

SPECTRAL PROPERTIES OF RENORMALIZATION FOR AREA-PRESERVING MAPS

DENIS GAIDASHEV

Department of Mathematics, Uppsala University
Uppsala, Sweden

TOMAS JOHNSON

Fraunhofer-Chalmers Research Centre for Industrial Mathematics
SE-412 88 Gothenburg, Sweden

(Communicated by Rafael de la Llave)

ABSTRACT. Area-preserving maps have been observed to undergo a universal period-doubling cascade, analogous to the famous Feigenbaum-Coulet-Tresser period doubling cascade in one-dimensional dynamics. A renormalization approach has been used by Eckmann, Koch and Wittwer in a computer-assisted proof of existence of a conservative renormalization fixed point.

Furthermore, it has been shown by Gaidashev, Johnson and Martens that *infinitely renormalizable maps* in a neighborhood of this fixed point admit invariant Cantor sets with vanishing Lyapunov exponents on which dynamics for any two maps is smoothly conjugate.

This rigidity is a consequence of an interplay between the decay of geometry and the convergence rate of renormalization towards the fixed point.

In this paper we prove a result which is crucial for a demonstration of rigidity: that an upper bound on this convergence rate of renormalizations of infinitely renormalizable maps is sufficiently small.

CONTENTS

Introduction	3652
1. Renormalization for area-preserving reversible twist maps	3653
2. Statement of main results	3660
3. Coordinate changes and renormalization eigenvalues	3661
4. Analyticity and compactness of renormalization	3662
5. Strong contraction on the stable manifold	3663
6. Spectral properties of \mathcal{R}_c . Proof of main theorem	3669
Acknowledgments	3674
REFERENCES	3674

2010 *Mathematics Subject Classification.* 37E20, 37E40.

Key words and phrases. Renormalization, area-preserving maps, period-doubling, hyperbolicity, computer-assisted proof, rigidity.

Introduction. Following the pioneering discovery of the Feigenbaum-Coulet-Tresser period doubling universality in unimodal maps, universality — independence of the quantifiers of the geometry of orbits and bifurcation cascades in families of maps of the choice of a particular family — has been demonstrated to be a rather generic phenomenon in dynamics.

Universality problems are typically approached via *renormalization*. In a renormalization setting one introduces a *renormalization* operator on a functional space, and demonstrates that this operator has a *hyperbolic fixed point*. This approach has been very successful in one-dimensional dynamics, and has led to explanation of universality in unimodal maps, critical circle maps and holomorphic maps with a Siegel disk. There is, however, at present no complete understanding of universality in conservative systems, other than in the case of the universality for systems “near integrability” [1, 2, 24, 25, 26, 12, 27, 23].

Period-doubling renormalization for two-dimensional maps has been extensively studied in [6, 5, 28]. Specifically, the authors of [5] have considered strongly dissipative Hénon-like maps of the form

$$F(x, y) = (f(x) - \epsilon(x, y), x), \quad (1)$$

where $f(x)$ is a unimodal map (subject to some regularity conditions), and ϵ is small. Whenever the one-dimensional map f is *renormalizable*, one can define a renormalization of F , following [5], as

$$R_{dCLM}[F] = H^{-1} \circ F \circ F|_U \circ H,$$

where U is an appropriate neighborhood of the critical value $v = (f(0), 0)$, and H is an explicit non-linear change of coordinates. [5] demonstrates that the degenerate map $F_*(x, y) = (f_*(x), x)$, where f_* is the Feigenbaum-Collet-Tresser fixed point of one-dimensional renormalization, is a hyperbolic fixed point of R_{dCLM} . Furthermore, according to [5], for any infinitely-renormalizable map of the form (1), there exists a hierarchical family of “pieces” $\{B_\sigma^n\}$, organized by inclusion in a dyadic tree, such that the set

$$\mathcal{C}_F = \bigcap_n \bigcup_\sigma B_\sigma^n$$

is an attracting Cantor set on which F acts as an *adding machine*. Compared to the Feigenbaum-Collet-Tresser one-dimensional renormalization, the new striking feature of the two dimensional renormalization for highly dissipative maps (1), is that *the restriction of the dynamics to this Cantor set is not rigid*. Indeed, if the average Jacobians of F and G are different, for example, $b_F < b_G$, then the conjugacy $F|_{\mathcal{C}_F} \overset{\approx}{\underset{h}{\sim}} G|_{\mathcal{C}_G}$ is not smooth, rather it is at best a Hölder continuous function with a definite upper bound on the Hölder exponent: $\alpha \leq \frac{1}{2} \left(1 + \frac{\log b_G}{\log b_F}\right) < 1$.

The theory has been also generalized to other combinatorial types in [19, 20], and also to three dimensional dissipative Hénon-like maps in [30].

Finally, the authors of [5] show that the geometry of these Cantor sets is rather particular: the Cantor sets have universal bounded geometry in “most” places, however there are places in the Cantor set where the geometry is unbounded. Rigidity and universality as we know from one-dimensional dynamics has a probabilistic nature for strongly dissipative Hénon like maps. See [29] for a discussion of probabilistic universality and probabilistic rigidity.

It turns out that the period-doubling renormalization for area-preserving maps is very different from the dissipative case.

A universal period-doubling cascade in families of area-preserving maps was observed by several authors in the early 80's [8, 21, 3, 4, 7, 9]. The existence of a hyperbolic fixed point for the period-doubling renormalization operator

$$R_{EKW}[F] = \Lambda_F^{-1} \circ F \circ F \circ \Lambda_F,$$

where $\Lambda_F(x, u) = (\lambda_F x, \mu_F u)$ is an F -dependent *linear* change of coordinates, has been proved with computer-assistance in [10].

We have proved in [15] that *infinitely renormalizable* maps in a neighborhood of the fixed point of [10] admit a “stable” Cantor set, that is the set on which the Lyapunov exponents are zero. We have also shown in the same publication that the conjugacy of stable dynamics is at least bi-Lipschitz on a submanifold of locally infinitely renormalizable maps of a finite codimension. Furthermore, [16] improves this conclusion in the following way.

Rigidity for Area-preserving Maps. *The period doubling Cantor sets of area-preserving maps in the universality class of the Eckmann-Koch-Wittwer renormalization fixed point are smoothly conjugate.*

A crucial ingredient of the proof in [16] is a new tight bound on the spectral radius of the renormalization operator. The goal of the present paper is to prove this new bound.

We demonstrate that the spectral radius of the action of the derivative $\mathcal{D}R_{EKW}$, evaluated at the Eckmann-Koch-Wittwer fixed point F_{EKW} , restricted to the tangent space $T_{F_{EKW}}\mathcal{W}$ of the stable manifold \mathcal{W} of the infinitely renormalizable maps, is equal exactly to the absolute value of the “horizontal” scaling parameter

$$\rho_{\text{spec}}(\mathcal{D}R_{EKW}[F_{EKW}]|_{T_{F_{EKW}}\mathcal{W}}) = |\lambda_{F_{EKW}}| = 0.2488\dots$$

Furthermore, we show that the single eigenvalue $\lambda_{F_{EKW}}$ in the spectrum of $\mathcal{D}R_{EKW}[F_{EKW}]$ corresponds to an eigenvector, generated by a very specific coordinate change. To eliminate this *irrelevant* eigenvalue from the renormalization spectrum, we introduce an F -dependent *nonlinear* coordinate change S_F into the period-doubling renormalization scheme

$$R_c[F] := \Lambda_F^{-1} \circ S_F^{-1} \circ F \circ F \circ S_F \circ \Lambda_F,$$

compute the spectral radius of the restriction of the spectrum of $\mathcal{D}R_c[F^*]$ to its stable subspace $T_{F^*}\mathcal{W}$ at the fixed point F^* of R_c , and obtain the following spectral bound, which is of crucial importance to our proof of rigidity.

Main Theorem.

$$\rho_{\text{spec}}(\mathcal{D}R_c[F^*]|_{T_{F^*}\mathcal{W}}) \leq 0.1258544921875.$$

1. Renormalization for area-preserving reversible twist maps. An “area-preserving map” will mean an exact symplectic diffeomorphism of a subset of \mathbb{R}^2 onto its image.

Recall, that an area-preserving map that satisfies the twist condition

$$\partial_u(\pi_x F(x, u)) \neq 0$$

everywhere in its domain of definition can be uniquely specified by a generating function S :

$$\begin{pmatrix} x \\ -S_1(x, y) \end{pmatrix} \xrightarrow{F} \begin{pmatrix} y \\ S_2(x, y) \end{pmatrix}, \quad S_i \equiv \partial_i S. \tag{2}$$

Furthermore, we will assume that F is reversible, that is

$$T \circ F \circ T = F^{-1}, \quad \text{where } T(x, u) = (x, -u). \tag{3}$$

For such maps it follows from (2) that

$$S_1(y, x) = S_2(x, y) \equiv s(x, y),$$

and

$$\begin{pmatrix} x \\ -s(y, x) \end{pmatrix} \xrightarrow{F} \begin{pmatrix} y \\ s(x, y) \end{pmatrix}. \tag{4}$$

It is this “little” s that will be referred to below as “the generating function”.

The period-doubling phenomenon can be illustrated with the area-preserving Hénon family (cf. [4]) :

$$H_a(x, u) = (-u + 1 - ax^2, x).$$

Maps H_a have a fixed point $((-1 + \sqrt{1+a})/a, (-1 + \sqrt{1+a})/a)$ which is stable (elliptic) for $-1 < a < 3$. When $a_1 = 3$ this fixed point becomes hyperbolic: the eigenvalues of the linearization of the map at the fixed point bifurcate through -1 and become real. At the same time a stable orbit of period two is “born” with $H_a(x_{\pm}, x_{\mp}) = (x_{\mp}, x_{\pm})$, $x_{\pm} = (1 \pm \sqrt{a-3})/a$. This orbit, in turn, becomes hyperbolic at $a_2 = 4$, giving birth to a period 4 stable orbit. Generally, there exists a sequence of parameter values a_k , at which the orbit of period 2^{k-1} turns unstable, while at the same time a stable orbit of period 2^k is born (passes from \mathbb{C}^2 to \mathbb{R}^2). The parameter values a_k accumulate on some a_{∞} . The crucial observation is that the accumulation rate

$$\lim_{k \rightarrow \infty} \frac{a_k - a_{k-1}}{a_{k+1} - a_k} = 8.721\dots \tag{5}$$

is universal for a large class of families, not necessarily Hénon.

Furthermore, the 2^k periodic orbits scale asymptotically with two scaling parameters

$$\lambda = -0.249\dots, \quad \mu = 0.061\dots \tag{6}$$

To explain how orbits scale with λ and μ we will follow [4]. Consider an interval (a_k, a_{k+1}) of parameter values in a “typical” family F_a . For any value $\alpha \in (a_k, a_{k+1})$ the map F_{α} possesses a stable periodic orbit of period 2^k . We fix some α_k within the interval (a_k, a_{k+1}) in some consistent way; for instance, by requiring that $DF_{\alpha_k}^{2^k}$ at a point in the stable 2^k -periodic orbit is conjugate, via a diffeomorphism H_k , to a rotation with some fixed rotation number r . Let p'_k be some unstable periodic point in the 2^{k-1} -periodic orbit, and let p_k be the further of the two stable 2^k -periodic points that bifurcated from p'_k . Denote with $d_k = |p'_k - p_k|$, the distance between p_k and p'_k . The new elliptic point p_k is surrounded by (infinitesimal) invariant ellipses; let c_k be the distance between p_k and p'_k in the direction of the minor semi-axis of an invariant ellipse surrounding p_k , see Figure 1. Then,

$$\frac{1}{\lambda} = - \lim_{k \rightarrow \infty} \frac{d_k}{d_{k+1}}, \quad \frac{\lambda}{\mu} = - \lim_{k \rightarrow \infty} \frac{\rho_k}{\rho_{k+1}}, \quad \frac{1}{\lambda^2} = \lim_{k \rightarrow \infty} \frac{c_k}{c_{k+1}},$$

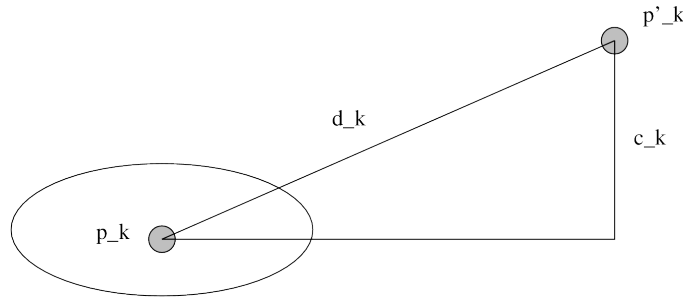


FIGURE 1. The geometry of the period doubling. p_k is the further elliptic point that has bifurcated from the hyperbolic point p'_k .

where ρ_k is the ratio of the smaller and larger eigenvalues of $DH_k(p_k)$ ¹

This universality can be explained rigorously if one shows that the *renormalization operator*

$$R_{EKW}[F] = \Lambda_F^{-1} \circ F \circ F \circ \Lambda_F, \quad (7)$$

where Λ_F is some F -dependent coordinate transformation, has a fixed point, and the derivative of this operator is hyperbolic at this fixed point.

