we shall simply denote $\xi^\varepsilon \to \xi$) if the following two conditions are satisfied:*

1. *There exists a constant M such that for every $\varepsilon > 0$, one has*

$$\|\xi^\varepsilon\|_{\varepsilon,p} := \Big(\varepsilon^2 \sum_{x\in\Omega_\varepsilon} \sum_{i=1}^{4} \xi^\varepsilon_i(x)^p\Big)^{1/p} \leq M, \tag{3.17}$$

2. *for every $\varphi \in C(\overline{\Omega}, \mathbb{R}^4)$, one has*

$$\lim_{\varepsilon\to 0^+} \varepsilon^2 \sum_{x\in\Omega_\varepsilon} \sum_{i=1}^{4} \varphi_i(x)\xi^\varepsilon_i(x) = \int_\Omega \varphi(x)\cdot\xi(x)dx.$$

Definition 3.3. *For $\varepsilon > 0$, let $F^\varepsilon : \mathbb{R}^{4\#\Omega_\varepsilon}_+ \to \mathbb{R}\cup\{+\infty\}$ and $F : L^p_+ \to \mathbb{R}\cup\{+\infty\}$ then the family of functionals $(F^\varepsilon)_\varepsilon$ is said to Γ-converge (for the weak L^p topology) to F if and only if the following two conditions are satisfied:*

1. (Γ-liminf inequality) *for every $\xi \in L^p_+$ and every family $\xi^\varepsilon \in \mathbb{R}^{4\Omega_\varepsilon}_+$ such that $\xi^\varepsilon \to \xi$ one has*

$$\liminf_{\varepsilon\to 0^+} F^\varepsilon(\xi^\varepsilon) \geq F(\xi),$$

2. (Γ-limsup inequality) *for every $\xi \in L^p_+$, there exists a family $\xi^\varepsilon \in \mathbb{R}^{4\Omega_\varepsilon}_+$ such that $\xi^\varepsilon \to \xi$ and*

$$\limsup_{\varepsilon\to 0^+} F^\varepsilon(\xi^\varepsilon) \leq F(\xi).$$

Our main result whose full proof will be given in the next section, then reads

Theorem 3.4. *Under assumptions (3.1), (3.2), (3.9), the family of functionals J^ε defined by (3.6) Γ-converges (for the weak L^p topology) to the functional J defined by (3.16).*

By very classical arguments from general Γ-convergence theory, we obtain the following convergence result :

Corollary 3.5. *Under assumptions (3.1), (3.2), (3.9), one has:*

$$\min_{\xi^\varepsilon \in \mathbb{R}_+^{4\#\Omega_\varepsilon}} J^\varepsilon(\xi^\varepsilon) \to \min_{\xi \in L_+^p} J(\xi)$$

(the fact that infima actually are achieved being part of the statement). Moreover, if for each $\varepsilon > 0$, ξ^ε solves (3.6), then $\xi^\varepsilon \to \xi$ where ξ is the minimizer of J over L_+^p .

Proof. First, thanks to the estimate (4.7) that follows from Lemma 4.2 proved below and assumption (3.9), we deduce an equi-coercivity estimate, namely that there exists M such that for every ε and $\xi^\varepsilon \in \mathbb{R}_+^{4\#\Omega_\varepsilon}$:

$$J^\varepsilon(\xi^\varepsilon) \geq \lambda(\|\xi^\varepsilon\|_{\varepsilon,p}^p - 1) - M\|\xi^\varepsilon\|_{\varepsilon,p}.$$

where $\|\xi^\varepsilon\|_{\varepsilon,p}$ is defined by (3.17). Not only this proves that the infimum of J^ε over $\mathbb{R}_+^{4\#\Omega_\varepsilon}$ is attained (this is a finite dimensional minimization problem with a continuous and coercive objective function) at some ξ^ε but also that $\|\xi^\varepsilon\|_{\varepsilon,p}$ is uniformly bounded with respect to ε, in particular if we define for ε the $\mathbb{R}^4$-valued Radon measure M_ε by

$$\langle M_\varepsilon, \varphi \rangle := \varepsilon^2 \sum_{x \in \Omega_\varepsilon} \sum_{i=1}^4 \varphi_i(x)\xi_i^\varepsilon(x), \forall \varphi \in C(\overline{\Omega}, \mathbb{R}^4)$$

thanks to Hölder inequality, we have for every $\varepsilon > 0$ and $\varphi \in C(\overline{\Omega}, \mathbb{R}^4)$

$$|\langle M_\varepsilon, \varphi \rangle| \leq C\|\varphi\|_{\varepsilon,p'} \tag{3.18}$$

where $p' = p/(p-1)$ is the conjugate exponent of p and the semi-norm $\|.\|_{\varepsilon,p'}$ is defined in a similar way as in (3.17). Since there is a constant still denoted C such that $\|\varphi\|_{\varepsilon,p'} \leq C\|\varphi\|_\infty$ for every $\varphi \in C(\overline{\Omega}, \mathbb{R}^4)$, we deduce from (3.18) and Banach-Alaoglu's theorem that there exists a (not relabeled) subsequence M_ε and M, an $\mathbb{R}^4$-valued Radon measure to which M_ε weakly star converges. We therefore deduce from (3.18) that for every $\varphi \in C(\overline{\Omega}, \mathbb{R}^4)$, we have

$$|\langle M, \varphi \rangle| \leq C \lim_{\varepsilon \to 0^+} \|\varphi\|_{\varepsilon,p'} = C\|\varphi\|_{L^{p'}}$$

which proves that in fact M admits an L^p representative that we denote ξ, of course $\xi \in L_+^p$ since componentwise nonnegativity is stable under weak limits and $\xi^\varepsilon \to \xi$ in the sense of definition 3.2. It remains to prove that ξ minimizes J over L_+^p. First we know from the Γ-liminf inequality that

$$J(\xi) \leq \liminf_\varepsilon J^\varepsilon(\xi^\varepsilon) = \liminf_{\varepsilon \to 0^+} \min_{\xi^\varepsilon \in \mathbb{R}_+^{4\#\Omega_\varepsilon}} J^\varepsilon.$$

Let then $\zeta \in L_+^p$, we deduce from the Γ-limsup inequality the existence, for each $\varepsilon > 0$, of a $\zeta_\varepsilon \in \mathbb{R}_+^{4\#\Omega_\varepsilon}$ such that $\zeta^\varepsilon \to \zeta$ in the sense of definition 3.2 and

$$\limsup_\varepsilon J^\varepsilon(\zeta^\varepsilon) \leq J(\zeta)$$

and since ξ^ε minimizes J^ε we thus get

$$J(\xi) \leq \liminf_\varepsilon J^\varepsilon(\xi^\varepsilon) \leq \limsup_\varepsilon J^\varepsilon(\xi^\varepsilon) \leq \limsup_\varepsilon J^\varepsilon(\zeta^\varepsilon) \leq J(\zeta)$$

from which we deduce that ξ minimizes J over L^p_+ (and therefore the existence of a minimizer to the limit problem) as well as

$$\min_{L^p_+} J \leq \liminf_{\varepsilon\to 0^+} \min_{\xi^\varepsilon\in\mathbb{R}_+^{4\#\Omega_\varepsilon}} J^\varepsilon \leq \limsup_{\varepsilon\to 0^+} \min_{\xi^\varepsilon\in\mathbb{R}_+^{4\#\Omega_\varepsilon}} J^\varepsilon \leq J(\zeta),\ \forall\zeta\in L^p_+$$

which also proves the convergence of the values of the discrete minimization problems to the value of the continuous one. Finally, we have convergence of the whole family ξ^ε and not only of a subsequence by the uniqueness of the minimizer ξ of J of L^p_+ (since J is strictly convex). □

4. **Proof of the Γ-convergence result.** Recall that in all what follows, we will always assume (3.1), (3.2), (3.9).

