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# COMPLIANCE ESTIMATES FOR TWO-DIMENSIONAL PROBLEMS WITH DIRICHLET REGION OF PRESCRIBED LENGTH

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Abstract. In this paper we prove some lower bounds for the compliance functional, in terms of the 1-dimensional Hausdorff measure of the Dirichlet region and the number of its connected components. When the measure of the Dirichlet region is large, these estimates are asymptotically optimal and yield a proof of a conjecture by Buttazzo and Santambrogio.

## 1. Introduction.

Given a bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary and a compact set $\Sigma \subset \overline{\Omega}$ with positive capacity, one can consider the so called "compliance" of $\Sigma$, defined as the quantity

$$C(\Sigma, \Omega) = \int_\Omega u_\Sigma(x)\, dx, \tag{1}$$

where the state function $u_\Sigma$ is the unique solution of the variational problem

$$\min_{u \in V_\Sigma} \left( \int_\Omega |\nabla u(x)|^2\, dx - 2 \int_\Omega u(x)\, dx \right) \tag{2}$$

with

$$V_\Sigma = \{u \in H^1(\Omega) \,|\, u = 0 \text{ on } \Sigma\}. \tag{3}$$

The assumption that $\Sigma$ has positive capacity guarantees that the Dirichlet condition along $\Sigma$ is well posed in $H^1$ (and this makes the energy functional coercive on $V_\Sigma$ via a Poincaré inequality, see for instance Corollary 4.5.2 in [5]).

Note that the state function $u_\Sigma$ satisfies $-\Delta u_\Sigma = 1$ in $\Omega \setminus \Sigma$, a homogeneous Dirichlet condition on $\Sigma$, and a homogeneus Neumann condition along $\partial\Omega \setminus \Sigma$. It is a possible model for the deflection of a structure (e.g. a membrane over $\Omega$) subject to a uniform force acting in the vertical direction, and glued to the ground along $\Sigma$. In this case the compliance represents the work done by the force $f$ at the equilibrium, hence, the smaller the compliance, the more rigid the structure.

It is also natural to consider the compliance in a $H^1_0$ setting, namely

$$C_0(\Sigma, \Omega) := \int_\Omega u_{\Sigma,0}(x)\, dx,$$

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where the function $u_{\Sigma,0}$ solves the variational problem

$$\min_{u\in V_\Sigma\cap H_0^1(\Omega)}\left(\int_\Omega |\nabla u(x)|^2\,dx - 2\int_\Omega u(x)\,dx\right), \tag{4}$$

to be compared with (2): the energy functional is the same, but now the admissible functions are required to be in $H_0^1(\Omega)\cap V_\Sigma$ (or, which is the same, in $H_0^1(\Omega\setminus\Sigma)$), not merely in $V_\Sigma$: now $\Sigma$ can be interpreted as the reinforcement of a membrane that is already glued along $\partial\Omega$, by imposing an additional Dirichlet condition along $\Sigma$.

When the domain $\Omega$ is fixed, it is natural to consider the compliance as a functional depending on the unknown set $\Sigma$ subject to suitable constraints, and try to minimize the compliance among all admissible configurations. This leads to several interesting variational problems and questions (see [1]). In particular, in [1] it was proved that the variational problem

$$\min\{C_0(\Sigma,\Omega)\,|\,\Sigma\subseteq\overline{\Omega},\quad \Sigma\text{ compact and connected, }\mathcal{H}^1(\Sigma)\le L\} \tag{5}$$

has at least one solution $\Sigma_L$ for every $L>0$, and the asymptotics of the optimal configurations $\Sigma_L$ as $L\to\infty$ was investigated via $\Gamma$-convergence (the functional setting of [1] is in fact more general, and non-constant external forces are also considered). In (5) and throughout the paper, $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure: the condition that $\mathcal{H}^1(\Sigma)\le L$ therefore represents a *length constraint* on the admissible configurations.