It has been argued in [7] that Λ_F is a diagonal linear transformation. Furthermore, such Λ_F has been used in [9] and [10] in a computer assisted proof of existence of a reversible renormalization fixed point F_{EKW} and hyperbolicity of the operator R_{EKW} .

We will now derive an equation for the generating function of the renormalized map $\Lambda_F^{-1} \circ F \circ F \circ \Lambda_F$.

Applying a reversible F twice we get

$$\begin{pmatrix} x' \\ -s(Z, x') \end{pmatrix} \xrightarrow{F} \begin{pmatrix} Z \\ s(x', Z) \end{pmatrix} = \begin{pmatrix} Z \\ -s(y', Z) \end{pmatrix} \xrightarrow{F} \begin{pmatrix} y' \\ s(Z, y') \end{pmatrix}.$$

According to [7] Λ_F can be chosen to be a linear diagonal transformation:

$$\Lambda_F(x, u) = (\lambda x, \mu u).$$

We, therefore, set $(x', y') = (\lambda x, \lambda y)$, $Z(\lambda x, \lambda y) = z(x, y)$ to obtain:

$$\begin{pmatrix} x \\ -\frac{1}{\mu}s(z, \lambda x) \end{pmatrix} \xrightarrow{\Lambda_F} \begin{pmatrix} \lambda x \\ -s(z, \lambda x) \end{pmatrix} \xrightarrow{F \circ F} \begin{pmatrix} \lambda y \\ s(z, \lambda y) \end{pmatrix} \xrightarrow{\Lambda_F^{-1}} \begin{pmatrix} y \\ \frac{1}{\mu}s(z, \lambda y) \end{pmatrix}, \quad (8)$$

where $z(x, y)$ solves

$$s(\lambda x, z(x, y)) + s(\lambda y, z(x, y)) = 0. \quad (9)$$

If the solution of (9) is unique, then $z(x, y) = z(y, x)$, and it follows from (8) that the generating function of the renormalized F is given by

$$\tilde{s}(x, y) = \mu^{-1}s(z(x, y), \lambda y). \quad (10)$$

One can fix a set of normalization conditions for \tilde{s} and z which serve to determine scalings λ and μ as functions of s . For example, the normalization $s(1, 0) = 0$ is

¹ Derivatives with respect to elements of the functional space, typically s , will be denoted \mathcal{D} , ex. $\mathcal{D}R_{EKW}$, while derivatives of maps from \mathbb{C}^2 to \mathbb{C}^2 will be denoted D , as it has been done here: DH_k .

reproduced for \tilde{s} as long as $z(1, 0) = z(0, 1) = 1$. In particular, this implies that

$$s(Z(\lambda, 0), 0) = 0,$$

which serves as an equation for λ . Furthermore, the condition $\partial_1 s(1, 0) = 1$ is reproduced as long as $\mu = \partial_1 z(1, 0)$.

We will now summarize the above discussion in the following definition of the renormalization operator acting on generating functions originally due to the authors of [9] and [10]:

Definition 1.1. Define the prerenormalization of s as

$$\mathcal{P}_{EKW}[s] = s \circ G[s], \tag{11}$$

where

$$0 = s(x, Z(x, y)) + s(y, Z(x, y)), \tag{12}$$

$$G[s](x, y) = (Z(x, y), y). \tag{13}$$

The renormalization of s will be defined as

$$\mathcal{R}_{EKW}[s] = \frac{1}{\mu} \mathcal{P}_{EKW}[s] \circ \lambda, \tag{14}$$

where

$$\lambda(x, y) = (\lambda x, \lambda y), \quad \mathcal{P}_{EKW}[s](\lambda, 0) = 0 \quad \text{and} \quad \mu = \lambda \partial_1 \mathcal{P}_{EKW}[s](\lambda, 0).$$

Definition 1.2. The Banach space of functions $s(x, y) = \sum_{i,j=0}^{\infty} c_{ij}(x-\beta)^i(y-\beta)^j$, analytic on a bi-disk

$$\mathcal{D}_\rho(\beta) = \{(x, y) \in \mathbb{C}^2 : |x - \beta| < \rho, |y - \beta| < \rho\},$$

for which the norm

$$\|s\|_\rho = \sum_{i,j=0}^{\infty} |c_{ij}| \rho^{i+j}$$

is finite, will be referred to as $\mathcal{A}^\beta(\rho)$.

$\mathcal{A}_s^\beta(\rho)$ will denote its symmetric subspace $\{s \in \mathcal{A}^\beta(\rho) : s_1(x, y) = s_1(y, x)\}$.

We will use the simplified notation $\mathcal{A}(\rho)$ and $\mathcal{A}_s(\rho)$ for $\mathcal{A}^0(\rho)$ and $\mathcal{A}_s^0(\rho)$, respectively.

As we have already mentioned, the following has been proved with the help of a computer in [9] and [10]:

Theorem 1. *There exist a polynomial $s_{0.5} \in \mathcal{A}_s^{0.5}(\rho)$ and a ball $\mathcal{B}_\varrho(s_{0.5}) \subset \mathcal{A}_s^{0.5}(\rho)$, $\varrho = 6.0 \times 10^{-7}$, $\rho = 1.6$, such that the operator \mathcal{R}_{EKW} is well-defined and analytic on $\mathcal{B}_\varrho(s_{0.5})$.*

Furthermore, its derivative $\mathcal{DR}_{EKW}|_{\mathcal{B}_\varrho(s_{0.5})}$ is a compact linear operator, and has exactly two eigenvalues

$$\delta_1 = 8.721\dots, \quad \text{and} \quad \delta_2 = \frac{1}{\lambda_*}$$

of modulus larger than 1, while

$$\text{spec}(\mathcal{DR}_{EKW}|_{\mathcal{B}_\varrho(s_{0.5})}) \setminus \{\delta_1, \delta_2\} \subset \{z \in \mathbb{C} : |z| \leq \nu\},$$

where

$$\nu < 0.85. \quad (15)$$

Finally, there is an $s^{EKW} \in \mathcal{B}_\rho(s_{0.5})$ such that

$$\mathcal{R}_{EKW}[s^{EKW}] = s^{EKW}.$$

The scalings λ_* and μ_* corresponding to the fixed point s^{EKW} satisfy

$$\lambda_* \in [-0.24887681, -0.24887376], \quad \mu_* \in [0.061107811, 0.061112465]. \quad (16)$$

Remark 1.3. The bound (15) is not sharp. In fact, a bound on the largest eigenvalue of $\mathcal{DR}_{EKW}(s^{EKW})$, restricted to the tangent space of the stable manifold, is expected to be quite smaller.

The size of the neighborhood in $\mathcal{A}_s^\beta(\rho)$ where the operator \mathcal{R}_{EKW} is well-defined, analytic and compact has been improved in [13]. Here, we will cite a somewhat different version of the result of [13] which suits the present discussion (in particular, in the Theorem below some parameter, like ρ in $\mathcal{A}_s^\beta(\rho)$, are different from those used in [13]). We would like to emphasize that all parameters and bounds used and reported in the Theorem below, and, indeed, throughout the paper, are numbers representable on the computer.

Theorem 2. *There exists a polynomial $s^0 \in \mathcal{A}(\rho)$, $\rho = 1.75$, such that the following holds.*

i) *The operator \mathcal{R}_{EKW} is well-defined and analytic in $\mathcal{B}_R(s^0) \subset \mathcal{A}(\rho)$ with*

$$R = 0.00426483154296875.$$

ii) *For all $s \in \mathcal{B}_R(s^0)$ with real Taylor coefficients, the scalings $\lambda = \lambda[s]$ and $\mu = \mu[s]$ satisfy*

$$\begin{aligned} 0.0000253506004810333 &\leq \mu \leq 0.121036529541016, \\ -0.27569580078125 &\leq \lambda \leq -0.222587585449219. \end{aligned}$$

iii) *The operator \mathcal{R}_{EKW} is compact in $\mathcal{B}_R(s^0) \subset \mathcal{A}(\rho)$, with $\mathcal{R}_{EKW}[s] \in \mathcal{A}(\rho')$, $\rho' = 1.0699996948242188\rho$.*

Definition 1.4. The set of reversible twist maps F of the form (4) with $s \in \mathcal{B}_\rho(\tilde{s}) \subset \mathcal{A}_s^\beta(\rho)$ will be referred to as $\mathcal{F}_\rho^{\beta,\rho}(\tilde{s})$:

$$\mathcal{F}_\rho^{\beta,\rho}(\tilde{s}) = \{F : (x, -s(y, x)) \mapsto (y, s(x, y)) \mid s \in \mathcal{B}_\rho(\tilde{s}) \subset \mathcal{A}_s^\beta(\rho)\}. \quad (17)$$

We will also use the notation

$$\mathcal{F}_\rho^\rho(\tilde{s}) \equiv \mathcal{F}_\rho^{0,\rho}(\tilde{s}).$$

We will finish our introduction into period-doubling for area-preserving maps with a summary of properties of the fixed point map. In [14] we have described the domain of analyticity of maps in some neighborhood of the fixed point. Additional properties of the domain are studied in [22]. Before we state the results of [14], we will fix a notation for spaces of functions analytic on a subset of \mathbb{C}^2 .

Definition 1.5. Denote $\mathcal{O}_2(\mathcal{D})$ the Banach space of maps $F : \mathcal{D} \mapsto \mathbb{C}^2$, analytic on an open simply connected set $\mathcal{D} \subset \mathbb{C}^2$, continuous on $\partial\mathcal{D}$, equipped with a finite max supremum norm $\|\cdot\|_{\mathcal{D}}$:

$$\|F\|_{\mathcal{D}} = \max \left\{ \sup_{(x,u) \in \mathcal{D}} |F_1(x, u)|, \sup_{(x,u) \in \mathcal{D}} |F_2(x, u)| \right\}.$$

The Banach space of functions $y : \mathcal{A} \mapsto \mathbb{C}$, analytic on an open simply connected set $\mathcal{A} \subset \mathbb{C}^2$, continuous on $\partial\mathcal{A}$, equipped with a finite supremum norm $\|\cdot\|_{\mathcal{A}}$ will be denoted $\mathcal{O}_1(\mathcal{A})$: $\|y\|_{\mathcal{D}} = \sup_{(x,u) \in \mathcal{D}} |y(x,u)|$.

