NETWORKS AND HETEROGENEOUS MEDIA
doi:10.3934/nhm.2012.7.137

Volume **7**, Number **1**, March **2012**
pp. **137–150**

POSITIVE SPEED OF PROPAGATION IN A SEMILINEAR PARABOLIC INTERFACE MODEL WITH UNBOUNDED RANDOM COEFFICIENTS

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(Communicated by Cyrill Muratov)

Abstract. We consider a model for the propagation of a driven interface through a random field of obstacles. The evolution equation, commonly referred to as the Quenched Edwards-Wilkinson model, is a semilinear parabolic equation with a constant driving term and random nonlinearity to model the influence of the obstacle field. For the case of isolated obstacles centered on lattice points and admitting a random strength with exponential tails, we show that the interface propagates with a finite velocity for sufficiently large driving force. The proof consists of a discretization of the evolution equation and a supermartingale estimate akin to the study of branching random walks.

1. **Introduction, model, and the main result.** In this article, we consider a parabolic model for the evolution of an interface in a random medium. The interface at time t is assumed to be the graph of a function. The local velocity of the interface is governed by line tension and a competition between a constant external driving force $F > 0$ and a heterogeneous random field $f\colon \mathbf{R}\times\mathbf{R}\times\Omega \to \mathbf{R}$. This field describes the environment of the interface. More precisely, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space. We consider the evolution equation

$$\begin{aligned} u_t(x,t,\omega) &= u_{xx}(x,t,\omega) - f(x,u(x,t,\omega),\omega) + F \\ u(x,0,\omega) &= 0 \end{aligned} \tag{1}$$

for $t \geq 0$, $x \in \mathbf{R}$, $\omega \in \Omega$. The heterogeneous field $f \geq 0$ thus plays the role of obstacles that impede the free propagation of the interface.

2000 *Mathematics Subject Classification.* Primary: 35K58; Secondary: 35R60, 35B40, 60H15.
Key words and phrases. Interfaces, heterogeneous media, random media, semilinear parabolic equations with randomness, asymptotic behavior of non-negative solutions.

We are particularly interested in the macroscopic behavior of solutions to (1) and their dependence on the parameter F. Specifically, assume that the random field f is not uniformly bounded from above, i.e., there exist obstacles of arbitrarily large strength. Can one now find a deterministic constant F^*, such that the interface will propagate to infinity for $F \geq F^*$? The question of non-existence of a stationary solution in such a model – for obstacles with exponential tails in the distribution of their strength – was answered by Coville, Dirr, and Luckhaus in [2]. The present article extends this result and proves finite speed of propagation for large enough driving force in the same setting.

In [4], the opposite question was answered, namely whether interfaces become stuck (i.e., whether non-negative stationary solutions to the evolution equation (1) exist) for small but positive external load F. The work there is based on a percolation result [3]. Together, the results show a transition from a viscous kinetic relation at the microscopic level to a stick-slip behavior, leading to a rate independent hysteresis at the macro-level. Problems of the form of (1) find substantial interest in the physics community, see for example [5, 1]. Further connections to physics and to homogenization problems in degenerate elliptic equations can be found in [2].

In order to introduce the main result of this article, we first fix the nature of the random field f.

Assumption 1. Let $f_{ij}(\omega)$, $i, j \in \mathbf{Z}$, be iid (independent and identically distributed) non-negative random variables with a finite exponential moment, i.e., $\mathbf{E}\exp\{\lambda f_{00}(\omega)\} = \beta < \infty$ for some $\lambda > 0$. Furthermore, set $\phi\colon \mathbf{R}^2 \to \mathbf{R}$, $\phi \in C^1(\mathbf{R}^2)$ such that $0 \leq \phi \leq 1$ and

$$\operatorname{supp}\phi \subset [-\delta, \delta]^2 \quad \text{with } \delta < 1/2. \tag{2}$$

The random field f shall then be given as

$$f(x, y, \omega) := \sum_{i,j \in \mathbf{Z}} f_{ij}(\omega)\phi(x - i, y - j - 1/2). \tag{3}$$

This means that the random field consists of obstacles centered at points of the square lattice (shifted by $1/2$ in the y-(propagation) direction) with random strength with exponential tail of the distribution, but uniform shape[1]. We show the following result about the propagation velocity of the interface.

Theorem 1.1. *Let $u\colon \mathbf{R} \times [0, \infty) \times \Omega \to \mathbf{R}$ be a solution to (1) and f as in Assumption 1. Let*

$$U\colon [0, \infty) \times \Omega \to \mathbf{R}$$
$$U\colon (t, \omega) \mapsto \int_0^1 u(\xi, t, \omega)\,\mathrm{d}\xi.$$

Then there exists a non-decreasing function $V\colon [0, \infty) \to [0, \infty)$, that is not identically zero and only depends on the parameters λ, β and δ, such that

$$\mathbf{E}\frac{U(t)}{t} \geq V(F) \quad \textit{for all } t > 0. \tag{4}$$

Remark 1. It will be evident from the proof that a viable choice for V is

$$V(F) = \frac{1}{4(1+\delta)} \sup_{\mu > \tilde{\lambda}} \frac{1}{\mu}\Big(\tilde{\lambda}((1-2\delta)F - 2) - \log p(\tilde{\lambda}, \mu) - \log\tilde{\beta}\Big),$$

[1]As one can easily see from the proofs, the assumption of uniformity of ϕ can be relaxed, as long as certain obvious uniform bounds are still adhered to.

where

$$p(l,m) := \frac{1}{1-\mathrm{e}^{-l}} + \frac{1}{1-\mathrm{e}^{l-m}}.$$

The parameter $\tilde\lambda$ can be chosen arbitrarily with $\tilde\lambda \in (0,\lambda)$ and the parameter $\tilde\beta$ is given as

$$\tilde\beta := \mathrm{e}^{\tilde\lambda} \inf_{c > \exp\{(\log\beta)\frac{180\tilde\lambda}{\lambda}\}} \left(c + \int_c^\infty \frac{\beta \mathrm{e}^{-\frac{\lambda}{180\tilde\lambda}\log x}}{1-\beta\mathrm{e}^{-\frac{\lambda}{180\tilde\lambda}\log x}}\,\mathrm{d}x \right).$$

Remark 2. Theorem 1.1 includes the result of non-existence of non-negative stationary solutions by Coville, Dirr and Luckhaus [2], since existence of a non-negative stationary solution to (1) would violate the fact that $\limsup_{t\to\infty} u(t,x)/t \geq V(F)$ for all $x \in \mathbf{R}$ almost surely. See Subsection 3.2 for a proof of this corollary of Theorem 1.1.