In [1] it was proved that, if $\Sigma_L$ is a solution of (5), then $C_0(\Sigma_L,\Omega)$ decays as $L^{-2}$ as $L\to\infty$, but no sharp lower bound for $C_0(\Sigma_L,\Omega)$ was available. However, the following conjecture was formulated:

**Conjecture 1** (Buttazzo and Santambrogio, [1])**.** *Let $\Omega=(0,1)^2$ be the unit square. Then*

$$\inf\left(\liminf_{n\to\infty} L_n^2 C_0(\Sigma_n,\Omega)\right)=\frac{1}{12}, \tag{6}$$

*where the infimum is taken over all sequences of positive numbers $L_n\to\infty$ and all sequences $\{\Sigma_n\}$ of compact connected sets $\Sigma_n\subseteq\overline{\Omega}$ such that $\mathcal{H}^1(\Sigma_n)\le L_n$.*

**Remark 1.** With the notation of [1], the conjecture on p. 772 in [1] reads "$\theta = 1/24$", where $2\theta$ is defined as the left hand side of (6). Here, however, for notational reasons we prefer to reformulate the conjecture in the equivalent form given above.

The fact that the infimum in (6) does not exceed 1/24 was proved in [1] by constructing a suitable sequence of configurations $\Sigma_n$, which were conjectured to be (asymptotically) optimal. The validity of this conjecture makes it possible to explicitly determine the $\Gamma$-limit functional obtained in [1], which was therein computed up to a moltiplicative constant (see [1] for more details).

In this paper we prove some lower bounds for the compliances $C(\Sigma,\Omega)$ and $C_0(\Sigma,\Omega)$ in terms of the length $\mathcal{H}^1(\Sigma)$ and the number of connected components of $\Sigma$. These estimates are asymptotically optimal as $\mathcal{H}^1(\Sigma)\to\infty$ and, in the particular case where $\Sigma$ is connected, yield a proof of the conjecture.

**Definition 1.1.** Let $N\ge 1$ be an integer. We say that a set $\Sigma\subset\mathbb{R}^2$ is an $N$-continuum if the following conditions are satisfied:

(i) $\Sigma$ can be partitioned as

$$\Sigma=\bigcup_{i=1}^N \Sigma_i,\quad \Sigma_i\cap\Sigma_j=\emptyset\quad \forall i\ne j,$$

where each $\Sigma_i$ is a nonempty, compact, connected set.

(ii) There holds $0 < \mathcal{H}^1(\Sigma) < +\infty$, that is, the 1-dimensional Hausdorff measure of $\Sigma$ is finite and strictly positive.

Within this framework, the sets $\Sigma_i$ are called the (connected) components of $\Sigma$.

Note that every $N$-continuum $\Sigma$ has positive capacity, hence the compliances $C(\Sigma,\Omega)$ and $C_0(\Sigma,\Omega)$ are well defined if $\Sigma \subseteq \overline{\Omega}$.

The main results of this paper can be stated as follows.

**Theorem 1.2.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected domain with Lipschitz boundary, and let $\Sigma \subset \overline{\Omega}$ be an $N$-continuum for some $N \geq 1$. Then the following estimate for the compliance holds true:*

$$C(\Sigma,\Omega) \geq \frac{2}{3}(L\alpha^3 + N\pi\alpha^4), \tag{7}$$

*where $L = H^1(\Sigma)$ and*

$$\alpha = \frac{\operatorname{meas}(\Omega)}{L + \sqrt{L^2 + N\pi \operatorname{meas}(\Omega)}} \tag{8}$$

*is the positive root of the equation $2L\alpha + N\pi\alpha^2 = \operatorname{meas}(\Omega)$.*

To estimate the capacity $C_0$ in the $H_0^1$ setting, we have the following similar result.

**Theorem 1.3.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded connected domain with Lipschitz boundary, let $M$ denote the number of connected components of $\partial\Omega$, and let $\Sigma \subset \overline{\Omega}$ be an $N$-continuum for some $N \geq 1$. Then the following estimate for the compliance holds true:*

$$C_0(\Sigma,\Omega) \geq \frac{2}{3}(L\alpha^3 + (M+N)\pi\alpha^4), \tag{9}$$

*where now $L = H^1(\Sigma \cup \partial\Omega)$ and*

$$\alpha = \frac{\operatorname{meas}(\Omega)}{L + \sqrt{L^2 + (M+N)\pi \operatorname{meas}(\Omega)}} \tag{10}$$

*is the positive root of the equation $2L\alpha + (M+N)\pi\alpha^2 = \operatorname{meas}(\Omega)$.*

Finally, we have the following

**Theorem 1.4.** *Conjecture 1 is true.*

We point out that estimates (7) and (9) are sharp when $\mathcal{H}^1(\Sigma)$ is large, and asymptotically optimal when $\mathcal{H}^1(\Sigma) \to \infty$ (see Example 1). On the other hand, when $\mathcal{H}^1(\Sigma) \to 0$ the compliance $C(\Sigma,\Omega) \to \infty$: optimal estimates in this case require a different approach and will be the subject of a forthcoming paper.