If \mathcal{D} is a bidisk $\mathcal{D}_\rho \subset \mathbb{C}^2$ for some ρ , then we use the notation

$$\|\cdot\|_\rho \equiv \|\cdot\|_{\mathcal{D}_\rho}.$$

The next Theorem describes the analyticity domains for maps in a neighborhood of the Eckmann-Koch-Wittwer fixed point map, and those for functions in a neighborhood of the Eckmann-Koch-Wittwer fixed point generating function. The Theorem has been proved in two different versions: one for the space $\mathcal{A}_s^{0.5}$ (1.6) (the functional space in the original paper [10]), the other for the space \mathcal{A}_s (1.75) — the space in which we will obtain a bound on the renormalization spectral radius in the stable manifold in this paper. To state the Theorem in a compact form, we introduce the following notation:

$$\rho_{0.5} = 1.6, \quad \rho_0 = 1.75, \quad \varrho_{0.5} = 6.0 \times 10^{-7}, \quad \varrho_0 = 5.79833984375 \times 10^{-4},$$

while $s_{0.5}$ (as in Theorem 1) and s_0 will denote the approximate renormalization fixed points in spaces $\mathcal{A}_s^{0.5}$ (1.6) and \mathcal{A}_s (1.75), respectively.

Theorem 3. *There exists a polynomial s_β such that the following holds for all $F \in \mathcal{F}_{\varrho_\beta}^{\beta, \rho_\beta}(s_\beta)$, $\beta = 0.5$ or $\beta = 0$.*

i) There exists a simply connected open set $\mathcal{D} = \mathcal{D}(\beta, \varrho_\beta, \rho_\beta) \subset \mathbb{C}^2$ such that the map F is in $\mathcal{O}_2(\mathcal{D})$.

ii) There exist simply connected open sets $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}(\beta, \varrho_\beta, \rho_\beta) \subset \mathcal{D}$, such that $\tilde{\mathcal{D}} \cap \mathbb{R}^2$ is a non-empty simply connected open set, and such that for every $(x, u) \in \tilde{\mathcal{D}}$ and $s \in \mathcal{B}_{\varrho_\beta}(s_\beta) \subset \mathcal{A}_s^\beta(\rho_\beta)$, the equation

$$0 = u + s(y, x) \tag{18}$$

has a unique solution $y[s](x, u) \in \mathcal{O}_1(\tilde{\mathcal{D}})$. The map

$$\mathcal{S} : s \mapsto y[s] \tag{19}$$

is analytic as a map from $\mathcal{B}_{\varrho_\beta}(s_\beta)$ to $\mathcal{O}_1(\tilde{\mathcal{D}})$.

Furthermore, for every $x \in \pi_x \tilde{\mathcal{D}}$, there is a function $U \in \mathcal{O}_1(\mathcal{D}_{\rho_\beta}(\beta))$, that satisfies

$$y[s](x, U(x, v)) = v.$$

The map

$$Y : y[s] \mapsto U$$

is analytic as a map from $\mathcal{O}_1(\mathcal{D}_{\rho_\beta}(\beta))$ to $\mathcal{B}_{\varrho_\beta}(s_\beta)$.

Remark 1.6. It is not too hard to see that the subsets $\mathcal{F}_{\varrho_\beta}^{\beta, \rho_\beta}(s_\beta)$, $\beta = 0$ or 0.5 , are analytic Banach submanifolds of the spaces $\mathcal{O}_2(\tilde{\mathcal{D}}(\beta, \varrho_\beta, \rho_\beta))$. Indeed, the map

$$\mathcal{I} : s \mapsto (y[s], s \circ h[s]), \tag{20}$$

where $y[s](x, u)$ is the solution of the equation (18), and $h[s](x, u) = (x, y[s](x, u))$, is analytic as a map from $\mathcal{B}_{\varrho_\beta}(s_\beta)$ to $\mathcal{O}_2(\tilde{\mathcal{D}}(\beta, \varrho_\beta, \rho_\beta))$ according to Theorem 3, and has an analytic inverse

$$\mathcal{I}^{-1} : F \mapsto \pi_u F \circ g[F], \tag{21}$$

on $\mathcal{I}(\mathcal{B}_{\varrho_\beta}(s_\beta))$, where $g[F](x, y) = (x, U(x, y))$, and U is as in Theorem 3. Furthermore, the derivative of the map \mathcal{S} , defined in (19),

$$D\mathcal{S}[s]v(x, y) = -\frac{v(y, x)}{s_1(y, x)}, \quad v \in T_s\mathcal{B}_{\varrho_\beta}(s_\beta),$$

is injective, which also implies injectivity of the derivative of \mathcal{I} . Thus \mathcal{I} is an analytic embedding, which implies the claim.

We are now ready to give a definition of the Eckmann-Koch-Wittwer renormalization operator for maps of the subset of a plane. Notice, that the condition $\mathcal{P}_{EKW}[s](\lambda, 0) = 0$ from Definition 1.1 is equivalent to

$$F(F(\lambda, -s(z(\lambda, 0), \lambda))) = (0, 0),$$

or, using the reversibility

$$\lambda = \pi_x F(F(0, 0)).$$

On the other hand,

$$-s(z(y(x, u), x), x) = -\mathcal{P}_{EKW}[s](y(x, u), x) = u,$$

and

$$\begin{aligned} \partial_u \mathcal{P}_{EKW}[s](y(x, u), x) &= \mathcal{P}_{EKW}[s]_1(y(x, u), x) y_2(x, u) \\ &= \mathcal{P}_{EKW}[s]_1(y(x, u), x) \pi_x(F \circ F)_2(x, u) = -1, \end{aligned}$$

then

$$\mathcal{P}_{EKW}[s]_1(\lambda, 0) \pi_x(F \circ F)_2(0, 0) = -1,$$

and

$$\mu = \frac{-\lambda}{\pi_x(F \circ F)_2(0, 0)}.$$

Definition 1.7. We will refer to the composition $F \circ F$ as the *prerenormalization* of F , whenever this composition is defined:

$$P_{EKW}[F] = F \circ F. \quad (22)$$

Set

$$R_{EKW}[F] = \Lambda^{-1} \circ P_{EKW}[F] \circ \Lambda,$$

where

$$\Lambda(x, u) = (\lambda x, \mu u), \quad \lambda = \pi_x P_{EKW}[F](0, 0), \quad \mu = \frac{-\lambda}{\pi_x P_{EKW}[F]_2(0, 0)},$$

whenever these operations are defined. $R_{EKW}[F]$ will be called the (EKW-)renormalization of F .

Remark 1.8. Suppose that for some choice of β , ϱ_β and ρ_β , the operator \mathcal{R}_{EKW} and the analytic embedding map \mathcal{I} , described in Remark 1.6, are well-defined on some $\mathcal{B}_{\varrho_\beta}(s_\beta) \subset \mathcal{A}_s^\beta(\rho_\beta)$. Then,

$$R_{EKW} = \mathcal{I} \circ \mathcal{R}_{EKW} \circ \mathcal{I}^{-1}$$

on $\mathcal{F}_{\varrho_\beta}^{\beta, \rho_\beta}(s_\beta)$.

2. Statement of main results. Consider the coordinate transformation

$$S_t(x, u) = \left(x + tx^2, \frac{u}{1 + 2tx} \right), \quad S_t^{-1}(y, v) = \left(\frac{\sqrt{1 + 4ty} - 1}{2t}, v\sqrt{1 + 4ty} \right), \quad (23)$$

for $t \in \mathbb{C}$, $|t| < 4/(\rho + |\beta|)$ (recall Definition 1.2).

We will now introduce two renormalization operators, one - on the generating functions, and one - on the maps, which incorporates the coordinate change S_t as an additional coordinate transformation.

Definition 2.1. Given $c \in \mathbb{R}$, set, formally,

$$\mathcal{P}_c[s](x, y) = (1 + 2t_c y)s(G(\xi_{t_c}(x, y))), \quad \text{and} \quad \mathcal{R}_c[s] = \mu^{-1}\mathcal{P}_c[s] \circ \lambda,$$

with G is as in (13), and

$$\xi_t(x, y) = (x + tx^2, y + ty^2), \quad t_c[s] = \frac{1}{4} \frac{c - (s \circ G)_{(0,3)}}{(s \circ G)_{(0,2)}}, \quad (24)$$

where λ and μ solve the following equations:

$$\mathcal{P}_c[s](\lambda[s], 0) = 0, \quad \mu[s] = \lambda[s]\partial_1 \mathcal{P}_c[s](\lambda[s], 0). \quad (25)$$

Definition 2.2. Given $c \in \mathbb{R}$, set, formally,

$$P_c[F] = S_{t_c}^{-1} \circ F \circ F \circ S_{t_c}, \quad R_c[F] = \Lambda_F^{-1} \circ P_c[F] \circ \Lambda_F, \quad (26)$$

where S_{t_c} is as in (23), $\Lambda_F(x, u) = (\lambda[F]x, \mu[F]u)$, and

$$t_c[F] = \frac{1}{4} \frac{c - (\pi_u(F \circ F))_{(0,3)}}{(\pi_u(F \circ F))_{(0,2)}}, \quad \lambda[F] = \pi_x P_c[F](0, 0), \quad \mu[F] = \frac{-\lambda[F]}{\pi_x P_c[F]_2(0, 0)}.$$

We are now ready to state our main theorem. Below, and through the paper, $s_{(i,j)}$ stands for the (i, j) -th component of a Taylor series expansion of an analytic function of two variables.

Main Theorem. (*Existence and Spectral properties*). *There exists a polynomial $s_0 : \mathbb{C}^2 \mapsto \mathbb{C}$, such that*

- i) *The operators \mathcal{R}_{exw} and \mathcal{R}_{c_0} , where $c_0 = (s_0 \circ G[s_0])_{(0,3)}$, are well-defined, analytic and compact in $\mathcal{B}_{\varrho_0}(s_0) \subset \mathcal{A}_s(\rho)$, with*

$$\rho = 1.75, \quad \varrho_0 = 5.79833984375 \times 10^{-4}.$$

- ii) *There exists a function $s^* \in \mathcal{B}_r(s_0) \subset \mathcal{A}_s(\rho)$ with*

$$r = 6.0 \times 10^{-12},$$

such that

$$\mathcal{R}_{c_0}[s^*] = s^*.$$

- iii) *The linear operator $\mathcal{DR}_{c_0}[s^*]$ has two eigenvalues outside of the unit circle:*

$$8.72021484375 \leq \delta_1 \leq 8.72216796875, \quad \delta_2 = \frac{1}{\lambda_*},$$

where

$$-0.248875313689 \leq \lambda_* \leq -0.248886108398438.$$

- iv) *The complement of these two eigenvalues in the spectrum is compactly contained in the unit disk:*

$$\text{spec}(\mathcal{DR}_{c_0}[s^*]) \setminus \{\delta_1, \delta_2\} \subset \{z \in \mathbb{C} : |z| \leq 0.1258544921875 \equiv \nu\}.$$

The Main Theorem implies that there exist codimension 2 local stable manifolds $\mathcal{W}_{\mathcal{R}_{c_0}}(s^*) \subset \mathcal{A}_s(1.75)$, such that the contraction rate in $\mathcal{W}_{\mathcal{R}_{c_0}}(s^*)$ is bounded from above by ν :

$$\|\mathcal{R}_{c_0}^n[s] - \mathcal{R}_{c_0}^n[\tilde{s}]\|_\rho = O(\nu^n)$$

for any two s and \tilde{s} in $\mathcal{W}_{\mathcal{R}_{c_0}}(s^*)$.

Definition 2.3. (Infinitely renormalizable maps). The set of reversible twist maps of the form (4) such that $s \in \mathcal{W}_{\mathcal{R}_{c_0}}(s^*) \subset \mathcal{A}_s(1.75)$ will be denoted W , and referred to as infinitely renormalizable maps.

Naturally, these sets are invariant under renormalization if ϱ is sufficiently small.

Notice, that, among other things, this Theorem restates the result about existence of the Eckmann-Koch-Wittwer fixed point and renormalization hyperbolicity of Theorem 1 in a setting of a different functional space. We do not prove that the fixed point s^* , after a small adjustment corresponding to the coordinate change S_t , coincides with s^{EKW} from Theorem 1, although the computer bounds on these two fixed points differ by a tiny amount on any bi-disk contained in the intersection of their domains.

The fact that the operator R_{c_0} as in (26) contains an additional coordinate change does not cause a problem: conceptually, period-doubling renormalization of a map is its second iterate conjugated by a coordinate change, which does not have to be necessarily linear.

