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SHOCK FORMATION IN A TRAFFIC FLOW MODEL WITH ARRHENIUS LOOK-AHEAD DYNAMICS

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Abstract. We consider a nonlocal traffic flow model with Arrhenius look-ahead dynamics. We provide a complete local theory and give the blowup alternative of solutions to the conservation law with a nonlocal flux. We show that the finite time blowup of solutions must occur at the level of the first order derivative of the solution. Furthermore, we prove that finite time singularities do occur for several types of physical initial data by analyzing the solutions on different characteristic lines. These results are new and are consistent with the blowups observed in previous numerical simulations on the nonlocal traffic flow model [6].

1. Introduction.

Traffic management has become a key quality of life issue for national, state and local authorities across the world. The energy, environmental and economic crises many of our citizens are experiencing is motivating the scientific community to engage in research aimed at addressing the negative impacts of traffic congestion towards achieving sustainable mobility goals. Tremendous efforts have been devoted to model traffic congestion [1, 2, 3, 4, 5, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17]. There are many important approaches to the modeling of traffic phenomena: microscopic models, mesoscopic models and macroscopic models. Macroscopic models describe traffic phenomena through parameters which characterize collective traffic properties.

Consider the following macroscopic traffic flow model with a nonlocal flux

$$\begin{cases} \partial_t u + \partial_x (u(1-u)e^{-J\circ u}) = 0, \quad \text{in } (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R} \end{cases} \tag{1.1}$$

where the function $u(t,x)$ represents the density of traffic flow, the kernel J acts only on the spatial variable x:

$$(J \circ u)(t,x) = \int_x^\infty J(y-x)u(t,y)dy$$

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and

$$J(r) = \begin{cases} \frac{J_0}{\gamma}, & \text{if } 0 \le r \le \gamma \\ 0, & \text{otherwise} \end{cases}$$

is an anisotropic short range inter-vehicle interaction potential, $\gamma > 0$ is proportional to the look-ahead distance and $J_0 > 0$ is the interaction strength. We suppress the dependence of J on γ and J_0 for simplicity of notation.

The nonlocal traffic flow model (1.1) based on stochastic microscopic dynamics with Arrhenius look-ahead dynamics was derived in [16]. It takes into account interactions of every vehicle with other vehicles ahead. Numerical simulations in [6] indicated that, when $\gamma > 0$, there are shock formations in finite time in the solutions to (1.1) which corresponds to congestion formation in traffic flow.

When the look-ahead distance $\gamma \to +\infty$, the global flux in (1.1) becomes a non-global one $u(1-u)$. The model (1.1) is then reduced to the classical Lightwill-Whitham-Richards(LWR) model [11, 15]

$$\partial_t u + \partial_x(u(1-u)) = 0, \quad \text{in } (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{1.2}$$

If, on the other hand, $\gamma \to 0$, then the global flux in (1.1) is again reduced to a non-global one $u(1-u)exp(-J_0 u)$ where $exp(-J_0 u)$ is a slow down factor in the limiting low visibility. This flux is concave if $J_0 < 3$ and changes concavity when $J_0 \ge 3$. It is well-known that the LWR model (1.2) can describe the formation of shock waves in traffic flow.

In the current paper, we consider the case $0 < \gamma < +\infty$ and normalize the look-ahead distance γ to be 1 for simplicity of presentation. We also take the interaction strength J_0 to be 1. Our goal is to study the finite time blow up of the solutions to the nonlocal model (1.1). Indeed, it is shown that the finite time blow up must occur at the level of the first order derivative of the solution and all L^p, $1 \le p \le \infty$ norms of the solution remain finite near the blowup time. This suggests that the finite time blow up is a shock wave. Despite the nonlocal nature of the problem, we identify several scenarios of blowups for physical initial data. The list is certainly not exhaustive, nevertheless it is consistent with the blowups observed in numerical simulations in [6, 16]. Our results confirm that there are finite time blowup in the nonlocal model (1.1).

The outline of the paper is the following. In Section 2, we prove the local well-posedness and regularity of solutions to the model (1.1) by establishing the *a priori* estimates and by considering the mollified problems of (1.1). We also establish the blowup alternative which quantifies the nature of blowup, and prove an interesting maximum principle which shows that the L^∞ norm of the solution cannot increase in time. Section 3 is devoted to the study of the finite time singularities formation in solutions to (1.1). Various finite time singularities scenarios are analyzed by using method of characteristics. In Section 4, we give some concluding remarks.

2. **Local wellposedness and regularity.**

In this section, we study the local well-posedness and regularity of solutions to the nonlocal model (1.1).

Theorem 2.1. *Let $u_0 \in H^m$ and $m \ge 2$ is an integer. Then there exists $T = T(\|u_0\|_{H^m}) > 0$ and a unique solution u to (1.1) such that $u \in C([0,T), H^m) \cap C^1([0,T), H^{m-1})$.*

Proof of Theorem 2.1. The argument is a variation of the standard energy estimates. To simplify the presentation, we shall only present the proof for $m = 2$.

The case $m \geq 3$ is only slightly more complicated with some necessary changes in numerology.

We first carry out the *a priori* estimates[1] At the end of the proof, we sketch the standard mollification and contraction arguments. For clarity of presentation, we divide the proof into several steps.

