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A SUFFICIENT CONDITION FOR CLASSIFIED NETWORKS TO POSSESS COMPLEX NETWORK FEATURES

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Abstract. We investigate network features for complex networks. A sufficient condition for the limiting random variable to possess the scale free property and the high clustering property is given. The uniqueness and existence of the limit of a sequence of degree distributions for the process is proved. The limiting degree distribution and a lower bound of the limiting clustering coefficient of the graph-valued Markov process are obtained as well.

1. **Introduction.** Networks appear in various systems in the world, a class of which is the random networks. During the past decades, scientists in different fields have done many different work [12, 8, 7, 11] and achieved a lot of empirical and simulative results on the various real-world random networks, such as social networks, biological networks, electronic communication networks *etc.* These results reveal that different networks may have some striking similarities in their underlying structures, such as scale-free behaviors for the degree distribution and high clustering properties (see for example [2]). These are two main network features in the literature. The formation mechanism of random complex network features has become a topic of interest much work on which has been done in recent years. The small-world model [16], which was proposed by Watts and Strogatz, describes the high clustering property. The scale-free model [1], which was proposed by Albert and Barabási, describes the scale-free property. In 2002, Holme and Kim provided a random network model with the two main properties [10]. However, all of these models lack strict mathematical descriptions. As a matter of fact, from a mathematical point of view, a network is a graph consisting of nodes connected by edges. Up until now, most of the network models in the literature can be viewed as a Markov process

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with the state space consisting of graphs. Aldous and Kendall [4], Ganesh and Xue [9] studied the small-world property of networks by Markov processes. Bollobás and Szymański used a graph process as a mathematical model of random networks, and proved some results on the tails of the degree distributions of the random networks [6, 15]. Bartel et al [5], Sorensen [14], Ou[13] investigated the modified method of cluster analysis based on graph theory. However, many problems remain open. For example, is the limit of the sequence of the degree distribution for the stochastic graph-valued process unique? Does the limit of the clustering coefficient exist? What are the expressions of the limiting degree distribution and the clustering coefficient?

In this paper, we investigate a graph-valued Markov process and provide a sufficient condition that the limiting random variable of the graph-valued Markov process has the network features. We will prove the unique existence of the limiting degree distribution, and obtain a lower bound of the limiting clustering coefficient. One of the most significant differences between this article and other existing literature is that this article is the first to propose a sufficient condition of clustering, and to prove the existence of two random variable limits. The article is organized as follows. In the second section, we introduce the graph-valued Markov process. In the third section, we give a sufficient condition for the limiting random variable to possess network features. As an application of the main result, we provide two examples in the fourth section. We give the summary in the fifth section.

2. **Graph-valued Markov processes.** For convenience, we introduce some notations and terminology. Let $V = \{v_1, v_2, v_3, \dots\}$ denote a set of countably many distinct nodes, Ω denote the set of all undirected simple graphs with nodes from V, and G denote the set of all undirected simple graphs with finitely many nodes from V. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, where $\mathcal{F}$ is a σ-field on Ω and P is a probability measure on $\mathcal{F}$.

Let $\{X(t), t \in T\}$ be a graph-valued Markov process with time parameter $T = \{0, 1, 2, \dots\}$. Namely, for any fixed $\omega \in \Omega$, $X(t,\omega)$ denotes, at time t, a subgraph of the concrete graph ω. For $s \le t$, $x, y \in G$, denote the transition probability by $P(s,t;x,y) = P(X(t) = y|X(s) = x)$. Let $V(t,\omega)$ be the set of all the nodes of $X(t,\omega)$, $B(t,\omega)$ be the set of all the edges of $X(t,\omega)$, $N(t,\omega) = |V(t,\omega)|$ and $D(t,\omega) = 2|B(t,\omega)|$ be the number of nodes and the total degree of $X(t,\omega)$, respectively. For $v \in V(t,\omega)$, let $k_v(t,\omega)$ be the degree of node v and $e_v(t,\omega)$ be the number of the edges induced by the neighbor nodes of v. For $k \ge 0$, let $V_k(t,\omega)$ be the set of the nodes with degree k at time t and $N_k(t,\omega) = |V_k(t,\omega)|$. Define the degree distribution of $X(t)$ as:

$$P(k,t) = E\left(\frac{N_k(t,\omega)}{N(t,\omega)}\right), k \ge 0. \tag{1}$$

Denote the clustering coefficient of node v at time t by $c_v(t,\omega) = \frac{2e_v(t,\omega)}{k_v(t,\omega)(k_v(t,\omega)-1)}$, and define the clustering coefficient of $X(t)$ as:

$$C(t) = E\left(\frac{1}{N(t,\omega)} \sum_{v \in V(t,\omega)} c_v(t,\omega)\right). \tag{2}$$