Due to our stationarity and independence assumption on f, together with uniqueness of solutions (see [2], Lemma 3.2) it is clear that the processes $U_i(t,\omega) := \int_i^{i+1} u(\xi,t,\omega)\,\mathrm{d}\xi$, $i \in \mathbf{Z}$ are stationary and ergodic for each $t \geq 0$. In order to see this, note that the process is a function of iid random variables which are ergodic under shifts in the x-direction. Therefore, this theorem shows that the average area covered by the interface per unit time and per unit length is bounded from below by a positive constant deterministic velocity if the driving force is sufficiently large.

Furthermore, note that we have $u(x,t,\omega) \leq Ft$ for all x and all t. This can be seen by a comparison principle argument. If there was a time t_0 and a point x_0 where u passes the constant in space function Ft for the first time, we have $u_t(x_0,t_0,\omega) \leq F$. This is a contradiction.

We will prove Theorem 1.1 by first addressing a discretized version of the evolution equation in Section 2 and then showing the reduction of the continuum problem to the discrete problem in Section 3. In the final Section 4 we conclude by presenting a number of open questions.

2. **The discretized front propagation model.**

We consider the following discrete model, which arises as a lower bound for the velocity in the continuum problem (1) – as shown in Section 3 – but also is interesting to study in its own right. Let $u_i\colon [0,\infty) \to \mathbf{R}$ solve the equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} u_i(t) &= \Big(u_{i-1}(t) + u_{i+1}(t) - 2u_i(t) - \tilde f_i(u_i(t),\omega) + F \Big)^+ \\ u_i(0) &= 0, \qquad i \in \mathbf{Z}, \end{aligned} \tag{5}$$

where $F \geq 0$ and $\tilde f_i : \mathbf{R}\times\Omega \to [0,\infty)$, $i \in \mathbf{Z}$ are independent and identically distributed functions such that the map $(y,\omega) \mapsto \tilde f_0(y,\omega)$ is measurable with respect to the product of the Borel-σ algebra on $\mathbf{R}$ and $\mathcal{F}$ and the map $y \mapsto \tilde f_0(y,\omega)$ is locally Lipschitz for almost all $\omega \in \Omega$. These assumptions guarantee that the equation above has a unique solution which depends measurably on ω for each $t \geq 0$. Note that we do not assume that the map $y \mapsto \tilde f_0(y)$ is stationary.

Remark 3. The symbol $(\cdot)^+$ denotes taking the non-negative part of the term inside the parenthesis. One can easily see from the comparison principle for (discrete) elliptic equations that taking the non-negative part is only necessary if one can not ensure that the initial velocity is non-negative. For the continuous equation, non-negativity is shown in Proposition 2.

2.1. **Lower bound on the averaged velocity.** For this discretized version of the main evolution problem (1) we can show the analog to Theorem 1.1.

Theorem 2.1. *Assume – in addition – that there exists* $\tilde\lambda > 0$ *such that*

$$\tilde\beta := \sup_{n\in\mathbf{Z}} \mathbf{E} \sup_{n-.5<y\le n+.5} \exp\{\tilde\lambda \tilde f_0(y,\omega)\} < \infty.$$

Then there exists a non-decreasing function $W : [0,\infty) \to [0,\infty)$ *which is not identically zero and which depends on* $\tilde\lambda$ *and* $\tilde\beta$ *only, such that*

$$\mathbf{E}\dot u_0(t) \ge W(F) \ \text{ for all } t \ge 0 \text{ and hence } \mathbf{E}\frac{u_0(t)}{t} \ge W(F) \ \text{ for all } t > 0.$$

Specifically, we can choose

$$W(F) = \sup_{\mu>\tilde\lambda} \frac{1}{\mu}\Big(\tilde\lambda(F-2) - \log p(\tilde\lambda,\mu) - \log\tilde\beta\Big),$$

where p *is as in Remark 1. In fact, the function* V *there is just a rescaled version of* W*.*

Remark 4. The supremum inside the expectation in the definition of $\tilde\beta$ is not necessarily measurable. Strictly speaking, one should replace the expectation by the infimum of the expectation of all random variables dominating the supremum.

We will prove Theorem 2.1 using the following result:

Lemma 2.2. *Let* $\bar f_{ij} : \Omega \to [0,\infty)$, $i,j \in \mathbf{Z}$ *be random variables such that the functions* $\bar f_i : \Omega\times\mathbf{Z} \to [0,\infty)$ *defined as* $\bar f_i(\omega,j) := \bar f_{ij}(\omega)$ *are independent. Assume that there exists some* $\bar\lambda > 0$ *such that* $\bar\beta := \sup_{m,n\in\mathbf{Z}} \mathbf{E}\exp\{\bar\lambda \bar f_{mn}\} < \infty$*. Then, for each* $F > 0$*, there exists a set* Ω_0 *of full measure such that for any function* $w : \Omega\times\mathbf{Z} \to \mathbf{N}_0$ *and any* $\omega \in \Omega_0$*, we have*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \Big(w_{i-1} + w_{i+1} - 2w_i - \bar f_i(\omega, w_i) + F\Big)^+ \ge \bar W(F), \tag{6}$$

where

$$\bar W(F) := \sup_{\mu>\bar\lambda} \frac{1}{\mu}\Big(\bar\lambda F - \log p(\bar\lambda,\mu) - \log\bar\beta\Big),$$

and p *is defined as in Remark 1.*

Proof. Since the assertion holds trivially in case $\bar W(F) \le 0$, we can and will assume that $\bar W(F) > 0$. In this case, the supremum in the definition of $\bar W(F)$ is actually a maximum (observing that the function of μ converges to 0 as $\mu \to \infty$) and we choose a maximizer which we denote again by μ. Further, it suffices to show that for each fixed $w_{-1}, w_0 \in \mathbf{N}_0$ there exists a set Ω_0 of full measure such that for any function $w : \Omega\times\mathbf{Z} \to \mathbf{Z}$ attaining the prescribed values at -1 and 0, and all $\omega \in \Omega_0$ satisfying

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \Big(w_{i-1} + w_{i+1} - 2w_i - \bar f_i(\omega, w_i) + F\Big)^+ < \bar W(F),$$

we have $\liminf_{n\to\infty} w_n(\omega) < 0$.