`Step 1.` L^2_x estimate. Multiplying both sides of (1.1) by u and integrating by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t,\cdot)\|_2^2 &= \int_{\mathbb{R}} u(1-u)u_x e^{-J\circ u}dx \\ &= -\int_{\mathbb{R}} (\frac{1}{2}u^2 - \frac{1}{3}u^3)\partial_x e^{-J\circ u}dx \\ &= \int_{\mathbb{R}} (\frac{1}{2}u^2(t,x) - \frac{1}{3}u^3(t,x))e^{-J\circ u}(u(t,x+1)-u(t,x))dx\end{aligned}$$

where in the last equality we have used

$$(J\circ u)(t,x) = \int_x^{x+1} u(t,y)dy. \tag{2.1}$$

By using again (2.1), we have

$$\|e^{-J\circ u}\|_\infty \lesssim e^{\|u\|_\infty}.$$

Therefore

$$\left|\frac{d}{dt}\|u(t,\cdot)\|_2^2\right| \lesssim e^{\|u(t,\cdot)\|_\infty}\|u(t,\cdot)\|_\infty(1+\|u(t,\cdot)\|_\infty)\|u(t,\cdot)\|_2^2. \tag{2.2}$$

Here and below the notation $A \sim B$ means that A will be estimated in the same way as B.

`Step 2.` H^2_x estimate. In a similar way in getting the L^2_x estimate, by direct calculations, we have

$$\left|\frac{d}{dt}\|\partial_x^2 u(t,\cdot)\|_2^2\right| \lesssim \left|\int_{\mathbb{R}} \partial_x^3(u(1-u))\partial_x^2 u e^{-J\circ u}dx\right| \tag{2.3}$$

$$+\left|\int_{\mathbb{R}} \partial_x^2(u(1-u))\partial_x^2 u\partial_x e^{-J\circ u}dx\right| \tag{2.4}$$

$$+\left|\int_{\mathbb{R}} \partial_x(u(1-u))\partial_x^2 u\partial_x^2 e^{-J\circ u}dx\right| \tag{2.5}$$

$$+\left|\int_{\mathbb{R}} u(1-u)\partial_x^2 u\partial_x^3 e^{-J\circ u}dx\right|. \tag{2.6}$$

We need derivative estimates of the term $e^{-J\circ u}$. By (2.1), we have

$$\begin{aligned}\partial_x(e^{-J\circ u}) &\sim e^{-J\circ u}(u(t,x+1)-u(t,x)) \\ &\sim e^{-J\circ u}u.\end{aligned} \tag{2.7}$$

By the translation invariance of the Sobolev norms, it is clear that the term $e^{-J\circ u}u(t,x+1)$ will satisfy similar estimates as the term $e^{-J\circ u}u(t,x)$, therefore

[1]Actually the a priori estimates presented here also hold true in more general situations such as $\partial_t u + \partial_x(f(u)\exp(-J\circ u)) = 0$, where f is a smooth function satisfying $f(0) = 0$. The results also hold true when J is a linear interation potential. We thank the anonymous referees for pointing these out.

we only need to write them symbolically as $e^{-J\circ u}u$ in (2.7). By using this convention and avoiding cumbersome notations, we write

$$\partial_x^2(e^{-J\circ u}) \sim e^{-J\circ u}u^2 + e^{-J\circ u}\partial_x u, \tag{2.8}$$

$$\partial_x^3(e^{-J\circ u}) \sim e^{-J\circ u}(u^3 + u\partial_x u + \partial_{xx} u). \tag{2.9}$$

These expressions are quite useful for the estimates below.

We now estimate (2.3). By simple integration by parts and (2.7), we get

$$\begin{aligned}(2.3) \lesssim & \left|\int_{\mathbb{R}} (\partial_x^2 u)^2 \partial_x (ue^{-J\circ u}) dx\right| \\ & + \left|\int_{\mathbb{R}} (\partial_x^2 u)^2 \partial_x e^{-J\circ u} dx\right| \\ & + \left|\int_{\mathbb{R}} \partial_x u (\partial_x^2 u)^2 e^{-J\circ u} dx\right| \\ \lesssim & \|\partial_x^2 u\|_2^2 e^{\|u\|_\infty}\Big(\|\partial_x u\|_\infty + \|u\|_\infty + \|u\|_\infty^2\Big).\end{aligned}$$

For (2.4), we use (2.7) and get

$$\begin{aligned}(2.4) \lesssim & \left|\int_{\mathbb{R}} (\partial_x^2 u)^2 (1-u)\partial_x (e^{-J\circ u}) dx\right| \\ & + \left|\int_{\mathbb{R}} (\partial_x^2 u)^2 u \partial_x (e^{-J\circ u}) dx\right| \\ & + \left|\int_{\mathbb{R}} (\partial_x u)^2 \partial_x^2 u \partial_x (e^{-J\circ u}) dx\right| \\ \lesssim & \|\partial_x^2 u\|_2^2 e^{\|u\|_\infty} \|u\|_\infty (1 + \|u\|_\infty) \\ & + e^{\|u\|_\infty}\|u\|_\infty \|\partial_x^2 u\|_2 \|\partial_x u\|_\infty \|\partial_x u\|_2 \\ \lesssim & \|u\|_{H^2}^2 e^{\|u\|_\infty}\|u\|_\infty (1 + \|u\|_\infty + \|\partial_x u\|_\infty).\end{aligned}$$