We say that $\{X(t), t \in T\}$ possesses the scale-free behavior and high clustering coefficient property, if $\lim_{t\to\infty} P(k,t) \sim k^{-\gamma}$ and $\lim_{t\to\infty} C(t) > 0$, where γ is a constant and $\gamma > 1$ (see for example [2, 3]).

3. **A sufficient condition for a class of graph-valued Markov process to possess the two main network features.** We consider a class of graph-valued Markov process $\{X(t), t \in T\}$, on which we impose the following conditions:

(1) For $N(0,\omega) = n_0$, where n_0 is a positive integer, denote $V(0,\omega) = \{-n_0, \dots, -1\}$. For $i = -n_0, \dots, -1$, $k_i(0) > 0$.

(2) For any integer $t \geq 0$, let $V(t+1,\omega) = V(t,\omega) \cup \{t+1\}$, $B(t,\omega) \subset B(t+1,\omega)$ and for some integers $c > b \geq \frac{n_0(n_0-1)}{2}$, $d \geq 1$, there exist $0 \leq p_j \leq 1$ and integers m_j, for $1 \leq j \leq d$ such that

$$P\left(|B(t+1,\omega)| = c \big| |B(t,\omega)| = b\right) = \begin{cases} p_1, & c = b + m_1, \\ p_2, & c = b + m_2, \\ \cdots, & \\ p_d, & c = b + m_d, \\ 0, & \text{otherwise}, \end{cases}$$

where $\sum_{j=1}^{d} p_j = 1$;

(3) For $j = 1, 2, \dots d$, let $k_i^{(j)}(t)$ be the degree of node i at time t under the restriction of $\{k_i(i) = m_j\}$. $\{k_i^{(j)}(t), t = i, i+1, \dots\}$ is a Markov process with $P^{(j)}(k,i,t) = P(k_i^{(j)}(t) = k), t \geq i$. Moreover, define $P_i^{(j)}(k,t) = P(k_i^{(j)}(t+1) = k+1 \mid k_i^{(j)}(t) = k)$ and suppose that

$$P(k_i^{(j)}(t+1) = l \mid k_i^{(j)}(t) = k) = \begin{cases} 1 - P_i^{(j)}(k,t), & l = k, \\ P_i^{(j)}(k,t), & l = k+1, \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

We recommend readers to see the examples in Section 4 to understand the stochastic process defined above. Here is the main result of this paper.

Theorem 3.1. *That for $j = 1, 2, \dots, d$, $i \geq 1$, $P_i^{(j)}(k,t)$ is independent of i and there exists some $a_j > 0$ such that $\lim_{t\to\infty} tP_i^{(j)}(k,t) = \frac{k}{a_j}$, where $k \geq m_j$, also, there exists $b > 0$ such that $E(e_i(i)) > b$ is a sufficient condition for the limiting random variable of $\{X(t), t \in T\}$ to possess the two main network features, i.e. scale-free behavior and high clustering property.*

We adopt the following five lemmas to prove the theorem. For the Markov process $\{X(t), t \in T\}$ defined in this section, for $t \geq 1$, Eq.(1) can be rewritten as:

$$\begin{aligned} P(k,t) &= \frac{1}{t+n_0} \sum_{i=-n_0, i\neq 0}^{t} P(k,i,t) \\ &= \frac{1}{t+n_0} \left(\sum_{i=-n_0}^{-1} P(k,i,t) + \sum_{j=1}^{d} \sum_{i=1}^{t} P^{(j)}(k,i,t) \cdot P(k_i(i) = m_j) \right) \\ &= \frac{1}{t+n_0} \left(\sum_{i=-n_0}^{-1} P(k,i,t) + \sum_{j=1}^{d} p_j \sum_{i=1}^{t} P^{(j)}(k,i,t) \right), \end{aligned}$$

where $P(k,i,t) = P(k_i(t) = k)$. Denote $\overline{P^{(j)}}(k,t) = \frac{1}{t}\sum_{i=1}^{t} P^{(j)}(k,i,t)$. It is easy to prove that

$$\lim_{t\to\infty} P(k,t) = \lim_{t\to\infty} \sum_{j=1}^{d} p_j \overline{P^{(j)}}(k,t). \tag{4}$$