## 2. **Some auxiliary results.**

In the sequel we will use the following characterization of the compliance, as the solution of a maximization problem:

$$C(\Sigma,\Omega) = \max_{u \in V_\Sigma} \left( 2\int_\Omega u(x)\,dx - \int_\Omega |\nabla u(x)|^2\,dx \right), \tag{11}$$

where $V_\Sigma$ is defined in (3). Indeed, the above maximum problem clearly has the the same solution $u_\Sigma$ as the minimization problem (2), and $u_\Sigma$ is characterized by the Euler equation

$$\int_\Omega \nabla u_\Sigma \nabla \varphi = \int_\Omega \varphi \quad \forall \varphi \in V_\Sigma.$$

Choosing $\varphi = u_\Sigma$ one obtains the equilibrium condition

$$\int_\Omega u_\Sigma(x)\,dx = \int_\Omega |\nabla u_\Sigma(x)|^2\,dx,$$

whence

$$C(\Sigma,\Omega) := \int_\Omega u_\Sigma(x)\,dx = 2\int_\Omega u_\Sigma(x)\,dx - \int_\Omega |\nabla u_\Sigma(x)|^2\,dx$$

and the representation formula (11) follows.

Throughout the paper, if $\Sigma$ is a nonempty closed set in $\mathbb{R}^2$, we denote by

$$d_\Sigma(x) = \min_{y\in\Sigma} |y-x|, \quad x \in \mathbb{R}^2 \tag{12}$$

the distance function to $\Sigma$. Moreover, we denote by $\operatorname{meas}(E)$ the two-dimensional Lebesgue measure of a measurable set $E \subset \mathbb{R}^2$, and by $\mathcal{H}^1(E)$ the one-dimensional Hausdorff measure of a set $E \subset \mathbb{R}^2$.

A crucial role in our arguments is played by the level sets of the distance function and, in particular, we will need the following result.

**Lemma 2.1** (Area of tubular neighbours)**.** *Let $\Omega \subset \mathbb{R}^2$ be a measurable set, and let $\Sigma \subset \mathbb{R}^2$ be an $N$-continuum for some $N \geq 1$. For $t \geq 0$, let*

$$A_t = \{x \in \Omega \mid d_\Sigma(x) < t\},$$

*where $d_\Sigma(x)$ is the distance function to $\Sigma$. Then the following estimate holds true, for the measure of $A_t$:*

$$\operatorname{meas}(A_t) \leq \min\{\operatorname{meas}(\Omega), 2\mathcal{H}^1(\Sigma)t + N\pi t^2\} \quad \forall t > 0.$$

**Remark 2.** It is easy to see that this estimate is sharp. For instance, if $\Sigma$ is the union of $N$ pairwise disjoint segments, $\Omega$ is an open set containing $\Sigma$, and $t > 0$ is small enough, then equality is attained, as one can easily check by elementary computations. The same is true if segments are replaced by arcs of smooth curves (see [4]).

*Proof.* Since by construction $A_t \subseteq \Omega$, the inequality $\operatorname{meas}(A_t) \leq \operatorname{meas}(\Omega)$ is trivial. On the other hand, letting $\Sigma_i$ $(1 \leq i \leq N)$ denote the components of $\Sigma$ and letting

$$A_t^i = \{x \in \Omega \mid d_{\Sigma_i}(x) < t\},$$

the inequality

$$\operatorname{meas}(A_t^i) \leq 2\mathcal{H}^1(\Sigma_i)t + \pi t^2 \quad \forall t > 0$$

was proved in [3] as a technical lemma. Finally, since $d_\Sigma(x) = \min_i d_{\Sigma_i}(x)$, we see that

$$A_t = \bigcup_{i=1}^{n} A_t^i \quad \forall t > 0,$$

and hence

$$\operatorname{meas}(A_t) \leq \sum_{i=1}^{N} \operatorname{meas}(A_t^i) \leq \sum_{i=1}^{N} \bigl(2\mathcal{H}^1(\Sigma_i)t + \pi t^2\bigr) = 2\mathcal{H}^1(\Sigma)t + N\pi t^2.$$