For (2.5), we use (2.8) to give us

$$\begin{aligned}(2.5) \lesssim & e^{\|u\|_\infty}\|\partial_x^2 u\|_2 \|\partial_x u\|_2 (\|u\|_\infty^2 + \|\partial_x u\|_\infty)(1 + \|u\|_\infty) \\ \lesssim & e^{\|u\|_\infty}\|u\|_{H^2}^2 (1 + \|u\|_\infty)(\|u\|_\infty^2 + \|\partial_x u\|_\infty).\end{aligned}$$

Finally, we use (2.9) to estimate (2.6) as

$$\begin{aligned}(2.6) \lesssim & e^{\|u\|_\infty}\|\partial_x^2 u\|_2 \|u\|_\infty (1 + \|u\|_\infty) \\ & (\|\partial_x^2 u\|_2 + \|u\|_2 \|u\|_\infty^2 + \|\partial_x u\|_2 \|u\|_\infty) \\ \lesssim & e^{\|u\|_\infty}\|u\|_{H^2}^2 (1 + \|u\|_\infty)^4.\end{aligned}$$

Collecting all the estimates, we obtain

$$\begin{aligned}&\left|\frac{d}{dt}\|\partial_x^2 u(t,\cdot)\|_2^2\right| \\ \lesssim & e^{\|u(t,\cdot)\|_\infty}\|u(t,\cdot)\|_{H^2}^2 \Big(\|\partial_x u(t,\cdot)\|_\infty (1 + \|u(t,\cdot)\|_\infty) + (1 + \|u(t,\cdot)\|_\infty)^4\Big).\end{aligned} \tag{2.10}$$

Adding together (2.2) and (2.10), we get

$$\begin{aligned}&\left|\frac{d}{dt}\|u(t,\cdot)\|_{H^2}^2\right|\\ &\lesssim e^{\|u(t)\|_\infty}\|u(t,\cdot)\|_{H^2}^2\Big(\|\partial_x u(t,\cdot)\|_\infty(1+\|u(t,\cdot)\|_\infty)+(1+\|u(t,\cdot)\|_\infty)^4\Big)\end{aligned} \tag{2.11}$$

here and in the rest of the paper we adopt the following equivalent definition of H^2 norm $\|f\|_{H^2}=\sqrt{\|f\|_{L^2}^2+\|\frac{d^2}{dx^2}f\|_{L^2}^2}$ for any $f\in H^2$.

`Step 3`. *A priori* H^2 bound. By (2.11) and using the Sobolev embedding $H^1(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$, we obtain

$$\left|\frac{d}{dt}\|u(t,\cdot)\|_{H^2}\right|\le C_1e^{C_2\|u(t,\cdot)\|_{H^2}}$$

where C_1 and C_2 are absolute constants. Rewrite the above inequality as

$$\left|\frac{d}{dt}e^{-C_2\|u(t,\cdot)\|_{H^2}}\right|\le C_1C_2, \tag{2.12}$$

and integrate over the time interval $[0,t]$ for $t\le T$ to obtain

$$e^{-C_2\|u(t,\cdot)\|_{H^2}}\ge e^{-C_2\|u_0\|_{H^2}}-C_1C_2t,\quad \forall\, 0\le t\le T.$$

Now if we choose $T>0$ such that

$$C_1C_2T\le\frac{1}{2}e^{-C_2\|u_0\|_{H^2}},$$

then for all $t\le T$,

$$e^{-C_2\|u(t,\cdot)\|_{H^2}}\ge\frac{1}{2}e^{-C_2\|u_0\|_{H^2}}.$$

Therefore

$$\sup_{0\le t\le T}\|u(t,\cdot)\|_{H^2}\le\|u_0\|_{H^2}+\frac{\ln 2}{C_2}.$$

This is the *a priori* H^2 bound that we needed.

`Step 4.` Mollification and contraction. We use the Littlewood-Paley projectors which is introduced now. Let $\phi(\cdot)$ be an even C^∞ function supported in the ball $\{\xi\in\mathbb{R}:\ |\xi|\le\frac{11}{10}\}$ and equal to 1 on the ball $\{\xi\in\mathbb{R}:\ |\xi|\le 1\}$. For each dyadic number $N>0$ and any $f\in S'(\mathbb{R})$ (here $S'(\mathbb{R})$ denotes the space of tempered distributions on $\mathbb{R}$), we introduce the standard Littlewood-Paley projectors $P_{\le N}$ and P_N by

$$\begin{aligned}\widehat{P_{\le N}f}(\xi)&=\phi(\xi/N)\hat{f}(\xi),\\ \widehat{P_N f}(\xi)&=\psi(\xi/N)\hat{f}(\xi)=(\phi(\xi/N)-\phi(2\xi/N))\hat{f}(\xi).\end{aligned}$$

For each dyadic $N>0$, we consider the mollified problem of (1.1)

$$\begin{cases}\partial_t u^{(N)}+\partial_x P_{\le N}\Big(P_{\le N}u^{(N)}(1-P_{\le N}u^{(N)})e^{-J\circ u^{(N)}}\Big)=0,\\ u^{(N)}(0,x)=u_0(x).\end{cases} \tag{2.13}$$

By standard ordinary differential equation theory in Banach spaces (cf. Theorem 3.1 of [10]) and the *a priori* estimates established in Step 2, one can show that for some $T = T(\|u_0\|_{H^2}) > 0$, there exists a unique solution $u^{(N)} \in C([0,T],\, H^2)$ to (2.13). Furthermore

$$\sup_{N\geq 1}\sup_{0\leq t\leq T}\|u^{(N)}(t,\cdot)\|_{H^2} \leq C(\|u_0\|_{H^2}) \tag{2.14}$$

where $C(\|u_0\|_{H^2})$ denotes a constant depending only on $\|u_0\|_{H^2}$.