Since the existence of $\lim\limits_{t\to\infty} \overline{P^{(j)}}(k,t)$ implies that of $\lim\limits_{t\to\infty} P(k,t)$, it is sufficient to prove the former limit. For $i \geq 1$, and $t > i$, denote the first-passage probability of the Markov chain $\{k_i^{(j)}(t), t = t_i, t_i+1, \dots\}$ to degree k by:

$$\begin{aligned} &f^{(j)}(k,i,t) \\ =&\begin{cases} P(k_i^{(j)}(t) = k, k_i^{(j)}(l) \neq k, l = t_i, t_i+1, \dots, t-1), & m_j < k \leq m_j + t - i, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 3.2. *For $i \geq 1$ and $k > m_j, j = 1, 2, \dots, d$, we have*

$$f^{(j)}(k,i,s) = P^{(j)}(k-1,i,s-1)P_i^{(j)}(k-1,s-1), s > i, \tag{5}$$

$$P^{(j)}(k,i,t) = \sum_{s=i+k-m_j}^{t} f^{(j)}(k,i,s) \prod_{h=s}^{t-1}(1 - P_i^{(j)}(k,h)), t > i, \tag{6}$$

where we use the convention that $\prod\limits_{h\in\emptyset} = 1$.

Proof. Consider Eq.(5). The degree of any node is always nondecreasing and increases at most by 1 at each time. Thus, it follows from the Markovian properties that

$$\begin{aligned} &f^{(j)}(k,i,s) \\ =&P(k_i^{(j)}(s) = k, k_i^{(j)}(l) \neq k, l = t_i, t_i+1, \dots, s-1) \\ =&P(k_i^{(j)}(s) = k, k_i^{(j)}(s-1) = k-1, k_i^{(j)}(l) \neq k, l = t_i, t_i+1, \dots, s-2) \\ =&P(k_i^{(j)}(s) = k, k_i^{(j)}(s-1) = k-1) \\ =&P(k_i^{(j)}(s-1) = k-1) \cdot P(k_i^{(j)}(s) = k \mid k_i^{(j)}(s-1) = k-1) \\ =&P^{(j)}(k-1,i,s-1)P_i^{(j)}(k-1,s-1). \end{aligned}$$

Note that the earliest time that the degree of node i reaches k is at time $k+i-m_j$, and the latest time to do so is at time t. After the node degree becomes k, it will not increase any further. Thus, Eq.(6) is proved. □

Lemma 3.3. (Stolz-Cesáro Theorem) *Let $\{\frac{x_t}{y_t}, t \geq 1\}$ be a sequence, and assume that $\{y_t, t \geq 1\}$ is a monotone increasing sequence with $y_t \to \infty$ as $t \to \infty$. If the limit $\lim\limits_{t\to\infty} \frac{x_{t+1}-x_t}{y_{t+1}-y_t} = l$ exists with $-\infty \leq l \leq +\infty$, then $\lim\limits_{t\to\infty} \frac{x_t}{y_t} = l$.*

Lemma 3.4. *The limit $\lim\limits_{t\to\infty} \overline{P^{(j)}}(m_j,t)$ exists. Moreover,*

$$\overline{P^{(j)}}(m_j) = \lim_{t\to\infty} \overline{P^{(j)}}(m_j,t) = \frac{a_j}{m_j + a_j}.$$