4.1. **Γ-liminf inequality.** For (small) $\varepsilon > 0$, let $\xi^\varepsilon \in \mathbb{R}_+^{4\#\Omega_\varepsilon}$ and $\xi \in L^p_+$ such that $\xi^\varepsilon \to \xi$ (in the sense of definition 3.2), recall that our aim is to prove that

$$\liminf_{\varepsilon\to 0^+} J^\varepsilon(\xi^\varepsilon) \geq J(\xi). \tag{4.1}$$

As far as the local term I_0^ε is concerned, by our convexity, growth and continuity assumptions, we easily get

Lemma 4.1. *Under the previous assumptions, one has*

$$\liminf_{\varepsilon\to 0^+} I_0^\varepsilon(\xi^\varepsilon) \geq I_0(\xi). \tag{4.2}$$

Proof. Let $\delta > 0$, we claim that there exists $\varphi = (\varphi_1, ..., \varphi_4)$ continuous on $\overline{\Omega}$ such that

$$I_0(\xi) \leq \delta + \sum_{i=1} \int_\Omega [\varphi_i(x)\xi_i(x) - G_i(x, \varphi_i(x))]dx$$

where we recall that $G_i(x, .)$ is the Legendre Transform of $H_i(x, .)$. Indeed, without imposing continuity, this is just convex duality, now the fact that φ_i can be chosen continuous follows from the continuity for the $L^{p'}$ topology ($p' := p/(p-1)$ the conjugate exponent of p) of $\varphi \mapsto \sum_{i=1} \int_\Omega G_i(x, \varphi_i(x))dx$ and the density of continuous functions in $L^{p'}$.

Now using Young's inequality yields for every $\varepsilon > 0$, and $x \in \Omega_\varepsilon$:

$$\sum_{i=1}^4 H_i(x, \xi_i^\varepsilon(x)) \geq \sum_{i=1}^4 [\varphi_i(x)\xi_i^\varepsilon(x) - G_i(x, \varphi_i(x))]$$

from which we easily deduce that

$$\liminf_{\varepsilon\to 0^+} I_0^\varepsilon(\xi^\varepsilon) \geq I_0(\xi) - \delta$$

and since $\delta > 0$ is arbitrary, we get the claim. □

To deal with the nonlocal term, we shall need some compactness for the minimal length terms, this will follow from the following discretization of Morrey's inequality:

Lemma 4.2. *Let $\theta^\varepsilon \in \mathbb{R}_+^{\Omega_\varepsilon}$ and $\varphi^\varepsilon \in \mathbb{R}^{\Omega_\varepsilon}$ such that*

$$|\varphi^\varepsilon(x) - \varphi^\varepsilon(y)| \leq \varepsilon\theta^\varepsilon(x), \text{ for every } x \in \Omega_\varepsilon \text{ and every } y \text{ neighbor of } x \tag{4.3}$$

then there is a constant C such that for every $(x, y) \in \Omega_\varepsilon \times \Omega_\varepsilon$, one has

$$|\varphi^\varepsilon(x) - \varphi^\varepsilon(y)| \leq C\|\theta^\varepsilon\|_{\varepsilon,p}(|x_1 - y_1| + |x_2 - y_2|)^\alpha$$

where $\alpha := 1 - 2/p$ *and*

$$\|\theta^\varepsilon\|_{\varepsilon,p} = \Big(\varepsilon^2 \sum_{x\in\Omega_\varepsilon} \theta^\varepsilon(x)^p\Big)^{1/p}.$$

Proof. For $x \in \Omega_\varepsilon$, divide the cell $x + \varepsilon[0,1]^2$ into two triangles and extend φ^ε on these triangles by linear interpolation. Still denoting φ^ε this interpolation, we then have $\varphi^\varepsilon \in W^{1,p}$ with $\|\nabla\varphi^\varepsilon\|_{L^p} \le C\|\theta^\varepsilon\|_{\varepsilon,p}$ so that the desired result follows from Morrey's inequality. □

To shorten notations, let us define for every $(x,y) \in \Omega_\varepsilon \times \Omega_\varepsilon$:

$$c^\varepsilon(x,y) := \varepsilon T^\varepsilon_{\xi^\varepsilon}(x,y) = \inf_{\sigma\in C^\varepsilon_{x,y}} \int_0^1 \Phi(\dot{\widetilde{\sigma}}(t)) \cdot \xi^\varepsilon(\widetilde{\sigma}(t))dt. \tag{4.4}$$

By definition, if $x_0 \in \Omega_\varepsilon$ and x and y are neighbors in Ω_ε, we have

$$c^\varepsilon(x_0, x) \le c^\varepsilon(x_0, y) + \varepsilon \max_{i=1,..,4} \xi^\varepsilon_i(y) \tag{4.5}$$

we therefore deduce from Lemma 4.2 that there is a constant C such that for every $\varepsilon > 0$ and every $(x, y, x_0, y_0) \in {\Omega_\varepsilon}^4$, one has

$$|c^\varepsilon(x,y)-c^\varepsilon(x_0,y_0)| \le C\|\xi^\varepsilon\|_{\varepsilon,p}(|x-x_0|^\alpha+|y-y_0|^\alpha) \le C'(|x-x_0|^\alpha+|y-y_0|^\alpha) \tag{4.6}$$

where $C' := C\sup_\varepsilon \|\xi^\varepsilon\|_{\varepsilon,p}$ (recalling that $(\|\xi^\varepsilon\|_{\varepsilon,p})_\varepsilon$ is bounded). Since $c^\varepsilon(x,x) = 0$ and Ω is bounded, this implies in particular that there is some constant $M > 0$ such that

$$I^\varepsilon_1(\xi^\varepsilon) = \sum_{(x,y)\in\Omega_\varepsilon\times\Omega_\varepsilon} c^\varepsilon(x,y)\gamma^\varepsilon(x,y) \le M\|\xi^\varepsilon\|_{\varepsilon,p}. \tag{4.7}$$

The Hölder estimate (4.1) enables us to extend c^ε(and slightly abusing notations we still denote by c^ε this extension) to the whole of $\overline{\Omega} \times \overline{\Omega}$ by setting

$$c^\varepsilon(x,y) := \sup_{(x_0,y_0)\in\Omega_\varepsilon\times\Omega_\varepsilon} \{c^\varepsilon(x_0,y_0)-C'(|x-x_0|^\alpha+|y-y_0|^\alpha)\},\ \forall (x,y) \in \overline{\Omega}\times\overline{\Omega}. \tag{4.8}$$

By construction, the extensions c^ε still satisfy the uniform Hölder estimate on the whole of $\overline{\Omega} \times \times\overline{\Omega}$ and since c^ε vanishes on the diagonal of $\Omega_\varepsilon \times \Omega_\varepsilon$, we deduce from Arzelà-Ascoli Theorem that the family $(c^\varepsilon)_\varepsilon$ is precompact in $C(\overline{\Omega} \times \overline{\Omega})$, taking a subsequence if necessary, we may therefore assume that there is some $c \in C(\overline{\Omega} \times \overline{\Omega})$ such that

$$c^\varepsilon \to c \text{ in } C(\overline{\Omega} \times \overline{\Omega}), \text{ and thus } c(x,x) = 0,\ \forall x \in \overline{\Omega} \tag{4.9}$$

so that thanks to assumption (3.1):

$$I^\varepsilon_1(\xi^\varepsilon) = \sum_{(x,y)\in\Omega_\varepsilon\times\Omega_\varepsilon} c^\varepsilon(x,y)\gamma^\varepsilon(x,y) \to \int_{\overline{\Omega}\times\overline{\Omega}} c d\gamma. \tag{4.10}$$

Thanks to Lemma 4.1, to prove (4.1), it is therefore enough to prove that

$$c \le \overline{c}_\xi \text{ on } \overline{\Omega} \times \overline{\Omega}. \tag{4.11}$$