3. Coordinate changes and renormalization eigenvalues. Let \mathcal{D} and $\tilde{\mathcal{D}}$ be as in the Theorem 3. Consider the action of the operator

$$R_*[F] = \Lambda_*^{-1} \circ F \circ \Lambda_* \quad (27)$$

on $\mathcal{O}_2(\mathcal{D})$, where

$$\Lambda_*(x, u) = (\lambda_*x, \mu_*u),$$

with λ_* and μ_* being the fixed scaling parameters corresponding to the Collet-Eckmann-Koch as in Theorem 1.

According to Theorem 1 this operator is analytic and compact on the subset $\mathcal{F}_\varrho^{0.5, 1.6}(s_{0.5})$, $\varrho = 6.0 \times 10^{-7}$, of $\mathcal{O}_2(\mathcal{D})$, and has a fixed point F_{EKW} . In this paper, we will prove the existence of a fixed point s^* of the operator \mathcal{R}_{EKW} in a Banach space different from that in Theorem 1. Therefore, we will state most of our results concerning the spectra of renormalization operators for general spaces $\mathcal{A}_s^\beta(\rho)$ and sets $\mathcal{F}_{\varrho^\beta}^{\beta, \rho^\beta}(s^*)$, under the hypotheses of existence of a fixed point s^* , and analyticity and compactness of the operators in some neighborhood of the fixed point. Later, a specific choice of parameters β , ρ and ϱ will be made, and the hypotheses - verified.

Let $S = id + \sigma$ be a coordinate transformation of the domain \mathcal{D} of maps F , satisfying

$$DS \circ F = DS.$$

In particular, these transformations preserve the subset of area-preserving maps. Notice, that

$$(id + \epsilon\sigma)^{-1} \circ F \circ (id + \epsilon\sigma) = F + \epsilon(-\sigma \circ F + DF \cdot \sigma) + O(\epsilon^2) \equiv F + \epsilon h_{F, \sigma} + O(\epsilon^2).$$

Suppose that the operator R_* has a fixed point F^* in some neighborhood $\mathcal{B} \subset \mathcal{O}_2(\mathcal{D})$, on which R_* is analytic and compact. Consider the action $DR_*[F]h_{F, \sigma}$ of

the derivative of this operator.

$$\begin{aligned}
 \mathcal{D}R_*[F]h_{F,\sigma} &= \partial_\epsilon (\Lambda_*^{-1} \circ (F + \epsilon h_{F,\sigma}) \circ (F + \epsilon h_{F,\sigma}) \circ \Lambda_*)|_{\epsilon=0} \\
 &= \partial_\epsilon (\Lambda_*^{-1} \circ (id + \epsilon\sigma)^{-1} \circ F \circ F \circ (id + \epsilon\sigma) \circ \Lambda_*)|_{\epsilon=0} \\
 &= \Lambda_*^{-1} \cdot [-\sigma \circ F \circ F + D(F \circ F) \cdot \sigma] \circ \Lambda_* \\
 &= \Lambda_*^{-1} \cdot h_{F \circ F, \sigma} \circ \Lambda_*.
 \end{aligned} \tag{28}$$

Specifically, if $F = F^*$, one gets

$$\mathcal{D}R_*[F^*]h_{F^*,\sigma} = h_{F^*,\tau}, \quad \tau = \Lambda_*^{-1} \cdot \sigma \circ \Lambda_*,$$

and clearly, $h_{F^*,\sigma}$ is an eigenvector, if $\tau = \kappa\sigma$, of eigenvalue κ . In particular,

$$\kappa = \lambda_*^i \mu_*^j, \quad i \geq 0, \quad j \geq 0$$

is an eigenvalue of multiplicity (at least) 2 with eigenvectors $h_{F^*,\sigma}$ generated by

$$\sigma_{i,j}^1(x, u) = (x^{i+1}u^j, 0), \quad \sigma_{i,j}^2(x, u) = (0, x^i u^{j+1}), \tag{29}$$

while

$$\kappa = \mu_*^j \lambda_*^{-1}, \quad j \geq 0, \quad \text{and} \quad \kappa = \lambda_*^i \mu_*^{-1}, \quad i \geq 0,$$

are each eigenvalues of multiplicity (at least) 1, generated by

$$\sigma_{-1,j}^1(x, u) = (u^j, 0), \quad \text{and} \quad \sigma_{i,-1}^2(x, u) = (0, x^i), \tag{30}$$

respectively.

We will denote S_t^σ , $S_0^\sigma = Id$, the coordinate transformation generated by a function σ as in (29)-(30).

In particular, the coordinate transformations $S_t^{\sigma_{i,0}^i}$, $i = 1, 2$, correspond to the rescalings of the coordinates x and u . The next Lemma is a specification to our case of the well-known fact that rescalings correspond to the eigenvalue 1. We will omit its lengthy proof (which we have, nevertheless carried out in detail).

Lemma 3.1. *Suppose that there are β , ϱ , ρ , λ_* , μ_* and a function $s^* \in \mathcal{A}_s^\beta(\rho)$ such that the operator R_{EKW} is analytic and compact as maps from $\mathcal{F}_\varrho^{\beta,\rho}(s^*)$ to $\mathcal{O}_2(\mathcal{D})$, and*

$$R_{EKW}[F^*] = R_*[F^*] = F^*,$$

where F^* is generated by s^* .

Then, there exists a neighborhood $\mathcal{B}(F^*) \subset \mathcal{F}_\varrho^{\beta,\rho}(s^*)$, in which R_* is analytic and compact, and

$$\text{spec}(\mathcal{D}R_*[F^*]|_{T_{F^*}\mathcal{B}(F^*)}) = \text{spec}(\mathcal{D}R_{EKW}[F^*]|_{T_{F^*}\mathcal{F}_\varrho^{\beta,\rho}(s^*)}) \cup \{1\}.$$

4. Analyticity and compactness of renormalization. We will quote a version of a lemma from [13] which we will require to demonstrate analyticity and compactness of the operator \mathcal{R}_c . The proof of the Lemma is computer-assisted. Notice, the parameters that enter the Lemma are different from those used in [13]. As before, the reported numbers are representable on a computer.

Lemma 4.1. *For all $s \in \mathcal{B}_R(s^0)$, where*

$$R = 5.47321968732772541 \times 10^{-3},$$

and s^0 is as in Theorem 2, the prerenormalization $\mathcal{P}_{EKW}[s]$ is well-defined and analytic function on the set

$$\mathcal{D}_r \equiv \mathcal{D}_r(0) = \{(x, y) \in \mathbb{C}^2 : |x| < r, |y| < r\}, \quad r = 0.51853174082497335,$$

with

$$\|Z\|_r \leq 1.63160151494042404.$$

We will now demonstrate analyticity and compactness of the modified renormalization operator \mathcal{R}_c , defined in 2.1, in a functional space, different from that used in [10], specifically, in the space $\mathcal{A}_s(1.75)$. It is in this space that we will eventually compute a bound on the spectral radius of the action of the modified renormalization operator on infinitely renormalizable maps.

Proposition 4.2. *There exists a polynomial $s_0 \in \mathcal{B}_R(s^0) \subset \mathcal{A}_s(1.75)$, where R and s^0 are as in Theorem 2, such that the operators \mathcal{R}_c , $c \in [c_0 - \delta, c_0 + \delta]$,*

$$c_0 = (s_0 \circ G[s_0])_{(0,3)} \quad \text{and} \quad \delta = 1.068115234375 \times 10^{-4},$$

are well-defined and analytic as maps from $\mathcal{B}_{\varrho_0}(s_0)$, $\varrho_0 = 5.79833984375 \times 10^{-4}$, to $\mathcal{A}_s(1.75)$.

Furthermore, the operators \mathcal{R}_c are compact in $B_R(s^0) \subset \mathcal{A}(\rho)$, with $\mathcal{R}_c[s] \in \mathcal{A}(\rho')$, $\rho' = 1.0699996948242188\rho$.

Proof. The polynomial s_0 has been computed as a high order numerical approximation of a fixed point s^* of \mathcal{R}_{EKW} .

First, we get a computer bound on t_c as in Definition 2.1 for all $s \in \mathcal{B}_{\varrho_0}(s_0)$ and $c \in [c_0 - \delta, c_0 + \delta]$:

$$|t| \leq 2.1095979213715 \times 10^{-6}. \quad (31)$$

The condition of the hypothesis that $s^* \in \mathcal{B}_\delta(s_0)$ is specifically required to be able to compute this estimate.

Notice that according to Definition 2.1 and Theorem 2, the maps $s \mapsto t$ and, hence, $s \mapsto \xi_t$ are analytic on a larger neighborhood $\mathcal{B}_R(s^0)$ of analyticity of \mathcal{R}_{EKW} . According to Theorem 2 and Lemma 4.1, the prerenormalization \mathcal{P}_{EKW} is also analytic as a map from $\mathcal{B}_R(s^0)$ to $\mathcal{A}_s(r)$, $r = 0.516235055482147608$. We verify that for all $s \in \mathcal{B}_\delta(s_0)$ and t as in (31) the following holds:

$$\{\xi_t(x, y) : (x, y) \in \mathcal{D}_{r'}\} \in \mathcal{D}_{r'}, \quad r' = |\lambda_-|\rho, \quad (32)$$

where $\lambda_- = -0.27569580078125$ is the lower bound from Theorem 2. Furthermore,

$$1 > 2|t|\rho$$

with t as in (31). Therefore, the map $s \mapsto \mathcal{P}[s]$ is analytic on $\mathcal{B}_\delta(s_0)$.

Since the inclusion of sets (32) is compact, $\mathcal{R}_c[s]$ has an analytic extension to a neighborhood of $\mathcal{D}_{1.75}$, $\mathcal{R}_c[s] \in \mathcal{A}_s(\rho')$, $\rho' > 1.75$. Compactness of the map $s \mapsto \mathcal{R}_c[s]$ now follows from the fact that the inclusions of spaces $\mathcal{A}_s(\rho') \subset \mathcal{A}_s(\rho)$ is compact. \square

5. Strong contraction on the stable manifold.

Lemma 5.1. *Suppose that β , ϱ and ρ are such that the operator*

$$\mathcal{R}_*[s] = \frac{1}{\mu_*} \mathcal{P}_{EKW}[s] \circ \lambda_*$$

has a fixed point $s^ \in \mathcal{B}_\varrho \subset \mathcal{A}_s^\beta(\rho)$, and \mathcal{R}_* is analytic and compact as a map from \mathcal{B}_ϱ to $\mathcal{A}_s^\beta(\rho)$.*

Then, the number λ_ is an eigenvalue of $\mathcal{D}\mathcal{R}_*[s^*]$, and the eigenspace of λ_* contains the eigenvector*

$$\psi_{s^*}(x, y) = s_1^*(x, y)x^2 + s_2^*(x, y)y^2 + 2s^*(x, y)y. \quad (33)$$

Proof. Consider the coordinate transformation (23),

$$\begin{aligned} S_\epsilon(x, u) &= \left(x + \epsilon x^2, \frac{u}{1 + 2\epsilon x} \right) \\ &= (x, u) + \epsilon \sigma_{1,0}^1(x, u) - 2\epsilon \sigma_{1,0}^2(x, u) + O(\epsilon^2), \end{aligned} \quad (34)$$

$$S_\epsilon^{-1}(y, v) = \left(\frac{\sqrt{1 + 4\epsilon y} - 1}{2\epsilon}, v\sqrt{1 + 4\epsilon y} \right), \quad (35)$$

for real ϵ , $|\epsilon| < 4/(\rho + |\beta|)$ (recall Definition 1.2).