By a direct calculation similar to the one leading to the *a priori* estimates in Step 2 and shrinking $T = T(\|u_0\|_{H^2})$ further if necessary, we have that the sequence $\{u^{(N)}\}$ is Cauchy in the space $C([0,T],\, L^2)$. Interpolating this with the bound (2.14), we obtain that $\{u^{(N)}\}$ is also Cauchy in the space $C([0,T],\, H^1)$. Hence we have that $\{u^{(N)}\}$ has a limit $u \in C([0,T],\, H^1) \cap L^\infty((0,T), H^1)$ which solves (1.1). We still have to show that $u \in C([0,T],\, H^2)$. Since u is weakly continuous on H^1 (i.e. let $\langle , \rangle$ denote the inner product on H^2, then the pairing $\langle u(t,\cdot), \phi(\cdot)\rangle$ for any $\phi \in H^2$ is a continuous function of t), it suffices for us to show the norm continuity, i.e. that $\|u(t,\cdot)\|_{H^2}$ is continuous in t. By using a similar estimate leading to (2.12), we have for any $0 \leq t_1,\, t_2 \leq T$,

$$\left| e^{-C_3\|u(t_1,\cdot)\|_{H^2}} - e^{-C_3\|u(t_2,\cdot)\|_{H^2}} \right| \leq C_4|t_1 - t_2|$$

where C_3, C_4 are absolute constants. Hence the norm continuity follows.

Concluding from the above four steps, we obtain the desired classical solution $u \in C([0,T], H^2)$ to (1.1) for some $T = T(\|u_0\|_{H^2}) > 0$. □

The proof of Theorem 2.1 actually yields more information about the local solution. The following corollary is immediate.

Corollary 2.2 (Blowup alternative)**.** *Let $u_0 \in H^m$ and $m \geq 2$ is an integer. Let u be the corresponding maximal-lifespan solution obtained in Theorem 2.1 with lifespan $[0,T)$. Then only one of the following occur*

- *$T = +\infty$ and u is a global solution;*
- *$0 < T < \infty$ and*

$$\lim_{\tau\to T}\int_0^\tau e^{\|u(t,\cdot)\|_\infty}\Big(\|\partial_x u(t,\cdot)\|_\infty(1+\|u(t,\cdot)\|_\infty) + (1+\|u(t,\cdot)\|_\infty)^4\Big)dt = +\infty. \tag{2.15}$$

In particular, we have

$$\limsup_{t\to T}(\|u(t,\cdot)\|_\infty + \|\partial_x u(t,\cdot)\|_\infty) = +\infty. \tag{2.16}$$

Proof. This follows from the proof of Theorem 2.1. For simplicity we take $m = 2$, the case $m > 2$ is only more complicated in numerology. Let u be the maximal-lifespan solution and assume its lifespan is $[0,T)$ with $T < \infty$. It suffices for us to recycle the bound (2.11). The Gronwall argument then gives (2.15). The assertion (2.16) is an immediate consequence of (2.15). □

In the theory of traffic flow, the function $u(t,x)$ represents the density which is normalized to the interval $[0,1]$, i.e., typically, we have $0 \leq u \leq 1$. The following lemma justifies this fact.

Lemma 2.3 (*A priori* L^∞ bound)**.** *Let $u_0 \in H^m$ and $m \geq 2$ is an integer. Assume that $0 \leq u_0(x) \leq 1$ for all $x \in \mathbb{R}$. Let u be the corresponding maximal-lifespan solution obtained in Theorem 2.1 with lifespan $[0,T)$. Then we have*

$$0 \leq u(t,x) \leq 1, \quad \forall\, x \in \mathbb{R},\, 0 \leq t < T.$$

Proof. This follows from the method of characteristics. We first show that $u(t,x) \geq 0$. Define the following family of characteristics lines:

$$\begin{cases} \frac{d}{dt}X(t,\alpha) = \Big((1-u)e^{-J\circ u}\Big)(t, X(t,\alpha)), \\ X(0,\alpha) = \alpha \in \mathbb{R}. \end{cases}$$

These characteristic lines are well-defined for all $0 \leq t < T$ since $u \in C([0,T),\, H^2)$ and the function $(1-u)e^{-J\circ u}$ is Lipschitz and in fact is bounded in C^1. We have that for all $0 \leq t < T$, $X(t,\cdot)$ is a diffeomorphism from $\mathbb{R}$ to $\mathbb{R}$. Furthermore

$$\frac{d}{dt}u(t, X(t,\alpha)) = -\Big(\partial_x((1-u)e^{-J\circ u})\Big)(t, X(t,\alpha))u(t, X(t,\alpha)).$$

Since $u(0, X(0,\alpha)) = u_0(\alpha) \geq 0$ for any $\alpha \in \mathbb{R}$, we conclude that $u(t, X(t,\alpha)) \geq 0$ for all $\alpha \in \mathbb{R}$, $0 \leq t < T$. Since $X(t,\cdot)$ has a smooth inverse, we obtain $u(t,x) \geq 0$ for all $x \in \mathbb{R}$ and $0 \leq t < T$.