The rest of this paragraph will be devoted to the proof of inequality (4.11), the strategy to prove (4.11) will consist in showing that c is a sort of subsolution in a very weak sense of an Hamilton-Jacobi equation and this is enough to conclude by some comparison principle, all this seems very classical except that we have to deal with the fact that ξ is only L^p_+. Let us start by remarking that, for fixed $x_0 \in \Omega$, $c(x_0,.) \in W^{1,p}(\Omega)$, indeed if $\varphi \in C^1_c(\Omega)$ and $i = 1$ or 2, denoting by (e_1, e_2) the canonical basis of $\mathbb{R}^2$, choosing $x_0^\varepsilon \in \Omega_\varepsilon$ such that $|x_0 - x_0^\varepsilon| \leq \sqrt{2}\varepsilon$ and using the uniform convergence of $c^\varepsilon(x_0^\varepsilon, .)$ to $c(x_0, .)$, it is easy to see that

$$\begin{aligned}\int_\Omega c(x_0,.)\partial_i\varphi &= \lim_{\varepsilon\to 0^+} \varepsilon^2 \sum_{x\in\Omega_\varepsilon} c^\varepsilon(x_0^\varepsilon, x)\frac{\varphi(x+\varepsilon e_i)-\varphi(x)}{\varepsilon}\\ &= \lim_{\varepsilon\to 0^+} \varepsilon^2 \sum_{x\in\Omega_\varepsilon} \frac{c^\varepsilon(x_0^\varepsilon, x-\varepsilon e_i)-c^\varepsilon(x_0^\varepsilon, x)}{\varepsilon}\varphi(x)\end{aligned}$$

and thanks to (4.5), Hölder's inequality and the fact that $(\|\xi^\varepsilon\|_{\varepsilon,p})_\varepsilon$ is bounded we obtain that

$$\Big|\int_\Omega c(x_0,.)\partial_i\varphi\Big| \leq C\|\varphi\|_{L^{p'}},\ \forall\varphi\in C_c^1(\Omega)$$

which proves that $c(x_0, .) \in W^{1,p}(\Omega)$ (and by a similar argument that $c(., y_0) \in W^{1,p}(\Omega)$ for every fixed $y_0 \in \Omega$).

Lemma 4.3. *Let $x_0 \in \Omega$, $\xi \in L^p_+$ and $\varphi \in W^{1,p}(\Omega)$ such that $\varphi(x_0) = 0$ (which makes sense since $p > 2$ so that φ is continuous). If for a.e. $x \in \Omega$ one has*

$$\nabla\varphi(x)\cdot u \leq \xi(x)\cdot\Phi(u),\ \forall u\in\mathbb{R}^2 \tag{4.12}$$

then $\varphi \leq \bar{c}_\xi(x_0, .)$ on Ω.

Proof. The result is obvious if $\varphi \in C^1(\overline{\Omega})$ and ξ is continuous on $\overline{\Omega}$, indeed, in this case, (4.12) holds pointwise, and if $x \in \Omega$ and σ is an absolutely continuous curve with values in $\overline{\Omega}$ connecting x_0 and x, by the chain rule we have

$$\varphi(x) = \int_0^1 \nabla\varphi(\sigma(t))\cdot\dot{\sigma}(t)dt \leq \int_0^1 \Phi(\dot{\sigma}(t))\cdot\xi(\sigma(t))dt$$

and taking the infimum in σ we obtain $\varphi \leq c_\xi(x_0, .)$ on Ω i.e. $\varphi \leq \bar{c}_\xi(x_0, .)$ thanks to Lemma 3.1. If φ is only $W^{1,p}$ and ξ only L^p_+, we first extend φ to a function in $W^{1,p}(\mathbb{R}^2)$ (recall that Ω is assumed to be smooth), we then extend ξ outside Ω by setting $\xi = |\nabla\varphi|(1, 1, 1, 1)$ so that if $x \in \mathbb{R}^2 \setminus \overline{\Omega}$ and $u \in S^1$ we have

$$\begin{aligned}\nabla\varphi(x)\cdot u &\leq |\nabla\varphi(x)| = |\nabla\varphi(x)|(u_1^2+u_2^2)\\ &\leq |\nabla\varphi(x)|(|u_1|+|u_2|) = \xi(x)\cdot\Phi(u)\end{aligned}$$

so that by the homogenity of (4.12) in u, (4.12) continues to hold outside Ω with the previous extension. We then consider a mollifying sequence $\rho_n(x) = n^2\rho(nx)$, $x \in \mathbb{R}^2$ where ρ is a smooth nonnegative function supported on the unit ball and such that $\int_{\mathbb{R}^2}\rho = 1$ and define $\xi_n := \rho_n \star \xi$ and $\varphi_n := \rho_n \star \varphi - (\rho_n \star \varphi)(x_0)$. By construction we have

$$\nabla\varphi_n(x)\cdot u \leq \xi_n(x)\cdot\Phi(u),\ \forall (x,u)\in\mathbb{R}^2\times\mathbb{R}^2$$

so that with the previous argument and the smoothness of φ_n and ξ_n we get $\varphi_n \leq c_{\xi_n}(x_0, .)$, using the convergence of φ_n to φ we thus get

$$\varphi = \limsup\varphi_n \leq \limsup c_{\xi_n}(x_0,.) \leq \bar{c}_\xi(x_0,.)$$

where the last inequality follows from the very definition of $\bar{c}_\xi$ as a supremum (3.13)-(3.14) and the precompactness of c_{ξ_n} in $C(\overline{\Omega}\times\overline{\Omega})$. □

The last ingredient to prove $c \leq \bar{c}_\xi$ and then to terminate the proof of the Γ-liminf inequality (4.1) is given by :

Lemma 4.4. *Let $x_0 \in \Omega$ and c be defined by (4.9), one has*

1. *for every $w \in C_c^\infty(\Omega, \mathbb{R}^2)$, the following inequality holds*

$$\int_\Omega \nabla_x c(x_0, x) \cdot w(x) dx \leq \int_\Omega \Phi(w(x)) \cdot \xi(x) dx \tag{4.13}$$

2. *$c \leq \overline{c}_\xi$ (and then, thanks to Lemma 4.1, the Γ-liminf inequality (4.1) holds).*

Proof. 1. Let $x_0^\varepsilon \in \Omega_\varepsilon$ be such that $|x_0 - x_0^\varepsilon| \leq \sqrt{2}\varepsilon$ so that $c^\varepsilon(x_0^\varepsilon, .)$ converges uniformly to $c(x_0, .)$. For $\varphi \in C_c^1(\Omega)$, $i = 1, 2$, and (e_1, e_2) the canonical basis of $\mathbb{R}^2$, we already know that

$$T\varphi := \int_\Omega \partial_i c(x_0, .)\, \varphi = \lim_{\varepsilon \to 0^+} T_\varepsilon \varphi \tag{4.14}$$

where

$$T_\varepsilon \varphi = \varepsilon^2 \sum_{x \in \Omega_\varepsilon} \frac{c^\varepsilon(x_0^\varepsilon, x + \varepsilon e_i) - c^\varepsilon(x_0^\varepsilon, x)}{\varepsilon} \varphi(x).$$

For $\varphi \in C_c(\Omega) \cap W^{1,\infty}(\Omega)$, approximating φ uniformly by smooth and compactly supported functions (again by convolution), it is easy to see that (4.14) also holds. In particular (4.14) applies to the components of $\Phi(w)$ (they are nonsmooth because of the positive part but still Lipschitz and compactly supported). We may thus write

$$\int_\Omega \nabla_x c(x_0, .) \cdot w = \int_\Omega (\partial_1 c(x_0, .)((w_1)_+ - (w_1)_-) + \partial_2 c(x_0, .)((w_2)_+ - (w_2)_-))$$

as the limit as $\varepsilon \to 0^+$ of

$$\varepsilon^2 \sum_{x \in \Omega^\varepsilon} \sum_{i=1}^4 \frac{c^\varepsilon(x_0^\varepsilon, x + \varepsilon v_i) - c^\varepsilon(x_0^\varepsilon, x)}{\varepsilon} (w(x) \cdot v_i)_+$$

now we use the inequality $c^\varepsilon(x_0^\varepsilon, x + \varepsilon v_i) - c^\varepsilon(x_0^\varepsilon, x) \leq \varepsilon \xi_i^\varepsilon(x)$ to obtain that the previous sum is bounded from above by

$$\varepsilon^2 \sum_{x \in \Omega_\varepsilon} \Phi(w(x)) \cdot \xi^\varepsilon(x)$$

passing to the limit in $\varepsilon \to 0^+$ thus exactly gives (4.13).