Let $s \in \mathcal{A}_s^\beta(\rho)$ be the generating function for some F , then the following demonstrates that $S_\epsilon^{-1} \circ F \circ S_\epsilon$ is reversible, area-preserving and generated by

$$\hat{s}(x, y) = s(x + \epsilon x^2, y + \epsilon y^2)(1 + 2\epsilon y) :$$

$$\begin{aligned} \left(\begin{array}{c} x \\ -s(y + \epsilon y^2, x + \epsilon x^2)(1 + 2\epsilon x) \end{array} \right) &\xrightarrow{S_\epsilon} \left(\begin{array}{c} x + \epsilon x^2 \\ -s(y + \epsilon y^2, x + \epsilon x^2) \end{array} \right) \\ &= \left(\begin{array}{c} x' \\ -s(y', x') \end{array} \right) \xrightarrow{F} \left(\begin{array}{c} y' \\ s(x', y') \end{array} \right) \\ &= \left(\begin{array}{c} y + \epsilon y^2 \\ s(x + \epsilon x^2, y + \epsilon y^2) \end{array} \right) \xrightarrow{S_\epsilon^{-1}} \left(\begin{array}{c} y \\ s(x + \epsilon x^2, y + \epsilon y^2)(1 + 2\epsilon y) \end{array} \right). \end{aligned}$$

Next,

$$\hat{s}(x, y) = s(x, y) + \epsilon s_1(x, y)x^2 + \epsilon s_2(x, y)y^2 + \epsilon 2s(x, y)y + O(\epsilon^2).$$

We will demonstrate that

$$\psi_{s^*}(x, y) = s_1^*(x, y)x^2 + s_2^*(x, y)y^2 + 2s^*(x, y)y.$$

is an eigenvector of $\mathcal{DR}_*[s^*]$ of the eigenvalue λ_* . Notice, that

$$\partial_1 \psi_s = \partial_1 \psi_s \circ I, \quad I(x, y) = (y, x),$$

and therefore, the function $s + \epsilon \psi_s \in \mathcal{A}_s^\beta(\rho)$.

Consider the midpoint equation

$$\begin{aligned} 0 = O(\epsilon^2) &+ s(x, Z(x, y) + \epsilon \mathcal{DZ}[s]\psi_s(x, y)) + s(y, Z(x, y) + \epsilon \mathcal{DZ}[s]\psi_s(x, y)) \\ &+ \epsilon \psi_s(x, Z(x, y)) + \epsilon \psi_s(y, Z(x, y)) \end{aligned}$$

for the generating function $s + \epsilon \psi_s$. We get that

$$\mathcal{DZ}[s]\psi_s(x, y) = -\frac{\psi_s(x, Z(x, y)) + \psi_s(y, Z(x, y))}{s_2(x, Z(x, y)) + s_2(y, Z(x, y))},$$

and

$$\begin{aligned} \mathcal{DP}_{EKW}[s]\psi_s(x, y) &= s_1(Z(x, y), y)\mathcal{DZ}[s]\psi_s(x, y) + \psi_s(Z(x, y), y) \\ &= -2s_1(Z(x, y), y)\frac{s(x, Z(x, y))Z + s(y, Z(x, y))Z}{s_2(x, Z(x, y)) + s_2(y, Z(x, y))} \\ &\quad -s_1(Z(x, y), y)\frac{s_2(x, Z(x, y))Z(x, y)^2 + s_2(y, Z(x, y))Z(x, y)^2}{s_2(x, Z(x, y)) + s_2(y, Z(x, y))} \\ &\quad +s_1(Z(x, y), y)Z(x, y)^2 \\ &\quad -s_1(Z(x, y), y)\frac{s_1(y, Z(x, y))y^2}{s_2(x, Z(x, y)) + s_2(y, Z(x, y))} + s_2(Z(x, y), y)y^2 \\ &\quad -s_1(Z(x, y), y)\frac{s_1(x, Z(x, y))x^2}{s_2(x, Z(x, y)) + s_2(y, Z(x, y))} \\ &\quad +2s(Z(x, y), y)y \end{aligned}$$

The terms on line 2 add up to zero (the numerator is equal to zero because of the midpoint equation), and so do those on lines 3 and 4. We can also use the equalities

$$\begin{aligned} s_2(x, Z(x, y)) + s_2(y, Z(x, y)) &= -\frac{s_1(y, Z(x, y))}{Z_2(x, y)} \\ \partial_2\mathcal{P}_{EKW}[s](x, y) &= s_2(Z(x, y), y) + s_1(Z(x, y), y)Z_2(x, y) \end{aligned}$$

(the first one being the midpoint equation differentiated with respect to y) to reduce the 5-th line to $\partial_2\mathcal{P}_{EKW}[s](x, y)y^2$. The 6-th line reduces to $\partial_1\mathcal{P}_{EKW}[s](x, y)x^2$ after we use the midpoint equation differentiated with respect to x :

$$s_2(x, Z(x, y)) + s_2(y, Z(x, y)) = -\frac{s_1(x, Z(x, y))}{Z_1(x, y)}.$$

To summarize,

$$\begin{aligned} \mathcal{DP}_{EKW}\psi_s(x, y) &= \partial_1\mathcal{P}_{EKW}[s](x, y)x^2 + \partial_2\mathcal{P}_{EKW}[s](x, y)y^2 + 2\mathcal{P}_{EKW}[s](x, y)y \\ &= \psi_{\mathcal{P}_{EKW}[s]}(x, y). \end{aligned} \tag{36}$$

Finally, we use the fact that

$$\lambda_*\partial_i\mathcal{P}_{EKW}[s](\lambda_*x, \lambda_*y) = \partial_i(\mathcal{P}[s](\lambda_*x, \lambda_*y))$$

to get

$$\mathcal{DR}_*[s^*]\psi_{s^*} = \lambda_*\psi_{s^*}.$$

□

The Lemma below, whose elementary proof we will omit, shows that λ_* is also in the spectrum of $\mathcal{DR}_*[F^*]$:

Lemma 5.2. *Suppose that β, ϱ and ρ are such that $s^* \in \mathcal{A}_s^\beta(\rho)$ is a fixed point of the operators \mathcal{R}_* and \mathcal{R}_{EKW} , and the operators \mathcal{R}_* and \mathcal{R}_{EKW} are analytic and compact as a map from $\mathcal{B}_\varrho(s^*)$ to $\mathcal{A}_s^\beta(\rho)$. Also, suppose that the map \mathcal{I} , described in Remark 1.6, is a well-defined analytic embedding on $\mathcal{B}_\varrho(s^*)$. Then,*

$$\begin{aligned} \text{spec}\left((\mathcal{DR}_*[F^*])|_{T_{F^*}\mathcal{F}_\varrho^{\beta, \rho}(s^*)}\right) &= \text{spec}(\mathcal{DR}_*[s^*]) \quad \text{and} \\ \text{spec}\left((\mathcal{DR}_{EKW}[F^*])|_{T_{F^*}\mathcal{F}_\varrho^{\beta, \rho}(s^*)}\right) &= \text{spec}(\mathcal{DR}_{EKW}[s^*]), \end{aligned}$$

in particular,

$$\lambda_* \in \text{spec}(\mathcal{DR}_*[F^*]) \quad \text{and} \quad \lambda_* \in \text{spec}(\mathcal{DR}_{EKW}[s^*]).$$

The convergence rate in the stable manifold of the renormalization operator plays a crucial role in demonstrating rigidity. It turns out that *the eigenvalue λ_* is the largest eigenvalues in the stable subspace of $\mathcal{DR}_{EKW}[F^*]$* , or equivalently $\mathcal{DR}_{EKW}[s^*]$. However, it's value $|\lambda_*| \approx 0.2488$ is not small enough to ensure rigidity. At the same time, the eigenspace of the eigenvalue λ_* is, in the terminology of the renormalization theory, *irrelevant* to dynamics (the associated eigenvector is generated by a coordinate transformation). We, therefore, would like to eliminate this eigenvalue via an appropriate coordinate change, as described above.

However, first we would like to identify the eigenvector corresponding to the eigenvalue λ_* for the operator \mathcal{R}_{EKW} . This vector turns out to be different from ψ_{s^*} .

Lemma 5.3. *Suppose that β, ϱ and ρ are such that the operator \mathcal{R}_{EKW} has a fixed point $s^* \in \mathcal{A}_s^\beta(\rho)$, and \mathcal{R}_{EKW} is analytic and compact as a map from $\mathcal{B}_\varrho(s^*)$ to $\mathcal{A}_s^\beta(\rho)$. Also, suppose that the map \mathcal{I} , described in Remark 1.6, is a well-defined analytic embedding on $\mathcal{B}_\varrho(s^*)$.*

Then, the number λ_ is an eigenvalue of $\mathcal{DR}_{EKW}[s^*]$, and the eigenspace of λ_* contains the eigenvector*

$$\psi_{s^*}^{EKW} = \psi_{s^*} + \tilde{\psi}, \tag{37}$$

where

$$\tilde{\psi}(x, y) = s^*(x, y) - (s_1^*(x, y)x + s_2^*(x, y)y).$$

Proof. Notice, that $\tilde{\psi}$ is of the form

$$\tilde{\psi} = \psi_u - \psi_x, \tag{38}$$

where

$$\psi_x(x, y) = s_1^*(x, y)x + s_2^*(x, y)y$$

is the eigenvector of $\mathcal{DR}_*[s^*]$ corresponding to the rescaling of the variables x and y , while

$$\psi_u(x, y) = s^*(x, y)$$

is the eigenvector corresponding to the rescaling of s . $\psi_x(x, y)$ and $\psi_u(x, y)$ correspond to the eigenvectors $h_{F^*, \sigma_{0,0}^1}$ and $h_{F^*, \sigma_{0,0}^2}$, respectively, of $\mathcal{DR}_0[F^*]$.

Recall, that $h_{F^*, \sigma_{0,0}^1}$ and $h_{F^*, \sigma_{0,0}^2}$ are eigenvectors of $\mathcal{DR}_0[F^*]$, with eigenvalue 1, and eigenvectors of $\mathcal{DR}_{EKW}[F^*]$ with eigenvalue 0.

By Lemma 5.1 ψ_{s^*} is an eigenvector of \mathcal{DR}_* , the corresponding eigenvector of \mathcal{DR}_* is $h_{F^*, \hat{\sigma}}$, where $\hat{\sigma} = \sigma_{1,0}^1 - 2\sigma_{1,0}^2$. Thus, $\psi_{s^*} + \tilde{\psi}$ corresponds to the vector

$$h_{\lambda_*}^{EKW} := h_{F^*, \hat{\sigma}} - h_{F^*, \sigma_{0,0}^1} + h_{F^*, \sigma_{0,0}^2}. \tag{39}$$

To finish the proof, it suffices to prove that

$$\mathcal{DR}_{EKW} h_{\lambda_*}^{EKW} = \lambda_* h_{\lambda_*}^{EKW}.$$

A straightforward computation (which is a part of the proof of Lemma 3.1) gives that $\mathcal{DR}_{EKW}[F^*]h_{\lambda_*}^{EKW} = \mathcal{DR}_{EKW}[F^*]h_{F^*, \hat{\sigma}}$. Therefore,

$$\begin{aligned} \mathcal{DR}_{EKW}[F^*]h_{\lambda_*}^{EKW} &= \mathcal{DR}_*[F^*]h_{F^*, \hat{\sigma}} + (\mathcal{Dt}[F^*]h_{F^*, \hat{\sigma}})h_{F^*, \sigma_{0,0}^1} + (\mathcal{Dr}[F^*]h_{F^*, \hat{\sigma}})h_{F^*, \sigma_{0,0}^2} \\ &= \lambda_* h_{F^*, \hat{\sigma}} + (\mathcal{Dt}[F^*]h_{F^*, \hat{\sigma}})h_{F^*, \sigma_{0,0}^1} + (\mathcal{Dr}[F^*]h_{F^*, \hat{\sigma}})h_{F^*, \sigma_{0,0}^2} \end{aligned}$$

The result follows if

$$\mathcal{Dt}[F^*]h_{F^*, \hat{\sigma}} = -\lambda_* \quad \text{and} \quad \mathcal{Dr}[F^*]h_{F^*, \hat{\sigma}} = \lambda_*.$$

Indeed, going through the calculations as in the proof of Lemma 3.1 we have that, if $h = h_{F^*, \sigma_{1,0}^1}$, then

$$\begin{aligned} \mathcal{D}P_{EKW}[F^*]h(x, u) &= \\ &= (-\pi_x P_{EKW}[F^*](x, u))^2 + \pi_x P_{EKW}[F^*]_1(x, u)x^2, \pi_u P_{EKW}[F^*]_1(x, u)x^2, \\ \pi_x \mathcal{D}P_{EKW}[F^*]h(0, 0) &= -(\pi_x P_{EKW}[F^*](0, 0))^2 = -\lambda_*^2, \\ \mathcal{D}t[F^*]h &= -\lambda_*, \\ \mathcal{D}r[F^*]h &= -\lambda_* + 2\pi_x P_{EKW}[F^*](0, 0) = \lambda_*. \end{aligned}$$

Here and below the subscript 1 or 2 denotes the partial derivative with respect to the corresponding argument.