Next we show that $u(t,x) \leq 1$. Define $v = 1-u$. Then for v we have the equation

$$\begin{cases} \partial_t v - \partial_x\Big(v(1-v)e^{-1}e^{J\circ v}\Big) = 0, \\ v(0,x) = v_0(x) = 1 - u_0(x). \end{cases}$$

Define the characteristic lines

$$\begin{cases} \frac{d}{dt}Y(t,\alpha) = -\Big((1-v)e^{-1}e^{J\circ v}\Big)(t, Y(t,\alpha)), \\ Y(0,\alpha) = \alpha \in \mathbb{R}. \end{cases}$$

Furthermore

$$\frac{d}{dt}v(t, Y(t,\alpha)) = \Big(\partial_x((1-v)e^{-1}e^{J\circ v})\Big)(t, Y(t,\alpha))v(t, Y(t,\alpha)).$$

Since $v_0(x) = 1 - u_0(x) \geq 0$ for all $x \in \mathbb{R}$, thus $v(t, Y(t,\alpha)) \geq 0$ and consequently $u(t,x) \leq 1$ for all $x \in \mathbb{R}$, $0 \leq t < T$. □

Lemma 2.3 can be slightly strengthened. The following lemma shows that if for some constant $0 < M \leq 1$, the initial data $0 \leq u_0(x) \leq M$, for any $x \in \mathbb{R}$, then the inequality $0 \leq u(t,x) \leq M$ holds for all t. While the proof of Lemma 2.3 is Lagrangian (based on method of characteristics), the proof of Lemma 2.4 will be Eulerian.

Lemma 2.4 (*A priori* L^∞ bound, another version)**.** *Let $u_0 \in H^m$ and $m \geq 2$ is an integer. Let u be the corresponding maximal-lifespan solution obtained in Theorem 2.1 with lifespan $[0,T)$. Assume that $0 \leq u_0(x) \leq M$ and $0 < M \leq 1$. Then for any $0 \leq t < T$, we have*

$$0 \leq u(t,x) \leq M, \quad \forall\, x \in \mathbb{R}.$$

Proof of Lemma 2.4. Let $T_1 < T$ be such that $T_1 < \infty$. We first prove the lemma on $[0,T_1]$ and then use the arbitrariness of T_1 to conclude the proof. Let $0 < \varepsilon < 1$ and define

$$w(t,x) = e^{-\varepsilon t}u(t,x), \quad \forall\, 0 \leq t \leq T_1,\ x \in \mathbb{R}.$$

We suppress the dependence of w on ε for simplicity of notation. We first show that there exists some point $(t_0,x_0) \in [0,T_1] \times \mathbb{R}$ such that

$$w(t_0,x_0) = \sup_{\substack{0 \leq t \leq T_1 \\ x \in \mathbb{R}}} w(t,x) = M_1 > 0. \tag{2.17}$$

Here M_1 is finite since $u \in C([0,T_1], H^m)$ with $m \geq 2$. Indeed suppose that supremum is not attained, then there exists $(t_n, x_n) \in [0,T_1] \times \mathbb{R}$ with $|x_n| \to \infty$, such that

$$w(t_n, x_n) \to M_1, \quad n \to \infty.$$

But this then contradicts to the fact that $w \in L_t^2 L_x^2([0,T_1] \times \mathbb{R})$ since $\partial_t w$ and ∇w are uniformly bounded (one can construct disjoint balls of size $O(1)$ around each (t_n, x_n) so that $w \gtrsim M_1/2$ in each ball). Hence (2.17) holds. Now we assert that $t_0 = 0$. Indeed if $0 < t_0 \leq T_1$, then

$$\partial_t w(t,x)\Big|_{(t,x)=(t_0,x_0)} \geq 0.$$

On the other hand, we have

$$\begin{aligned}
\partial_t w(t,x)\Big|_{(t,x)=(t_0,x_0)} &= -\varepsilon M_1 + e^{-\varepsilon t_0}(\partial_t u)(t_0,x_0) \\
&= -\varepsilon M_1 - (1-2u(t_0,x_0))(\partial_x w)(t_0,x_0) e^{-(J\circ u)(t_0,x_0)} \\
&\quad - u(t_0,x_0)(1-u(t_0,x_0)) e^{-(J\circ u)(t_0,x_0)}(M_1 - w(t_0, x_0+1)) \\
&\leq -\varepsilon M_1 < 0
\end{aligned}$$

where in the last inequality we have used the facts that $\partial_x w(t_0,x_0) = 0$, $w \leq M_1$ and that $0 \leq u \leq 1$. The above computation shows that the maximum cannot occur at $0 < t_0 \leq T_1$ and therefore we have $t_0 = 0$ which gives

$$u(t,x)e^{-\varepsilon t} \leq M, \quad \forall\, 0 \leq t \leq T_1,\ x \in \mathbb{R}.$$

Taking $\varepsilon \to 0$ and letting T_1 be arbitrary immediately yield the lemma . □

Corollary 2.2 in conjunction with Lemma 2.3 gives us the following useful

Corollary 2.5 (Blowup alternative, physical initial data)**.** *Let $u_0 \in H^m$ and $m \geq 2$ is an integer. Assume $0 \leq u_0(x) \leq 1$ for all $x \in \mathbb{R}$. Let u be the corresponding maximal-lifespan solution obtained in Theorem 2.1 with lifespan $[0,T)$. Then only one of the following occurs*

