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GLOBAL EXISTENCE AND LONG-TIME BEHAVIOR OF ENTROPY WEAK SOLUTIONS TO A QUASILINEAR HYPERBOLIC BLOOD FLOW MODEL

Tong Li

Department of Mathematics, University of Iowa,
Iowa City, IA 52242, USA

Kun Zhao

Department of Mathematics, University of Iowa,
Iowa City, IA 52242, USA

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Abstract. This paper is concerned with an initial-boundary value problem on bounded domains for a one dimensional quasilinear hyperbolic model of blood flow with viscous damping. It is shown that L^∞ entropy weak solutions exist globally in time when the initial data are large, rough and contains vacuum states. Furthermore, based on entropy principle and the theory of divergence measure field, it is shown that any L^∞ entropy weak solution converges to a constant equilibrium state exponentially fast as time goes to infinity. The physiological relevance of the theoretical results obtained in this paper is demonstrated.

1. **Introduction.** In 2007, approximately 620,000 patients died from cardiovascular diseases, the top killer in the US. Major concern has been hovering around among researchers and clinicians alike to develop models and methods for the prevention and treatment of cardiovascular diseases. A very important first step in the research of cardiovascular diseases is to obtain qualitative and quantitative descriptions of the human vascular system. To understand the fundamental mechanisms of this complex physiological system, mathematical modeling of the human vascular system was initiated in the 1950's. Among the commonly used models, the hyperbolic PDE model in [1, 4] has attracted considerable attention in recent years. It is well known that such a model is capable of capturing many complicated physiological phenomena associated with the human vascular system. In particular, this model has been demonstrated to be useful in fast real-time computations when quick answers are needed in the cases when the geometry of the patient's vessel can be approximated by a straight, narrow compliant wall channel. The majority of mathematical research conducted on this model has been focusing on numerical simulations and few rigorous analytical results are available [4, 18, 19]. In the current paper, we develop a rigorous analytical framework which will further our understanding of the quantitative and qualitative behavior of the solutions to the

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model and extract more crucial information of the underlying processes. It is urgent to explore the structure and morphology of blood vessels and the dynamics of the blood flow in order to develop new therapeutic strategies that target diagnosis and prevention of cardiovascular diseases such as stroke and heart failure.

Consider the quasilinear hyperbolic PDE model simulating blood flow through cylindrical sections of the cardiovascular system or through the network of blood vessel [4]

$$\begin{cases} A_t + m_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ m_t + \Big(\alpha \dfrac{m^2}{A}\Big)_x + \dfrac{A}{\rho} p(A)_x = -\mu \dfrac{m}{A} \end{cases} \tag{1}$$

which describes conservation of mass and balance of axial momentum in terms of the cross-sectional area $A(x,t)$ and the flow rate $m(x,t) = A(x,t)u(x,t)$, where u denotes the averaged axial velocity $v_x(x,r,t)$ across the cross-section of the vessel of radius $R(x,t)$:

$$u(x,t) = \frac{1}{R(x,t)^2} \int_0^{R(x,t)} 2r v_x(x,r,t) dr.$$

In system (1), $\rho > 0$ denotes fluid density which is assumed to be a constant throughout this paper. α is the ratio of the averaged axial momentum and the momentum of the averaged flow

$$\alpha = \frac{1}{R^2 u^2} \int_0^R 2r v_x^2 dr$$

and it easy to derive that $\alpha \geq 1$. $\mu > 0$ is proportional to the viscosity of the fluid

$$\mu = \frac{2\alpha}{\alpha - 1} \nu$$

where ν is the viscosity of the fluid. Fluid pressure is denoted by $p(A)$. It is through this term that modeling of vessel wall mechanics comes into play (c.f. [4]). We will be assuming

$$p(A) = G_0 \bigg(\Big(\frac{A}{A_r} \Big)^{\frac{\beta}{2}} - 1 \bigg)$$

where the constants $A_r > 0$ is the reference cross-sectional area, G_0 describes the stiffness of the vessel wall and $\beta > 0$ captures the linearity/nonlinearity of the stress-strain response. When $\beta = 1$, one gets the well-known Law of Laplace. The case $\beta = 2$ captures well the nonlinear pressure-radius relationship observed in experiments [29]. In this paper, we will consider the general case $\beta > 0$.

The model (1) was derived from the three dimensional Navier-Stokes equations of a viscous, incompressible, Newtonian fluid flow in a cylindrical tube by assuming axially symmetric flow in a cylindrical tube with elastic walls and with small aspect ratio (c.f. [1, 2, 3, 4, 6, 12, 13, 27]). For a detailed analysis of the conditions under which system (1) is a good approximation of the full, three dimensional model, we refer the readers to [5].

One of the most attracting features of (1) is its ability to describe the propagation of wave-like solutions, such as pulse generated by heart which is one of the most important physiological phenomena of the human vascular system. A huge body of literature is contributed to numerical simulations of this characteristic feature of the model [2, 3, 4, 5, 6, 12, 13, 27, 29].

Comparing to numerical simulations, breakthrough on analytical studies of the model has been impeded. Only mild progress has been made in this direction. In a recent work, Čanić and Kim [4] showed that, for $\alpha = 1$, when initial data are small and smooth, smooth solutions exist globally in time subject to a pulsatile boundary condition. The most recent findings in this area generalized this result by rigorously studying the critical thresholds and the influence of the viscous damping on the solutions to the Cauchy problem of the model with α is close to 1 in [18] with the new findings: (1) For $\mu > 0$ the class of data that gives rise to smooth solutions is richer than that for the case $\mu = 0$; and (2) For the physiologically relevant data that give rise to shock formation, shock formation is delayed in time for the case when $\mu > 0$. Our own studies (c.f. [19]) contributed to this contemporary body of knowledge by showing that when initial data are small and smooth, then smooth solutions to an initial-boundary value problem of the model exist globally in time, and converge to constant equilibrium states, determined by the initial data, exponentially as time goes to infinity, in the biologically relevant regime $\alpha \geq 1$. Although the work cited here provides some analytical results that target wellposedness and breakdown of smooth solutions of the model, detailed quantitative behavior of weak solutions to the model is completely unknown in the field.

In conservative form, system (1) reads

$$\begin{cases} A_t + m_x = 0, \\ m_t + \alpha\Big(\dfrac{m^2}{A}\Big)_x + P(A)_x = -\mu\dfrac{m}{A}. \end{cases} \tag{2}$$

The pressure $P(A)$ is determined by

$$P(A) \equiv P_0 A^\gamma \tag{3}$$

where

$$P_0 = \frac{\beta G_0}{(\beta+2)\rho A_r^{\beta/2}} \quad \text{and} \quad \gamma = \frac{\beta}{2} + 1 > 1.$$

System (2) is hyperbolic with two characteristic speeds

$$\lambda_1 = \alpha\frac{m}{A} - \sqrt{\alpha(\alpha-1)\frac{m^2}{A^2} + P'(A)}, \quad \lambda_2 = \alpha\frac{m}{A} + \sqrt{\alpha(\alpha-1)\frac{m^2}{A^2} + P'(A)},$$

and the corresponding right eigenvectors

$$\vec{r}_1 = (1, \lambda_1)^{\mathrm{T}}, \quad \vec{r}_2 = (1, \lambda_2)^{\mathrm{T}}.$$

Furthermore, (2) is strictly hyperbolic at the point away from vacuum where the two characteristics coincide. This simple looking system involves three interacting mechanisms: nonlinear convection, lower-order dissipation of damping, and resonance due to vacuum. By vacuum in current context we mean $A = 0$. Because of the hyperbolicity and nonlinearity of (2), it has been shown in [4, 18] that smooth solutions will break down due to shock formation if the initial data are large or rough. It is also known that, in the case for $\alpha = 1$ which reduces (2) to a gas dynamics type system, if the initial data contain vacuum states, smooth solutions will break down even *before shock formation* due to resonance caused by the vacuum (see e.g. [22, 23, 24]). In addition, although the viscous damping helps prevent development of singularity when the initial data are small, it destroys the self-similarity of the system, which is crucial for constructing large solutions. Therefore, when the initial data are large, rough and contain vacuum, the interaction of these mechanisms makes the problem physically interesting and mathematically challenging.