If $h = h_{F^*, \sigma_{1,0}^2}$, then

$$\begin{aligned} \mathcal{D}P_{EKW}[F^*]h(x, u) &= \\ &= (\pi_x P_{EKW}[F^*]_2(x, u)xu, -\pi_x P_{EKW}[F^*](x, u)\pi_u P_{EKW}[F^*](x, u) + \pi_u P_{EKW}[F^*]_2(x, u)xu), \\ \pi_x \mathcal{D}P_{EKW}[F^*]h(0, 0) &= 0 \\ \mathcal{D}t[F^*]h &= 0 \\ \mathcal{D}r[F^*]h &= 0 + \frac{\lambda_* (\pi_x P_{EKW}[F^*]_{2,2}(0, 0)0 + \pi_x P_{EKW}[F^*]_2(0, 0)0)}{\mu_* (\pi_x (F^* \circ F^*)_2(0, 0))^2} = 0. \end{aligned}$$

□

Recall, that according to Lemma 5.2, λ_* is an eigenvalue of $\mathcal{D}R_*[F^*]$ of multiplicity at least 1. According to Lemma 3.1, λ_* is in the spectrum of $\mathcal{D}R_{EKW}[F^*]$ and in the spectrum of $\mathcal{D}R_{EKW}[s^*]$.

Proposition 5.4. (*Elimination of an eigenvalue*). *Suppose that β, ρ, ϱ and the neighborhood $\mathcal{B}_\varrho(s^*) \subset \mathcal{A}_s^\beta(\rho)$ satisfy the hypothesis of Lemma 5.2. Furthermore, suppose that the operator $\mathcal{R}_{c^*}, c^* = (s^* \circ G)_{(0,3)}$, is analytic and compact in $\mathcal{B}_\varrho(s^*)$.*

Set

$$C = \frac{1}{2} \frac{c^*}{\mathcal{P}_{EKW}[s^*]_{0,2}}.$$

Then

$$\text{spec}(\mathcal{D}R_{EKW}[s^*]) \setminus \{\lambda_*, -\lambda_*(1 - C)\} \subset \text{spec}(\mathcal{D}R_{c^*}[s^*]),$$

and $\psi_{s^*}^{EKW}$ is an eigenvector of $\mathcal{D}R_{c^*}[s^*]$ associated with the eigenvalue λ_*C .

In addition,

$$\text{spec}(\mathcal{D}R_{c^*}[s^*]) \subset \text{spec}(\mathcal{D}R_{EKW}[s^*]) \cup \{\lambda_*C\},$$

and if $\lambda_* \notin \text{spec}(\mathcal{D}R_{c^*}[s^*])$, then λ_* has multiplicity 1 in $\text{spec}(\mathcal{D}R_{EKW}[s^*])$.

Proof. According to Proposition 4.2, under the hypothesis of the Lemma 5.2, \mathcal{R}_{EKW} and \mathcal{R}_{c^*} are analytic and compact as operators from $\mathcal{B}_\varrho(s^*)$ to $\mathcal{A}_s(1.75)$.

Recall, that $\psi_{s^*}^{EKW}$ is an eigenvector of $\mathcal{D}R_{EKW}[s^*]$ corresponding to the eigenvalue λ_* .

We consider the action of $\mathcal{D}R_{c^*}[s^*]$ on a vector ψ . First, notice the difference between the definition of λ in (1.1)

$$s(G(\lambda, 0)) = 0,$$

and in Definition 2.1

$$s(G(\lambda + t\lambda^2, 0)) = 0$$

(we will use the notation λ_{EKW} below to emphasize the difference). This implies that if $\mathcal{D}\lambda_{EKW}[s]\psi$ is an action of the derivative of $\lambda_{EKW}[s]$ on a vector ψ , then

$$\mathcal{D}\lambda[s^*]\psi = \mathcal{D}\lambda_{EKW}[s^*]\psi - \lambda_*^2 \mathcal{D}t_{c^*}[s^*]\psi$$

is that of the derivative of $\lambda[s]$.

Similarly,

$$\begin{aligned} \mathcal{D}\mu_{EKW}[s^*]\psi &= [\partial_1(s^* \circ G)(\lambda_*, 0) + \lambda_* \partial_1^2(s^* \circ G)(\lambda_*, 0)] \mathcal{D}\lambda_{EKW}[s^*]\psi \\ &\quad + \lambda_* \partial_1(\mathcal{D}\mathcal{P}_{EKW}[s^*]\psi)(\lambda_*, 0), \\ \mathcal{D}\mu[s^*]\psi &= [\partial_1(s^* \circ G)(\lambda_*, 0) + \lambda_* \partial_1^2(s^* \circ G)(\lambda_*, 0)] \mathcal{D}\lambda[s^*]\psi + \lambda_* \partial_1(\mathcal{D}\mathcal{P}_{EKW}[s^*]\psi)(\lambda_*, 0) \\ &\quad + \lambda_*^3 \partial_1^2(s^* \circ G)(\lambda_*, 0) \mathcal{D}t_{c^*}[s^*]\psi \\ &= \mathcal{D}\mu_{EKW}[s^*]\psi - \partial_1 \mathcal{P}_{EKW}[s^*](\lambda_*, 0) \lambda_*^2 \mathcal{D}t_{c^*}[s^*]\psi \\ &= \mathcal{D}\mu_{EKW}[s^*]\psi - \lambda_* \mu_* \mathcal{D}t_{c^*}[s^*]\psi. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{D}\mathcal{R}_{c^*}[s^*]\psi &= \mathcal{D}\mathcal{R}_{EKW}[s^*]\psi + 2\lambda_* (\mathcal{D}t_{c^*}[s^*]\psi) s^* \pi_y + \frac{1}{\mu_*} (\nabla \mathcal{P}_{EKW}[s^*] \cdot (D\xi_t \psi)) \circ \lambda_* \\ &\quad - \mathcal{D}t_{c^*}[s^*]\psi \frac{\lambda_*^2}{\mu_*} \nabla \mathcal{P}_{EKW}[s^*] \circ \lambda_* \cdot (\pi_x, \pi_y) + \lambda_* \mathcal{D}t_{c^*}[s^*]\psi s^* \\ &= \mathcal{D}\mathcal{R}_{EKW}[s^*]\psi - \lambda_* (\mathcal{D}t_{c^*}[s^*]\psi) \nabla s^* \cdot (\pi_x, \pi_y) \\ &\quad + \lambda_* (\mathcal{D}t_{c^*}[s^*]\psi) s^* + \lambda_* (\mathcal{D}t_{c^*}[s^*]\psi) \psi_{s^*} \\ &= \mathcal{D}\mathcal{R}_{EKW}[s^*]\psi + \lambda_* (\mathcal{D}t_{c^*}[s^*]\psi) \psi_{s^*}^{EKW} \end{aligned} \tag{40}$$

Now, let ψ be an eigenvector of $\mathcal{D}\mathcal{R}_{EKW}[s^*]$ of eigenvalue $\kappa \neq \lambda_*$ (that is, $\psi \neq \psi_{s^*}^{EKW}$). Consider the action of $\mathcal{D}\mathcal{R}_{c^*}[s^*]$ on $\psi + a\psi_{s^*}^{EKW}$.

$$\mathcal{D}\mathcal{R}_{c^*}[s^*](\psi + a\psi_{s^*}^{EKW}) = \kappa\psi + \lambda_* (\mathcal{D}t_{c^*}[s^*](\psi + a\psi_{s^*}^{EKW})) \psi_{s^*}^{EKW}.$$

Below we use the notation introduced in (38). In the following calculation we use (36) and

$$\begin{aligned} \mathcal{D}t_c[s^*]\psi_{s^*}^{EKW} &= \mathcal{D}t_c[s^*](\psi_{s^*} + \psi_u - \psi_x) \\ &= -\frac{1}{4} \frac{(\mathcal{D}\mathcal{P}_{EKW}[s^*](\psi_{s^*} + \psi_u - \psi_x))_{0,3}}{\mathcal{P}_{EKW}[s^*]_{0,2}} \\ &\quad - \frac{1}{4} \frac{(\mathcal{D}\mathcal{P}_{EKW}[s^*](\psi_{s^*} + \psi_u - \psi_x))_{0,2} (c - \mathcal{P}_{EKW}[s^*]_{0,3})}{(\mathcal{P}_{EKW}[s^*]_{0,2})^2} \\ &= -\frac{1}{4} \frac{(\psi_{\mathcal{P}_{EKW}[s^*]} + \mathcal{P}_{EKW}[s^*] - \nabla \mathcal{P}_{EKW}[s^*] \cdot (\pi_x, \pi_y))_{0,3}}{\mathcal{P}_{EKW}[s^*]_{0,2}} \\ &\quad - \frac{1}{4} \frac{(\psi_{\mathcal{P}_{EKW}[s^*]} + \mathcal{P}_{EKW}[s^*] - \nabla \mathcal{P}_{EKW}[s^*] \cdot (\pi_x, \pi_y))_{0,2} (c - \mathcal{P}_{EKW}[s^*]_{0,3})}{(\mathcal{P}_{EKW}[s^*]_{0,2})^2} \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{4} \frac{(\partial_2 \mathcal{P}_{EKW}[s^*])_{0,1} + 2(\mathcal{P}_{EKW}[s^*])_{0,2}}{\mathcal{P}_{EKW}[s^*]_{0,2}} - \frac{1}{4} \frac{(\mathcal{P}_{EKW}[s^*])_{0,3} - (\partial_2 \mathcal{P}_{EKW}[s^*])_{0,2}}{\mathcal{P}_{EKW}[s^*]_{0,2}} \\
&\quad - \frac{1}{4} \frac{\left((\partial_2 \mathcal{P}_{EKW}[s^*])_{0,0} + 2(\mathcal{P}_{EKW}[s^*])_{0,1} \right) (c - \mathcal{P}_{EKW}[s^*]_{0,3})}{(\mathcal{P}_{EKW}[s^*]_{0,2})^2} \\
&\quad - \frac{1}{4} \frac{\left((\mathcal{P}_{EKW}[s^*])_{0,2} - (\partial_2 \mathcal{P}_{EKW}[s^*])_{0,1} \right) (c - \mathcal{P}_{EKW}[s^*]_{0,3})}{(\mathcal{P}_{EKW}[s^*]_{0,2})^2} \\
&= -1 + \frac{1}{2} \frac{c^*}{\mathcal{P}_{EKW}[s^*]_{0,2}} - \frac{1}{4} \left(\frac{3\mathcal{P}_{EKW}[s^*]_{0,1}}{\mathcal{P}_{EKW}[s^*]_{0,2}} - 1 \right) \frac{c - c^*}{\mathcal{P}_{EKW}[s^*]_{0,2}} \\
&= -1 + C.
\end{aligned}$$

Denote $d \equiv \mathcal{D}t_c[s^*]\psi$, then

$$\begin{aligned}
\mathcal{DR}_{c^*}[s^*](\psi + a\psi_{s^*}^{EKW}) &= \kappa\psi + \lambda_*(d + a(-1 + C))\psi_{s^*}^{EKW} \\
&= \kappa \left(\psi + \frac{\lambda_*}{\kappa} (d + a(-1 + C))\psi_{s^*}^{EKW} \right),
\end{aligned}$$

and we see that the equation

$$a = \frac{\lambda_*}{\kappa} (d + a(-1 + C))$$

has a unique solution a if

$$\kappa \neq \lambda_*(-1 + C). \quad (41)$$

For such κ , the vector

$$\psi + \frac{\lambda_* d}{\kappa - \lambda_*(C - 1)} \psi_{s^*}^{EKW}$$

is an eigenvector of $\mathcal{DR}_{c^*}[s^*]$ associated with the eigenvalue κ .