- *$T = +\infty$ and u is a global solution;*
- *$0 < T < \infty$ and*

$$\lim_{\tau \to T} \int_0^\tau \|\partial_x u(t,\cdot)\|_\infty dt = +\infty. \tag{2.18}$$

In particular, we have

$$\limsup_{t \to T} \|\partial_x u(t,\cdot)\|_\infty = +\infty. \tag{2.19}$$

Proof. By Lemma 2.3, we have $0 \leq u(t,x) \leq 1$ for all $0 \leq t < T$. By (2.15), for any $\tau < T$, we have

$$\begin{aligned}
&\int_0^\tau e^{\|u(t,\cdot)\|_\infty}\Big(\|\partial_x u(t,\cdot)\|_\infty(1+\|u(t,\cdot)\|_\infty) + (1+\|u(t,\cdot)\|_\infty)^4\Big)dt \\
&\lesssim \int_0^\tau \|\partial_x u(t,\cdot)\|_\infty dt.
\end{aligned}$$

Hence (2.18) follows. The assertion (2.19) is obvious. □

Remark 2.6. We should stress that for physical data, $0 \le u_0(x) \le 1$, for all $x \in \mathbb{R}$, if the blowup happens at $t = T$, then it must occur at the level of the first order derivative of the solution and all L^p, $1 \le p \le \infty$ norms of the solution remain finite near the blowup time T. Indeed by Lemma 2.3,

$$\limsup_{t \to T} \|u(t,\cdot)\|_\infty \le 1.$$

By L^1-conservation, we have

$$\|u(t,\cdot)\|_1 = \|u_0\|_1, \quad \forall\, 0 \le t < T.$$

Hence by interpolation, we have for all $1 \le p \le \infty$,

$$\limsup_{t \to T} \|u(t,\cdot)\|_p \le C$$

for some constant $C > 0$. The analysis suggests that the blowup is a shock wave.

3. **Finite time singularities.**

In this section we prove that finite time singularities do occur for several types of physical initial data. The arguments are primarily based on comparisons of the speed of characteristic lines. Despite the nonlocal nature of the problem, we nail down several scenarios of blowups of solutions for physical initial data. The list is not exhaustive, nevertheless it is consistent with the blowups observed in numerical simulations in [6, 16]. Our results confirm that there will be blowup in finite time in the nonlocal model.

Theorem 3.1 (Existence of finite time blowups, scenario 1: collision with 0 or 1)**.** *Let $u_0 \in H^m(\mathbb{R})$ and $m \ge 2$ is an integer. Assume that $0 \le u_0(x) \le 1$ for all $x \in \mathbb{R}$. If there exist two points $-\infty < \alpha_1 < \alpha_2 < \infty$, such that $u_0(\alpha_1) = 0 < u_0(\alpha_2) = 1$, then u must blow up at some finite time $0 < T < \infty$, i.e.,*

$$\limsup_{t \to T} \|\partial_x u(t,\cdot)\|_\infty = +\infty. \tag{3.1}$$

Moreover, all the L^p, $1 \le p \le \infty$ norms of u remain finite as $t \to T$,

$$\limsup_{t \to T} \|u(t,\cdot)\|_p < \infty, \quad \forall\, 1 \le p \le \infty. \tag{3.2}$$

Proof of Theorem 3.1. We only need to prove the existence of finite time blowups. The assertions (3.1) and (3.2) follow directly from Corollary 2.5 and Remark 2.6.

To prove existence of singularities, we shall argue by contradiction. We assume that the corresponding maximal-lifespan solution u is global. We shall demonstrate that the characteristic lines emanating from the points $\alpha_1 < \alpha_2$ (with different definitions of characteristic lines) will eventually collide and a contradiction will be derived there. The contradiction shows that the solution is not a global solution. According to Corollary 2.5 and Remark (2.6), the solution must develop singularity at a finite time (3.1).

We first consider the characteristic line starting at the point α_1. Define

$$\begin{cases} \frac{d}{dt} X_1(t) = \Big((1-u)e^{-J \circ u}\Big)(t, X_1(t)), \\ X_1(0) = \alpha_1. \end{cases} \tag{3.3}$$

Clearly $X_1(t)$ is globally well-defined since u is global and the driving term $(1-u)e^{-J\circ u}$ is Lipschitz. By a direct calculation, we have

$$\begin{cases} \frac{d}{dt}u(t,X_1(t)) = -u(t,X_1(t))\left(\partial_x\Big((1-u)e^{-J\circ u}\Big)\right)(t,X_1(t)), \\ u(0,X_1(0)) = u_0(\alpha_1) = 0. \end{cases}$$

Hence $u(t,X_1(t)) \equiv 0$ for all $t \geq 0$. Plugging this into (3.3), we obtain

$$\frac{d}{dt}X_1(t) \geq e^{-\|u(t,\cdot)\|_\infty} \geq e^{-1}, \quad \forall\, t \geq 0$$

where we have used Lemma 2.3. Therefore

$$X_1(t) \geq \alpha_1 + e^{-1}t, \quad \forall\, t \geq 0. \tag{3.4}$$

Next we consider the characteristic line starting at the point $(0,\alpha_2)$. However, we shall use a different form of characteristic line than that starting at $(0,\alpha_1)$. Define $v(t,x) = 1-u(t,x)$, then for $v(t,x)$ we have the equation

$$\begin{cases} \partial_t v - \partial_x\Big(v(1-v)e^{-1}e^{J\circ v}\Big) = 0, \\ v(0,x) = 1-u_0(x). \end{cases}$$