In this paper, we consider (2) on bounded domains with zero flow rate $m = 0$ on the boundary. Without loss of generality, we consider the system on interval $[0, 1]$ supplemented by the initial and boundary conditions

$$\begin{cases} (m, A)(x, 0) = (m_0, A_0)(x) \in L^\infty([0, 1]), \ A_0(x) \geq 0, \\ m(0, t) = m(1, t) = 0, \quad t \geq 0. \end{cases} \tag{4}$$

We will study the global existence and large-time asymptotic behavior of large entropy weak solutions (see Definition 2.1) to the initial-boundary value problem (2)–(4). A real-world physiological scenario that corresponds to the entropy weak solutions with zero flow rate on the boundary of the domain is the blood flow in an ischemic area, where blood supply is inadequate due to blockage of the blood vessels causing irregularity in the shape of the blood vessels and the dynamics of the blood flow. It is also worthy of mentioning that the blockage of blood flow in patients is often needed in many surgical procedures. For example, in order to perform certain types of trauma surgery in the injured area of patients, surgical tubing is applied to stop bleeding. Therefore, the initial-boundary value problem (2)–(4) describes the blood flow within the blocked area.

We will first study the case $\alpha = 1$ and $\mu > 0$. In this case, (2) can be reduced to a gas dynamics type system with a damping term where A appears in the denominator. This damping term prevents us from using the arguments in [28] to construct global weak solutions to (2) if the initial data contain vacuum states. Indeed, in [28], the authors constructed global entropy weak solutions to the damped compressible Euler equations in bounded domains based on the Godunov (fractional step) scheme and the compensated compactness theory. One of the crucial steps in [28] is to analyze the Riemann invariants associated with the non-homogeneous system in order to derive the desired uniform L^∞ bound of the approximate solutions constructed via the Godunov scheme. However, with A standing in the denominator in the second equation of (2) and the possible appearance of vacuum states, the modified Riemann invariants of (2) have an obvious disadvantage that forces the size of the discrete time step to be zero. This is an unsurpassable obstacle if one persists to use the Godunov scheme to construct approximate solutions for (2). We also remark that, the dissipation term $-\mu\frac{m}{A}$ was excluded from the general scenario considered in [10], where the Godunov scheme was used to construct global entropy weak solutions to the compressible Euler equations. Therefore, we turn to other methods to tackle this problem. Among several approaches used to construct large solutions in hyperbolic conservation/balance laws, we use the method of vanishing viscosity and the theory of compensated compactness to prove the global existence of entropy weak solutions to (2)–(4). Such a program has been successfully implemented in numerous problems in the literature to study the global existence of entropy weak solutions containing vacuum. We refer the readers to [8, 10, 11, 20, 21] for the gas dynamics equations, and to [14] for general wave equations.

We then study the long-time behavior of the large entropy weak solutions to (2)–(4). Based on the entropy principle, we will show that the solutions converge to a constant state, determined by the initial data, exponentially as time goes to infinity due to the viscous damping and boundary effects. Since the solutions do not have any differentiability, the proof starts with the introduction of an antiderivative through mass conservation law in order to gain differentiability. Then, accurate estimation of the so-called entropy inequality, application of the theory of divergence-measure fields [7], and Poincaré's inequality will be implemented to the energy inequalities to generate a desired estimate leading to the exponential decay of the solution. An elementary lemma which is due to the convexity of the pressure $P(A)$ defined in (3) plays an important role in the control of singularity near vacuum. The appearance of A in the denominator brings considerable difficulties into the analysis which distinguishes the problem from the case for fractional damping and requires more elaboration. It should be pointed out that the long time behavior holds for large rough solutions. Hence, regardless of the wild behavior of the solution during intermediate time, it will eventually calm down and become stable. This is one of the remarkable advantages of our results.

The second part of the present paper is contributed to the study of the case $\alpha > 1$ and $\mu > 0$, which corresponds to the more physical case. Due to the change of the intrinsic system structure, the arguments used in the case $\alpha = 1$ will fail when $\alpha > 1$. For large rough solutions to (2), the intrinsic structural change caused by the value of α is drastic. The main reason is that the explicit expressions of the so-called Riemann invariants corresponding to the hyperbolic conservation laws (i.e., (2) with $\mu = 0$) are unknown when $\alpha > 1$. The Riemann invariants play crucial roles in the construction of large solutions to (2) and are fundamentally important in the development of the compensated compactness frameworks in [10, 11, 20, 21]. This structural change indeed brings tremendous difficulty in the analysis of the global existence of entropy weak solutions to (2). Therefore, we will not investigate the existence of entropy weak solutions when $\alpha > 1$ which is beyond the scope of the present paper. However, we will see in Section 3 that, by recalculating the entropy-entropy flux pairs and properly modifying of the definition of an entropy weak solution, and by assuming the existence of such a solution, the constant state determined by the initial data is still a global attractor for such a solution. Therefore, our result presents a general framework for long-time behavior of entropy weak solutions to (2) with $\alpha > 1$ and $\mu > 0$, which may be used in other problems.

The physiological relevance of the theoretical results obtained in this paper is demonstrated in the following. It is well-known that the blockage of blood flow in certain area occurs due to illness such as ischemia or due to necessary blockages in surgical procedures. At the boundary of the blocked area the blood flow velocity is zero which corresponds to our zero velocity boundary conditions. A question of practical importance is: how long does it take the blood flow rate in a blocked area to decrease to a critical level, below which the patient cannot recover? Our result on the asymptotic behavior provides qualitative answer which helps understanding the physiological scenarios and developing new and effective therapeutic strategies dealing with such problems.

The rest of the paper is organized as follows. We study global existence and asymptotic behavior of entropy weak solutions to (2)–(4) in Section 2 for the case $\alpha = 1, \mu > 0$. The investigation of (2)–(4) for physiologically reasonable values of $\alpha > 1$, will be carried out in Section 3. We will conclude the paper by a discussion in Section 4.

2. L^∞ entropy weak solutions ($\alpha = 1, \mu > 0$).

In this section, we study the existence and the global existence and asymptotic behavior of the L^∞ entropy weak solutions to (2)–(4) with $\alpha = 1$ and $\mu > 0$ for large, rough initial data containing vacuum. Allowing the presence of vacuum state in the initial data makes the problem physically interesting and mathematically challenging.

Since we are interested in the solutions to (2) for fixed values of $P_0 > 0$ and $\mu > 0$, without loss of generality, we take $P_0 = \frac{1}{\gamma}$ and $\mu > 0$ throughout this section. Hence, we consider

$$\begin{cases} A_t + m_x = 0, \\ m_t + \Big(\dfrac{m^2}{A}\Big)_x + P(A)_x = -\mu \dfrac{m}{A} \end{cases} \tag{5}$$

for $x \in [0,1]$, $\quad t > 0$, with the initial and boundary conditions

$$\begin{cases} m|_{x=0} = m|_{x=1} = 0, \quad t \geq 0, \\ (A,m)(x,0) = (A_0,m_0)(x) \in L^\infty([0,1]), \\ 0 \leq A_0(x) \leq M, \quad |m_0(x)| \leq M A_0(x), \quad x \in [0,1], \\ \displaystyle\int_0^1 A_0(x)dx = \bar{A} > 0 \end{cases} \tag{6}$$

for some constant $M > 0$ and we define $u_0 = m_0/A_0$, where the last condition is imposed to avoid the trivial case $A \equiv 0$.

We now give the definition of weak solutions to (5)–(6).

Definition 2.1. For every $T > 0$, we define an L^∞ weak solution to (5)–(6) to be a pair of bounded measurable functions $\vec{v}(x,t) = \big(A(x,t), m(x,t)\big)$ satisfying the following pair of integral identities:

$$\int_0^T \int_0^1 (A\psi_t + m\psi_x)\ dx\ dt + \int_0^1 A_0 \psi|_{t=0} dx = 0,$$

$$\int_0^T \int_0^1 \bigg(m\psi_t + \Big(\frac{m^2}{A} + P(A)\Big)\psi_x - \mu \frac{m}{A}\psi \bigg)\ dx\ dt + \int_0^1 m_0 \psi|_{t=0} dx = 0$$

for all $\psi \in C^\infty(I_T)$ satisfying $\psi(x,T) = 0$ for $0 \leq x \leq 1$ and $\psi(0,t) = \psi(1,t) = 0$ for $t \geq 0$, where $I_T = (0,1) \times (0,T)$, and $\frac{m}{A}$ is bounded when $A \to 0$. Moreover, the initial and boundary conditions in (6) are satisfied in the sense of trace and section as defined in [14].

