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QUASI-EFFECTIVE STABILITY FOR NEARLY INTEGRABLE HAMILTONIAN SYSTEMS

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Abstract. This paper concerns with the stability of the orbits for nearly integrable Hamiltonian systems. Based on Nekehoroshev's original works in [14], we present the definition of quasi-effective stability and prove a theorem on quasi-effective stability under the R*ü*ssmann's non-degeneracy. Our result gives a relation between KAM theorem and effective stability. A rapidly converging iteration procedure with two parameters is designed.

1. **Introduction and main result.** KAM theory and effective stability are two important contexts in the area of Hamiltonian dynamical systems. The former is established by Kolmogorov, Arnold and Moser, in 1954-1963s [7, 1, 13]. The latter is developed by Nekhoroshev in 1977 [14]. On the one hand, the classical KAM theory shows that under appropriate non-degeneracy such as the classical non-degeneracy or R*ü*ssmann's non-degeneracy of the integrable Hamiltonian, the nearly integrable systems persist or keep the majority of invariant tori of integrable systems. Hence, the majority of orbits, which is in the invariant tori, is perpetual stable. On the other hand, Nekhoroshev's theorem points out that under the steepness of the integrable systems the action variables slowly evolve over exponentially long time interval under sufficiently small Hamiltonian perturbations.

A question is whether there are any relations between the KAM theory and the effective stability. As is known to all, their similarities are that they can be used to describe the stability of orbits in the phase space for Hamiltonian systems. A common condition of them is convexity of the integrable Hamiltonian [9, 17]. In

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1995 Morbidelli and Giorgilli considered a kind of nearly integrable Hamiltonian systems, and found a connection between KAM theorem and effective stability in the sense of the diffusion speed [12]. Later on Delshams and Gutiérrez discussed the similar problem [4]. They investigated the quasiconvex systems, and gave a common approach to the proofs of KAM and Nekhoroshev's theorems by applying Nekhoroshev's iteration with some modifications.

An interesting topic is that under the conditions of KAM theorem, such as R$\ddot{u}$ssmann's non-degeneracy, one is wondering if there is a Nekhoroshev type result. In this paper we investigate stability of the orbits in nearly integrable Hamiltonian systems under R$\ddot{u}$ssmann's non-degeneracy and obtain a result about quasi-effective stability.

Consider a nearly integrable Hamiltonian system in the form

$$\dot{p} = -H_q(p,q),\ \dot{q} = H_p(p,q) \tag{1}$$

with the Hamiltonian

$$H(p,q) = h(p) + f_\epsilon(p,q),\ f_\epsilon(p,q) = \epsilon f_*(p,q,\epsilon) \tag{2}$$

for nonnegative small parameter ϵ. Here $p \in D$ are the action variables, D is some bounded domain in $\mathbb{R}^n$, while $q \in \mathbb{T}^n$ are the conjugate angle variables, $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ is a usual torus. Moreover, all our Hamiltonian functions are assumed to be real analytic in all arguments. The phase space of system (1) is $D\times\mathbb{T}^n \subset D\times\mathbb{R}^n$ with the standard symplectic structure $\sum_{j=1}^{n} \mathrm{d}p_j \wedge \mathrm{d}q_j$.

As $\epsilon = 0$, system (1) is said to be integrable, and its general solution is

$$p(t) = p_0,\ q(t) = q_0 + \omega(p_0)t \,(\mathrm{mod} 2\pi)$$

with $\omega(p_0) = h_p(p_0)$, which forms an invariant torus $\mathbf{T}_{p_0} = \{p_0\}\times\mathbb{T}^n$.

To state our results, we need some concepts. Throughout this paper we use Euclidean norm and the supremum norm, and denoted by $|\cdot|$ and $\|\cdot\|$, respectively. For an $m\times n$ matrix function $A(u)$ defined on some set D, let $\|A\| = \sup_{u\in D}\sup_{|z|=1}\|A(u)z\|$.