□

## 3. Proof of the main results.

**Proposition 1.** *Let $\Omega$, $\Sigma$, $L$ and $\alpha$ be as in the statement of Theorem 1.2. If $h : [0, +\alpha] \mapsto \mathbb{R}$ is any $C^{1,1}$ function such that*

$$h(0) = 0, \quad h' \geq 0, \quad h'' \leq 1 \quad \text{on } [0, \alpha], \tag{13}$$

*then we have the following estimate for the compliance:*

$$C(\Sigma, \Omega) \geq 2 \int_0^\alpha \big(2h(t) - h'(t)^2\big)(L + N\pi t)\, dt. \tag{14}$$

*Proof.* To estimate $C(\Sigma, \Omega)$ from below, we rely on its representation as a supremum given by (11), and we construct a competitor $u \in V_\Sigma$ which depends only on the distance function to $\Sigma$, as defined in (12).

Consider $h : [0, \alpha] \mapsto \mathbb{R}$ as in our statement, and extend $h$ letting $h(t) = h(\alpha) + h'(\alpha)(t - \alpha)$ for $t > \alpha$ (note that $h$ is $C^{1,1}$ on every interval $[0, a]$). It is well known that the function $d_\Sigma$ is Lipschitzian and satisfies $|\nabla d_\Sigma| = 1$ almost everywhere. Thus, since $d_\Sigma \equiv 0$ along $\Sigma$ and $h(0) = 0$ the composite function

$$u(x) := h(d_\Sigma(x)), \quad x \in \Omega,$$

belongs to $V_\Sigma$ and satisfies

$$|\nabla u(x)| = |h'(d_\Sigma(x))\nabla d_\Sigma(x)| = |h'(d_\Sigma(x))|$$

almost everywhere. Thus, using (11), we can estimate

$$C(\Sigma, \Omega) \geq \int_\Omega \big(2u(x) - |\nabla u(x)|^2\big)\, dx = \int_\Omega \big(2h(d_\Sigma(x)) - h'(d_\Sigma(x))^2\big)\, dx.$$

Setting

$$m := \max_{x \in \overline{\Omega}} d_\Sigma(x)$$

and using again the fact that $|\nabla d_\Sigma| = 1$ almost everywhere, we can perform a slicing along the level sets of the distance function, thus obtaining from the coarea formula (see [2])

$$C(\Sigma, \Omega) \geq \int_0^m \big(2h(t) - h'(t)^2\big) P(A_t, \Omega)\, dt,$$

where

$$A_t = \{x \in \Omega \,|\, d_\Sigma(x) < t\}$$

and $P(A_t, \Omega)$ is the perimeter of $A_t$ in $\Omega$ (see [2] for more details on perimeters). Still from the coarea formula, we have

$$\operatorname{meas}(A_t) = \int_0^t P(A_t, \Omega)\, dt \quad \forall t \in (0, m),$$

and hence

$$P(A_t, \Omega) = \frac{d}{dt} \operatorname{meas}(A_t) \quad \text{for a.e. } t \in (0, m).$$

Therefore, setting for simplicity $H(t) = 2h(t) - h'(t)^2$, integration by parts yields

$$\begin{aligned} C(\Sigma, \Omega) &\geq \int_0^m H(t) P(A_t, \Omega)\, dt \\ &= -\int_0^m H'(t) \operatorname{meas}(A_t)\, dt + \operatorname{meas}(A_m) H(m) - \operatorname{meas}(A_0) H(0) \\ &= -\int_0^m H'(t) \operatorname{meas}(A_t)\, dt + \operatorname{meas}(A_m) H(m) \end{aligned}$$

as $\operatorname{meas}(A_0) = 0$. Recalling that $L = \mathcal{H}^1(\Sigma)$ and introducing the auxiliary function

$$p(t) = \min\{\operatorname{meas}(\Omega), 2Lt + N\pi t^2\}, \tag{15}$$

from Lemma 2.1 and (15) we obtain the inequalities

$$\operatorname{meas}(A_t) \le p(t) \le \operatorname{meas}(\Omega), \quad \forall t \ge 0. \tag{16}$$

Note that (13) (and the way we extended $h$) guarantee that $H'(t) \ge 0$ for $t \ge 0$, and therefore we have

$$-H'(t)\operatorname{meas}(A_t) \ge -H'(t)p(t) \quad \forall t \in (0, m).$$

Plugging this estimate into the last integral, and integrating by parts again, from $p(0) = 0$ we obtain the inequality

$$\begin{aligned} C(\Sigma,\Omega) &\ge -\int_0^m H'(t)p(t)\,dt + \operatorname{meas}(A_m)H(m) \\ &= \int_0^m H(t)p'(t)\,dt + \big(\operatorname{meas}(A_m) - p(m)\big)H(m). \end{aligned} \tag{17}$$