An interesting feature of nonlinear hyperbolic conservation/balance laws is that when weak solution is considered, the uniqueness is usually lost. In order to select the physically relevant solutions, one often imposes entropy admissible conditions. We now define the entropy and entropy flux pairs.

Definition 2.2. A pair of mappings $\eta :\ \mathbb{R}^2 \to \mathbb{R}$ and $q :\ \mathbb{R}^2 \to \mathbb{R}$ is called an entropy-entropy flux pair for the hyperbolic system of balance laws $\vec{v}_t + \vec{f}(\vec{v})_x = \vec{g}(\vec{v})$ $(v \in \mathbb{R}^2)$ if it satisfies the following equation

$$\nabla q = \nabla \eta \nabla \vec{f}.$$

Let $\tilde{\eta}(A, m/A) = \eta(A,m)$. If $\tilde{\eta}(0,u) = 0$, then η is called a weak entropy. Among all entropies, the most natural entropy is the so-called mechanical energy

$$\eta_e(A,m) = \frac{m^2}{2A} + \frac{A^\gamma}{\gamma(\gamma-1)}$$

which plays a very important role in getting estimates from entropy dissipation. It is easy to check that η_e is a weak and convex entropy. Using Definition 2.2, we now give the definition of L^∞ entropy weak solutions to (5)–(6).

Definition 2.3. The weak solution $\vec{v}(x,t) = \big(A(x,t), m(x,t)\big)$ defined in Definition 2.1 is said to be entropy admissible if for any convex entropy η and the associated entropy flux q, the following entropy inequality holds

$$\eta_t + q_x + \eta_m \mu \frac{m}{A} \leq 0 \tag{7}$$

in the sense of distribution.

The following two theorems are the main results of this section. The first theorem gives the global existence of L^∞ entropy weak solutions to (5)–(6).

Theorem 2.4. *Suppose that the initial data $(A_0, m_0) \in L^\infty([0,1])$ satisfy the conditions*

$$0 \leq A_0(x) \leq M, \quad A_0 \not\equiv 0, \quad |m_0(x)| \leq M A_0(x)$$

for some positive constant M. Then, for $\gamma > 1$, the initial-boundary value problem (5)–(6) has a global entropy admissible weak solution $(A(x,t), m(x,t))$, as defined in Definitions 2.1–2.3, satisfying the following estimates:

$$0 \leq A(x,t) \leq C_1, \quad |m(x,t)| \leq C_1 A(x,t) \quad a.e. \ \ in \ \ [0,1] \times [0, +\infty)$$

for a constant $C_1 > 0$ which is independent of time.

Concerning the long-time behavior of the solution obtained in Theorem 2.4, we have

Theorem 2.5. *Suppose $\displaystyle\int_0^1 A_0(x)dx = \bar{A} > 0$. Let (A,m) be any L^∞ entropy weak solution to (5)–(6) defined in Definitions 2.1–2.3, satisfying the estimates*

$$0 \leq A(x,t) \leq C_1 < \infty, \quad |m(x,t)| \leq C_1 A(x,t), \quad a.e. \ \ in \ \ [0,1] \times [0, +\infty)$$

for some time-independent constant $C_1 > 0$. Then, there exist constants C_2, $\delta > 0$ depending on γ, $\bar{A}, C_1$, and initial data such that

$$\big\|(A - \bar{A}, m)(\cdot, t)\big\|^2_{L^2([0,1])} \leq C_2 e^{-\delta t} \quad \text{as } t \to \infty.$$

The proof of Theorem 2.4 is in the spirit of [8, 11, 14]. We construct the approximate solutions of (5)–(6) by the method of vanishing viscosity. The uniform ε-independent upper bound and the ε, T-dependent lower bound of the approximate solutions are established by using the invariant region theory [9, 25] and the arguments in [8, 11] respectively, which lead to the global existence of smooth approximations for any fixed diffusion rate $\varepsilon > 0$. The compensated compactness frameworks established in [10, 11, 20, 21] are then applied to the sequence of approximate solutions to obtain the strong convergence of the approximate solutions in order to get a global weak entropy solution to (5)–(6). The initial and boundary conditions are satisfied in the sense of trace and section which are clearly stated in [14], see also [15, 28, 30], and we will omit the details.

In the proof of Theorem 2.5, an elementary lemma, which is due to the convexity of the pressure $P(A)$ defined in (3), plays an important role in the control of singularities near a vacuum state. Due to the roughness of the solution, elementary energy estimates cannot be performed in this situation. Instead, we will start the proof with defining an anti-derivative through the mass conservation in order to gain differentiability. The first step of energy estimate will be carried out on the equation satisfied by the anti-derivative, which is a nonlinear wave equation with source terms. Then the entropy inequality satisfied by the weak solution will be implemented in order to deal with nonlinearities in the resulting energy inequality obtained from the first step. Although the initial and boundary conditions are satisfied in the weak sense, the theory of divergence measure fields [7] guarantees the eligibility of the calculations. Finally, Poincaré's inequality on bounded domains will be utilized to yield the exponential decay of the solution.

2.1. **Global existence.**

There are several approaches to construct entropy weak solutions to (5)–(6). Our proof is based on the approach of viscosity approximation as in [8, 11]. To construct global L^∞ entropy weak solutions to (5), the following program is to be carried out:

- Construct smooth approximate solutions via viscous perturbations of the hyperbolic system and obtain a uniform ε-independent L^∞ upper bound and a (ε, T)-dependent lower bound of the sequence of approximate solutions, to get global smooth solutions to the viscous equations.
- Show that $\eta_t(\vec{v}^\varepsilon) + q_x(\vec{v}^\varepsilon)$ is compact in H^{-1}_{loc} and apply the *div-curl* lemma in [26] to reduce the Young measure associated with the flux function to Dirac measure, to conclude that the sequence converges strongly in the L^∞ topology.

The first bullet can be accomplished by applying standard theory on parabolic equations together with the invariant region theory [9, 25]. When $\gamma > 1$, the compensated compactness frameworks established in [10, 11, 20, 21] are sufficient to conclude the second bullet.

Step 1. *Construction of approximate solutions and L^∞ bounds.* Following the general procedure of vanishing viscosity in [8, 11, 14], let us consider the artificial viscous approximation to the original hyperbolic system (5)

$$\begin{cases} A^\varepsilon_t + m^\varepsilon_x = \varepsilon A^\varepsilon_{xx}, \\ m^\varepsilon_t + \left(\dfrac{(m^\varepsilon)^2}{A^\varepsilon}\right)_x + P(A^\varepsilon)_x = -\mu \dfrac{m^\varepsilon}{A^\varepsilon} + \varepsilon m^\varepsilon_{xx} \end{cases} \tag{8}$$

where $x \in [0,1]$, $t > 0$ and $\varepsilon > 0$, with the initial and boundary conditions:

$$\begin{cases} m^\varepsilon|_{x=0} = m^\varepsilon|_{x=1} = 0, \\ A^\varepsilon_x|_{x=0} = A^\varepsilon_x|_{x=1} = 0; \\ (A^\varepsilon, m^\varepsilon)(x,0) = (A^\varepsilon_0, m^\varepsilon_0)(x) \end{cases} \tag{9}$$

where the initial data satisfy

$$\begin{aligned} &A^\varepsilon_0 = B^\varepsilon_0 + \varepsilon, \quad B^\varepsilon_0 \in C^\infty_0([0,1]), \quad 0 \le B^\varepsilon_0(x) \le \|A_0\|_{L^\infty}, \\ &m^\varepsilon_0 = A^\varepsilon_0 u^\varepsilon_0, \quad u^\varepsilon_0 \in C^\infty_0([0,1]), \quad |u^\varepsilon_0(x)| \le \|u_0\|_{L^\infty} \end{aligned} \tag{10}$$

and B^ε_0 converges to A_0 in the weak* topology of $L^\infty([0,1])$ and u^ε_0 converges to u_0 in the strong topology of $L^2([0,1])$ as $\varepsilon \to 0$. We remark that, the parabolic boundary conditions in (9) are compatible with the hyperbolic boundary condition (4) according to [14], and (4) will be recovered in the limiting process as $\varepsilon \to 0$. Under this setting, it holds that $(A^\varepsilon_0, m^\varepsilon_0)$ converges to (A_0, m_0) in the weak* topology of $L^\infty([0,1])$ as $\varepsilon \to 0$.