Proof. On one hand, for $i \geq 1$, $k_i^{(j)}(i) = m_j$ and Eq.(3) indicates that $P^{(j)}(m_j, t, t) = 1$ and for $t > i$, $P^{(j)}(m_j, i, t) = P^{(j)}(m_j, i, t-1)(1 - P_i^{(j)}(m_j, t-1))$. Thus

$$\begin{aligned}
\overline{P^{(j)}}(m_j, t) &= \frac{1}{t}\sum_{i=1}^{t} P^{(j)}(m_j, i, t) \\
&= \frac{1}{t}\left(\sum_{i=1}^{t-1} P^{(j)}(m_j, i, t) + P^{(j)}(m_j, t, t)\right) \\
&= \frac{1}{t}\left(\sum_{i=1}^{t-1} P^{(j)}(m_j, i, t-1)(1 - P_i^{(j)}(m_j, t-1)) + 1\right).
\end{aligned}$$

From the condition of the theorem, we know that

$$\begin{aligned}
\overline{P^{(j)}}(m_j, t) &= \frac{1}{t}(1 - P_i^{(j)}(m_j, t-1))\sum_{i=1}^{t-1} P^{(j)}(m_j, i, t-1) + \frac{1}{t} \\
&= \frac{t-1}{t}(1 - P_i^{(j)}(m_j, t-1))\frac{1}{t-1}\sum_{i=1}^{t-1} P^{(j)}(m_j, i, t-1) + \frac{1}{t} \\
&= \frac{t-1}{t}(1 - P_i^{(j)}(m_j, t-1))\overline{P^{(j)}}(m_j, t-1) + \frac{1}{t}.
\end{aligned}$$

Then, by iteration,

$$\begin{aligned}
&\overline{P^{(j)}}(m_j, t) \\
=& \prod_{h=1}^{t-1}(1 - P_i^{(j)}(m_j, h))\frac{h}{h+1}\left(\overline{P^{(j)}}(m_j, 1) + \sum_{l=1}^{t-1}\frac{\frac{1}{l+1}}{\prod_{l'=1}^{l}(1 - P_i^{(j)}(m_j, l'))\frac{l'}{l'+1}}\right) \\
=& \frac{1}{t}\prod_{h=1}^{t-1}(1 - P_i^{(j)}(m_j, h))\left(\overline{P^{(j)}}(m_j, 1) + \sum_{l=1}^{t-1}\prod_{l'=1}^{l}(1 - P_i^{(j)}(m_j, l'))^{-1}\right).
\end{aligned}$$

On the other hand, if we define

$$x_t = \overline{P^{(j)}}(m_j, 1) + \sum_{l=1}^{t-1}\prod_{l'=1}^{l}(1 - P_i^{(j)}(m_j, l'))^{-1},$$

$$y_t = t\prod_{h=1}^{t-1}\left(1 - P_i^{(j)}(m_j, h)\right)^{-1} > 0.$$

Then we have

$$x_{t+1} - x_t = \prod_{l'=1}^{t}(1 - P_i^{(j)}(m_j, l'))^{-1},$$

$$y_{t+1} - y_t = (1 + tP_i^{(j)}(m_j, t)) \cdot \prod_{h=1}^{t}(1 - P_i^{(j)}(m_j, h))^{-1} > 0,$$

$$\frac{x_{t+1} - x_t}{y_{t+1} - y_t} = \frac{1}{1 + tP_i^{(j)}(m_j, t)}.$$

Therefore,

$$\lim_{t\to\infty}\frac{x_{t+1} - x_t}{y_{t+1} - y_t} = \lim_{t\to\infty}\frac{1}{1 + tP_i^{(j)}(m_j, t)} = \frac{a_j}{m_j + a_j}.$$

Since $y_t > 0$ and $y_{t+1} - y_t > 0$, $\{y_t, t \geq 1\}$ is a strictly monotone increasing nonnegative sequence, we have $y_t \to \infty$ as $t \to \infty$.

By Lemma 3.3, we have

$$\lim_{t\to\infty} \overline{P^{(j)}}(m_j, t) = \lim_{t\to\infty} \frac{x_t}{y_t} = \lim_{t\to\infty} \frac{x_{t+1} - x_t}{y_{t+1} - y_t} = \frac{a_j}{m_j + a_j}.$$

This completes the proof. □

Lemma 3.5. *For $k > m_j$, $\overline{P^{(j)}}(k) = \lim_{t\to\infty} \overline{P^{(j)}}(k, t) = \frac{k-1}{k+a_j} \overline{P^{(j)}}(k - 1)$.*