We would like to get rid of the quantity $m$, which is unknown and depends on the geometry of $\Sigma$ and $\Omega$. This can be done observing that

$$p(t) = \begin{cases} 2Lt + N\pi t^2 & \text{if } 0 \le t \le \alpha, \\ \operatorname{meas}(\Omega) & \text{if } t > \alpha \end{cases} \tag{18}$$

according to the definition of $\alpha$ as the positive root of $2L\alpha + N\pi\alpha^2 = \operatorname{meas}(\Omega)$.

Now, since clearly $\operatorname{meas}(A_m) = \operatorname{meas}(\Omega)$, setting $t = m$ in (16) we see that

$$p(m) = \operatorname{meas}(A_m) = \operatorname{meas}(\Omega). \tag{19}$$

It is also clear from the definition of $\alpha$ that $p(\alpha) = \operatorname{meas}(\Omega)$ and $p(t) < \operatorname{meas}(\Omega)$ for $t \in (0, \alpha)$ which, combined with $p(m) = \operatorname{meas}(\Omega)$, gives

$$0 < \alpha \le m. \tag{20}$$

Finally, we see from (18) and (20), that

$$p'(t) = \begin{cases} 2L + 2N\pi t & \text{if } 0 \le t < \alpha, \\ 0 & \text{if } \alpha < t \le m. \end{cases}$$

Plugging this formula into (17) and using (19) immediately yields (14), and the lemma is proved. □

**Remark 3.** To turn (14) into a concrete lower bound for the compliance, one has to make concrete choices of the function $h(t)$, satisfying (13).

Forgetting, for a while, the constraints in (13), a priori the best choice is the function which solves the variational problem

$$\max\left\{\int_0^\alpha \big(2h(t) - h'(t)^2\big)(L + N\pi t)\,dt, \quad h(0) = 0\right\}. \tag{21}$$

The Euler equation is

$$\frac{d}{dt}\big(h'(t)(L + N\pi t)\big) = -(L + N\pi t), \quad 0 < t < \alpha,$$

with boundary conditions $h(0) = 0$, $h'(\alpha) = 0$ and explicit solution given by

$$h(t) = \frac{(L + N\pi\alpha)^2}{2N^2\pi^2}\ln(1 + N\pi t/L) - \frac{t^2}{4} - \frac{Lt}{2N\pi}. \tag{22}$$

It turns out that this function satisfies the conditions in (13), hence one can plug it into (14) and obtain an explicit estimate for $C(\Sigma, \Omega)$ after working out the resulting integral. Clearly, this is the best explicit estimate one can obtain from (14).

However, the resulting expression involving $L$, $\alpha$, $N$ and $\text{meas}(\Omega)$ is quite complicated (yet it has interesting applications in the case, not considered in this paper, where $L$ is small).

The estimate in (7) has a neater form, and will be obtained by a choice of $h(t)$ which is much simpler than the best function in (22): the two estimates, however, are asymptotically equivalent when $L \to \infty$.

*Proof of Theorem 1.2.* We apply Proposition 1, choosing $h(t) = \alpha t - t^2/2$, which clearly satisfies (13). Since

$$\int_0^\alpha \big(2h(t) - h'(t)^2\big)(L + N\pi t)\, dt = \frac{L\alpha^3 + N\pi\alpha^4}{3},$$

(7) is established. □

The choice of $h(t)$ in the previous proof is motivated by the fact that this function solves the variational problem

$$\max\left\{ L\int_0^\alpha \big(2f(t) - f'(t)^2\big)\, dt, \quad f(0) = 0 \right\},$$

obtained from (21) replacing the coefficient $L + N\pi t$ with the constant $L$. When $L \gg 1$, we have $\alpha \ll 1$, and hence $L + N\pi t \sim L$, for $t \in (0, \alpha)$. This is the reason why the choice $h(t) = \alpha t - t^2/2$, although not the best possible, yields an estimate which is asymptotically optimal when $L \to \infty$.