The corresponding Riemann invariants of the hyperbolic system associated with (8) are

$$w^\varepsilon = \frac{m^\varepsilon}{A^\varepsilon} + \frac{(A^\varepsilon)^\theta}{\theta}, \quad z^\varepsilon = \frac{m^\varepsilon}{A^\varepsilon} - \frac{(A^\varepsilon)^\theta}{\theta}, \quad \text{where} \quad \theta = \frac{\gamma - 1}{2}.$$

Due to the dissipative structure of (8) and the theory of invariant region by Chueh, Conley and Smoller [9] and Proposition 4.2 by Marcati and Rubino [25], the set

$$\Sigma = \{(A^\varepsilon, m^\varepsilon) \mid 0 \le w^\varepsilon - z^\varepsilon,\ z^\varepsilon \ge z^\varepsilon_0,\ w^\varepsilon \le w^\varepsilon_0\}$$

is an invariant region for (8).

From the invariant region Σ, we derive

$$0 \le A^{\varepsilon}(x,t) \le M_1, \quad |m^{\varepsilon}(x,t)| \le M_1 A^{\varepsilon}(x,t) \tag{11}$$

for some constant $M_1 > 0$ independent of ε. Hence, the uniform ε-independent L^{∞} upper bound of the approximate solutions is achieved.

The local smooth solutions of (8)–(9) was obtained in [8]. In order to extend any local smooth solution of (8)–(9) to a global one, we need to show that A^{ε} is bounded away from zero, i.e.,

$$A^{\varepsilon}(x,t) \ge \delta(\varepsilon, T) > 0, \quad \forall (x,t) \in (0,1) \times (0,T) \tag{12}$$

where $T > 0$ is the lifespan of any local smooth solution, and $\delta(\varepsilon, T) > 0$ is a constant depending on ε and T.

The proof of (12) is in the spirit of [8]. Consider

$$A^{\varepsilon}_t + (A^{\varepsilon} u^{\varepsilon})_x = \varepsilon A^{\varepsilon}_{xx} \tag{13}$$

where $u^{\varepsilon}(x,t)$ is a known function (local smooth solution to the approximate problem) satisfying $|u^{\varepsilon}(x,t)| \le M_1$, see (11). Multiplying (13) by $v'(A^{\varepsilon})$ with $v(A^{\varepsilon}) = 1/A^{\varepsilon}$ we obtain

$$v_t - \varepsilon v_{xx} = (u^{\varepsilon} v)_x + v''(A^{\varepsilon})(A^{\varepsilon} u A^{\varepsilon}_x - \varepsilon (A^{\varepsilon}_x)^2). \tag{14}$$

Notice that the last term on the RHS of (14) satisfies

$$v''(A^{\varepsilon})(A^{\varepsilon} u A^{\varepsilon}_x - \varepsilon (A^{\varepsilon}_x)^2) = 2(A^{\varepsilon})^{-3}(A^{\varepsilon} u A^{\varepsilon}_x - \varepsilon (A^{\varepsilon}_x)^2) \le \frac{(u^{\varepsilon})^2}{2A^{\varepsilon}\varepsilon} = \frac{v(u^{\varepsilon})^2}{2\varepsilon}.$$

Then it holds that

$$v_t - \varepsilon v_{xx} \le (u^{\varepsilon} v)_x + \frac{v(u^{\varepsilon})^2}{2\varepsilon}.$$

By the comparison principle, we know that the solution to (14) is dominated by the solution to the following initial-boundary value problem with homogeneous Neumann boundary conditions

$$\begin{cases} g_t - \varepsilon g_{xx} = (u^{\varepsilon} g)_x + \dfrac{g(u^{\varepsilon})^2}{2\varepsilon}, \\ g(x,0) = v(A^{\varepsilon}_0(x)), \\ g_x|_{x=0} = g_x|_{x=1} = 0. \end{cases} \tag{15}$$

Therefore, to show (12), all we have to do is to prove that

$$g(x,t) \le N(\varepsilon, T), \quad \forall\ (x,t) \in [0,1] \times [0,T] \tag{16}$$

for some positive constant $N(\varepsilon, T)$ depending on ε and T.

In view of the initial condition (10), we know that $0 < g(x,0) = v(A^{\varepsilon}_0(x)) \le 1/\varepsilon$. From local existence results of (15), we know that there must be a $t_0 \in (0,T]$, such that

$$\sup_{0 \le t \le t_0} \|g(\cdot, t)\|_{L^{\infty}(0,1)} \le 3/\varepsilon. \tag{17}$$

Following the arguments in Section 4 of [8], we consider the operator $\mathcal{L}$ in $\mathcal{G}_\tau = L^\infty([t_0, t_0+\tau]\times[0,1])$, with $t_0+\tau \le T$, given by

$$\mathcal{L}(h) = K^\varepsilon(\cdot, t-t_0) * \tilde{g}(\cdot, t_0) + \int_{t_0}^t K^\varepsilon(\cdot, t-s) * \frac{\tilde{h}(\bar{u}^\varepsilon)^2}{2\varepsilon}(\cdot, s)ds - \int_{t_0}^t \partial_x K^\varepsilon(\cdot, t-s) * (\bar{u}^\varepsilon \tilde{h})(\cdot, s)ds$$

where K^ε is the fundamental solution of the heat equation $w_t = \varepsilon w_{xx}$, that is

$$K^\varepsilon(x,t) = \frac{1}{\sqrt{4\pi\varepsilon t}} e^{-\frac{x^2}{4\varepsilon t}}, \quad \|K^\varepsilon\|_1 = 1, \quad \|K^\varepsilon_x\|_1 = \frac{1}{\sqrt{\pi\varepsilon t}}$$

where $\tilde{\xi}$ denotes the even, periodic (with period 2) extension of a function $\xi : [0,1] \to \mathbb{R}$ satisfying $\xi_x|_{x=0} = \xi_x|_{x=1} = 0$, that is

$$\begin{cases} \tilde{\xi}(x) = \xi(x), & 0 \le x \le 1, \\ \tilde{\xi}(-x) = \tilde{\xi}(x), & x \in \mathbb{R}, \\ \tilde{\xi}(x+2n) = \tilde{\xi}(x), & x \in \mathbb{R}, \quad n \in \mathbb{Z} \end{cases}$$

and $\bar{\xi}$ denotes the odd, periodic (with period 2) extension of a function $\xi : [0,1] \to \mathbb{R}$ satisfying $\xi|_{x=0} = \xi|_{x=1} = 0$, that is

$$\begin{cases} \bar{\xi}(x) = \xi(x), & 0 \le x \le 1, \\ \bar{\xi}(-x) = -\bar{\xi}(x), & x \in \mathbb{R}, \\ \bar{\xi}(x+2n) = \bar{\xi}(x), & x \in \mathbb{R}, \quad n \in \mathbb{Z}. \end{cases}$$

The operator $\mathcal{L}$ is a contraction mapping in $\mathcal{G}_\tau$ if

$$2\max\left\{\frac{M_1^2}{2},\ \frac{2M_1}{\sqrt{\pi}},\ 1\right\}\sqrt{\frac{\tau}{\varepsilon}} < 1$$

where M_1 is a constant such that $|u^\varepsilon| \le M_1$. Now we take

$$\tau_0 = \frac{\varepsilon}{9\bar{M}^2} > 0 \tag{18}$$

where

$$\bar{M} = 2\max\left\{\frac{M_1^2}{2},\ \frac{2M_1}{\sqrt{\pi}},\ 1\right\}.$$

Consider the problem (15) on $[0, t_0+\tau_0]$. Let

$$G(t_0) = \sup_{0\le t\le t_0} \|g(\cdot,t)\|_{L^\infty(0,1)}.$$

Then from (17), we have

$$G(t_0) \le \frac{3}{\varepsilon}. \tag{19}$$

We now show that there exists a constant $N(t_0,\tau_0) > G(t_0)$ such that, if $h \in \mathcal{G}_{\tau_0}$ and satisfies

$$\|h(\cdot,t)\|_{L^\infty(0,1)} \le N(t_0,\tau_0), \quad 0 \le t \le t_0+\tau_0, \tag{20}$$

then $\mathcal{L}(h)$ also satisfies the same estimate. Indeed,

$$\begin{aligned}\|\mathcal{L}(h)(\cdot,t)\|_{L^\infty(0,1)} &\le G(t_0) + \frac{M_1^2}{2} N(t_0,\tau_0)\frac{\tau_0}{\varepsilon} + \frac{2M_1}{\sqrt{\pi}} N(t_0,\tau_0)\sqrt{\frac{\tau_0}{\varepsilon}} \\ &\le G(t_0) + \bar{M} N(t_0,\tau_0)\sqrt{\frac{\tau_0}{\varepsilon}} \\ &= G(t_0) + \frac{1}{3} N(t_0,\tau_0)\end{aligned}$$

where we have used (18). Therefore, we conclude that the assertion (20) is true by choosing

$$N(t_0,\tau_0) = 3G(t_0).$$

Since $\mathcal{L}$ is a contraction mapping in $\mathcal{G}_{\tau_0}$, the estimate (20) must also hold for its fixed point. It is easy to see that its fixed point is the solution of equation (15).