Proof. For $k > m_j$, note that $P^{(j)}(k, i, t) = 0$ for $i > t + m_j - k$. This is because that, in this case, even if node i increases its degree by 1 at each time, it cannot reach degree k at time t. We use the same notation $\overline{P^{(j)}}(k, t)$ as in the proof of Lemma 3.4. It follows from Lemma 3.2 that

$$
\begin{aligned}
&\overline{P^{(j)}}(k,t)\\
=&\frac{1}{t}\sum_{i=1}^{t+m_j-k} P^{(j)}(k,i,t)\\
=&\frac{1}{t}\sum_{i=1}^{t+m_j-k}\sum_{s=i+k-m_j}^{t} f^{(j)}(k,i,s)\prod_{l'=s}^{t-1}(1-P_i^{(j)}(k,l'))\\
=&\frac{1}{t}\sum_{i=1}^{t+m_j-k}\sum_{s=i+k-m_j}^{t} P^{(j)}(k-1,i,s-1)P_i^{(j)}(k-1,s-1)\prod_{l'=s}^{t-1}(1-P_i^{(j)}(k,l'))\\
=&\frac{1}{t}\sum_{s=k-m_j+1}^{t}\sum_{i=1}^{s+m_j-k} P^{(j)}(k-1,i,s-1)P_i^{(j)}(k-1,s-1)\prod_{l'=s}^{t-1}(1-P_i^{(j)}(k,l'))\\
=&\frac{1}{t}\sum_{s=k-m_j+1}^{t}\sum_{i=1}^{s-1} P^{(j)}(k-1,i,s-1)P_i^{(j)}(k-1,s-1)\prod_{l'=s}^{t-1}(1-P_i^{(j)}(k,l'))\\
=&\frac{1}{t}\sum_{s=k-m_j+1}^{t}(s-1)\overline{P^{(j)}}(k-1,s-1)P_i^{(j)}(k-1,s-1)\prod_{l'=s}^{t-1}(1-P_i^{(j)}(k,l'))\\
=&\frac{1}{t}((k-m_j)\overline{P^{(j)}}(k-1,k-m_j)P_i^{(j)}(k-1,k-m_j)\prod_{l'=k-m_j+1}^{t-1}(1-P_i^{(j)}(k,l'))\\
&+\sum_{s=k-m_j+2}^{t}(s-1)\overline{P^{(j)}}(k-1,s-1)P_i^{(j)}(k-1,s-1)\prod_{l'=s}^{t-1}(1-P_i^{(j)}(k,l'))\\
=&\frac{1}{t}((k-m_j)\overline{P^{(j)}}(k-1,k-m_j)P_i^{(j)}(k-1,k-m_j)\prod_{l'=k-m_j+1}^{t-1}(1-P_i^{(j)}(k,l'))\\
&+\sum_{s=k-m_j+2}^{t}(s-1)\overline{P^{(j)}}(k-1,s-1)P_i^{(j)}(k-1,s-1)\prod_{h=k-m_j+1}^{t-1}(1-P_i^{(j)}(k,h))
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \prod_{l'=k-m_j+1}^{s-1} (1-P_i^{(j)}(k,l')^{-1}))\\
=&\frac{1}{t}\prod_{h=k-m_j+1}^{t-1}(1-P_i^{(j)}(k,h))((k-m_j)\overline{P^{(j)}}(k-1,k-m_j)P_i^{(j)}(k-1,k-m_j)\\
&+\sum_{l=k-m_j+1}^{t-1} l\overline{P^{(j)}}(k-1,l)P_i^{(j)}(k-1,l)\prod_{l'=k-m_j+1}^{l}(1-P_i^{(j)}(k,l'))^{-1}),
\end{aligned}
$$

where the sixth equality is because that $P_i^{(j)}(k-1,s-1)$ is independent of i. Let

$$
\begin{aligned}
x_t =&(k-m_j)\overline{P^{(j)}}(k-1,k-m_j)P_i^{(j)}(k-1,k-m_j)\\
&+\sum_{l=k-m_j+1}^{t-1} l\overline{P^{(j)}}(k-1,l)P_i^{(j)}(k-1,l)\prod_{l'=k-m_j+1}^{l}(1-P_i^{(j)}(k,l'))^{-1},
\end{aligned}
$$