2. First, using (4.13) with $w = \theta v$ for $v \in C_c^\infty(\Omega, \mathbb{R}^2)$ and an arbitrary scalar function $\theta \in C_c^\infty(\Omega, \mathbb{R})$, $\theta \geq 0$, we deduce from the homogeneity of Φ that

$$\nabla_x c(x_0, x) \cdot v(x) \leq \Phi(v(x)) \cdot \xi(x) \text{ a.e. on } \Omega \tag{4.15}$$

Now let x be a Lebesgue point of both ξ and $\nabla_x c(x_0, .)$, $u \in S^1$ and choose $v \in C_c^\infty(\Omega, \mathbb{R}^2)$ such that $v = u$ in some neighbourhood of x, integrating inequality (4.15) over $B_r(x)$ dividing by πr^2 and letting $r \to 0^+$ we exactly get

$$\nabla_x c(x_0, x) \cdot u \leq \Phi(u) \cdot \xi(x) \text{ a.e. on } \Omega$$

which thanks to Lemma 4.3 gives $c(x_0, .) \leq \overline{c}_\xi(.)$. □

4.2. **Γ-limsup inequality.** Given, $\xi \in L^p_+$, it remains now to prove the Γ-limsup inequality i.e. the existence of a family $\xi^\varepsilon \in \mathbb{R}_+^{4\#\Omega_\varepsilon}$ such that

$$\xi^\varepsilon \to \xi, \quad \text{and} \ \limsup_{\varepsilon \to 0^+} J^\varepsilon(\xi^\varepsilon) \le J(\xi). \tag{4.16}$$

The proof is much simpler than that of Γ-liminf inequality. We shall prove the result first in the case where ξ is continuous and then treat the general case by a density argument:

Step 1 : ξ is continuous

For $\varepsilon > 0$, let us define for every $x \in \Omega_\varepsilon$ and $i = 1, ..., 4$,

$$\xi_i^\varepsilon(x) := \frac{1}{\varepsilon} \int_{[x, x+\varepsilon v_i]} \xi_i = \int_0^1 \xi_i(x + s\varepsilon v_i) ds$$

We also extend ξ^ε in a piecewise constant way to the whole of Ω by setting $\xi^\varepsilon = \xi^\varepsilon(x)$ on the square having the neighbors of x in Ω_ε as vertices. Doing so, we obviously have

$$\|\xi^\varepsilon - \xi\|_{L^p} \to 0, \ \text{and} \ I_0^\varepsilon(\xi^\varepsilon) \to I_0(\xi) \ \text{as} \ \varepsilon \to 0^+. \tag{4.17}$$

Note in particular that $\xi^\varepsilon \to \xi$ in the weak sense of definition 3.2. For $\varepsilon > 0$, and $(x, y) \in \Omega_\varepsilon$, let us define

$$c^\varepsilon(x, y) := \varepsilon T^\varepsilon_{\xi^\varepsilon}(x, y) = \inf_{\sigma \in C^\varepsilon_{x,y}} \int_0^1 \Phi(\dot{\widetilde{\sigma}}(t)) \cdot \xi^\varepsilon(\widetilde{\sigma}(t)) dt$$

by the same arguments as in the proof of the Γ-liminf inequality, thanks to the fact that ξ^ε is bounded in L^p and again to Lemma 4.2, we may extend c^ε to the whole of $\overline{\Omega} \times \overline{\Omega}$ and thus obtain a bounded and equi-Hölder family still denoted c^ε, passing up to a subsequence we may also assume that c^ε converges to some c in $C(\overline{\Omega} \times \overline{\Omega})$. We then have $\liminf_{\varepsilon \to 0^+} I_1^\varepsilon(\xi^\varepsilon) = \int_{\overline{\Omega} \times \overline{\Omega}} c d\gamma$ so that to prove (4.16) it is enough to prove that $c \ge c_\xi = \overline{c}_\xi$ and to see that this inequality holds it is enough to remark that by construction for $(x, y) \in \Omega_\varepsilon \times \Omega_\varepsilon$ one has

$$c^\varepsilon(x, y) = \inf_{\sigma \in C^\varepsilon_{x,y}} \int_0^1 \Phi(\dot{\widetilde{\sigma}}(t)) \cdot \xi(\widetilde{\sigma}(t)) dt \ge c_\xi(x, y)$$

using the uniform convergence of c^ε to c we indeed obtain $c \ge c_\xi = \overline{c}_\xi$.

Step 2 : the general case where ξ is only L^p_+

Thanks to lemma 3.1 we can find for each n, $\xi_n \in C(\overline{\Omega}, \mathbb{R}^4_+)$ such that

$$\|\xi_n - \xi\|_{L^p} + \|c_{\xi_n} - \overline{c}_\xi\|_{L^\infty} + |I_0(\xi_n) - I_0(\xi)| \le \frac{1}{n} \tag{4.18}$$

for each ε we then construct a piecewise constant ξ_n^ε approximation of ξ_n as in step 1. Thanks to step 1, we deduce that for each n there is some $\varepsilon_n > 0$ (that we may choose nonincreasing and such that $\varepsilon_n \to 0$ as $n \to \infty$) such that for $0 < \varepsilon \le \varepsilon_n$ one has

$$\|\xi_n^\varepsilon - \xi_n\|_{L^p} + |I_0^\varepsilon(\xi_n^\varepsilon) - I_0(\xi_n)| \le \frac{1}{n} \ \text{and} \ I_1^\varepsilon(\xi_n^\varepsilon) \ge I_1(\xi_n) - \frac{1}{n} \tag{4.19}$$

For $\varepsilon > 0$ let $n_\varepsilon := \sup\{n \ : \ \varepsilon_n \ge \varepsilon\}$ and $\xi^\varepsilon := \xi^\varepsilon_{n_\varepsilon}$, by construction with (4.18) and (4.19), we have

$$\|\xi^\varepsilon - \xi\|_{L^p} + |I_0^\varepsilon(\xi^\varepsilon) - I_0(\xi)| \le \frac{2}{n_\varepsilon} \to 0 \ \text{as} \ \varepsilon \to 0^+ \tag{4.20}$$

as well as

$$I_1^\varepsilon(\xi^\varepsilon) \geq I_1(\xi_{n_\varepsilon}) - \frac{1}{n_\varepsilon} = \int_{\overline{\Omega}\times\overline{\Omega}} c_{\xi_{n_\varepsilon}} d\gamma - \frac{1}{n_\varepsilon}$$

using the fact that $c_{\xi_{n_\varepsilon}}$ converges to $\overline{c}_\xi$ we thus get

$$\liminf_{\varepsilon} I_1(\xi^\varepsilon) \geq I_1(\xi)$$

with (4.20) this proves the Γ-limsup inequality (4.16).

5. **Optimality conditions and continuous Wardrop equilibria.** Our aim now is to give optimality conditions for the limit problem:

$$\inf_{\xi\in L_+^p} J(\xi) \text{ with } J(\xi) := \sum_{i=1}^4 \int_\Omega H_i(x, \xi_i(x))dx - \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi d\gamma, \tag{5.1}$$

through some dual formulation that can be interpreted in terms of continuous Wardrop equilibria i.e. that is in some sense the continuous version of the finite dimensional optimization problem that consists in minimizing (2.4) subject to the mass conservation conditions (2.2) and (2.3). This dual formulation will involve probability measures on set of paths (i.e. the macroscopic version of the flows $w^\varepsilon(\sigma)$ of the network model of section 2.1) and will turn out to be an anisotropic variant of the problem studied in details in Carlier, Jimenez and Santambrogio [6]. Let $C := W^{1,\infty}([0,1], \overline{\Omega})$, viewed as a subset of $C([0,1], \mathbb{R}^2)$, i.e. equipped with the uniform topology, and slightly abusing notations let us denote by $\mathcal{M}_1^+(C)$ the set of Borel probability measures Q on $C([0,1], \mathbb{R}^2)$ such that $Q(C) = 1$. Let us define then the set of probability measures on paths that are consistent with the transport plan γ:

$$\mathcal{Q}(\gamma) := \{Q \in \mathcal{M}_1^+(C) \ : \ (e_0, e_1)_\# Q = \gamma\}$$

where e_0 and e_1 are evaluations at time 0 and 1 and $(e_0, e_1)_\# Q$ denotes the image measure of Q by (e_0, e_1). Thus $Q \in \mathcal{Q}(\gamma)$ means that

$$\int_C \varphi(\sigma(0), \sigma(1))dQ(\sigma) = \int_{\overline{\Omega}\times\overline{\Omega}} \varphi(x,y)d\gamma(x,y), \ \forall \varphi \in C(\mathbb{R}^2, \mathbb{R}).$$