To see that this is true, we proceed as follows. For $n \in \mathbf{N}_0$ we call each function $w : \{-1, 0, ..., n\} \to \mathbf{Z}$ resp. $w : \{-1, 0, ...\} \to \mathbf{Z}$ starting with the prescribed values

w_{-1}, w_0 a *path* of length n resp. a *path*. To each path (of length n), we associate $v_n := w_n - w_{n-1}$ and

$$s_n := \sum_{i=0}^{n-1} \Big(w_{i-1} + w_{i+1} - 2w_i - \bar{f}_i(\omega, w_i) + F\Big)^+, \qquad n \in \mathbf{N}_0.$$

Define

$$Y_n := \sum \exp\{\bar{\lambda} v_n - \mu s_n\},$$

where the sum is taken over all paths of length n. Let $\mathcal{F}_k$ denote the σ-algebra generated by $\bar{f}_0, ..., \bar{f}_{k-1}$. We now claim that

$$\mathbf{E}\big(Y_{n+1} | \mathcal{F}_n\big) \le \gamma Y_n \text{ a.s.},$$

where

$$\gamma = \bar{\beta} \exp\{-\bar{\lambda} F\} \Big(\frac{1}{1 - \mathrm{e}^{-\bar{\lambda}}} + \frac{1}{1 - \mathrm{e}^{\bar{\lambda} - \mu}}\Big)$$

showing that $Z_n := Y_n / \gamma^n$ is a (nonnegative) supermartingale. The proof of the claim is a straightforward computation:

$$\begin{aligned}
&\mathbf{E}\big(Y_{n+1} | \mathcal{F}_n\big) \\
&\quad = \sum \exp\{-\mu s_n\} \mathbf{E}\Big(\sum_{j \in \mathbf{Z}} \exp\{\bar{\lambda} j - \mu (j - v_n - \bar{f}_n(\omega, w_n) + F)^+\} | \mathcal{F}_n\Big) \\
&\quad = \sum \exp\{-\mu s_n\} \\
&\qquad \mathbf{E}\Big(\sum_{j \ge \lceil v_n + \bar{f}_n(\omega, w_n) - F \rceil} \exp\{(\bar{\lambda} - \mu) j + \mu (v_n + \bar{f}_n(\omega, w_n) - F)\} \\
&\qquad\quad + \sum_{j < \lceil v_n + \bar{f}_n(\omega, w_n) - F \rceil} \exp\{\bar{\lambda} j\} | \mathcal{F}_n\Big) \\
&\quad = \sum \exp\{-\mu s_n\} \\
&\qquad \mathbf{E}\Big(\frac{\exp\{(\bar{\lambda} - \mu)\lceil v_n + \bar{f}_n(\omega, w_n) - F \rceil\}}{1 - \exp\{\bar{\lambda} - \mu\}} \\
&\qquad\quad \exp\{\mu (v_n + \bar{f}_n(\omega, w_n) - F)\} \\
&\qquad\quad + \exp\{\bar{\lambda}\big(\lceil v_n + \bar{f}_n(\omega, w_n) - F \rceil - 1\big)\} \frac{1}{1 - \exp\{-\bar{\lambda}\}} | \mathcal{F}_n\Big) \\
&\quad \le \sum \exp\{-\mu s_n\} \\
&\qquad \mathbf{E}\Big(\exp\{\bar{\lambda}\big(v_n + \bar{f}_n(\omega, w_n) - F)\}\Big(\frac{1}{1 - \mathrm{e}^{\bar{\lambda} - \mu}} + \frac{1}{1 - \mathrm{e}^{-\bar{\lambda}}}\Big) | \mathcal{F}_n\Big) \\
&\quad \le \gamma Y_n,
\end{aligned}$$

where the first sum is extended over all paths of length n. The operator $\lceil a \rceil = \inf\{z \in \mathbf{Z} : z \ge a\}$ denotes taking the integer ceiling of $a \in \mathbf{R}$.

By the supermartingale convergence theorem, there exists a set Ω_0 of full measure such that $\sup_{n \in \mathbf{N}_0} Y_n / \gamma^n$ is finite for all $\omega \in \Omega_0$. On Ω_0, we therefore have

$$\limsup_{n \to \infty} \frac{1}{n} \sup\{\bar{\lambda} v_n - \mu s_n\} \le \limsup_{n \to \infty} \frac{1}{n} \log Y_n \le \log \gamma,$$

where the sup is extended over all paths of length n. Therefore, for each path

$$\bar{\lambda} \limsup_{n \to \infty} \frac{v_n}{n} < \log \gamma + \mu \bar{W}(F) = 0 \text{ on the set } \Big\{\limsup_{n \to \infty} \frac{s_n}{n} < \bar{W}(F)\Big\} \cap \Omega_0.$$

In particular, we have $\liminf_{n\to\infty} w_n(\omega) < 0$ on that set (in fact even for lim sup instead of lim inf), so the statement of the lemma follows. □

Remark 5. We note the following properties of the functions W and $\bar{W}$.

1. The functions W and $\bar{W}$ defined in Theorem 2.1 and Lemma 2.2 are nonnegative (let $\mu \to \infty$). Further, $\bar{W}(F) > 0$ whenever $F > \frac{1}{\bar{\lambda}}\big(\log \bar{\beta} + \log\big(1 + (1 - \exp\{-\bar{\lambda}\})^{-1}\big)\big)$, $W(F) > 0$ whenever $F - 2$ satisfies the same property.
2. If we choose $\mu = \bar{\lambda} + 1/F$, then we see that $F - \bar{W}(F) \lesssim \frac{1}{\bar{\lambda}} \log F$ as $F \to \infty$.

Remark 6. The set Ω_0 in the previous lemma can actually be chosen independently of F since both sides of (6) depend continuously on F.

Now we turn to Theorem 2.1. The proof will follow immediately from the following statement.