$$
y_t =t\prod_{h=k-m_j+1}^{t-1}(1-P_i^{(j)}(k,h))^{-1}>0.
$$

It follows that

$$
x_{t+1}-x_t =t\overline{P^{(j)}}(k-1,t)P_i^{(j)}(k-1,t)\prod_{l'=k-m_j+1}^{t}(1-P_i^{(j)}(k,l'))^{-1},
$$

$$
y_{t+1}-y_t =(1+tP_i^{(j)}(k,t))\prod_{h=k-m_j+1}^{t}(1-P_i^{(j)}(k,h))^{-1}>0,
$$

$$
\frac{x_{t+1}-x_t}{y_{t+1}-y_t} =\frac{tP_i^{(j)}(k-1,t)}{1+tP_i^{(j)}(k,t)}\overline{P^{(j)}}(k-1,t).
$$

Thus

$$
\lim_{t\to\infty}\frac{x_{t+1}-x_t}{y_{t+1}-y_t}=\lim_{t\to\infty}\frac{tP_i^{(j)}(k-1,t)}{1+tP_i^{(j)}(k,t)}\overline{P^{(j)}}(k-1,t)=\frac{k-1}{k+a_j}\overline{P^{(j)}}(k-1).
$$

Similar to the proof of Lemma 3.4, $\{y_t, t\geq 1\}$ is a strictly monotone increasing nonnegative sequence, we have $y_t\to\infty$ as $t\to\infty$. From Lemma 3.3, we arrive at

$$
\overline{P^{(j)}}(k)=\lim_{t\to\infty}\overline{P^{(j)}}(k,t)=\lim_{t\to\infty}\frac{x_t}{y_t}=\lim_{t\to\infty}\frac{x_{t+1}-x_t}{y_{t+1}-y_t}=\frac{k-1}{k+a_j}\overline{P^{(j)}}(k-1).
$$

Therefore,

$$
\overline{P^{(j)}}(k)=\frac{k-1}{k+a_j}\overline{P^{(j)}}(k-1).
$$

□

Lemma 3.6. *Let* $m'=\min\{m_1,\ldots,m_d\}$, $P(m')=\lim_{t\to\infty}P(m',t)$. *We have*

$$
\lim_{t\to\infty}C(t)>\frac{2b}{m'(m'-1)}P(m').
$$

Proof. On one hand, for $t \geq 1$, Eq.(2) can be rewritten as

$$\begin{aligned} C(t) =& \frac{1}{n_0+t} E\left(\sum_{i=-n_0, i\neq 0}^{t} c_i(t,\omega)\right) \\ =& \frac{1}{n_0+t}\left(E\left(\sum_{i=-n_0}^{-1} c_i(t,\omega)\right) + E\left(\sum_{i=1}^{t} c_i(t,\omega)\right)\right). \end{aligned}$$

On the other hand, since $e_i(t,\omega)$ is a nondecreasing sequence in t, if $t > i \geq 1$, then $e_i(t,\omega) \geq e_i(i,\omega)$, thus $\lim\limits_{t\to\infty} E(e_i(t)) \geq E(e_i(i)) > b$. Let $m'' = \max\{m_1, \ldots, m_d\}$. We have that

$$\begin{aligned} \lim_{t\to\infty} C(t) &= \lim_{t\to\infty} \frac{1}{n_0+t} E\left(\sum_{i=1}^{t} c_i(t,\omega)\right) \\ &= \lim_{t\to\infty} \frac{1}{n_0+t} E\left(\sum_{i=1}^{t} E\left(c_i(t,\omega)|k_i(t)\right)\right) \\ &= \lim_{t\to\infty} \frac{1}{n_0+t} \sum_{i=1}^{t} \sum_{k=m'}^{t+m''-1} E\left(\frac{2e_i(t,\omega)}{k(k-1)}\right) \cdot P(k_i(t)=k) \\ &> \lim_{t\to\infty} \frac{1}{n_0+t} \sum_{k=m'}^{t+m''-1} \sum_{i=1}^{t} \frac{2b}{k(k-1)} \cdot P(k_i(t)=k) \\ &= \lim_{t\to\infty} \sum_{k=m'}^{t+m''-1} \frac{2b}{k(k-1)} \cdot P(k,t) \\ &> \frac{2b}{m'(m'-1)} P(m'). \end{aligned}$$