Next, observe that the constant $\bar{M}$ is independent of ε and τ_0. Then, by a bootstrap argument one can show that

$$\|g(\cdot,t)\|_{L^\infty} \le 3^n G(t_0), \quad \forall\, 0 \le t \le T$$

where $n = \lceil \frac{T-t_0}{\tau_0} \rceil$. Recalling (19), we then have

$$\|g(\cdot,t)\|_{L^\infty} \le \frac{3^{n+1}}{\varepsilon}, \quad \forall\, 0 \le t \le T$$

which implies that

$$A^\varepsilon(x,t) = \frac{1}{v(x,t)} \ge \frac{1}{g(x,t)} \ge \frac{\varepsilon}{3^{n+1}} > 0, \quad (x,t) \in [0,1]\times[0,T].$$

Thus by choosing $\delta(\varepsilon,T) = \frac{\varepsilon}{3^{n+1}}$, (12) is proved.

This, together with (11), gives

$$0 < A^\varepsilon(x,t) \le M_1, \quad |m^\varepsilon(x,t)| \le M_1 A^\varepsilon(x,t). \tag{21}$$

The global existence of smooth solutions to (8)–(9) follows from the above local existence result and the *a priori* estimate (21). This completes the first step.

Step 2. H^{-1}_{loc} *compactness and strong convergence.* In order to obtain global solutions to (5), it suffices to show the strong convergence of the sequence of approximate solutions $(A^\varepsilon, m^\varepsilon)$ as $\varepsilon \to 0$, extracting to a subsequence if necessary. However, with the uniform L^∞ estimate of the approximate solutions in hand, one can only guarantee the convergence in the weak* topology, which is insufficient to handle the nonlinear terms in (5). One then applies the compensated compactness frameworks established in [10, 11, 20, 21] for the gas dynamics equations to conclude that there exist functions $(A,m)(x,t) \in L^\infty((0,1)\times(0,\infty))$ such that

$$(A^\varepsilon, m^\varepsilon) \to (A,m) \ \text{ a.e. in } (0,1)\times(0,\infty) \text{ as } \varepsilon \to 0,$$

and satisfy (c.f. (21))

$$0 \le A(x,t) \le M_1, \quad |m(x,t)| \le M_1 A(x,t), \quad \text{a.e. in } (0,1)\times(0,\infty).$$

One also defines $u(x,t) = m(x,t)/A(x,t)$ a.e.. It is straightforward to verify that (A,m) is a weak solution to the original system (5) and satisfies the entropy inequality (7) in the sense of distribution. Furthermore, the solution satisfies the initial and boundary conditions in the sense of trace and section, see [14, 15, 28, 30]. This completes the proof of Theorem 2.4.

2.2. **Long-time behavior.** We now study the long-time behavior of the entropy weak solution obtained in Theorem 2.4. The following lemma will play an important role in controlling the singularity of the solution near a vacuum state (c.f. [16, 28]).

Lemma 2.6. *Let* $0 \le A \le \Lambda < \infty$ *and* $0 < a < \bar{A} < \infty$. *Then there are positive constants* C_3, C_4, C_5 *depending only on* Λ, *a and* $\bar{A}$ *such that*

(1) $C_3(A-\bar{A})^2 \le P(A) - P(\bar{A}) - P'(\bar{A})(A-\bar{A})$;

(2) $C_4(A-\bar{A})^2 \le \left[P(A) - P(\bar{A})\right](A-\bar{A})$;

(3) $P(A) - P(\bar{A}) - P'(\bar{A})(A-\bar{A}) \le C_5\left[P(A) - P(\bar{A})\right](A-\bar{A})$

where P *is defined in (3).*

The lemma is easily proved by using the convexity of P and $\bar{A} > a > 0$.

We now turn to the proof of Theorem 2.5. The proof follows closely the corresponding one in [28], carried out for a different damping term. We separate the proof into several steps.

Step 1. *Definition of an anti-derivative and reformulation.* Since (A, m) is conjectured to converge to $(\bar{A}, 0)$, we set

$$w = A - \bar{A}$$

which satisfy

$$\begin{cases} w_t + m_x = 0, \\ m_t + \left(\dfrac{m^2}{A}\right)_x + \left[P(A) - P(\bar{A})\right]_x + \mu \dfrac{m}{A} = 0 \end{cases} \tag{22}$$

where $x \in [0,1]$, $\quad t > 0$, and

$$\int_0^1 w(x,t)dx = 0.$$

Define anti-derivative

$$y = -\int_0^x w(\sigma,t)d\sigma$$

which implies that

$$y_x = -w = \bar{A} - A, \quad y_t = m. \tag{23}$$

Since

$$\int_0^1 A(x,t)dx = \int_0^1 A_0(x)dx = \bar{A}$$

we have

$$y(0) = y(1) = 0.$$

Therefore the second equation of (22) turns into

$$y_{tt} + \left(\frac{y_t^2}{A}\right)_x + \left[P(A) - P(\bar{A})\right]_x + \mu\frac{y_t}{A} = 0. \tag{24}$$

We shall work on (24) in what follows.

Step 2. *Preliminary estimates.* Taking L^2 inner product of (24) with y, we have

$$\begin{aligned} &\frac{d}{dt}\left(\int_0^1 y_t y dx\right) - \int_0^1 y_t^2 dx + \int_0^1 \left[P(A) - P(\bar{A})\right](A-\bar{A})dx \\ &= \int_0^1 \frac{y_t^2}{A} y_x dx - \mu\int_0^1 \frac{y y_t}{A} dx. \end{aligned}$$

Since by the definition of entropy weak solutions, Definitions 2.1–2.3, $A, u = m/A$, $m = y_t \in L^\infty[0,1]$ and $y_x = \bar{A} - A$, we have

$$\frac{d}{dt}\left(\int_0^1 y_t y dx\right) + \int_0^1 \left[P(A) - P(\bar{A})\right](A - \bar{A})dx = \int_0^1 y_t^2 \frac{\bar{A}}{A} dx - \mu \int_0^1 \frac{y y_t}{A} dx. \quad (25)$$

Since $0 \le A \le C_1$, the first term on the RHS of (25) is estimated in the following

$$\int_0^1 y_t^2 \frac{\bar{A}}{A} dx = \int_0^1 y_t^2 \frac{\bar{A}A}{A^2} dx \le \int_0^1 \bar{A} C_1 \frac{y_t^2}{A^2} dx. \quad (26)$$

The second term on the RHS of (25) is estimated by Cauchy-Schwartz inequality as

$$\left|-\mu \int_0^1 \frac{y y_t}{A} dx\right| \le \frac{\mu^2}{C_4} \int_0^1 \frac{y_t^2}{A^2} dx + \frac{C_4}{4} \int_0^1 y^2 dx, \quad (27)$$

where C_4 is given in Lemma 2.6.