Definition 1. ([14]) System (1) is said to be effective stable in $E\times\mathbb{T}^n$, if there exist positive constants a, b, c and ϵ_0 such that, as $0 \le \epsilon \le \epsilon_0$, for all $(p_0,q_0) \in E\times\mathbb{T}^n$, one has $|p(t)-p_0| \le c\epsilon^b$ with $(p(0),q(0)) = (p_0,q_0)$, provided $|t| \le \exp(c\epsilon^{-a})$. Here a and b are called stable exponents, $T(\epsilon) = \exp(c\epsilon^{-a})$ stable time, $R(\epsilon) = c\epsilon^b$ stable radius.

Definition 2. An orbit $(p(t),q(t))$ starting from (p_0,q_0) of system (1) is said to be of near-invariant tori on exponentially long time, if there exist positive constants a, b, c, ϵ_0 and constant $d \ge 0$, and the function ω_{**} defined on $E\times\mathbb{T}^n$ such that $|p(t)-p_0| \le c\epsilon^b$ and $|q(t)-q_0-t\omega_{**}(p_0,q_0)| \le c\epsilon^d$, provided $0 \le \epsilon \le \epsilon_0$ and $|t| \le \exp(c\epsilon^{-a})$.

Definition 2 is a notion of stability of orbits. This definition is established by Morbidelli and Giorgilli [10, 11, 12], and Perry and Wiggins ([15]), and Delshams and Gutiérrez [4], respectively. They deal with two different cases of invariant tori of the integrable system. In [15] and [11], the property of near-invariant tori are expressed in terms of the distance to a given KAM torus. In [10] and [12] and [4], the invariant tori are considered only under the frequency vector satisfying the finite inequalities of small denominators. This paper concerns the above two cases.

Definition 3. System (1) is said to be quasi-effective stable if there exist positive constants a, b, c, d and ϵ_0 such that, for any $\epsilon \in (0, \epsilon_0]$, there is an open subset E_ϵ of D suiting the following

(1) $\text{meas} E_\epsilon = \text{meas} D - O(\epsilon^d)$.
(2) For all $(p_0, q_0) \in E_\epsilon \times T^n$, the orbit $(p(t), q(t))$ starting from (p_0, q_0) satisfies the estimate

$$|p(t) - p_0| \leq c\epsilon^b,$$

provided $|t| \leq \exp(c\epsilon^{-a})$.

Here a and b are called stable exponents of the system, $T(\epsilon) = \exp(c\epsilon^{-a})$ stable time, $R(\epsilon) = c\epsilon^b$ stable radius.

It directly follows from the above definitions that the effective stability implies quasi-effective stability.

Let B be a bounded subset of $\mathbb{C}^n$. For a given constant $\delta > 0$, denote $B + \delta = \{x \in \mathbb{C}^n : \text{dist}(x, B) < \delta\}$ and $B - \delta = \{x \in B : \text{dist}(x, \partial B) > \delta\}$, respectively, which are used by Arnol'd in [1] . Write $\text{Re}(B) = B \cap \mathbb{R}^n$.

Note that real analytic property of Hamiltonian $H(p, q)$ implies that there exists a positive constant δ such that it is analytic in $(D \times \mathbb{T}^n) + \delta$. Moreover, on $(D \times \mathbb{T}^n) + \delta$, for ϵ with $0 \leq \epsilon \leq 1$,

$$\max\{\|p\|, \|f_\epsilon\|, \|h\|, \|\omega\|, \|\omega_p\|\} \leq \frac{1}{2}M \tag{3}$$

for some positive constant M. Here $\omega(p) = h_p(p)$. Assume that $\omega(p)$ satisfies R$\ddot{u}$ssmann's nondegenerate condition as follows

(H1)

$$\text{rank}\left\{\omega, \frac{\partial^\alpha \omega}{\partial p^\alpha} : \forall \alpha \in \mathbb{Z}^n_+, \ |\alpha| < n - 1\right\} = n, \ \forall p \in \text{Re}(D + \delta), \tag{4}$$

where $\mathbb{Z}^n_+$ denotes the subset of $\mathbb{Z}^n$ with nonnegative integer components; $\frac{\partial^\alpha \omega}{\partial p^\alpha} = \frac{\partial^{|\alpha|} \omega}{\partial p_1^{\alpha_1} \cdots \partial p_n^{\alpha_n}}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Now we describe the main result of this paper.