Note that Q plays the same role as the paths-flows in the network model and the condition $Q \in \mathcal{Q}(\gamma)$ is the continuous analogue of the mass conservation condition (2.2). Let us now define the analogue of the arc flows induced by $Q \in \mathcal{Q}(\gamma)$; for $i = 1, .., 4$ let us define the nonnegative measure on $\overline{\Omega}$, m_i^Q, by

$$\int_\Omega \varphi(x) dm_i^Q(x) = \int_C \Big(\int_0^1 \varphi(\sigma(t))(\dot\sigma(t)\cdot v_i)_+ dt \Big) dQ(\sigma), \ \forall \varphi \in C(\overline{\Omega}, \mathbb{R}).$$

Thus, the $\mathbb{R}^4$-valued measure $m^Q = (m_1^Q, m_2^Q, m_3^Q, m_4^Q)$ can be defined by

$$\int_{\overline{\Omega}} \xi \cdot dm^Q = \int_C L_\xi(\sigma)dQ(\sigma), \forall \xi \in C(\overline{\Omega}, \mathbb{R}_+^4)$$

where

$$L_\xi(\sigma) := \int_0^1 \xi(\sigma(t)) \cdot \Phi(\dot\sigma(t))dt = \int_0^1 \sum_{i=1}^4 \xi_i(\sigma(t))(\dot\sigma(t)\cdot v_i)_+ dt = \sum_{i=1}^4 L_{\xi_i}^i(\sigma). \tag{5.2}$$

Let us now recall that $H_i(x,.)$ is the convex conjugate of $G_i(x,.)$ where $G_i(x,.)$ is the primitive of the function $g_i(x,.)$ that relates the metric at x in direction v_i to the flow in this direction. The p growth assumption (3.9) on H_i then can be translated into a similar q-growth condition on G_i for $q = p/(p-1)$ the conjugate exponent of p. In fact, we will slightly strengthen assumption (3.9) by further assuming that $g_i(x,.)$ is continuous, everywhere positive and increasing in its second argument (so that $G_i(x,.)$ is strictly convex) and such there exists a and b such that $b \geq a > 0$ and

$$am^{q-1} \leq g_i(x,m) \leq b(m^{q-1}+1),\ \forall (i,x,m) \in \{1,..,4\} \times \overline{\Omega} \times \mathbb{R}_+,\ \text{with } q \in (1,2). \tag{5.3}$$

Let us define then

$$\mathcal{Q}^q(\gamma) := \{Q \in \mathcal{Q}(\gamma)\ :\ m^Q \in L^q(\Omega, \mathbb{R}^4)\} \tag{5.4}$$

and assume

$$\mathcal{Q}^q(\gamma) \neq \emptyset \tag{5.5}$$

this assumption is satisfied in particular when γ is supported by finitely many points and $q < 2$ (see [3]). Let $Q \in \mathcal{Q}^q(\gamma)$ and ξ and $\widetilde{\xi}$ be in $C(\overline{\Omega}, \mathbb{R}^4_+)$, we have

$$\int_C |L_\xi(\sigma) - L_{\widetilde{\xi}}(\sigma)| dQ(\sigma) \leq \|\xi - \widetilde{\xi}\|_{L^p} \|m^Q\|_{L^q}$$

which proves that if $\xi \in L^p_+$ and $(\xi_n)_n$ is a sequence in $C(\overline{\Omega}, \mathbb{R}^4_+)$ that converges in L^p to ξ, then L_{ξ_n} is a Cauchy sequence in $L^1(C,Q)$ and its limit, again denoted L_ξ does not depend on the approximating sequence $(\xi_n)_n$. As in [6], this enables us to define L_ξ in an $L^1(C,Q)$ sense for every $\xi \in L^p_+$ and $Q \in \mathcal{Q}^q(\gamma)$. For every $\xi \in L^p_+$ and $Q \in \mathcal{Q}^q(\gamma)$, one can show that

$$\int_\Omega \xi \cdot m^Q = \int_C L_\xi(\sigma) dQ(\sigma),\ \overline{c}_\xi(\sigma(0), \sigma(1)) \leq L_\xi(\sigma) \text{ for } Q\text{-a.e. } \sigma \in C \tag{5.6}$$

we refer to [6] for a proof when ξ is only L^p_+. Let $\xi \in L^p_+$ and $Q \in \mathcal{Q}^q(\gamma)$, we first deduce from the fact that $H_i(x,.)$ and $G_i(x,.)$ are conjugates that

$$\sum_{i=1}^4 \int_\Omega H_i(x, \xi_i(x)) dx \geq \int_\Omega \xi(x) \cdot m^Q(x) dx - \sum_{i=1}^4 \int_\Omega G_i(x, m_i^Q(x)) dx \tag{5.7}$$

using the fact that $Q \in \mathcal{Q}^q(\gamma)$ and (5.6), we also have

$$\int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi d\gamma = \int_C \overline{c}_\xi(\sigma(0), \sigma(1)) dQ(\sigma) \leq \int_C L_\xi(\sigma) dQ(\sigma) = \int_\Omega \xi \cdot m^Q \tag{5.8}$$

so that

$$\inf_{\xi \in L^p_+} J(\xi) \geq \sup_{Q \in \mathcal{Q}^q(\gamma)} -\sum_{i=1}^4 \int_\Omega G_i(x, m_i^Q(x)) dx. \tag{5.9}$$

We shall from now on call

$$\sup_{Q \in \mathcal{Q}^q(\gamma)} -\sum_{i=1}^4 \int_\Omega G_i(x, m_i^Q(x)) dx. \tag{5.10}$$

the *dual* of (5.1). Let us also remark the analogy between the continuous problem (5.10) and the finite-dimensional that consists in minimizing (2.4) subject to the mass conservation conditions (2.2) and (2.3). The precise relations between (5.10) and (5.1) and the connection with Wardrop-like equilibria are given by the following properties which are quite simple extensions of the results of [6] to the anisotropic setting:

Theorem 5.1. *Under assumptions (5.3) and (5.5), we have:*

1. *(5.10) admits solutions,*

2. *$\overline{Q} \in \mathcal{Q}^q(\gamma)$ solves (5.10) if only if*

$$\int_C \Big(L_{\xi_{\overline{Q}}}(\sigma) - \overline{c}_{\xi_{\overline{Q}}}(\sigma(0), \sigma(1)\Big) d\overline{Q}(\sigma) = 0 \tag{5.11}$$

where $\xi_{\overline{Q}} := (g_1(., m_1^{\overline{Q}}(.)), ..., g_4(., m_4^{\overline{Q}}(.)))$.

3. *there is no duality gap : the supremum of (5.10) equals the infimum of (5.1) and moreover if $\overline{Q}$ solves (5.10) then $\xi_{\overline{Q}}$ solves (5.1).*

Proof. We will only sketch the proof and refer to [6] for detailed proofs which can straightforwardly be adapted to the anisotropic case.

1. Let $(Q_n)_n$ be a maximizing sequence for (5.10), we may reparameterize paths by arclength (so that euclidean length becomes the Lipschitz constant of the curve), the corresponding measures on curves still form a maximizing sequence again denoted (Q_n), since (m^{Q_n}) is bounded in L^q, this gives a bound on $\int_C \mathrm{Lip}(\sigma) dQ_n(\sigma)$ and thus thanks to Ascoli and Prokhorov's theorems, this also gives some tightness of (Q_n), arguing as in Lemma 2.8 of [6] we find a $Q \in \mathcal{M}_1^+(C)$ to which, up to a subsequence, (Q_n) weakly star converges in $\mathcal{M}(C([0,1], \mathbb{R}^2)$. We may also assume that m^{Q_n} converges weakly in L^q to some m and arguing as in Lemma 2.9 of [6] we obtain that $m_i^Q \le m_i$ for $i = 1, ..., 4$, in particular $Q \in \mathcal{Q}^q(\gamma)$ and since $G_i(x, .)$ is nondecreasing and convex, we have

$$\int_\Omega G_i(x, m_i^Q(x))dx \le \int_\Omega G_i(x, m_i(x))dx \le \liminf_n \int_\Omega G_i(x, m_i^{Q_n}(x))dx$$

which proves that Q solves (5.10).