Lemma 2.3. *Let $\tilde{f}_i$ and W as in the statement of Theorem 2.1. Then almost surely we have for all non-negative sequences $u_i \in \mathbf{R}$, $u_i \geq 0$ for all $i \in \mathbf{Z}$ that*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \Big(u_{i-1} + u_{i+1} - 2u_i - \tilde{f}_i(u_i, \omega) + F\Big)^+ \geq W(F).$$

Proof. We define the associated discrete path w_i, $i \in \{-1, 0, ...\}$ taking values in $\mathbf{N}_0$ by rounding u_i to the closest integer (rounding up in case of ties). To apply Lemma 2.2, we define $\bar{f}_{ij} := \sup_{y\in(j-.5,j+.5]} \tilde{f}_i(y, \omega)$. Then we have

$$\Big(u_{i-1} + u_{i+1} - 2u_i - \tilde{f}_i(u_i, \omega) + F\Big)^+ \geq \\ \Big(w_{i-1} + w_{i+1} - 2w_i - \bar{f}_i(w_i, \omega) + F - 2\Big)^+$$

(we subtract 2 in order to compensate the deviations between the u_i and the w_i) and therefore – by Lemma 2.2 – we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \Big(u_{i-1} + u_{i+1} - 2u_i - \tilde{f}_i(u_i, \omega) + F\Big)^+ \geq \bar{W}(F - 2) = W(F)$$

for $\omega \in \Omega_0$ and Ω_0 of full measure and independent of the choice of the u_i. □

Proof of Theorem 2.1. Assume that the (first) statement in the theorem is untrue. Then there exist $F \geq 0$ and some t_0 such that $\mathbf{E}\dot{u}_0(t_0) < W(F)$. By our stationarity and independence assumptions on the field f, the processes $u_i(t_0)$, $\dot{u}_i(t_0)$, $i \in \mathbf{Z}$ are stationary and ergodic and take values in $[0, \infty)$. We write u_i instead of $u_i(t_0)$. By Birkhoff's ergodic theorem, we have $\mathbf{E}\dot{u}_0 = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \Big(u_{i-1} + u_{i+1} - 2u_i - \tilde{f}_i(u_i, \omega) + F\Big)^+ < W(F)$ almost surely. This is a contradiction to Lemma 2.3. □

2.2. **Almost sure statements about the propagation velocity.** In the discrete case, it is also possible to show some statements about the velocity of the interface that hold almost surely.

Proposition 1. *Consider our standard discrete set-up from Theorem 2.1 with the following relaxed assumptions on the $\tilde{f}_i$: the $\tilde{f}_i$ are nonnegative random functions (no independence or stationarity assumptions). Then we have $\lim_{t\to\infty} \frac{u_i(t) - u_j(t)}{t} = 0$ for all $i, j \in \mathbf{Z}$ and all $\omega \in \Omega$.*

Proof. Let

$$H(t) := u_{i-1}(t) + u_{i+1}(t) - 2u_i(t) + F.$$

We then have

$$H(t) \geq h(t) := u_{i-1}(t) + u_{i+1}(t) - 2u_i(t) + F - \tilde{f}_i(u_i(t), \omega).$$

If $h(t) < 0$, we have $\frac{\mathrm{d}}{\mathrm{d}t} u_i(t) = 0$ and thus $\frac{\mathrm{d}}{\mathrm{d}t} H(t) = \frac{\mathrm{d}}{\mathrm{d}t} h(t) \geq 0$. Since $H(t)$ is a continuous function of time, we have $H(t) \geq 0$ for all t.

Now let $\gamma > 0$. The non-negativity of $H(t)$ implies that

$$u_{i-1}(t) + u_{i+1}(t) - 2u_i(t) > -\gamma t \qquad \text{whenever } t > \frac{F}{\gamma},\ i \in \mathbf{Z}. \tag{7}$$

Fix $\Gamma > 0$ and $n \in \mathbf{N}$ and define $\gamma := \Gamma/n$. Let $t > F/\gamma$ and assume that there exists some $i \in \mathbf{Z}$ such that $\frac{u_{i+1}(t) - u_i(t)}{t} \geq \Gamma$. Since $F \geq u_j(t)/t \geq 0$[2] for all j, (7) implies that

$$F \geq \frac{u_{i+n}(t) - u_i(t)}{t} = \sum_{k=0}^{n-1} \frac{u_{i+k+1}(t) - u_{i+k}(t)}{t} \geq \sum_{k=0}^{n} (\Gamma - k\gamma) = \Gamma \frac{n+1}{2}.$$

This inequality can only hold in case $n \leq \frac{2F}{\Gamma} - 1$. The same is true in case $\frac{u_{i+1}(t) - u_i(t)}{t} \leq -\Gamma$. Therefore, $\frac{|u_{i+1}(t) - u_i(t)|}{t} \leq \Gamma$ for all $i \in \mathbf{Z}$ whenever $t > 2F^2/\Gamma^2$. Since $\Gamma > 0$ was arbitrary, the assertion follows. $\square$

Remark 7. Under the assumptions of Theorem 2.1, the processes

$$\begin{aligned} &i \mapsto \limsup_{t\to\infty} u_i(t)/t \quad \text{and} \\ &i \mapsto \liminf_{t\to\infty} u_i(t)/t \end{aligned}$$

are ergodic. By the previous proposition, these processes do not depend on i. These two facts together imply that there exist deterministic numbers $0 \leq c_1 \leq c_2 \leq F$ such that $\liminf_{t\to\infty} u_i(t)/t = c_1$ and $\limsup_{t\to\infty} u_i(t)/t = c_2$ almost surely for each $i \in \mathbf{Z}$. In general, c_1 and c_2 will not coincide. We know that $c_2 \geq W(F)$ since, by Fatou's Lemma,

$$\begin{aligned} c_2 &= \mathbf{E} \limsup_{t\to\infty} u_0(t)/t = F - \mathbf{E} \liminf_{t\to\infty} \big(F - u_0(t)/t\big) \\ &\geq F - \liminf_{t\to\infty} \mathbf{E}\big(F - u_0(t)/t\big) \geq W(F). \end{aligned}$$

We conjecture that we also have $c_1 \geq W(F)$.

3. **Discretization of the continuum problem.** We now return to the continuum problem. In the first part we prove the statement about the expected value of the velocity employing a discretization of the problem. In the second part we extend the almost sure result from subsection 2.2 to the continuum model.