On the other hand if $\psi = \psi_{s^*}^{EKW}$, then

$$\mathcal{DR}_{c^*}[s^*]\psi_{s^*}^{EKW} = \lambda_*\psi_{s^*}^{EKW} + \lambda_*(-1 + C)\psi_{s^*}^{EKW} = \lambda C\psi_{s^*}^{EKW}.$$

□

Remark 5.5. A computer estimate of C for all $s \in \mathcal{B}_{\varrho_0}(s_0)$ demonstrates that $|\lambda_* C| < 0.00124359130859375$, i.e. the eigenvalue corresponding to $\psi_{s^*}^{EKW}$ is almost eliminated.

At the same time, our computer estimates on the spectrum in the proof of the Main Theorem demonstrate that $-\lambda_*(1 - C)$ is not an eigenvalue of $\mathcal{DR}_{EKW}[s^*]$.

So far we were not able to make any claims about the multiplicity of the eigenvalue λ_* in the spectrum of $\mathcal{DR}_{EKW}[s^*]$. However, we will demonstrate in Section 6 that it is indeed equal to 1.

6. Spectral properties of \mathcal{R}_c . Proof of main theorem. We will now describe our computer-assisted proof of the Main Theorem.

To implement the operator $\mathcal{DR}_{c^*}[s^*]$ on the computer, we would have to know bounds on c^* . This is certainly possible, but unnecessary: for the purposes of proving rigidity, it suffices to work with an operator \mathcal{R}_c where c is close to c^* . At the same time, using an exact representable value of c in a computer assisted proof,

as opposed to using an interval bound on c^* , greatly reduces sizes of error bounds in the proof.

We choose the following value of c :

$$c_0 = (s_0 \circ G[s_0])_{(0,3)},$$

where s_0 is our polynomial approximation for the fixed point.

The operator \mathcal{R}_{c_0} differs from \mathcal{R}_{c^*} only in the “tiny amount” by which the eigendirection $\psi_{s^*}^{EKW}$ is “eliminated”. As long as the spectral radius of the operator \mathcal{DR}_{c_0} is sufficiently small, neither the difference between c_0 and c^* , nor the difference between the spectra of \mathcal{DR}_{c_0} and \mathcal{DR}_{c^*} plays an especially important role in our rigidity proof in [16].

We will now describe a rigorous computer upper bound on the spectrum of the operator $\mathcal{DR}_{c_0}[s^*]$.

Proof of part ii) of Main Theorem.

Step 1). Recall the Definition 1.2 of the Banach subspace $\mathcal{A}_s(\rho)$ of $\mathcal{A}(\rho)$. We will now choose a new bases $\{\psi_{i,j}\}$ in $\mathcal{A}_s(\rho)$. Given $s \in \mathcal{A}_s(\rho)$ we write its Taylor expansion in the form

$$s(x, y) = \sum_{(i,j) \in I} s_{i,j} \psi_{i,j}(x, y),$$

where $\psi_{i,j} \in \mathcal{A}_s(\rho)$:

$$\begin{aligned} \tilde{\psi}_{i,j}(x, y) &= x^{i+1}y^j, \quad i = -1, \quad j \geq 0, \\ \tilde{\psi}_{i,j}(x, y) &= x^{i+1}y^j + \frac{i+1}{j+1}x^{j+1}y^i, \quad i > -1, \quad j \geq i, \\ \psi_{i,j} &= \frac{\tilde{\psi}_{i,j}}{\|\tilde{\psi}_{i,j}\|_\rho}, \quad i \geq -1, \quad j \geq \max\{0, i\}, \end{aligned}$$

and the index set I of these basis vectors is defined as

$$I = \{(i, j) \in \mathbb{Z}^2 : i \geq -1, \quad j \geq \max\{0, i\}\}.$$

Denote $\tilde{\mathcal{A}}_s(\rho)$ the set of all sequences

$$\tilde{s} = \left\{ s_{i,j} : s_{i,j} \in \mathbb{C}, \sum_{(i,j) \in I} |s_{i,j}| < \infty \right\}.$$

Equipped with the l_1 -norm

$$|s|_1 = \sum_{(i,j) \in I} |s_{i,j}|, \tag{42}$$

$\tilde{\mathcal{A}}_s(\rho)$ is a Banach space, which is isomorphic to $\mathcal{A}_s(\rho)$. Clearly, the isomorphism $J : \mathcal{A}_s(\rho) \mapsto \tilde{\mathcal{A}}_s(\rho)$ is an isometry:

$$\|\cdot\|_\rho = |\cdot|_1.$$

We divide the set I in three disjoint parts:

$$\begin{aligned} I_1 &= \{(i, j) \in I : i + j < N\}, \\ I_2 &= \{(i, j) \in I : N \leq i + j < M\}, \\ I_3 &= \{(i, j) \in I : i + j \geq M\}, \end{aligned}$$

with

$$N = 22, \quad M = 60.$$

We will denote the cardinality of the first set as $D(N)$, the cardinality of $I_1 \cup I_2$ as $D(M)$.

We assign a single index to vectors $\psi_{i,j}$, $(i, j) \in I_1 \cup I_2$, as follows:

$$k(-1, 0) = 1, \quad k(-1, 1) = 2, \quad \dots, \quad k(-1, M) = M + 1, \quad k(0, 0) = M + 2,$$

$$k(0, 1) = M + 3, \quad \dots, \quad k\left(\left[\frac{M-1}{2}\right], M-1 - \left[\frac{M-1}{2}\right]\right) = D(M).$$

This correspondence $(i, j) \mapsto k$ is one-to-one, we will, therefore, also use the notation $(i(k), j(k))$.

For any $s \in \mathcal{A}_s(\rho)$, we define the following projections on the subspaces of the linear subspace $E_{D(N)}$ spanned by $\{\psi_k\}_{k=1}^{D(N)}$.

$$\Pi_k s = s_{i(k),j(k)} \psi_k, \quad \Pi_{E_{D(N)}} s = \sum_{m \leq D(N)} \Pi_m s.$$

Fix

$$c_0 = (s_0 \circ G[s_0])_{0,3},$$

where s_0 is some good numerical approximation of the fixed point. Denote for brevity $\mathcal{L}_c^s \equiv \mathcal{DR}_c[s]$. We can now write a matrix representation of the finite-dimensional linear operator

$$\Pi_{E_{D(N)}} \mathcal{L}_{c_0}^{s_0} \Pi_{E_{D(N)}}$$

as

$$D_{n,m} = \Pi_m \mathcal{L}_{c_0}^{s_0} \psi_n.$$

Step 2). We compute a set of eigenvectors e_k of the matrix D numerically, and form a $D(N) \times D(N)$ matrix A whose columns are the numerically computed eigenvectors e_k . We would now like to find a rigorous bound \mathbf{B} on the inverse B of A .

Let B_0 be an approximate inverse of A . Consider the operator C in the Banach space of all $D(N) \times D(N)$ matrices (isomorphic to $\mathbb{R}^{D(N)^2}$) equipped with the l_1 -norm, given by

$$C[B] = (A + \mathbb{I})B - \mathbb{I}.$$

Notice, that if B is a fixed point of C then $AB = \mathbb{I}$. Consider a ‘‘Newton map’’ for C :

$$N[z] = z + C[B_0 - B_0 z] - B_0 + B_0 z.$$

If z is a fixed point of N , then $B_0 - B_0 z$ is a fixed point of C . Furthermore,

$$DN[z] = \mathbb{I} - AB_0$$

is constant. We therefore, estimate l_1 matrix norms

$$\|N[0]\|_1 \leq \epsilon, \quad \|\mathbb{I} - AB_0\|_1 \leq \mathcal{L},$$

and obtain via the Contraction Mapping Principle, that the inverse of A is contained in the l_1 δ -neighborhood of B_0 , with

$$\delta = \|B_0\|_1 \frac{\epsilon}{1 - \mathcal{L}}.$$

Step 3). Define the linear operator

$$\mathcal{A} = A \Pi_{E_{D(N)}} \bigoplus (\mathbb{I} - \Pi_{E_{D(N)}}),$$

and its inverse

$$\mathcal{B} = B \Pi_{E_{D(N)}} \bigoplus (\mathbb{I} - \Pi_{E_{D(N)}}).$$

Consider the action of the operator $\mathcal{L}_{c_0}^s$ in the new basis

$$[e_1, e_2, \dots, e_{D(N)}] \equiv [\psi_1, \psi_2, \dots, \psi_{D(N)}]A, \text{ and } e_k \equiv \psi_k \text{ for } k > D(N). \tag{43}$$

To be specific, we consider a new Banach space $\hat{\mathcal{A}}_s(\rho)$: the space of all functions

$$s = \sum_k c_k e_k,$$

analytic on a bi-disk \mathcal{D}_ρ , for which the norm

$$\|s\|_1 = \sum_k |c_k|$$

is finite. For any $s \in \hat{\mathcal{A}}_s(\rho)$, we define the following projections on the basis vectors.

$$P_i s = c_i e_i, \quad P_{>k} s = \left(\mathbb{I} - \sum_{i=1}^k P_i \right) s. \tag{44}$$

Clearly, the Banach spaces $\mathcal{A}_s(\rho)$ and $\hat{\mathcal{A}}_s(\rho)$ are isomorphic, while the norms $\|\cdot\|_\rho$ and $\|\cdot\|_1$ are equivalent. We can use (43) to compute the equivalence constant α in

$$\alpha \|\cdot\|_1 \geq \|\cdot\|_\rho = |\cdot|_1$$

(recall, norms $\|\cdot\|_\rho$ and $|\cdot|_1$, defined in (42) are equal):

$$|s|_1 \leq \|A\|_1 \sum_{1 \leq k \leq D(N)} |c_k| + \sum_{i > D(N)} |c_i| \leq \alpha \|s\|_1,$$

where $\alpha = \max\{\|A\|_1, 1\}$. The constant has been rigorously evaluated on the computer:

$$\alpha \leq 49.435546875. \tag{45}$$

The operator $\mathcal{L}_{c_0}^s$ is ‘‘almost’’ diagonal in this new basis for all $s \in \mathcal{B}_r(s_0) \subset \mathcal{A}_s(\rho)$ with

$$r = 6.0 \times 10^{-12}.$$

We proceed to quantify this claim.

$$\begin{aligned} \|P_2 \mathcal{L}_{c_0}^s e_1\|_1 &\leq 5.19007444381714 \times 10^{-4}, & \|P_1 \mathcal{L}_{c_0}^s e_2\|_1 &\leq 1.76560133695602 \times 10^{-4}, \\ \|P_{>2} \mathcal{L}_{c_0}^s e_1\|_1 &\leq 3.5819411277771 \times 10^{-3}, & \|P_{>2} \mathcal{L}_{c_0}^s e_2\|_1 &\leq 1.49521231651306 \times 10^{-3}, \\ \|P_1 \mathcal{L}_{c_0}^s P_{>2}\|_1 &\leq 1.22539699077606 \times 10^{-4}, & \|P_2 \mathcal{L}_{c_0}^s P_{>2}\|_1 &\leq 8.2328915596008310 \times 10^{-5}, \end{aligned}$$

for all $s \in \mathcal{B}_r(s_0) \subset \mathcal{A}_s(\rho)$.