2. Assume first that $\overline{Q} \in \mathcal{Q}^q(\gamma)$ satisfies (5.11), and let $Q \in \mathcal{Q}^q(\gamma)$, by convexity of $G_i(x, .)$, (5.6) and (5.11) we have

$$\begin{aligned}
\sum_{i=1}^4 \int_\Omega G_i(x, m_i^Q(x))dx - \sum_{i=1}^4 \int_\Omega G_i(x, m_i^{\overline{Q}}(x))dx &\ge \int_\Omega \xi_{\overline{Q}}(\cdot m^Q - m^{\overline{Q}}) \\
&= \int_C L_{\xi_{\overline{Q}}}(\sigma) dQ(\sigma) - \int_C L_{\xi_{\overline{Q}}}(\sigma) d\overline{Q}(\sigma) \\
&\ge \int_C \overline{c}_{\xi_{\overline{Q}}}(\sigma(0), \sigma(1)) dQ(\sigma) - \int_C \overline{c}_{\xi_{\overline{Q}}}(\sigma(0), \sigma(1)) d\overline{Q}(\sigma) \\
&= \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\xi_{\overline{Q}}} d\gamma - \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\xi_{\overline{Q}}} d\gamma = 0
\end{aligned}$$

so that $\overline{Q}$ solves (5.10). Now assume that $\overline{Q} \in \mathcal{Q}^q(\gamma)$ solves (5.10), let $Q \in \mathcal{Q}^q(\gamma)$ and $\varepsilon \in (0,1)$, dividing the inequality

$$\sum_{i=1}^4 \int_\Omega G_i(x, (1-\varepsilon) m_i^{\overline{Q}}(x) + \varepsilon m_i^Q(x))dx - \sum_{i=1}^4 \int_\Omega G_i(x, m_i^{\overline{Q}}(x))dx \ge 0$$

by ε and letting $\varepsilon \to 0^+$ we get

$$\int_\Omega \xi_{\overline{Q}} \cdot m^{\overline{Q}} = \int_C L_{\xi_{\overline{Q}}} d\overline{Q} \le \int_\Omega \xi_{\overline{Q}} \cdot m^Q = \int_C L_{\xi_{\overline{Q}}} dQ, \ \forall Q \in \mathcal{Q}^q(\gamma)$$

arguing as in Proposition 3.9 of [6] we obtain that the infimum of the right-hand side of the previous inequality is in fact $\int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\xi_{\overline{Q}}} d\gamma$ so that

$$\int_C L_{\xi_{\overline{Q}}} d\overline{Q} = \int_C \overline{c}_{\xi_{\overline{Q}}}(\sigma(0), \sigma(1)) d\overline{Q}(\sigma).$$

3. Let $\overline{Q}$ solve (5.10) then, by construction taking $\xi = \xi_{\overline{Q}}$ and $m = m^{\overline{Q}}$, inequality (5.7) becomes an equality, and (5.8) as well thanks to (5.11), which proves that (5.9) is in fact an equality and $\xi_{\overline{Q}}$ solves (5.1). □

Of course, the optimality condition (5.11) for (5.10) can naturally be interpreted in terms of Wardrop equilibria. Indeed, ξ_Q being the metric induced by Q we may define continuous Wardrop equilibria as the set of Q's in $\mathcal{Q}^q(\gamma)$ such that Q gives full mass to geodesics for the congested metric ξ_Q, where such geodesics are by definition paths σ such that $L_{\xi_Q}(\sigma) = \bar{c}_{\xi_Q}(\sigma(0), \sigma(1))$. Condition (5.11) therefore exactly says that $\overline{Q}$ solves (5.10) if and only if it is a continuous Wardrop equilibrium. In particular there exist continuous Wardrop equilibria as soon as (5.3) and (5.5) hold.

A natural question now is whether the discrete problems corresponding to (2.4) i.e.:

$$\inf_{\mathbf{m}^\varepsilon, \mathbf{w}^\varepsilon} \varepsilon^2 \sum_{x \in \Omega_\varepsilon} \sum_{i=1}^{4} G_i\Big(x, \frac{m_i^\varepsilon(x)}{\varepsilon}\Big) \tag{5.12}$$

subject to the mass conservation constraints (2.2)-(2.3) converge in some sense to the continuous problem

$$\inf_{Q \in \mathcal{Q}^q(\gamma)} \sum_{i=1}^{4} \int_\Omega G_i(x, m_i^Q(x))dx. \tag{5.13}$$

Let $\mathbf{m}^\varepsilon = (m_i^\varepsilon(x))_{x,i} \in \Omega_\varepsilon \times \{1, ...4\}$ and $\mathbf{w}^\varepsilon = (w^\varepsilon(\sigma))_{\sigma \in C^\varepsilon}$ be a solution of the discrete problem (5.12) and define the discrete measure over C^ε:

$$Q^\varepsilon := \sum_{\sigma \in C^\varepsilon} w^\varepsilon(\sigma) \delta_\sigma$$

as well as

$$\widetilde{Q}^\varepsilon := \sum_{\sigma \in C^\varepsilon} w^\varepsilon(\sigma) \delta_{\widetilde{\sigma}}$$

where $\widetilde{\sigma} \in C$ denotes the constant speed reparameterization of the path σ. Since for every $i \in \{1, ..., 4\}$ and $\xi_i \in C(\overline{\Omega}, \mathbb{R}_+)$, using definition (5.2) one has $L^i_{\xi_i}(\sigma) = L^i_{\xi_i}(\widetilde{\sigma})$ and thus also $m^{Q^\varepsilon} = m^{\widetilde{Q}^\varepsilon}$. Let us also remark that the measure $\widetilde{Q}^\varepsilon$ contains all the information on $(\mathbf{m}^\varepsilon, \mathbf{w}^\varepsilon)$.

Theorem 5.2. *Under assumptions (3.1), (3.2), (3.9), (5.3) and (5.5), defining $\widetilde{Q}^\varepsilon$ as above, up to (a not relabeled) subsequence $(\widetilde{Q}^\varepsilon)_\varepsilon > 0$ converges weakly to some solution $Q \in \mathcal{Q}^q(\gamma)$ of (5.13) in the sense that*

$$\int_{C([0,1],\mathbb{R}^2)} \Phi(\sigma) d\widetilde{Q}^\varepsilon(\sigma) \to \int_{C([0,1],\mathbb{R}^2)} \Phi(\sigma) dQ(\sigma) \text{ as } \varepsilon \to 0^+$$

for every $\Phi \in C_b(C([0,1], \mathbb{R}^2), \mathbb{R})$.

Proof. We know by duality, from corollary 3.5 and theorem 5.1, that the value of (5.12) converges to the value of (5.13) in particular, thanks to (5.3) this gives a bound on the discrete L^q norm of $\mathbf{m}^\varepsilon$. Arguing as in the proof of corollary 3.5 and section 4.1, we deduce that there is some $m = (m_1, ..., m_4) \in L^q_+$ such that up to a subsequence $\varepsilon^{-1}\mathbf{m}^\varepsilon$ weakly converges in L^q to m in the sense of definition 3.2 (up to changing p to its conjugate q) and

$$\int_\Omega \sum_{i=1}^4 G_i(x, m_i(x))dx \le \liminf_{\varepsilon\to 0^+} \varepsilon^2 \sum_{x\in\Omega_\varepsilon}\sum_{i=1}^4 G_i\Big(x, \frac{m_i^\varepsilon(x)}{\varepsilon}\Big). \tag{5.14}$$