[2] The comparison principle argument used for the continuum equation can be applied also here.

3.1. **Proof of the main theorem.** For the discretization, we largely rely on the ideas put forward by Coville, Dirr and Luckhaus in [2]. As seen there, we first introduce a modified problem, basically turning off the driving force in the rows where the obstacles lie. Let $A := \mathbf{R} \setminus \{\bigcup_{i\in\mathbf{Z}}(i-\delta, i+\delta)\}$. We restrict F to act on the set A. Let thus $\tilde{u}\colon \mathbf{R}\times[0,\infty)\times\Omega$ solve the modified problem

$$\begin{aligned}\tilde{u}_t(x,t,\omega) &= \tilde{u}_{xx}(x,t,\omega) - f(x,\tilde{u}(x,t,\omega),\omega) + F\chi_A(x) \\ \tilde{u}(x,0,\omega) &= 0\end{aligned} \tag{8}$$

for $t \geq 0$, $x \in \mathbf{R}$, $\omega \in \Omega$. Existence and uniqueness of classical solutions for both the original problem (1) as well as for the modified problem are proved in [2], Lemma 3.2. Note that, as it is done in the reference, it is necessary to discard in space exponentially growing unphysical solutions to the heat equation on $\mathbf{R}$. Since we have

$$-f(x,y,\omega) + F\chi_A(x) \leq -f(x,y,\omega) + F \quad \text{for all} \quad x,y \in \mathbf{R}, \omega \in \Omega,$$

it follows that

$$\mathbf{E}U(t,\omega) = \mathbf{E}\int_0^1 u(\xi,t,\omega)\,\mathrm{d}\xi \geq \mathbf{E}\tilde{U}(t,\omega) := \mathbf{E}\int_0^1 \tilde{u}(\xi,t,\omega)\,\mathrm{d}\xi. \tag{9}$$

by the comparison principle for parabolic equations [6]. It is thus sufficient to prove Theorem 1.1 replacing u with a solution of the modified problem.

The following proposition shows that the velocity of a solution of (8) will be non-negative for all times.

Proposition 2. *Let $\tilde{u}$ be a solution of (8). We have*

$$\tilde{u}_t(x,t,\omega) \geq 0 \quad \textit{for all} \quad t > 0, x \in \mathbf{R}, \omega \in \Omega.$$

Proof. Since there are no obstacles on the line $\{y = 0\} \subset \mathbf{R}^2$, we have $\tilde{u}_t(x,0,\omega) \geq 0$ for all $x \in \mathbf{R}$, $\omega \in \Omega$. In fact, there even exists $\epsilon > 0$, so that $\tilde{u}_t(x,t,\omega) > 0$ for all $t \in (0,\epsilon)$, $x \in \mathbf{R}$, $\omega \in \Omega$, as one can easily see from the solution of the parabolic equation which is still linear for sufficiently small time t. Assume now that the proposition is untrue. Due to the fact that the solution $\tilde{u}$ is classical, there would have to exist a minimal $t_0 > 0$ and $x_0 \in \mathbf{R}$, $\omega_0 \in \Omega$ such that

$$\tilde{u}_t(x_0,t_0,\omega_0) = 0.$$

and $\epsilon > 0$ so that $\tilde{u}_t(x_0, t_0+\tau, \omega_0) < 0$ for all $\tau \in (0,\epsilon)$. Differentiating the equation with respect to t at this point yields[3]

$$\tilde{u}_{tt}(x_0,t_0,\omega_0) = \tilde{u}_{xxt}(x_0,t_0,\omega_0) - f_y(x_0,\tilde{u}(x_0,t_0,\omega_0),\omega_0)\tilde{u}_t(x_0,t_0,\omega_0).$$

The f-term vanishes due to the assumptions on t_0, x_0, and ω_0. We note that due to the fact that, by the assumption that t_0 is the first time that u_t is becoming negative anywhere, $\tilde{u}_t(x,t_0,\omega_0) \geq 0$ for all $x \in \mathbf{R}$. Thus, $\tilde{u}_{tt}(x_0,t_0,\omega_0)$ is positive contradicting the negativity of $\tilde{u}_t$. □

Next, we define an associated discrete process to the evolution equation (8). Let

$$\hat{u}_i := \tilde{u}(i-\delta) + 2\delta\tilde{u}_x(i-\delta) \quad \text{for } i \in \mathbf{Z}. \tag{10}$$

For notational simplicity, we have omitted the time- and ω-dependence of the terms.

[3] This additional time derivative might not be smooth, but the argument holds unchanged by considering the equation in the sense of viscosity solutions.

The next step is to use the estimates on the discrete Laplacian found in [2] and adapt them to our case of an additional positive velocity on the right hand side of the equation.

Proposition 3. *We have, for $\hat{u}_i$ defined as in (10), $\delta < 1/2$ as in equation (2),*

$$\hat{u}_{i-1} - 2\hat{u}_i + \hat{u}_{i+1} \le (1+2\delta)\left[\tilde{u}_x(i+\delta) - \tilde{u}_x(i-\delta) - \int_{i-\delta}^{i+\delta} \tilde{u}_t(\xi)\,\mathrm{d}\xi\right] - (1-2\delta)F + 2(1+\delta)\int_{i-1-\delta}^{i+1-\delta} \tilde{u}_t(\xi)\,\mathrm{d}\xi. \tag{11}$$

for each $i \in \mathbf{Z}$. The estimate holds for any time and on all of Ω.

Proof. Fix $i \in \mathbf{Z}$ and let

$$\tilde{v}(x) := \tilde{u}(x) - \int_{i-1-\delta}^{x} \int_{i-1-\delta}^{y} \tilde{u}_t(\xi)\,\mathrm{d}\xi\,\mathrm{d}y \tag{12}$$

for $x \in [i-1-\delta, i+1-\delta]$. We see that $\tilde{v}(x)$ solves

$$\tilde{v}_{xx} = \begin{cases} -F & \text{on } (i-1+\delta, i-\delta) \cup (i+\delta, i+1-\delta) \\ f(x, \tilde{u}(x), \omega) & \text{otherwise.} \end{cases}$$

To this function, we apply the estimate of the discrete Laplacian found in [2], Lemma 4.1, to obtain

$$\hat{v}_{i-1} - 2\hat{v}_i + \hat{v}_{i+1} \le (1+2\delta)\left[\tilde{v}_x(i+\delta) - \tilde{v}_x(i-\delta)\right] - (1-2\delta)F. \tag{13}$$

Here, $\hat{v}$ is discretized in the same way as $\hat{u}$, i.e, $\hat{v}_j = \tilde{v}(j-\delta) + 2\delta\tilde{v}_x(j-\delta)$, $j \in \{i-1, i, i+1\}$.