□

Proof of Theorem 3.1. From Lemma 3.4 and Lemma 3.5, for $k \geq m_j$, the limit $\lim\limits_{t\to\infty} \overline{P^{(j)}}(k,t)$ exists. So from Eq. (4), the limit $\lim\limits_{t\to\infty} P(k,t)$ exists. Together with Lemma 3.6, this completes the proof. □

Corollary 1. *If a_j is a nonnegative integer, then*

$$\overline{P^{(j)}}(k) = \prod_{i=0}^{a_j} \frac{m_j+i}{k+i} \cdot \frac{a_j}{m_j+a_j}.$$

Proof. This result can be obtained from Lemma 3.4 and Lemma 3.5 easily. □

4. **Examples.** As an application of the previous result, we consider two complex networks in this section. We use the same notation as in the above.

Example 1 The network starts from a complete graph with n_0 ($n_0 \geq 3$) nodes. At every time step, we add a new node with 2 edges into the network. The edges link to the existing nodes under the following mechanism: we choose an edge from the edges of the new node randomly. For convenience, we label this edge l_1, and label the other edge l_2. Assume that l_1 links to node i with preferential probability $\pi(k_i) = \frac{k_i(t-1)}{D(t-1)}$ at time t, $i = -n_0, \ldots, t-1; i \neq 0$. With probability p ($0 < p < 1$) l_2

links to a node which is chosen from the neighbors of i randomly, or with probability $1-p$, from the nodes except for node i preferentially.

From the above described model we know that: $N(0) = n_0$, $D(0) = n_0(n_0-1)$, for $-n_0 \leq i \leq -1$, $k_i(0) = n_0 - 1 > 0$, for $t \geq 0$, $V(t+1,\omega) = V(t,\omega) \cup \{t+1\}$, $B(t) \subset B(t+1)$, $|B(t+1,\omega) \backslash B(t,\omega))| = 2$, $N(t) = n_0 + t$, $D(t) = n_0(n_0-1) + 4t$, and for $i \geq -n_0, i \neq 0$, $\{k_i(t), t = t_i, t_i+1, \dots\}$ is a Markov process, and

$$P(k_i(t+1) = l \mid k_i(t) = k) = \begin{cases} 1 - P_i(k,t), & l = k, \\ P_i(k,t), & l = k+1, \\ 0, & \text{otherwise.} \end{cases}$$

For $i \geq 1$, $k_i(t_i) = 2$.

For $i \geq 1$, let $\Upsilon_i(t)$ be the set of the neighbor nodes of i at time t, $\Gamma_i(t)$ be the set of nodes which are neither i nor a neighbor node of i at time t. For a given ω_0, suppose that the degree of node i at time t is k, where $t \geq i$, and $2 \leq k \leq 2+t-i$. Node i is possibly connected by the newly added node $t+1$ at time $t+1$, which consists of three cases:

Case one: l_1 links to node i, this case happens with probability

$$q_1 = \frac{k}{D(t)}.$$

Case two: l_1 links to node h ($h \in \Upsilon_i(t,\omega_0)$), and l_2 links to node i, this case happens with probability

$$q_2 = \sum_{h \in \Upsilon_i(t,\omega_0)} \frac{k_h(t,\omega_0)}{D(t)} \cdot \left(p \frac{1}{k_h(t,\omega_0)} + (1-p) \frac{k}{D(t)} \right).$$

Case three: l_1 links to node g ($g \in \Gamma_i(t,\omega_0)$), and l_2 links to node i, this case happens with probability

$$q_3 = \sum_{g \in \Gamma_i(t,\omega_0)} \frac{k_g(t,\omega_0)}{D(t)} \cdot (1-p) \frac{k}{D(t)}.$$

It follows that

$$q = q_1 + q_2 + q_3 = \frac{k}{2t + \frac{n_0(n_0-1)}{2}} - (1-p) \left(\frac{k}{4t + n_0(n_0-1)} \right)^2$$

is independent of ω_0. Therefore

$$P_i(k,t) = \frac{k}{2t + \frac{n_0(n_0-1)}{2}} - (1-p) \left(\frac{k}{4t + n_0(n_0-1)} \right)^2$$

is independent of i. Moreover, $\lim_{t\to\infty} tP_i(k,t) = \frac{k}{2}$.