Let $i \in \{1, ..., 4\}$ and $\xi_i \in C(\overline{\Omega}, \mathbb{R}_+)$, using definition (5.2) and (2.3), rearranging terms, one easily gets

$$\begin{aligned}\int_{\overline{\Omega}} \xi_i(x) dm_i^{\widetilde{Q}^\varepsilon}(x) &= \sum_{\sigma\in C^\varepsilon} w^\varepsilon(\sigma) L^i_{\xi_i}(\sigma)\\
&= \sum_{x\in\Omega_\varepsilon}\Big(\int_{[x,x+\varepsilon v_i]}\xi_i\Big)\Big(\sum_{\sigma\in C^\varepsilon\,:\,[x,x+\varepsilon v_i]\subset\sigma} w^\varepsilon(\sigma)\Big)\\
&= \sum_{x\in\Omega_\varepsilon}\Big(\xi_i(x) + O(\omega_{\xi_i}(\varepsilon))\Big)\Big(\sum_{\sigma\in C^\varepsilon\,:\,[x,x+\varepsilon v_i]\subset\sigma} \varepsilon w^\varepsilon(\sigma)\Big)\\
&= \varepsilon^2 \sum_{x\in\Omega_\varepsilon} \xi_i(x)\frac{m_i^\varepsilon(x)}{\varepsilon} + O(\omega_{\xi_i}(\varepsilon))\end{aligned}$$

where ω_{ξ_i} denotes a modulus of continuity of ξ_i. Since $\varepsilon^{-1}\mathbf{m}^\varepsilon$ weakly converges in L^q to m in the sense of definition 3.2, this shows that $m_i^{\widetilde{Q}^\varepsilon}$ weakly star converges to m. Proceeding as in [6] (see lemmas 2.7, 2.8 and 2.9), we can find $Q \in \mathcal{M}_1^+(C)$ such that (up to a subsequence) $(\widetilde{Q}^\varepsilon)_\varepsilon$ converges weakly to Q and $m_i^Q \le m_i$ for $i = 1, ..., 4$. Clearly, $Q \in \mathcal{Q}^q(\gamma)$ and since $G_i(x, .)$ is nondecreasing, recalling (5.14), we have

$$\begin{aligned}\int_\Omega \sum_{i=1}^4 G_i(x, m_i^Q(x))dx &\le \int_\Omega \sum_{i=1}^4 G_i(x, m_i(x))dx\\
&\le \liminf_{\varepsilon\to 0^+} \varepsilon^2 \sum_{x\in\Omega_\varepsilon}\sum_{i=1}^4 G_i\Big(x, \frac{m_i^\varepsilon(x)}{\varepsilon}\Big)\end{aligned}$$

and the desired conclusion follows from the fact that the right-hand side is the value of the infimum in (5.13). □

6. **The long-term variant.** Instead of prescribing a transport plan γ^ε in the discrete problem (2.5), we could have fixed only its marginals i.e. the distribution of sources and of sinks (or supply and demand), respectively given by the nonnegative numbers $(\mu_0^\varepsilon(x))_{x\in\Omega_\varepsilon}$ and $(\mu_1^\varepsilon(y))_{y\in\Omega_\varepsilon}$ that satisfy the compatibility condition

$$\sum_{x\in\Omega_\varepsilon} \mu_0^\varepsilon(x) = \sum_{y\in\Omega_\varepsilon} \mu_1^\varepsilon(y) > 0.$$

In the equilibrium problem this means that the transport plan now becomes part of the unknown and should be determined by some additional optimality requirement, namely that it is an optimal transport plan between the prescribed marginals for the transport cost induced by the congested metric itself. In other words, in the long term variant, in addition to traffic congestion we are also facing an optimal transportation problem. We refer the reader to Villani's book [8] for a recent account of optimal transport theory and its numerous applications. More precisely, using the same notations as in section 2, the definition of an equilibirium is exactly the same as in definition 2.2 except that one replaces the mass conservation condition (2.2) by

$$\mu_0^\varepsilon(x) := \sum_{\sigma \in C^\varepsilon_{x,.}} w^\varepsilon(\sigma),\ \mu_1^\varepsilon(y) := \sum_{\sigma \in C^\varepsilon_{.,y}} w^\varepsilon(\sigma) \tag{6.1}$$

for every $(x, y) \in \Omega_\varepsilon$ where we have denoted by $C^\varepsilon_{x,.}$ (respectively $C^\varepsilon_{.,y}$) the set of simple paths on the network that start at x (respectively end at y). It is then easy to check that equilibria are obtained by minimizing the functional defined by (2.4) but now subject to (6.1) and (2.3). The dual formulation then reads as the following variant of (2.5):

$$\inf_{\mathbf{t}^\varepsilon \in \mathbb{R}_+^{4\#\Omega_\varepsilon}} \Big\{ \sum_{x \in \Omega_\varepsilon} \sum_{i=1}^4 H_i^\varepsilon(x, t_i^\varepsilon(x)) - \inf_{\gamma^\varepsilon \in \Pi(\mu_0^\varepsilon, \mu_1^\varepsilon)} \sum_{(x,y) \in {\Omega_\varepsilon}^2} \gamma^\varepsilon(x, y) T^\varepsilon_{\mathbf{t}^\varepsilon}(x, y) \Big\} \tag{6.2}$$

where H and $T^\varepsilon_{\mathbf{t}^\varepsilon}$ are defined as before by (2.6) and (2.7) and $\Pi(\mu_0^\varepsilon, \mu_1^\varepsilon)$ denotes the set of (discrete) transport plans between μ_0^ε and μ_1^ε i.e. the set of collection of nonnegative reals $(\gamma^\varepsilon(x, y))_{(x,y)\in{\Omega_\varepsilon}^2}$ such that

$$\sum_{y \in \Omega_\varepsilon} \gamma^\varepsilon(x, y) = \mu_0^\varepsilon(x),\ \sum_{x \in \Omega_\varepsilon} \gamma^\varepsilon(x, y) = \mu_1^\varepsilon(y),\ (x, y) \in \Omega_\varepsilon \times \Omega_\varepsilon.$$

As a normalization, we may assume that the common mass of μ_0^ε and μ_1^ε is 1 and identify them with the discrete probability measures:

$$\mu_0^\varepsilon := \sum_{x \in \Omega_\varepsilon} \mu_0^\varepsilon(x)\delta_x,\ \mu_1^\varepsilon := \sum_{y \in \Omega_\varepsilon} \mu_1^\varepsilon(y)\delta_y.$$

Thus, in (6.2) the second term in the criterion is the value of the optimal transport problem between μ_0^ε and μ_1^ε for the transport cost $T^\varepsilon_{\mathbf{t}^\varepsilon}$.

Let us now assume that assumptions (3.2) and (3.9) hold and let us replace (3.1) by the assumption that μ_0^ε and μ_1^ε weakly star converge to some probability measures μ_0 and μ_1 on $\overline{\Omega}$:

$$\lim_{\varepsilon \to 0^+} \sum_{x \in \Omega_\varepsilon} (\varphi(x)\mu_0^\varepsilon(x) + \psi(x)\mu_1^\varepsilon(x)) = \int_{\overline{\Omega}} \varphi d\mu_0 + \int_{\overline{\Omega}} \psi d\mu_1,\ \forall (\varphi, \psi) \in C(\overline{\Omega})^2. \tag{6.3}$$

With assumptions (3.2) and (3.9) and setting $\xi^\varepsilon := \varepsilon^{-1}\mathbf{t}^\varepsilon$ as previously, we may rewrite (6.2) as:

$$\inf_{\xi^\varepsilon \in \mathbb{R}_+^{4\#\Omega_\varepsilon}} F^\varepsilon(\xi^\varepsilon) := I_0^\varepsilon(\xi^\varepsilon) - F_1^\varepsilon(\xi^\varepsilon) \tag{6.4}$$

where I_0^ε is defined by (3.7) and

$$F_1^\varepsilon(\xi^\varepsilon) := \inf_{\gamma^\varepsilon \in \Pi(\mu_0^\varepsilon, \mu_1^\varepsilon)} \varepsilon \sum_{(x,y) \in {\Omega_\varepsilon}^2} \gamma^\varepsilon(x, y) T^\varepsilon_{\xi^\varepsilon}(x, y). \tag{6.5}$$

We then define the limit functional by

$$F(\xi) := I_0(\xi) - F_1(\xi),\ \text{where } F_1(\xi) := \inf_{\gamma \in \Pi(\mu_0, \mu_1)} \int_{\overline{\Omega} \times \overline{\Omega}} \overline{c}_\xi d\gamma;\ \forall \xi \in L_+^p \tag{6.6}$$

where, as before, I_0 is defined by (3.10), $\overline{c}_\xi$ is defined by (3.13) and $\Pi(\mu_0, \mu_1)$ is the set of transport plans between μ_0 and μ_1 i.e. the set of probability measures having μ_0 and μ_1 as marginals. We then have the following Γ-convergence result:

Theorem 6.1. *Under assumptions (6.3), (3.2), (3.9), the family of functionals F^ε defined by (6.4) Γ-converges (for the weak L^p topology) to the functional F defined by (6.6).*