The next step is to estimate the difference between $\hat{v}$ and $\hat{u}$. First, we note that

$$\left[\tilde{v}_x(i+\delta) - \tilde{v}_x(i-\delta)\right] = \left[\tilde{u}_x(i+\delta) - \tilde{u}_x(i-\delta) - \int_{i-\delta}^{i+\delta} \tilde{u}_t(\xi)\,\mathrm{d}\xi\right]. \tag{14}$$

It is clear that $\hat{v}_{i-1} = \hat{u}_{i-1}$. Furthermore, we have, by the non-negativity of $\tilde{u}_t$ (see Proposition 2), that

$$\tilde{u}(i-\delta) \ge \tilde{v}(i-\delta)$$

and

$$\tilde{u}_x(i-\delta) \ge \tilde{v}_x(i-\delta),$$

which yields, by the definition of the discretization,

$$\hat{u}_i \ge \hat{v}_i,$$

and thus

$$\hat{u}_i - \hat{u}_{i-1} \ge \hat{v}_i - \hat{v}_{i-1}. \tag{15}$$

For the second term in the discrete Laplacian, we first estimate, by (12), and again using positivity of $\tilde{u}_t$,

$$\begin{aligned}\tilde{u}(i+1-\delta) &= \tilde{v}(i+1-\delta) + \int_{i-1-\delta}^{i+1-\delta} \int_{i-1-\delta}^{x} \tilde{u}_t(\xi)\,\mathrm{d}\xi\,\mathrm{d}x \\ &\le \tilde{v}(i+1-\delta) + 2\int_{i-1-\delta}^{i+1-\delta} \tilde{u}_t(\xi)\,\mathrm{d}\xi.\end{aligned}$$

In addition, we see that

$$\tilde{u}_x(i+1-\delta) = \tilde{v}_x(i+1-\delta) + \int_{i-1-\delta}^{i+1-\delta} \tilde{u}_t(\xi)\,\mathrm{d}\xi$$

and find, combining those two estimates and using again the definition of the discretization,

$$\hat{u}_{i+1} - \hat{u}_i \leq \hat{v}_{i+1} - \hat{v}_i + 2(1+\delta)\int_{i-1-\delta}^{i+1-\delta} \tilde{u}_t(\xi)\,\mathrm{d}\xi. \tag{16}$$

Now, subtracting (15) from (16), we get

$$\hat{u}_{i-1} - 2\hat{u}_i + \hat{u}_{i+1} \leq \hat{v}_{i-1} - 2\hat{v}_i + \hat{v}_{i+1} + 2(1+\delta)\int_{i-1-\delta}^{i+1-\delta} \tilde{u}_t(\xi)\,\mathrm{d}\xi. \tag{17}$$

The proposition follows by inserting (13) and (14) into (17). $\square$

Corollary 1. *Using the non-negativity of $\tilde{u}_t$ shown in Proposition 2 we can deduce from Proposition 3 that*

$$\begin{aligned} &2(1+\delta)\int_{i-1-\delta}^{i+1-\delta} \tilde{u}_t(\xi)\,\mathrm{d}\xi \geq \\ &\quad \Bigg(\hat{u}_{i-1} - 2\hat{u}_i + \hat{u}_{i+1} - (1+2\delta)\left[\tilde{u}_x(i+\delta) - \tilde{u}_x(i-\delta) - \int_{i-\delta}^{i+\delta} \tilde{u}_t(\xi)\,\mathrm{d}\xi\right] \\ &\quad + (1-2\delta)F\Bigg)^+, \end{aligned}$$

i.e., the integrated velocity can be estimated by the positive part of the discretized problem.

Next, we come to estimate the effect of the obstacles. We procede as in [2]. Let

$$k(i) := \tilde{u}_x(i-\delta) - \tilde{u}_x(i+\delta).$$

The following proposition gives an estimate for k in terms of the obstacles passed by the function $\tilde{u}$.

Proposition 4. *Given $\tilde{u}(x) = \tilde{u}(x,t,\omega)$, solution of the evolution equation (8) at fixed time and fixed $\omega \in \Omega$, let $i \in \mathbf{Z}$,*
$M := \max\{|\tilde{u}_x(i-\delta)|, |\tilde{u}_x(i+\delta)|\}$. We then have

$$k(i) \leq \frac{18\delta}{M} \sum_{\hat{u}_i - 4\delta M \leq j \leq \hat{u}_i + 4\delta M} f_{ij}(\omega) + \int_{i-\delta}^{i+\delta} \tilde{u}_t(\xi)\,\mathrm{d}\xi.$$

Proof. This follows immediately from [2], Lemma 4.2, after subtracting and adding the effect of the positive time derivative on a snapshot of the function $\tilde{u}$. The proof in [2] only uses convexity of the function $\tilde{u}$ inside $(i-\delta, i+\delta)$, which also holds in our case. $\square$

We now need a uniform estimate on some exponential moment of the average of the random variables in the estimate of k. The following proposition provides this result.