Define $X_2(t)$ by the following

$$\begin{cases} \frac{d}{dt}X_2(t) = -\Big((1-v)e^{-1}e^{J\circ v}\Big)(t,X_2(t)), \\ X_2(0) = \alpha_2. \end{cases} \tag{3.5}$$

By using the regularity of v and u, it is not difficult to check that $X_2(t)$ are globally well-defined. By a direct computation, we have

$$\begin{cases} \frac{d}{dt}v(t,X_2(t)) = v(t,X_2(t))\left(\partial_x\Big((1-v)e^{-1}e^{J\circ v}\Big)\right)(t,X_2(t)), \\ v(0,X_2(0)) = 1-u_0(\alpha_2) = 0. \end{cases}$$

Hence $v(t,X_2(t)) \equiv 0$ for all $t \geq 0$. Plugging this into (3.5), we obtain

$$\frac{d}{dt}X_2(t) \leq -e^{-1}, \quad \forall\, t \geq 0$$

where we have used again Lemma 2.3. Integrating in the time variable, we get

$$X_2(t) \leq \alpha_2 - e^{-1}t, \quad \forall\, t \geq 0. \tag{3.6}$$

We are now ready to arrive at a contradiction. By (3.4), (3.6) and the assumption that $\alpha_1 < \alpha_2$, we conclude that there must exist some $t_0 > 0$ such that $X_1(t_0) = X_2(t_0) = x_0$. But then by using the definition of characteristic lines, we have that $0 = u(t_0,X_1(t_0)) = u(t_0,x_0)$ and $1 = u(t_0,X_2(t_0)) = u(t_0,x_0)$, which is obviously a contradiction. □

Theorem 3.1 is consistent with the finite time blowups observed in numerical simulations in [6, 16]. The proof of Theorem 3.1 also indicates that the finite time singularity is a shock wave. Are there initial data which does not satisfies the condition in Theorem 3.1 but still may develop finite time singularities? The following theorem shows that indeed there exist blowups in this class of data at the expense of some slope conditions.

We now introduce

Definition 3.2 (Slope condition)**.** We say a function $f:\mathbb{R}\to\mathbb{R}$ satisfies the slope condition if there exist two points $-\infty<\alpha_1<\alpha_2<\infty$ and a number $\lambda>1$ such that the following inequalities hold:

$$2f(\alpha_2)(1-\lambda^{-1})-2f(\alpha_1)(\lambda-1)\geq e(\alpha_2-\alpha_1), \tag{3.7}$$

$$f(\alpha_1)\leq\frac{1}{2\lambda}, \tag{3.8}$$

$$f(\alpha_2)\geq\frac{1}{2}\lambda, \tag{3.9}$$

$$f(\alpha_2)\neq f(\alpha_1)\lambda^2. \tag{3.10}$$

When such points $\alpha_1<\alpha_2$ and number $\lambda>1$ exist, we say that f satisfies the slope condition with respect to points $\alpha_1<\alpha_2$ and parameter $\lambda>1$.

Remark 3.3. It is fairly easy to construct functions which satisfy the slope condition. For example, we can fix the constant $\lambda>1$, and let $0\leq y_1\leq\frac{1}{2\lambda}\leq\frac{1}{2}\lambda\leq y_2$ and $y_2\neq\lambda^2y_1$. Then choose numbers $\alpha_1<\alpha_2$ with the difference $\alpha_2-\alpha_1$ so small such that

$$2(\frac{y_2}{\lambda}-y_1)(\lambda-1)\geq e(\alpha_2-\alpha_1).$$

It is then easy to choose a function f which passes through the two points (α_1,y_1), (α_2,y_2). Such a function satisfies the slope condition by construction.

We now establish the second scenario of blowups under the above slope condition.

Theorem 3.4 (Existence of finite time blowups, scenario 2: collision under the slope condition)**.** *Let $u_0\in H^m(\mathbb{R})$, $m\geq 2$ is an integer. Assume that $0\leq u_0(x)\leq 1$ for all $x\in\mathbb{R}$. Suppose u_0 satisfies the slope conditions (3.7)–(3.10) with respect to some points $\alpha_1<\alpha_2$ and some parameter $\lambda>1$. Denote the corresponding maximal-lifespan solution as $u=u(t,x)$. Then u must blow up at some time T, i.e.,*

$$\limsup_{t\to T}\|\partial_x u(t,\cdot)\|_\infty=+\infty \tag{3.11}$$

where $0<T<\log\lambda$.

Moreover, all the L^p, $1\leq p\leq\infty$ norms of u remain finite as $t\to T$,

$$\limsup_{t\to T}\|u(t,\cdot)\|_p<\infty,\quad\forall\, 1\leq p\leq\infty. \tag{3.12}$$

Proof of Theorem 3.4. It suffices for us to prove the existence of blowup. The assertions (3.11) and (3.12) follow from Corollary 2.5 and Remark 2.6.

We shall again argue by contradiction. Assume that the corresponding maximal-lifespan solution u is a global solution. We shall show that a collision of characteristics must occur on or before $t_1=\log\lambda$ and a contradiction is derived there. The contradiction shows that u is not a global solution. According to Corollary 2.5 and Remark (2.6), the solution must develop finite time singularity there (3.11).