Combining (25)–(27), we then have

$$\begin{aligned} &\frac{d}{dt}\left(\int_0^1 y_t y dx\right) + \int_0^1 \left[P(A) - P(\bar{A})\right](A - \bar{A})dx \\ &\le \int_0^1 \left(\bar{A}C_1 + \frac{\mu^2}{C_4}\right) \frac{y_t^2}{A^2} dx + \frac{C_4}{4} \int_0^1 y^2 dx. \end{aligned} \quad (28)$$

By splitting the second term on the LHS of (28) into two parts and using Lemma 2.6 (2), we have

$$\begin{aligned} &\frac{C_4}{2} \int_0^1 (A - \bar{A})^2 dx + \frac{1}{2} \int_0^1 \left[P(A) - P(\bar{A})\right](A - \bar{A})dx \\ &\le \int_0^1 \left[P(A) - P(\bar{A})\right](A - \bar{A})dx. \end{aligned} \quad (29)$$

Since $y(0) = y(1) = 0$ and $y_x = \bar{A} - A$, by Poincaré's inequality, we have

$$\frac{C_4}{2} \int_0^1 y^2 dx \le \frac{C_4}{2} \int_0^1 (A - \bar{A})^2 dx. \quad (30)$$

Plugging (30) into (29), we have

$$\begin{aligned} &\frac{C_4}{2} \int_0^1 y^2 dx + \frac{1}{2} \int_0^1 \left[P(A) - P(\bar{A})\right](A - \bar{A})dx \\ &\le \int_0^1 \left[P(A) - P(\bar{A})\right](A - \bar{A})dx, \end{aligned} \quad (31)$$

which, together with (28), gives

$$\begin{aligned} &\frac{d}{dt}\left(\int_0^1 y_t y dx\right) + \frac{C_4}{4} \int_0^1 y^2 dx + \frac{1}{2} \int_0^1 \left[P(A) - P(\bar{A})\right](A - \bar{A})dx \\ &\le \int_0^1 \left(\bar{A}C_1 + \frac{\mu^2}{C_4}\right) \frac{y_t^2}{A^2} dx. \end{aligned} \quad (32)$$

Step 3. *Estimates derived from entropy dissipation.* In order to deal with the nonlinearity on the RHS of (32), we now use the entropy inequality (7), rather than the usual energy method. Let

$$\eta_e = \frac{m^2}{2A} + \frac{P(A)}{\gamma - 1}, \qquad q_e = \frac{m^3}{2A^2} + \frac{A^{\gamma-1} m}{\gamma - 1} \quad (33)$$

be the mechanical energy and the related flux. We define

$$\eta_* = \eta_e - \frac{1}{\gamma-1}P'(\bar{A})(A-\bar{A}) - \frac{1}{\gamma-1}P(\bar{A}).$$

Thus, by the definition of entropy weak solutions, (η_e, q_e) satisfies the entropy inequality (7) in the sense of distribution. Then it holds

$$\eta_{*,t} + \frac{1}{\gamma-1}[P'(\bar{A})(A-\bar{A})]_t + q_{e,x} + \mu\frac{m^2}{A^2} \leq 0$$

in the sense of distribution.

Since $\bar{A}$ is a constant, we get

$$\eta_{*,t} + \frac{P'(\bar{A})}{\gamma-1}(A-\bar{A})_t + q_{e,x} + \mu\frac{m^2}{A^2} \leq 0.$$

By the conservation of mass and theory of divergence-measure fields [7], we have

$$\frac{d}{dt}\left(\int_0^1 \eta_* dx\right) + \mu\int_0^1 \frac{m^2}{A^2}dx \leq 0,$$

i.e.,

$$\frac{d}{dt}\left(\int_0^1 \eta_* dx\right) + \mu\int_0^1 \frac{y_t^2}{A^2}dx \leq 0. \tag{34}$$

Choosing

$$K = \max\left\{2C_1,\ \frac{\gamma-1}{C_3},\ \left(\bar{A}C_1 + \frac{\mu^2}{C_4} + 1\right)\mu^{-1}\right\}$$

and adding (32) to (34) $\times K$, we have

$$\begin{aligned}&\frac{d}{dt}\left(\int_0^1 (K\eta_* + yy_t)dx\right) + \frac{C_4}{4}\int_0^1 y^2 dx\\ &\quad + \frac{1}{2}\int_0^1 \left[P(A) - P(\bar{A})\right](A-\bar{A})dx + \int_0^1 \left(K\mu - \bar{A}C_1 - \frac{\mu^2}{C_4}\right)\frac{y_t^2}{A^2}dx \leq 0.\end{aligned} \tag{35}$$

Step 4. *Exponential decay estimates.* Our next goal is to compare the terms inside the temporal derivative with $\|y_t\|_{L^2}^2$ and $\|A-\bar{A}\|_{L^2}^2$, and with the diffusion terms in order to show the exponential decay of the solution.

Clearly, by using the definition of η_*, Lemma 2.6 (1), (23), and the upper bound of A, we have

$$\int_0^1 (K\eta_* + yy_t)dx \geq \int_0^1 \left(\frac{K}{2C_1}y_t^2 + yy_t + \frac{KC_3}{\gamma-1}(A-\bar{A})^2\right)dx.$$

By the definition of the constant K and the Poincaré's inequality, we have

$$\begin{aligned}
&\int_0^1 \left(\frac{K}{2C_1} y_t^2 + y y_t + \frac{KC_3}{\gamma - 1}(A - \bar{A})^2 \right) dx \\
=&\int_0^1 \left(\frac{K}{4C_1} y_t^2 + \frac{K}{4C_1} y_t^2 + y y_t + \frac{KC_3}{2(\gamma - 1)}(A - \bar{A})^2 + \frac{KC_3}{2(\gamma - 1)}(A - \bar{A})^2 \right) dx \\
\geq&\int_0^1 \left(\frac{K}{4C_1} y_t^2 + \frac{1}{2} y_t^2 + y y_t + \frac{KC_3}{2(\gamma - 1)} y^2 + \frac{KC_3}{2(\gamma - 1)}(A - \bar{A})^2 \right) dx \\
\geq&\int_0^1 \left(\frac{K}{4C_1} y_t^2 + \frac{1}{2} y_t^2 + y y_t + \frac{1}{2} y^2 + \frac{KC_3}{2(\gamma - 1)}(A - \bar{A})^2 \right) dx \\
=&\int_0^1 \left(\frac{K}{4C_1} y_t^2 + \frac{1}{2}(y_t + y)^2 + \frac{KC_3}{2(\gamma - 1)}(A - \bar{A})^2 \right) dx.
\end{aligned}$$

Therefore, we have

$$\int_0^1 (K\eta_* + y y_t) dx \geq \int_0^1 \left(\frac{K}{4C_1} y_t^2 + \frac{1}{2}(y_t + y)^2 + \frac{KC_3}{2(\gamma - 1)}(A - \bar{A})^2 \right) dx. \tag{36}$$

On the other hand, Lemma 2.6 (3) implies

$$\begin{aligned}
&\int_0^1 \frac{K}{\gamma - 1} \left[P(A) - P(\bar{A}) - P'(\bar{A})(A - \bar{A}) \right] dx \\
\leq&\frac{C_5 K}{\gamma - 1} \int_0^1 \left[P(A) - P(\bar{A}) \right] (A - \bar{A}) dx.
\end{aligned}$$

Moreover, since

$$\begin{aligned}
\int_0^1 \left(\frac{K}{2A} y_t^2 + y y_t \right) dx &\leq \int_0^1 \left(\frac{C_1 K}{2A^2} y_t^2 + \frac{1}{2} y^2 + \frac{1}{2} y_t^2 \right) dx \\
&\leq \int_0^1 \left(\frac{C_1 K}{2A^2} y_t^2 + \frac{1}{2} y^2 + \frac{C_1^2}{2A^2} y_t^2 \right) dx \\
&= \int_0^1 \left(\frac{C_1(K + C_1)}{2} \frac{y_t^2}{A^2} + \frac{1}{2} y^2 \right) dx,
\end{aligned}$$

we have

$$\begin{aligned}
&\int_0^1 (K\eta_* + y y_t) \, dx \\
\leq& C_6 \left(\int_0^1 y^2 dx + \int_0^1 \left[P(A) - P(\bar{A}) \right] (A - \bar{A}) dx + \int_0^1 \frac{y_t^2}{A^2} dx \right) \\
\leq& \frac{C_6}{C_7} \left(\frac{C_4}{4} \int_0^1 y^2 dx + \frac{1}{2} \int_0^1 \left[P(A) - P(\bar{A}) \right] (A - \bar{A}) dx \right. \\
&\left. + \int_0^1 \left(K\mu - \bar{A}C_1 - \frac{\mu^2}{C_4} \right) \frac{y_t^2}{A^2} dx \right)
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
C_6 &= \max \left\{ \frac{C_5 K}{\gamma - 1}, \ \frac{C_1(K + C_1)}{2}, \ \frac{1}{2} \right\}, \\
C_7 &= \min \left\{ \frac{C_4}{4}, \ \frac{1}{2}, \ \left(K\mu - \bar{A}C_1 - \frac{\mu^2}{C_4} \right) \right\}.
\end{aligned}$$

Let $C_8 = C_7/C_6$. We then get from (37) that

$$C_8 \int_0^1 (K\eta_* + yy_t)\,dx \le \bigg(\frac{C_4}{4} \int_0^1 y^2 dx + \frac{1}{2} \int_0^1 \left[P(A) - P(\bar{A})\right](A - \bar{A})dx + \int_0^1 \left(K\mu - \bar{A}C_1 - \frac{\mu^2}{C_4} \right) \frac{y_t^2}{A^2} dx \bigg)$$

which, together with (35), implies that

$$\frac{d}{dt} \int_0^1 (K\eta_* + yy_t)\,dx + C_8 \int_0^1 (K\eta_* + yy_t)\,dx \le 0.$$

Hence, we have

$$\int_0^1 (K\eta_* + yy_t)\,dx \le C_9 \exp\{-C_8 t\}.$$

By virtue of (36), we finally get

$$\int_0^1 \left(m^2 + (A - \bar{A})^2\right) dx \le C_{10} \exp\{-C_8 t\}.$$

This completes the proof of Theorem 2.5.