To describe the clustering coefficient, suppose that node $j \geq 1$ was added into the network at time j and two edges of node j are connected to node i_1 and node i_2, respectively.

1. If node i_2 is chosen from the neighbors of node i_1 randomly, then i_1 and i_2 are linked to each other and $P_1(e_j(j) = 1) = p$;

2. If node i_2 is chosen from the network except for node i_1 preferentially, note that if i_2 is a neighbor of i_1, then i_1 and i_2 are also linked to each other. Thus $P_2(e_j(j) = 1) > 0$.

Therefore, $P(e_j(j) = 1) = P_1(e_j(j) = 1) + P_2(e_j(j) = 1) > p$. Furthermore, we have

$$E(e_j(j)) = 1 \cdot P(e_j(j) = 1) > p > 0.$$

It is easy to see that this model satisfies all the conditions of Theorem 3.1. So the limiting network possesses the two main network features. Moreover, $P(k) = \frac{12}{k(k+1)(k+2)}$, for $k > 1$.

Example 2 The network starts from a complete graph with n_0 ($n_0 \geq 3$) nodes. At every time step, with probability p, we add a new node with m_1 edges into the network. The edges link to the existing nodes under the following mechanism: we choose an edge from the m_1 edges randomly, and link it to node i with preferential probability $\pi(k_i) = \frac{k_i(t-1)}{D(t-1)}$. Each of the other $m_1 - 1$ edges is randomly linked to a neighbor of node i with probability λ, or is linked to an existing node except for i preferentially with probability $1 - \lambda$. With probability $1 - p$, we add a new node with m_2 edges into the network, and the m_2 edges are linked to m_2 existing nodes preferentially.

From the above described model we know that: $N(0) = n_0$, $D(0) = n_0(n_0 - 1)$, for $-n_0 \leq i \leq -1$, $k_i(0) = n_0 - 1 > 0$. For $t \geq 0$, $V(t+1, \omega) = V(t, \omega) \cup \{t + 1\}$, $B(t) \subset B(t + 1)$,

$$P(|B(t+1,\omega)| = c \mid |B(t,\omega)| = b) = \begin{cases} p, & c = b + m_1, \\ 1 - p, & c = b + m_2, \end{cases}$$

$N(t) = n_0 + t$, and for $i \geq -n_0, i \neq 0, j = 1, 2$, $\{k_i^{(j)}(t), t = t_i, t_i + 1, \dots\}$ is a Markov process. Moreover

$$P(k_i^{(j)}(t+1) = l \mid k_i^{(j)}(t) = k) = \begin{cases} 1 - P_i^{(j)}(k,t), & l = k, \\ P_i^{(j)}(k,t), & l = k+1, \\ 0, & \text{otherwise.} \end{cases}$$

For $i \geq 1$, $j = 1, 2$, $k_i^{(j)}(i) = m_j$.

It is easy to obtain that $P_i^{(j)}(k,t) = \frac{k}{2t + \frac{n_0(n_0-1)}{m_j}}$ is independent of i. Moreover, $\lim_{t\to\infty} t \cdot P_i^{(j)}(k,t) = \frac{k}{2}$ and $E(e_i(i)) \geq (m_1 - 1)\lambda p$. Thus this model satisfies all the conditions of Theorem 3.1. So the limiting network possesses the two main network features. Moreover, by Eq. (4) and Corollary 1, we know that $P(k) = p \cdot \frac{2m_1(m_1+1)}{k(k+1)(k+2)}$ for $m_1 \leq k < m_2$, and $P(k) = p \cdot \frac{2m_1(m_1+1)}{k(k+1)(k+2)} + (1-p) \cdot \frac{2m_2(m_2+1)}{k(k+1)(k+2)}$ for $k \geq m_2$.

5. **Conclusion.** This paper gives a sufficient condition for the limiting random variable of a class of graph-valued Markov process $\{X(t), t \in T\}$ to possess the scale free property and the high clustering property. However, this condition is relatively strict. Hence, further work need to be done on simplifying the conditions, or on finding some necessary and sufficient conditions for the limiting random variable of $\{X(t), t \in T\}$ to possess the network features.

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