Proof. The proof can be achieved exactly as that of Theorem 6.1, except for the proof of the inequality

$$F_1(\xi) \geq \limsup_{\varepsilon} F_1^\varepsilon(\xi^\varepsilon)$$

as soon as $\xi^\varepsilon \to \xi$ for which we use Lemma 6.2 given below. From section 4.1, we already know that $c^\varepsilon := \varepsilon T^\varepsilon_{\xi^\varepsilon}$ has a subsequence that converges uniformly to some $c \leq \overline{c}_\xi$. Now, let $\gamma \in \Pi(\mu_0, \mu_1)$ be such that

$$F_1(\xi) = \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi d\gamma$$

thanks to Lemma 6.2 we may choose $\gamma^\varepsilon \in \Pi(\mu_0^\varepsilon, \mu_1^\varepsilon)$ that weakly star converges to γ as $\varepsilon \to 0$, we therefore get

$$\begin{aligned}\limsup_{\varepsilon} F_1^\varepsilon(\xi^\varepsilon) &\leq \limsup_{\varepsilon} \sum_{(x,y)\in\Omega_\varepsilon\times\Omega_\varepsilon} \gamma^\varepsilon(x,y)c^\varepsilon(x,y) \\ &= \int_{\overline{\Omega}\times\overline{\Omega}} c d\gamma \leq \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi d\gamma = F_1(\xi).\end{aligned}$$

□

Lemma 6.2. *Let μ and ν be probability measures on $\overline{\Omega}$, $(\mu_n)_n$, $(\nu_n)_n$ be sequences of probability measures on $\overline{\Omega}$ weakly star converging to μ and ν and let $\gamma \in \Pi(\mu, \nu)$ be a transport plan between μ and ν. There exists a sequence of transport plans $\gamma_n \in \Pi(\mu_n, \nu_n)$ that weakly star converges to γ.*

Proof. Let us recall that the 1-Wasserstein distance between μ and μ_n is defined by

$$W_1(\mu, \mu_n) := \inf_{\theta\in\Pi(\mu,\mu_n)} \int_{\overline{\Omega}\times\overline{\Omega}} |x - x'| d\theta(x, x')$$

and let $\theta \in \Pi(\mu, \mu_n)$ be an optimal plan in the minimization problem above. Similarly, let $\eta \in \Pi(\nu_n, \nu)$ be an optimal plan in the problem defining $W_1(\nu, \nu_n)$. Let us disintegrate θ as $\theta = \mu \otimes \theta^x$ and η as $\eta = \nu \otimes \eta^y$ and finally let $\gamma_n := \int_{\overline{\Omega}\times\overline{\Omega}} \theta^x \otimes \eta^y d\gamma(x, y)$ i.e.

$$\int_{\overline{\Omega}\times\overline{\Omega}} \varphi(x', y')d\gamma_n(x', y') = \int_{\overline{\Omega}\times\overline{\Omega}\times\overline{\Omega}\times\overline{\Omega}} \varphi(x', y')d\pi(x, x', y, y'),\ \forall \varphi \in C(\overline{\Omega} \times \overline{\Omega}))$$

where

$$\begin{aligned}&\int_{\overline{\Omega}\times\overline{\Omega}\times\overline{\Omega}\times\overline{\Omega}} \psi(x, x', y, y')d\pi(x, x, y, y') \\ &= \int_{\overline{\Omega}\times\overline{\Omega}} \Big(\int_{\overline{\Omega}\times\overline{\Omega}} \psi(x, x', y, y'))d\theta^x(x')d\eta^y(y')\Big) d\gamma(x, y),\ \forall \psi \in C(\overline{\Omega}^4).\end{aligned}$$

By construction, γ_n belongs to $\Pi(\mu_n, \nu_n)$ and

$$\begin{aligned} W_1(\gamma_n, \gamma) &\leq \int_{\overline{\Omega}\times\overline{\Omega}\times\overline{\Omega}\times\overline{\Omega}} (|x-x'|+|y-y'|) d\pi(x,x',y,y') \\ &= \int_{\overline{\Omega}\times\overline{\Omega}} |x-x'| d\theta^x(x') d\mu(x) + \int_{\overline{\Omega}\times\overline{\Omega}} |y-y'| d\eta^y(y') d\nu(y) \\ &= \int_{\overline{\Omega}\times\overline{\Omega}} |x-x'| d\theta(x,x') + \int_{\overline{\Omega}\times\overline{\Omega}} |y-y'| d\eta(y,y') \\ &= W_1(\mu_n,\mu) + W_1(\nu_n,\nu) \end{aligned}$$

and we conclude thanks to the well-known fact that W_1 is a distance that metrizes the weak-star topology on the set of probability measures on $\overline{\Omega}$ (see for instance [8]). □

Let us finally mention that the continuous long term problem

$$\inf_{\xi\in L^p_+} \Big\{ \sum_{i=1}^4 \int_\Omega H_i(x,\xi_i(x))dx - \inf_{\gamma\in\Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_\xi d\gamma \Big\} \tag{6.7}$$

admits a dual formulation that reads as

$$\sup_{Q\in\mathcal{Q}^q(\mu_0,\mu_1)} -\sum_{i=1}^4 \int_\Omega G_i(x, m_i^Q(x))dx \tag{6.8}$$

where

$$\begin{aligned} \mathcal{Q}^q(\mu_0,\mu_1) &:= \{Q \in \mathcal{M}_1^+(C) \ : \ e_{0\#}Q = \mu_0, \ e_{1\#}Q=\mu_1, \ m^Q \in L^q\} \\ &= \bigcup_{\gamma\in\Pi(\mu_0,\mu_1)} \mathcal{Q}^q(\gamma). \end{aligned}$$

Provided $\mathcal{Q}^q(\mu_0,\mu_1) \neq \emptyset$ and (5.3) holds, one can generalize theorem 5.1 to the long-term models as follows:

- the supremum in (6.8) is achieved and coincides with the infimum of (6.7),
- $\overline{Q} \in \mathcal{Q}^q(\mu_0,\mu_1)$ solves (6.8) if and only if it is a long-term equilibrium : it gives full mass to the geodesics for the congested metric $\xi_{\overline{Q}}$ generated by $\overline{Q}$ in the sense that (5.11) holds and in addition $\overline{\gamma} := (e_0, e_1)_\# Q$ solves the optimal transport problem:

$$\inf_{\gamma\in\Pi(\mu_0,\mu_1)} \int_{\overline{\Omega}\times\overline{\Omega}} \overline{c}_{\xi_{\overline{Q}}}(x,y) d\gamma(x,y),$$

- if $\overline{Q}$ solves (6.8) then $\xi_{\overline{Q}}$ solves (6.7).

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REFERENCES

[1] J.-B. Baillon and R. Cominetti, *Markovian traffic equilibrium*, Math. Prog., **111** (2008), 33–56.

[2] M. Beckmann, C. McGuire and C. Winsten, "Studies in Economics of Transportation," Yale University Press, New Haven, 1956.

[3] F. Benmansour, G. Carlier, G. Peyré and F. Santambrogio, *Numerical approximation of continuous traffic congestion equilibria*, Netw. Heterog. Media, **4** (2009), 605–623.

[4] A. Braides, "Γ-Convergence for Beginners," Oxford Lecture Series in Mathematics and its Applications, **22**, Oxford University Press, Oxford, 2002.

[5] L. Brasco, G. Carlier and F. Santambrogio, *Congested traffic dynamics, weak flows and very degenerate elliptic equations*, Journal de Mathématiques Pures et Appliquées (9), **93** (2010), 652–671.

[6] G. Carlier, C. Jimenez and F. Santambrogio, *Optimal transportation with traffic congestion and Wardrop equilibria*, SIAM J. Control Optim., **47** (2008), 1330–1350.

[7] G. Dal Maso, "An Introduction to Γ−Convergence," Progress in Nonlinear Differential Equations and their Applications, **8**, Birkhäuser Boston, Inc., Boston, MA, 1993.

[8] C. Villani, "Topics in Optimal Transportation," Graduate Studies in Mathematics, **58**, American Mathematical Society, Providence, RI, 2003.

[9] J. G. Wardrop, *Some theoretical aspects of road traffic research*, Proc. Inst. Civ. Eng., **2** (1952), 325–378.

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