Next, we adjust the basis by an almost diagonal near identity linear operator $\mathcal{M} = \mathcal{M}(s)$: $[\hat{e}_1, \hat{e}_2, \dots] \equiv [e_1, e_2, \dots]\mathcal{M}$, so that the operator $\mathcal{L}_{c_0}^s$ would be block-diagonal in the new basis:

$$\mathcal{L}_{c_0}^s = \delta_1(s) \hat{P}_1 \oplus \delta_2(s) \hat{P}_2 \oplus \mathcal{Y}_{c_0}^s,$$

where \hat{P}_i and $\hat{P}_{>k}$ are defined similarly to (44) for the new basis \hat{e}_i , and $\mathcal{Y}_{c_0}^s = \hat{P}_{>2} \mathcal{L}_{c_0}^s \hat{P}_{>2}$. We have estimated

$$\|\mathcal{Y}_{c_0}^s\|_1 < 0.1258544921875. \tag{46}$$

for all $s \in \mathcal{B}_r(s_0)$. This is also an upper bound on the norm of the operator $P_{>2} \mathcal{L}_{c_0}^s P_{>2}$ for all $s \in \mathcal{B}_r(s_0)$.

Step 4). We will now demonstrate existence of a fixed point s^* in $\mathcal{B}_r(s_0) \in \mathcal{A}_s(\rho)$, of the operator \mathcal{R}_{c_0} , where

$$c_0 = (s_0 \circ G[s_0])_{0,3}.$$

We will use the Contraction Mapping Principle in the following form. Define the following linear operator on $\hat{\mathcal{A}}_s(\rho)$

$$M \equiv [\mathbb{I} - K]^{-1}, \text{ where } Kh \equiv \hat{\delta}_1 P_1 h + \hat{\delta}_2 P_2 h,$$

and $\hat{\delta}_1$ and $\hat{\delta}_2$ are defined via

$$P_1 \mathcal{L}_{c_0}^{s_0} e_1 = \hat{\delta}_1 e_1, \quad P_2 \mathcal{L}_{c_0}^{s_0} e_2 = \hat{\delta}_2 e_2.$$

Consider the operator

$$\mathcal{N}[h] = h + \mathcal{R}_{c_0}[s_0 + Mh] - (s_0 + Mh)$$

on $\hat{\mathcal{A}}_s(\rho)$. The operator \mathcal{N} is analytic on $\mathcal{B}_{\|M\|_1^{-1}\alpha^{-1}r}(0)$, where α is the norm equivalence constant (45), and

$$\|M\|_1 = \max \left\{ \left| \frac{1}{1 - \hat{\delta}_1} \right|, \left| \frac{1}{1 - \hat{\delta}_2} \right|, 1 \right\} = 1.$$

Notice, that if h^* is a fixed point of \mathcal{N} , then $s_0 + Mh^*$ is a fixed point of \mathcal{R}_{c_0} . The norm of the derivative of the operator \mathcal{N} is bounded from above by a number close to the norm of $\mathcal{Y}_{c_0}^s$, indeed,

$$\begin{aligned} \mathcal{DN}[h] &= \mathbb{I} + \mathcal{DR}_{c_0}[s_0 + Mh] \cdot M - M = [M^{-1} + \mathcal{DR}_{c_0}[s_0 + Mh] - \mathbb{I}] \cdot M \\ &= [\mathbb{I} - K + \mathcal{DR}_{c_0}[s_0 + Mh] - \mathbb{I}] \cdot M = [\mathcal{DR}_{c_0}[s_0 + Mh] - K] \cdot M \end{aligned}$$

which implies that

$$\|\mathcal{DN}[h]\|_1 \leq \|\mathcal{L}_{c_0}^s - K\|_1 \|M\|_1 = \|\mathcal{L}_{c_0}^s - K\|_1,$$

here $s = s_0 + Mh$. The last norm is close to (46). We have found a conservative estimate on $\|\mathcal{DN}[h]\|_1$ for all $h \in \mathcal{B}_{\alpha^{-1}r}(0)$:

$$\|\mathcal{DN}[h]\|_1 =: L \leq 0.15625.$$

At the same time

$$\|\mathcal{N}[0]\|_1 = \|\mathcal{R}_{c_0}[s_0] - s_0\|_1 =: \epsilon \leq 4.9560546875 \times 10^{-16}.$$

We can now see that the hypothesis of the Contraction Mapping Principle is indeed verified:

$$\epsilon \leq 4.9560546875 \times 10^{-16} < 1.015625 \times 10^{-13} < (1 - L)\alpha^{-1}r,$$

and therefore, the neighborhood $\mathcal{B}_{\epsilon/(1-L)}(0) \subset \mathcal{B}_{0.005\alpha^{-1}r}(0)$ contains a fixed point h^* of \mathcal{N} , i.e. the neighborhood $\mathcal{B}_{0.005r}(s_0) \subset \mathcal{B}_r(s_0) \subset \mathcal{A}_s(\rho)$ contains a fixed point $s^* = s_0 + Mh^*$ of \mathcal{R}_{c_0} .

We quote here for reference purposes the bounds on the values of the scalings $\lambda[s^*]$ and $\mu[s^*]$:

$$\lambda[s^*] = [-0.248875288734817765, -0.248875288702286711], \quad (47)$$

$$\mu[s^*] = [0.0611101382055370338, 0.0611101382190655586]. \quad (48)$$

Step 5). Notice, that in general,

$$(s^* \circ G[s^*])_{0,3} \neq c_0,$$

therefore

$$t_{c_0}[s^*] \neq 0.$$

However, $t_{c_0}[s^*]$ is a small number which we have estimated to be

$$|t_{c_0}[s^*]| < 7.89560771750566329 \times 10^{-12}. \quad (49)$$

Consider the map F^* generated by s^* . Recall that by Theorem 3, there exists a simply connected open set \mathcal{D} such that $F_{c_0}^* \in \mathcal{O}_2(\mathcal{D})$. The fixed point equation for the map $F_{c_0}^*$ is as follows:

$$\Lambda_{F_{c_0}^*}^{-1} \circ S_{t_{c_0}[s^*]}^{-1} \circ F_{c_0}^* \circ F_{c_0}^* \circ S_{t_{c_0}[s^*]} \circ \Lambda_{F_{c_0}^*} = F_{c_0}^*.$$

□

Acknowledgments. This work was started during a visit by the authors to the Institut Mittag-Leffler (Djursholm, Sweden) as part of the research program on “Dynamics and PDEs”. The hospitality of the institute is gratefully acknowledged. The second author was funded by a postdoctoral fellowship from the Institut Mittag-Leffler.

REFERENCES

- [1] J. J. Abad and H. Koch, [Renormalization and periodic orbits for Hamiltonian flows](#), *Comm. Math. Phys.*, **212** (2000), 371–394.
- [2] J. J. Abad, H. Koch and P. Wittwer, [A renormalization group for Hamiltonians: Numerical results](#), *Nonlinearity*, **11** (1998), 1185–1194.
- [3] G. Benettin et al, [Universal properties in conservative dynamical systems](#), *Lettere al Nuovo Cimento*, **28** (1980), 1–4.
- [4] T. Bountis, [Period doubling bifurcations and universality in conservative Systems](#), *Physica*, **3** (1981), 577–589.
- [5] A. de Carvalho, M. Lyubich and M. Martens, [Renormalization in the Hénon family, I: Universality but non-rigidity](#), *J. Stat. Phys.*, **121** (2005), 611–669.
- [6] P. Collet, J.-P. Eckmann and H. Koch, [Period doubling bifurcations for families of maps on \$\mathbb{R}^n\$](#) , *J. Stat. Phys.*, **3D** (1980).
- [7] P. Collet, J.-P. Eckmann and H. Koch, [On universality for area-preserving maps of the plane](#), *Physica D*, **3** (1981), 457–467.
- [8] B. Derrida and Y. Pomeau, [Feigenbaum’s ratios of two dimensional area preserving maps](#), *Phys. Lett. A*, **80** (1980), 217–219.
- [9] J.-P. Eckmann, H. Koch and P. Wittwer, [Existence of a fixed point of the doubling transformation for area-preserving maps of the plane](#), *Phys. Rev. A*, **26** (1982), 720–722.
- [10] J.-P. Eckmann, H. Koch and P. Wittwer, [A computer-assisted proof of universality for area-preserving maps](#), *Memoirs of the American Mathematical Society*, **47** (1984), vi+122 pp.
- [11] H. Epstein, [New proofs of the existence of the Feigenbaum functions](#), *Commun. Math. Phys.*, **106** (1986), 395–426.
- [12] D. Gaidashev, [Renormalization of isoenergetically degenerate Hamiltonian flows and associated bifurcations of invariant tori](#), *Discrete Contin. Dyn. Syst.*, **13** (2005), 63–102.
- [13] D. Gaidashev, [Period doubling renormalization for area-preserving maps and mild computer assistance in contraction mapping principle](#), *Int. Journal of Bifurcations and Chaos*, **21** (2011), 3217–3230.
- [14] D. Gaidashev and T. Johnson, [Dynamics of the universal area-preserving map associated with period doubling: Hyperbolic sets](#), *Nonlinearity*, **22** (2009), 2487–2520.
- [15] D. Gaidashev and T. Johnson, [Dynamics of the universal area-preserving map associated with period doubling: Stable sets](#), *J. Mod. Dyn.*, **3** (2009), 555–587.
- [16] D. Gaidashev, T. Johnson and M. Martens, [Rigidity for infinitely renormalizable area-preserving maps](#), *Duke Mathematical Journal*, **165** (2016), 129–159.
- [17] D. Gaidashev and H. Koch, [Renormalization and shearless invariant tori: Numerical results](#), *Nonlinearity*, **17** (2004), 1713–1722.
- [18] D. Gaidashev and H. Koch, [Period doubling in area-preserving maps: An associated one-dimensional problem](#), *Ergod. Th. & Dynam. Sys.*, **31** (2011), 1193–1228.
- [19] P. Hazard, [Hénon-like maps with arbitrary stationary combinatorics](#), *Ergod. Th. & Dynam. Sys.*, **31** (2011), 1391–1443.
- [20] P. E. Hazard, M. Lyubich and M. Martens, [Renormalisable Henon-like maps and unbounded geometry](#), *Nonlinearity*, **25** (2012), 397–420.

- [21] R. H. G. Helleman, Self-generated chaotic behavior in nonlinear mechanics, in *Fundamental Problems in Statistical Mechanics* (ed. E. G. D. Cohen), North-Holland, Amsterdam, 1980, 165–233.
- [22] T. Johnson, No elliptic islands for the universal area-preserving map, *Nonlinearity*, **24** (2011), 2063–2078.
- [23] K. Khanin, J. Lopes Dias and J. Marklof, Multidimensional continued fractions, dynamic renormalization and KAM theory, *Comm. Math. Phys.*, **270** (2007), 197–231.
- [24] H. Koch, On the renormalization of Hamiltonian flows, and critical invariant tori, *Discrete Contin. Dyn. Syst.*, **8** (2002), 633–646.
- [25] H. Koch, A renormalization group fixed point associated with the breakup of golden invariant tori, *Discrete Contin. Dyn. Syst.*, **11** (2004), 881–909.
- [26] H. Koch, Existence of critical invariant tori, *Ergod. Th. & Dynam. Sys.*, **28** (2008), 1879–1894.
- [27] S. Kocić, Renormalization of Hamiltonians for Diophantine frequency vectors and KAM tori, *Nonlinearity*, **18** (2005), 2513–2544.
- [28] M. Lyubich and M. Martens, Renormalization in the Hénon family, II: Homoclinic tangle, *Invent. Math.*, **186** (2011), 115–189.
- [29] M. Lyubich and M. Martens, Probabilistic universality in two-dimensional dynamics, e-print [arXiv:1106.5067](https://arxiv.org/abs/1106.5067), (2011).
- [30] Y. W. Nam, Renormalization for three-dimensional Hénon-like maps, e-print [arXiv:1408.4289](https://arxiv.org/abs/1408.4289), (2014).
- [31] Programs available at <http://www2.math.uu.se/~gaidash>.

Received December 2014; revised November 2015.

E-mail address: gaidash@math.uu.se

E-mail address: tomas.johnson@fcc.chalmers.se