Define the characteristic lines[2]

$$\begin{cases}\frac{d}{dt}X(t,\alpha)=\Big((1-2u)e^{-J\circ u}\Big)(t,X(t,\alpha)),\\ X(0,\alpha)=\alpha\in\mathbb{R}.\end{cases} \tag{3.13}$$

[2]Actually the characteristics defined by (3.13) can also be used to give an alternative proof of Theorem 3.1 and the idea is essentially the same.

It is not difficult to check that $X(t,\alpha)$ are well-defined on the maximal lifespan of u.

By a direct calculation, we have

$$\begin{cases} \frac{d}{dt}u(t,X(t,\alpha)) = \Big(u(1-u)e^{-J\circ u}\Big)(t,X(t,\alpha))\Big(u(t,X(t,\alpha)) - u(t,X(t,\alpha)+1)\Big), \\ u(0,X(0,\alpha)) = u_0(\alpha). \end{cases}$$

Since $0 \le u \le 1$, we get

$$\left|\Big((1-u)e^{-J\circ u}\Big)(t,X(t,\alpha))(u(t,X(t,\alpha)) - u(t,X(t,\alpha)+1))\right| \le 1.$$

By a Gronwall argument, we then obtain that for $t \ge 0$,

$$u_0(\alpha)e^{-t} \le u(t,X(t,\alpha)) \le u_0(\alpha)e^{t}. \tag{3.14}$$

Now we shall invoke the slope conditions (3.7)–(3.10) with respect to points $\alpha_1 < \alpha_2$ and parameter $\lambda = e^{t_1} > 1$ where $t_1 = \log \lambda$.

Consider first the characteristic starting at the point α_1. For any $0 \le t \le t_1$, we have by (3.8)

$$u_0(\alpha_1)e^{t} \le \frac{1}{2}.$$

Therefore by (3.13) and (3.14),

$$\begin{cases} \frac{d}{dt}X(t,\alpha_1) \ge (1-2u_0(\alpha_1)e^{t})e^{-1} > 0, \\ u(t,X(t,\alpha_1)) \le u_0(\alpha_1)e^{t}, \quad \forall\, 0 \le t \le t_1. \end{cases} \tag{3.15}$$

Next consider the characteristic line starting at the point α_2. By (3.9), we have

$$u_0(\alpha_2)e^{-t} \ge \frac{1}{2}, \quad \forall\, 0 \le t \le t_1.$$

Hence by (3.13) and (3.14),

$$\begin{cases} \frac{d}{dt}X(t,\alpha_2) \le (1-2u_0(\alpha_2)e^{-t})e^{-1} < 0, \\ u(t,X(t,\alpha_2)) \ge u_0(\alpha_2)e^{-t}, \quad \forall\, 0 \le t \le t_1. \end{cases} \tag{3.16}$$

Collecting the estimates (3.15), (3.16) and integrating in time, we obtain for any $0 \le t \le t_1$,

$$\begin{aligned} X(t,\alpha_1) - X(t,\alpha_2) &\ge \alpha_1 - \alpha_2 + \int_0^t (1-2u_0(\alpha_1)e^{s})e^{-1}ds \\ &\quad - \int_0^t (1-2u_0(\alpha_2)e^{-s})e^{-1}ds \\ &\ge \alpha_1 - \alpha_2 + 2e^{-1}u_0(\alpha_2)(1-e^{-t}) - 2e^{-1}u_0(\alpha_1)(e^{t}-1) \\ &=: F(t). \end{aligned}$$

By using (3.8) and (3.9), it is not difficult to check that

$$\begin{aligned} F'(t) &= 2e^{-1}\Big(u_0(\alpha_2)e^{-t} - u_0(\alpha_1)e^{t}\Big) \\ &\ge 0, \quad \forall\, 0 \le t \le t_1. \end{aligned}$$

By (3.7), we have

$$X(t_1,\alpha_1) - X(t_1,\alpha_2) \ge 0.$$

Therefore there must exist some t_2 with $0 < t_2 \leq t_1$ such that

$$X(t_2, \alpha_1) = X(t_2, \alpha_2) = x_2.$$

This shows that characteristic lines emanating from α_1 and α_2 must collide at time $t = t_2$.

But by (3.14) and (3.8)–(3.9), we have

$$u(t, x_2) = u(t, X(t_2, \alpha_1)) \leq u_0(\alpha_1)e^{t_1} \leq \frac{1}{2},$$
$$u(t, x_2) = u(t, X(t_2, \alpha_2)) \geq u_0(\alpha_2)e^{-t_1} \geq \frac{1}{2}.$$

Hence the equality

$$u_0(\alpha_2) = u_0(\alpha_1)e^{2t_1} = u_0(\alpha_1)\lambda^2$$

must hold. But this contradicts to (3.10). Hence we have arrived at the desired contradiction. Consequently a shock must occur on or before $t_1 = \log \lambda$. □

4. **Concluding remarks.** We studied the finite time blow up of the solutions to the nonlocal model (1.1). Indeed, it is shown that the finite time blow up must occur at the level of the first order derivative of the solution. This suggests that the finite time blow up is a shock wave. Despite the nonlocal nature of the problem, we identify several scenarios of blowups for physical initial data. The list is certainly not exhaustive, nevertheless it is consistent with the blowups observed in numerical simulations in [6, 16]. Our results confirm that there are finite time blowup in the nonlocal model (1.1). We will investigate the finite time blow up of solutions for more initial data in the the future.

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