3. The case $\alpha > 1$ and $\mu > 0$.

This section is dedicated to the study of the case $\alpha > 1$ and $\mu > 0$. Physically speaking, this case is more realistic compared with (5). From the mathematical point of view, this case is significantly different from (5) due to the following reasons. When $\alpha > 1$, the Riemann invariants change expressions (in fact, the explicit expressions of the Riemann invariants are still unknown), which brings tremendous difficulties in the analysis of global existence of entropy weak solutions. Moreover, the functions (η_e, q_e) given in (33) are no longer an entropy-entropy flux pair for (5) when $\alpha > 1$. Hence, one has to seek new entropy-entropy flux pairs in order to study the long-time behavior of L^∞ entropy weak solutions.

We consider the following initial-boundary value problem:

$$\begin{cases} A_t + m_x = 0, \\ m_t + \alpha\Big(\dfrac{m^2}{A}\Big)_x + P(A)_x = -\mu\dfrac{m}{A} \end{cases} \tag{38}$$

where $x \in [0,1]$, $\quad t > 0$ and

$$\begin{cases} m|_{x=0} = m|_{x=1} = 0; \\ (A, m)(x, 0) = (A_0, m_0)(x) \in L^\infty([0,1]), \\ 0 < N_0 \le A_0(x) \le N_1, \quad |m_0(x)| \le N_1 A_0(x), \\ \displaystyle\int_0^1 A_0(x)dx = A_* > 0 \end{cases} \tag{39}$$

where $\alpha > 1$ and $\mu > 0$. Notice that we require $A_0(x)$ to be bounded away from zero in this case.

When $\alpha > 1$, the framework given in Section 2 is no longer valid for (38). First, it has been demonstrated in [18] that if $\alpha > 1$ is close to 1, then the associated Riemann invariants of (38) take the form

$$w_{\alpha>1} = w_{\alpha=1} + O(\alpha - 1)\frac{u}{A}, \quad z_{\alpha>1} = z_{\alpha=1} + O(\alpha - 1)\frac{u}{A}$$

where $w_{\alpha=1}$ and $z_{\alpha=1}$ are the Riemann invariants associated with (5). We remark that the explicit expressions of $w_{\alpha=1}$ and $z_{\alpha=1}$ play crucial roles in the invariant region theory and the compensated compactness frameworks established in [9] and [10, 11, 20, 21] respectively. The arguments therein depend strongly on the analysis of the structure of the region enclosed by the curves determined by $w_{\alpha=1}$ and $z_{\alpha=1}$ in the phase plane. When the expressions of the Riemann invariants change, especially, when the explicit expressions of the Riemann invariants are unknown, we are uncertain whether the invariant region theory and the compensated compactness frameworks still work or not. The slight difference in (38) indeed brings tremendous difficulty in the analysis of L^∞ entropy weak solutions. Therefore, we will not investigate the existence of entropy weak solutions of the system (38) in the present paper. Our goal is to establish a general framework of long-time behavior of such solutions assuming their existence.

Second, since the functions given in (33) are no longer an entropy-entropy flux pair for (38), we construct the entropy-entropy flux pairs for (38)

$$(\eta_e^{(i)}, q_e^{(i)}) = \left(\frac{m^2}{2A^{d_i}} + \frac{A^{\gamma-d_i+1}}{(\gamma-d_i)(\gamma-d_i+1)},\quad \frac{\alpha}{1+d_i}\frac{m^3}{A^{1+d_i}} + \frac{A^{\gamma-d_i}}{\gamma-d_i}m\right), \tag{40}$$

where $i = 1, 2$ and

$$d_1 = \frac{4\alpha - 1 + \sqrt{16\alpha^2 - 16\alpha + 1}}{2} > 1, \quad d_2 = \frac{4\alpha - 1 - \sqrt{16\alpha^2 - 16\alpha + 1}}{2} < 1. \tag{41}$$

Furthermore, by direct calculation one can show that $\eta_e^{(2)}$ is a convex entropy universally, while the convexity of $\eta_e^{(1)}$ depends on the range of the solution. We remark that, when $\alpha = 1$, $(\eta_e^{(2)}, q_e^{(2)})$ reduces to the usual mechanical energy and its associated flux, i.e., (33).

We shall use the new entropy-entropy flux pairs to redefine the entropy weak solutions to (38). Since $\eta_e^{(2)}$ is a convex entropy, it is natural to define the L^∞ entropy weak solutions to (38)–(39) in a similar fashion as in Definitions 2.1–2.3 as follows.

Definition 3.1. A pair of bounded measurable functions $\vec{v}(x,t) = \big(A(x,t), m(x,t)\big)$ satisfying

$$0 < \Lambda_1 \le A(x,t) \le \Lambda_2, \ |m(x,t)| \le \Lambda_2 A(x,t), \ 0 < \Lambda_1 \le \Lambda_2 \tag{42}$$

is said to be an L^∞ entropy weak solution to (38)–(39) if the two equations in (38) are satisfied in the sense of distribution and the initial and boundary conditions are satisfied in the sense of trace and section. Moreover, the following entropy inequality holds

$$\eta_{e,t}^{(2)} + q_{e,x}^{(2)} + \mu\frac{m}{A}\partial_m\eta_e^{(2)} \le 0$$

in the sense of distribution, where $\eta_e^{(2)}$ is given in (40).

Before stating our result, we need to redevelop Lemma 2.6 for the new entropy $\eta_e^{(2)}$. For this purpose, let

$$R(A) = \frac{A^{\gamma-d_2+1}}{(\gamma-d_2)(\gamma-d_2+1)}.$$

Then we have

Lemma 3.2. *Let $0 \le A \le \bar{\Lambda} < \infty$ and $0 < b < A_* < \infty$. There are positive constants D_1, D_2 depending only on $\bar{\Lambda}$, b, γ, d_2 and A_* such that*

(1) $R(A) - R(A_*) - R'(A_*)(A - A_*) \le D_1\big[P(A) - P(A_*)\big](A - A_*)$,

(2) $D_2(A - A_*)^2 \le R(A) - R(A_*) - R'(A_*)(A - A_*)$

where $P(A) = A^\gamma/\gamma$.

Proof. Consider

$$\begin{aligned}\Gamma(A) =& \frac{2\gamma}{A_*^{d_2}(\gamma - d_2 + 1)}\big[P(A) - P(A_*)\big](A - A_*)\\ &- \big[R(A) - R(A_*) - R'(A_*)(A - A_*)\big].\end{aligned}$$

Clearly, $\Gamma(A)$ is continuous for $A \ge 0$. Since

$$\Gamma(0) = \frac{A_*^{\gamma - d_2 + 1}}{\gamma - d_2 + 1} > 0,$$

there exists $h \in (0, \bar{\Lambda})$ such that

$$\Gamma(A) > \frac{1}{2}\Gamma(0) > 0, \quad for\ A \in [0, h].$$

For $\bar{\Lambda} \ge A > h > 0$, we see that

$$P'(h)(A - A_*)^2 \le \big[P(A) - P(A_*)\big](A - A_*)$$

and

$$R(A) - R(A_*) - R'(A_*)(A - A_*) \le \begin{cases} \dfrac{R''(h)}{2}(A - A_*)^2, & d_2 < \gamma \le d_2 + 1,\\ \dfrac{R''(\bar{\Lambda})}{2}(A - A_*)^2, & \gamma > d_2 + 1.\end{cases}$$

Choosing

$$D_1 = \max\left\{\frac{\gamma}{A_*},\ \frac{R''(h)}{2P'(h)},\ \frac{R''(\bar{\Lambda})}{2P'(h)}\right\},$$

we thus have

$$R(A) - R(A_*) - R'(A_*)(A - A_*) \le D_1\big[P(A) - P(A_*)\big](A - A_*).$$

The second inequality can be found in [16]. This completes the proof of Lemma 3.2. ∎

We then have the following main results of this section.