**Remark 4.** For fixed $L$, the quantity in the right hand side of (7) is *decreasing* as a function of $N$. Indeed, regarding for a while $N > 0$ as a *real* variable, let $\phi(N) = L\alpha^3 + N\pi\alpha^4$, where $\alpha = \alpha(N)$ is defined by (8). Recalling that

$$N\pi\alpha^2 = \text{meas}(\Omega) - 2L\alpha, \tag{23}$$

we see that

$$\phi(N) = L\alpha^3 + \alpha^2\big(\text{meas}(\Omega) - 2L\alpha\big) = \alpha^2\,\text{meas}(\Omega) - L\alpha^3.$$

Thus, differentiating with respect to $N$, we have

$$\phi'(N) = 2\alpha\alpha'\,\text{meas}(\Omega) - 3L\alpha^2\alpha'.$$

Since clearly $\alpha > 0$ and $\alpha' < 0$, the condition that $\phi'(N) < 0$ is equivalent to the condition that

$$2\,\text{meas}(\Omega) - 3L\alpha > 0,$$

which is true since $\text{meas}(\Omega) - 2L\alpha = N\pi\alpha^2 > 0$ by (23).

*Proof of Theorem 1.3.* Note that the number $M$ of connected components of $\partial\Omega$ is necessarily finite. Indeed, since $\partial\Omega$ is Lipschitz (and compact), it is the union of a finite number of curves which are graphs of Lipschitz functions, and each of these curves is connected. Moreover, $\mathcal{H}^1(\partial\Omega)$ is finite.

It easily follows that $\partial\Omega$ is an $M$-continum, hence the set $\Sigma' = \Sigma \cup \partial\Omega$ is a $P$-continuum with $P \leq M + N$ (equality holds only if $\Sigma \cap \partial\Omega = \emptyset$, but we are not assuming this). Now, recalling (3), for a function $u \in H^1(\Omega)$ we have the equivalent conditions

$$u \in V_\Sigma \cap H^1_0(\Omega) \iff u(x) = 0 \text{ for } \mathcal{H}^1\text{-a.e. } x \in \Sigma' \iff u \in V_{\Sigma'}$$

and hence we see that

$$C_0(\Sigma, \Omega) = C(\Sigma', \Omega). \tag{24}$$

Thus, recalling that $\Sigma'$ is a $P$-continuum, we may apply Theorem 1.2 to $\Sigma'$, thus finding that

$$C(\Sigma', \Omega) \geq \frac{2}{3}\big(L\alpha_P^3 + P\pi\alpha_P^4\big) \tag{25}$$

where $L = \mathcal{H}^1(\Sigma') = \mathcal{H}^1(\Sigma \cup \partial\Omega)$ and

$$\alpha_P = \frac{\operatorname{meas}(\Omega)}{L + \sqrt{L^2 + P\pi \operatorname{meas}(\Omega)}}.$$

Finally, since $P \leq M + N$, by Remark 4 and (25) we see that

$$C(\Sigma', \Omega) \geq \frac{2}{3}\big(L\alpha^3 + (M+N)\pi\alpha^4\big),$$

where now $\alpha$ is defined according to (10). As a consequence, (9) follows from (24). □

*Proof of Conjecture 1.* Let $\Omega$ be the unit square. As already mentioned in the Introduction, it suffices to prove the inequality "$\geq$" in (6) (see, however, Example 1 for the oppositve inequality) and hence, considering two sequences $\{\Sigma_n\}$, $\{L_n\}$ as in the statement of the conjecture, without loss of generality we may assume that, for each $n$, $\Sigma_n$ is a minimizer of the compliance, under the length constraint that $\mathcal{H}^1(\Sigma) \leq L_n$. As already pointed out in [1], the optimality of $\Sigma_n$ under this constraint entails that, in fact, $\mathcal{H}^1(\Sigma_n) = L_n$ (otherwise one could decrease the compliance, by attaching to $\Sigma_n$ any small piece of curve).

Since clearly each $\Sigma_n$ is a 1-continuum, we can apply Theorem 1.3, with $\Sigma = \Sigma_n$, $L = L_n$, $N = 1$ and $M = 1$, because $\partial\Omega$ is connected. As $\operatorname{meas}(\Omega) = 1$, it follows that

$$L_n^2 C_0(\Sigma_n, \Omega) \geq \frac{2}{3}\big(L_n^3\alpha_n^3 + 2\pi L_n^2\alpha_n^4\big),$$

where

$$\alpha_n = \frac{1}{L_n + \sqrt{L_n^2 + 2\pi}}.$$

As $L_n \to \infty$, we have that $\alpha_n \sim 1/(2L_n)$, whence

$$\liminf_{n\to\infty} L_n^2 C_0(\Sigma_n, \Omega) \geq \frac{2}{3}\left(\frac{1}{2}\right)^3 = \frac{1}{12},$$

and the claim follows from the arbitrariness of $\Sigma_n$ and $L_n$. □

We end the paper with an example (adapted from [1]), which shows that the estimates of Theorems 1.2 and 1.3 are sharp, when $L$ is large.