Proposition 5. *Let $X_1, X_2, \ldots$ be real valued iid random variables such that there exists $\lambda > 0$ with*

$$\mathbf{E}\exp\{\lambda X_1\} = \beta < \infty.$$

Then we have

$$\mathbf{E}\exp\{\tilde{\lambda}\sup_{N\in\mathbf{N}}\frac{1}{N}\sum_{j=1}^{N}X_j\}\leq\tilde{\beta}$$

for $0<\tilde{\lambda}<\lambda$ *and*

$$\tilde{\beta}=\inf_{c>\exp\{(\log\beta)\tilde{\lambda}/\lambda\}}\left(c+\int_c^\infty\frac{\beta\mathrm{e}^{-\frac{\lambda}{\tilde{\lambda}}\log x}}{1-\beta\mathrm{e}^{-\frac{\lambda}{\tilde{\lambda}}\log x}}\,\mathrm{d}x\right)<\infty.$$

Proof. Let $S:=\sup_{N\geq 1}\frac{1}{N}\sum_{j=1}^{N}X_j$. It is clear from the strong law of large numbers that this supremum is finite almost surely. We have

$$\begin{aligned}\mathbf{P}\left(S>u\right)&\leq\sum_{N=1}^{\infty}\mathbf{P}\left(\frac{1}{N}\sum_{j=1}^{N}X_j\geq u\right)\\&\leq\sum_{N=1}^{\infty}\mathbf{P}\left(\mathrm{e}^{\lambda\sum_{j=1}^{N}X_j}\geq\mathrm{e}^{\lambda Nu}\right)\\&\leq\sum_{N=1}^{\infty}\mathrm{e}^{-\lambda Nu}\left(\mathbf{E}\mathrm{e}^{\lambda X_1}\right)^N\\&=\sum_{N=1}^{\infty}\mathrm{e}^{-N\left(\lambda u-\log\mathbf{E}\mathrm{e}^{\lambda X_1}\right)}.\end{aligned}$$

Markov's inequality as well as independence and the identical distribution of the X_j was used in the last inequality. Setting $I(u):=\lambda u-\log\mathbf{E}\mathrm{e}^{\lambda X_1}=\lambda u-\log\beta$, we find that

$$\mathbf{P}\left(S>u\right)\leq\sum_{N=1}^{\infty}\mathrm{e}^{-NI(u)}=\frac{\mathrm{e}^{-I(u)}}{1-\mathrm{e}^{-I(u)}}$$

for u so large that the series converges, i.e, we have $I(u)>0$. Setting $0<\tilde{\lambda}<\lambda$ yields

$$\begin{aligned}\mathbf{E}\mathrm{e}^{\tilde{\lambda}S}&=\int_0^\infty\mathbf{P}\left(\mathrm{e}^{\tilde{\lambda}S}>x\right)\,\mathrm{d}x=\int_0^\infty\mathbf{P}\left(S>\frac{1}{\tilde{\lambda}}\log x\right)\,\mathrm{d}x\\&\leq c+\int_c^\infty\frac{\mathrm{e}^{-I(\frac{1}{\tilde{\lambda}}\log x)}}{1-\mathrm{e}^{-I(\frac{1}{\tilde{\lambda}}\log x)}}\,\mathrm{d}x.\end{aligned}$$

This holds for any $c>\exp\{(\log\beta)\tilde{\lambda}/\lambda\}$, since then $I(\frac{1}{\tilde{\lambda}}\log c)>0$. □

Remark 8. A similar result has been derived in [7], albeit without an explicit quantitative estimate.

We can now proceed to prove the main result.

Proof of Theorem 1.1. The proof is divided into three steps. We first use Proposition 5 to find a suitable set of random variables to dominate the effect of the precipitates. Then we show that the discretization of a solution to the continuous initial value problem (8) has to remain bounded from below. Finally we use the obstacles from the first step in a discretized problem and are able to estimate the integrated velocity.

Step 1: Set

$$g_{ij}(\omega) := 1 + (1+2\delta) \sup_{K\in\mathbf{N}} \frac{36\delta}{K} \sum_{j-4\delta K\leq l\leq j+4\delta K} f_{il}(\omega).$$

From Proposition 4 it is clear (note that $M \geq 1/2$ if $k \geq 1$) that $g_{ij} \geq k(i) - \int_{i-\delta}^{i+\delta} \tilde{u}_t(\xi)\,\mathrm{d}\xi$ for $j = \lceil \hat{u}_i - 1/2\rceil$, i.e., j is $\hat{u}_i$ rounded to the nearest integer. Using the bound $\delta < 1/2$ and counting the number of summands we find that

$$g_{ij} \leq 1 + 180 \sup_{N\in\mathbf{N}\cup\{0\}} \frac{1}{2N+1} \sum_{k=j-N}^{j+N} f_{ik}.$$

Since the f_{ik} are independent and identically distributed and we have by Assumption 1 that $\mathbf{E}e^{\lambda f_{ik}} = \beta < \infty$ it is now possible to give a bound on an exponential moment of g_{ij} using Proposition 5. Namely, we have

$$\mathbf{E}\exp\{\tilde{\lambda} g_{ij}\} \leq \tilde{\beta}$$

with $0 < \tilde{\lambda} < \lambda$ and

$$\tilde{\beta} := \mathrm{e}^{\tilde{\lambda}} \inf_{c>\exp\{(\log\beta)\frac{180\tilde{\lambda}}{\lambda}\}} \left(c + \int_c^\infty \frac{\beta \mathrm{e}^{-\frac{\lambda}{180\tilde{\lambda}}\log x}}{1-\beta\mathrm{e}^{-\frac{\lambda}{180\tilde{\lambda}}\log x}}\,\mathrm{d}x \right) < \infty.$$

Now let

$$\tilde{f}_i(y,\omega) := g_{i\lceil y-1/2\rceil}(\omega),$$

i.e, simply evaluate g_{ij} at $j = \lceil y - 1/2\rceil$, so j is y rounded to the nearest integer. It follows that

$$\tilde{u}_x(i-\delta) - \tilde{u}_x(i+\delta) - \int_{i-\delta}^{i+\delta} \tilde{u}_t(\xi)\,\mathrm{d}\xi \leq \tilde{f}_i(\hat{u},\omega). \tag{18}$$

Step 2: Now let $\tilde{u}(x,t_0,\omega)$ be a solution of the evolution equation (8) at time t_0 with fixed parameter F and let $\hat{u}_i$ for $i \in \mathbf{Z}$ be its discretization according to equation (10). We claim that there exists $C \in \mathbf{R}$, so that

$$\hat{u}_i \geq C \quad .$$

Indeed, we have $t_0 F \geq \tilde{u} \geq 0$, so $\inf_{i\in\mathbf{Z}}(\tilde{u}(i+1-\delta) - \tilde{u}(i-\delta)) \geq -t_0F$. But we also have that $\tilde{u}_{xx} \geq -F$, due to the non-negativity of $\tilde{u}_t$ and the non-negativity of f. It follows that

$$\begin{aligned} -t_0 F \leq \tilde{u}(i+1-\delta) - \tilde{u}(i-\delta) &= \int_{i-\delta}^{i+1-\delta} \tilde{u}_x(\xi)\,\mathrm{d}\xi \\ &\leq \sup_{\xi\in[i-\delta,i+1-\delta]} \tilde{u}_x(\xi) \\ &\leq \tilde{u}_x(i+1-\delta) + F. \end{aligned}$$

Thus, $\hat{u}_{i+1} \geq \tilde{u}(i+1-\delta) - 2\delta(1+t_0)F \geq -2\delta(1+t_0)F$ for all $i \in \mathbf{Z}$.