Theorem 3.3. *Let (A, m) be any L^∞ entropy weak solution of the initial-boundary value problem (38)–(39) defined in Definition 3.1. Then, there exist constants $D_3,\ \delta > 0$ depending on $\gamma, A_*, \Lambda_1, \Lambda_2, \alpha, \mu$, and initial data such that*

$$\big\|\big(A - A_*,\ m\big)(\cdot, t)\big\|^2_{L^2([0,1])} \le D_3 e^{-\delta t}. \tag{43}$$

Proof. The proof is in the spirit of the proof of Theorem 2.5. We only give a sketch here.

Following the arguments in the first step of the proof of Theorem 2.5, we arrive at

$$y_{tt} + \alpha\Big(\frac{m^2}{A}\Big)_x + \big[P(A) - P(A_*)\big]_x + \mu\frac{y_t}{A} = 0. \tag{44}$$

Taking L^2 inner product of (44) with y, we have

$$\begin{aligned}&\frac{d}{dt}\int_0^1 y_t y dx - \int_0^1 y_t^2 dx + \int_0^1 \big[P(A)-P(A_*)\big](A-A_*)dx\\ =&\alpha\int_0^1 \frac{m^2}{A} y_x dx - \mu\int_0^1 \frac{yy_t}{A}dx\end{aligned}$$

which gives, by the definition of y_x

$$\begin{aligned}&\frac{d}{dt}\int_0^1 y_t y dx + (\alpha-1)\int_0^1 y_t^2 dx + \int_0^1 \big[P(A)-P(A_*)\big](A-A_*)dx\\ =&\int_0^1 y_t^2 \frac{\alpha A_*}{A}dx - \mu\int_0^1 \frac{yy_t}{A}dx.\end{aligned} \tag{45}$$

Since $A \le \Lambda_2$, for the first term on the RHS of (45), we have

$$\begin{aligned}\int_0^1 y_t^2 \frac{\alpha A_*}{A}dx &= \int_0^1 y_t^2 \frac{\alpha A_* A^{d_2}}{A^{1+d_2}}dx\\ &\le \alpha A_* \Lambda_2^{d_2}\int_0^1 \frac{y_t^2}{A^{1+d_2}}dx.\end{aligned} \tag{46}$$

Since $d_2 < 1$ and $0 < \Lambda_1 \le A$, we have

$$\begin{aligned}\Big|-\mu\int_0^1 \frac{yy_t}{A}dx\Big| &= \mu\Big|\int_0^1 \frac{yy_t A^{\frac{d_2-1}{2}}}{A^{\frac{1+d_2}{2}}}dx\Big|\\ &\le \mu\Lambda_1^{\frac{d_2-1}{2}}\Big|\int_0^1 \frac{yy_t}{A^{\frac{1+d_2}{2}}}dx\Big|\\ &\le \frac{\mu^2\Lambda_1^{d_2-1}}{C_4}\int_0^1 \frac{y_t^2}{A^{1+d_2}}dx + \frac{C_4}{4}\int_0^1 y^2 dx\end{aligned} \tag{47}$$

where C_4 is given in Lemma 2.6.

Combining (45)–(47), we then have

$$\begin{aligned}&\frac{d}{dt}\int_0^1 y_t y dx + (\alpha-1)\int_0^1 y_t^2 dx + \int_0^1 \big[P(A)-P(A_*)\big](A-A_*)dx\\ \le&\Big(\alpha A_*\Lambda_2^{d_2} + \frac{\mu^2\Lambda_1^{d_2-1}}{C_4}\Big)\int_0^1 \frac{y_t^2}{A^{1+d_2}}dx + \frac{C_4}{4}\int_0^1 y^2 dx.\end{aligned} \tag{48}$$

Following the arguments in (29)–(32), we then have

$$\begin{aligned}&\frac{d}{dt}\int_0^1 y_t y dx + \frac{C_4}{4}\int_0^1 y^2 dx + (\alpha-1)\int_0^1 y_t^2 dx\\ &\qquad + \frac{1}{2}\int_0^1 \big[P(A)-P(A_*)\big](A-A_*)dx\\ \le&\Big(\alpha A_*\Lambda_2^{d_2} + \frac{\mu^2\Lambda_1^{d_2-1}}{C_4}\Big)\int_0^1 \frac{y_t^2}{A^{1+d_2}}dx.\end{aligned} \tag{49}$$

Let

$$\eta^* = \eta_e^{(2)} - \frac{1}{\gamma-1}R(A_*) - \frac{1}{\gamma-1}R'(A_*)(A-A_*).$$

Then, similar to (34), we have

$$\frac{d}{dt}\int_0^1 \eta^* dx + \int_0^1 \frac{y_t^2}{A^{1+d_2}}dx \le 0. \tag{50}$$

By combining (49) and (50) and following the arguments in the proof of Theorem 2.5, one can deduce (43). We omit the details here. This completes the proof of Theorem 3.3. □

Remark When the solution contains vacuum states, the situation becomes sophisticated. The framework of long-time behavior will be destroyed by the energy estimate (47) since $d_2 < 1$ and $A^{\frac{d_2-1}{2}}$ cannot be controlled when $A \to 0$. On the other hand, we calculate

$$\det(\mathrm{D}^2\eta_e^{(1)}) = \frac{d_1(1-d_1)}{2}m^2A^{-2d_1-2} + A^{\gamma-2d_1-1}.$$

It is easy to see that when $d_1 > 1$, $\det(\mathrm{D}^2\eta_e^{(1)})$ may change sign if one still considers the usual range of such kind of solutions given by $0 \le A(x,t) \le C$ and $|m(x,t)| \le CA(x,t)$. Therefore, $\eta_e^{(1)}$ may not be utilized to define such solutions. However, it is interesting to observe that if one considers such a solution satisfying

$$0 \le A(x,t) \le \Lambda_3, \quad |m(x,t)| \le \frac{\sqrt{2}}{\sqrt{d_1(d_1-1)}}A^{\frac{\gamma+1}{2}}, \tag{51}$$

then it is straightforward to check that $\eta_e^{(1)}$ is convex within the above range. Moreover, Lemma 3.2 is still valid for $\eta_e^{(1)}$ if one requires $\gamma > d_1$. And Theorem 3.3 still holds in this situation.

4. **Discussion.** Inspired by the relationship between the damped compressible Euler equations and the porous medium equation (c.f. [16, 17, 28]), we expect that the solution $A(x,t)$ to (1) will be captured by a porous medium-type equation and $m = Au$ should obey the classical Darcy's Law for large time. The porous medium-type equation and Darcy's Law in our case take the form

$$\begin{cases} A_t = Q(A)_{xx}, \quad x \in [0,1], \quad t > 0, \\ m = -Q(A)_x, \quad x \in [0,1], \quad t > 0, \\ A(x,0) = A_0(x), \quad x \in [0,1], \\ Q_x|_{x=0} = Q_x|_{x=1} = 0, \quad t \ge 0 \end{cases} \tag{52}$$

where the function $Q(A)$ is determined through the relation: $Q'(A) = AP'(A)$. In the case of bounded domains, using the arguments in [28, 31], one can show that A converges to a constant, which is its average over the domain, exponentially as time goes to infinity, and m goes to zero exponentially in time. The proof is based on either a dynamical system approach (ω-*limit*) or the energy method. Therefore, solutions to (1) and (52) converge to each other exponentially in time as time goes to infinity provided that the two systems carry the same initial mass.

It is interesting to study the Cauchy problem and the half-line problem of (1) and to investigate the propagation and stability of its asymptotic profiles, such as diffusion waves. Since the diffusion waves have explicit expressions in terms of the spatial and temporal variables, once the convergence of general solutions of the model to these profiles is investigated and explored in detail, it is expected that the knowledge will help to understand the evolution of general solutions of the model and to capture the detailed morphological behavior of blood vessels and the dynamical aspect of the blood flow, which in turn will contribute to the detection of physiological problems in real-world applications. Motivated by the studies in gas dynamics (c.f. [16, 17]), we expect that the solution to (1) shall be captured by the nonlinear diffusion wave generated by the porous medium equation when time goes to infinity. We leave the investigation for the future.

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E-mail address: tong-li@uiowa.edu
E-mail address: kun-zhao@uiowa.edu