**Example 1.** Let $\Omega = (0,1)^2$ be the unit square. Given an integer integer $n \geq 1$, divide the unit square $\Omega$ into $n$ equal rectangles ($n \geq 1$) of size $n^{-1} \times 1$ (in the natural way, with the long side in the vertical direction), and let $\Sigma_n = \overline{\Omega} \setminus A_n$, where $A_n$ is the union of the interiors of the $n$ rectangles. Note that $\Sigma_n$ is the union of the two horizontal sides of the unit square, and $n+1$ vertical segments of unit length (including the two vertical sides of the square), and therefore

$$L_n := \mathcal{H}^1(\Sigma_n) = n + 3. \tag{26}$$

Let $u_n$ be the state function associated to $C(\Sigma_n, \Omega)$, that is, the solution of the variational problem

$$\min_{u \in V_{\Sigma_n}} \left( \int_\Omega |\nabla u(x)|^2 \, dx - 2 \int_\Omega u(x) \, dx \right).$$

Since $\partial\Omega \subset \Sigma_n$, we have $V_{\Sigma_n} = V_{\Sigma_n} \cap H_0^1(\Omega)$, and hence we see that

$$C(\Sigma_n, \Omega) = C_0(\Sigma_n, \Omega) = \int_\Omega u_n(x, y) \, dxdy. \tag{27}$$

To estimate the last integral from above, consider the function

$$v_n : \mathbb{R}^2 \mapsto \mathbb{R}, \quad v_n(x, y) = n^{-2} v(nx),$$

where $v(x) = (x - x^2)/2$ for $x \in [0, 1]$ is extended to $\mathbb{R}$ by periodicity, with period 1. Observe that $v_n$ satisfies $-\Delta v_n = 1$ in $\Omega \setminus \Sigma_n$, a homogeneous Dirichlet condition along the $n + 1$ vertical segments of $\Sigma_n$, and a homogeneous Neumann condition along the two horizontal segments of $\Sigma_n$. On the other hand, $u_n$ satisfies $-\Delta u_n = 1$ in $\Omega \setminus \Sigma_n$, and a homogeneous Dirichlet condition along the whole $\Sigma_n$. From the maximum principle, it is easy to see that $u_n \le v_n$ in $\Omega \setminus \Sigma_n$, whence

$$\int_\Omega u_n(x, y) \, dxdy \le \int_\Omega v_n(x, y) \, dxdy =$$
$$= n \int_0^{1/n} n^{-2} v(nx) \, dx = n^{-2} \int_0^1 v(x) \, dx = \frac{1}{12n^2}.$$

Recalling (27), we have the upper bound

$$C(\Sigma_n, \Omega) = C_0(\Sigma_n, \Omega) \le \frac{1}{12n^2}. \tag{28}$$

On the other hand, recalling (26), from Theorem 1.2 (applied with $N = 1$) we have the lower bound

$$C(\Sigma_n, \Omega) \ge \frac{2}{3}\big((n+3)\alpha^3 + \pi\alpha^4\big), \quad \alpha = \frac{1}{n + 3 + \sqrt{(n+3)^2 + \pi}},$$

whereas from Theorem 1.3 (applied with $M = N = 1$) we have the lower bound

$$C_0(\Sigma_n, \Omega) \ge \frac{2}{3}\big((n+3)\alpha^3 + 2\pi\alpha^4\big), \quad \alpha = \frac{1}{n + 3 + \sqrt{(n+3)^2 + 2\pi}}.$$

As $n \to \infty$, our bounds can be written as

$$C(\Sigma_n, \Omega) \ge \frac{1}{12n^2} + O(1/n^3), \qquad C_0(\Sigma_n, \Omega) \ge \frac{1}{12n^2} + O(1/n^3),$$

which, combined with (28), shows that the estimates of Theorems 1.2 and (1.3) are sharp, when $L$ is large.

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