Step 3: Now we can finally apply Lemma 2.3 to the discrete process $\hat{u}_i$. It is clear that the estimate in the Lemma holds for any sequence that is uniformly bounded from below, the bound zero in the proof is arbitrary. We thus find that

$$W(F) \leq \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \Big(\hat{u}_{i-1} + \hat{u}_{i+1} - 2\hat{u}_i - \tilde{f}_i(\hat{u}_i,\omega) + F \Big)^+,$$

with W from Theorem 2.1 using the parameters $\tilde{\lambda}$ and $\tilde{\beta}$ from Step 1. Using Corollary 1 and equation (18), we see that

$$\begin{aligned}
W((1-2\delta)F) &\leq \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \Big(\hat{u}_{i-1} - 2\hat{u}_i + \hat{u}_{i+1} - \tilde{f}_i(\hat{u}_i,\omega) + (1-2\delta)F \Big)^+ \\
&\leq \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \Big(\hat{u}_{i-1} - 2\hat{u}_i + \hat{u}_{i+1} \\
&\qquad - \Big(\tilde{u}_x(i-\delta) - \tilde{u}_x(i+\delta) - \int_{i-\delta}^{i+\delta} \tilde{u}_t(\xi)\,\mathrm{d}\xi \Big) \\
&\qquad + (1-2\delta)F \Big)^+ \\
&\leq \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 2(1+\delta) \int_{i-1-\delta}^{i+1-\delta} \tilde{u}_t(\xi)\,\mathrm{d}\xi \\
&\leq 4(1+\delta)\,\mathbf{E} \frac{\mathrm{d}}{\mathrm{d}t} \tilde{U}(t).
\end{aligned}$$

Integrating $\frac{\mathrm{d}}{\mathrm{d}t}\tilde{U}(t)$ and using (9) proves the theorem. □

3.2. **Almost sure statements in the continuum model.** It is possible to show the analog of Proposition 1 in the continuum model.

Proposition 6. *Consider a solution of the model* (1)*, again with the relaxed assumption that f_{ij} are nonnegative random variables. Then we have* $\lim_{t\to\infty} \frac{u(t,x_2)-u(t,x_1)}{t} = 0$ *for all $x_1, x_2 \in \mathbf{R}$ and all $\omega \in \Omega$. Furthermore, the convergence is uniform for x_1, x_2 chosen on a compact interval.*

Proof. Let, without loss of generality $x_1 < x_2$ and let $u_j := u(x_1 + (j-1)(x_2 - x_1))$ for $j \in \mathbf{Z}$. Again define

$$H(t) := u_{j-1}(t) + u_{j+1}(t) - 2u_j(t) + 2(x_2 - x_1)^2 F, \quad \text{for } j \in \mathbf{Z}.$$

We have $H(t) \geq 0$. Indeed, assume that there exist $t_0 \geq 0, j_0 \in \mathbf{Z}$ with $H(t_0) < 0$, then at some point $\xi \in (x_1 + (j_0 - 1)(x_2 - x_1), x_1 + (j_0 + 1)(x_2 - x_1))$ we have $u_{xx}(t_0, \xi) + F < 0$. At this point, however, the propagation velocity of the interface would be negative, which violates the comparison principle (see Proposition 2). The rest of the proof continues as the proof of the discrete version of the proposition. □

Remark 9. The analog of Remark 7 also holds, i.e., under the assumptions of Theorem 1.1 there exist deterministic numbers $0 \leq c_1 \leq c_2 \leq F$ such that

$$\liminf_{t\to\infty} u(t,x)/t = c_1 \quad \text{and}$$
$$\limsup_{t\to\infty} u(t,x)/t = \limsup_{t\to\infty} \inf_{\xi\in K} u(t,\xi)/t = c_2$$

almost surely, for each $x \in \mathbf{R}$ and for any non empty compact set $K \subset \mathbf{R}$. This follows by using the ergodicity of the process $\int_j^{j+1} u(\xi,t)/t\,\mathrm{d}\xi$, $j \in \mathbf{Z}$ and by using the uniform convergence from Proposition 6.

Furthermore, we have $c_2 \geq V(F)$. Indeed, integrating over a spatial period and using uniform convergence, then using Fubini's theorem yields

$$\begin{aligned} c_2 &= \mathbf{E} \int_0^1 \limsup_{t\to\infty} u(t,\xi)/t \,\mathrm{d}\xi = F - \mathbf{E} \liminf_{t\to\infty} \left(F - \int_0^1 u(t,\xi)/t \,\mathrm{d}\xi \right) \\ &\geq F - \liminf_{t\to\infty} \mathbf{E} \left(F - \int_0^1 u(t,\xi)/t \,\mathrm{d}\xi \right) \geq V(F). \end{aligned}$$

The question whether $c_1 > 0$ remains open also here.

4. **Conclusion and open problems.** We have shown that interfaces in random media can cover a finite area per unit time on average, even if there is no uniform upper bound to the strength of obstacles. This is an extension to [2], where non-existence of a stationary solution was shown under the same assumptions. Many questions remain open, however. Amongst those are the conjecture stated in the previous section, namely whether an almost sure statement can be made about the inferior limit at a fixed point, even in the discrete problem.

Also open is the extension of the theorem to more than one dimension, as well as the question whether there exists a non-trivial interval so that if $F \in [F_0, F_1]$ we do not have a finite velocity, but also no stationary solution.

Acknowledgments. We are delighted to acknowledge useful discussions with Nicolas Dirr and are grateful for support from the DFG funded research group 'Analysis and Stochastics in Complex Physical Systems.'

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Received March 2011; revised September 2011.

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