Theorem A. *Under assumption (H1) system (1) is quasi-effective stable.*

Recently, many achievements have been made in studying KAM theory and the effective stability. For examples, Guzzo, Chierchia and Benettin have announced that they obtained optimal stability exponents under the steepness [6]; Bounemoura and Fischler make use of geometry of numbers to relate two dual Diophantine problems which correspond to the situations of KAM and Nekhoroshev theorems, respectively [2]. For the others, see [3, 5, 8, 19].

The paper is divided into five sections. In section 2 the stickiness of Diophantine invariant tori is considered and the theorem on property of near-invariant tori is described. Section 3 proposes an auxiliary proposition which plays a fundamental role in the proofs of theorems. Finally, the proofs of the theorems are placed in section 4 and section 5.

2. **Stickiness of Diophantine invariant tori.** So-called stickiness of an invariant torus means that all orbits starting near this torus are of near-invariant torus. In this section we consider the stickiness of Diophantine invariant tori.

For a given $p_0 \in D$, if $\omega(p_0)$ satisfies the following inequalities

$$|\langle k, \omega(p_0)\rangle| \geq \alpha |k|^{-\tau}, \ \forall k \in \mathbb{Z}^n \backslash \{0\} \tag{5}$$

for some positive constants α and τ, then $\mathbf{T}_{p_0}$ is said to be Diophantine.

According to KAM theory there is a nearly identity transformation Φ_ϵ which changes Diophantine invariant torus $\mathbf{T}_{p_0}$ of a integrable system into the the invariant torus $\Phi_\epsilon(\mathbf{T}_{p_0})$ of perturbed system (1) (ϵ is sufficiently small), and $\Phi_0(\mathbf{T}_{p_0}) = \mathbf{T}_{p_0}$.

Theorem B. *If $\omega(p_0)$ satisfies (5), then there is $\epsilon_0 > 0$ such that as $0 < \epsilon \leq \epsilon_0$, there exits a neighborhood O_ϵ of p_0 satisfying that for any $(p_*, q_*) \in O_\epsilon \times \mathbb{T}^n$, the orbit $(p(t), q(t))$ starting from (p_*, q_*) of system (1) is of near-invariant torus.*

3. **An auxiliary proposition.** We first construct a rapidly converging iteration scheme with two small parameters. This design is important to prove theorems. Notice that we only need finite iterations in the proof of theorems. Hence, instead of the Diophantine condition, we employ another weaker condition. Take a fixed $p_0 \in D$ and a given sufficiently small positive constant κ. Define two integers $L(\kappa)$ and $J(\kappa)$ depending on κ,

$$\begin{aligned} L(\kappa) &= \left[\frac{8}{3\kappa}\log\left(\frac{64}{\kappa}\left(\frac{16n}{e\kappa}\right)^n\right)\right] + 1, \\ J(\kappa) &= \left[\frac{\delta}{8\kappa}\right], \end{aligned} \tag{6}$$

where $[\cdot]$ denotes the integer part of a real number. Let

$$\begin{aligned} O(p_0, \epsilon) &= \{p \in D : |p - p_0| < K_1\sqrt{\epsilon}\}, \\ \mathcal{O}(0,1) &= \{p \in \mathbb{R}^n : |p| \leq K_1\} \end{aligned}$$

for some constant $K_1 > 0$. For the sake of convenience, by $c_1, c_2, \cdots$, denote the positive constants depending only on M, n, K_1 and τ.

We continue to assume

(H2) For constants $\alpha > 0$ and $\tau > 0$, $\omega(p_0)$ suits the inequalities

$$|\langle k, \omega(p_0)\rangle| \geq \alpha |k|^{-\tau} \tag{7}$$

for any $k \in \mathbb{Z}^n$ with $0 < |k| \leq L(\kappa)$.

Proposition 1. *Assume (H2). Then there is a positive constant ϵ_0 depending on M, n, K_1, τ, δ, α and κ such that, for all ϵ with $0 \leq \epsilon \leq \epsilon_0$, the following statements hold.*

1) *There exists a transformation Φ_* and a near-identity transformation Ψ of coordinates, defined on $\mathcal{O}(0,1) \times \mathbb{T}^n$, to reduce Hamiltonian (2) to the form*

$$H \circ \Phi_* \circ \Psi = N_* + \sqrt{\epsilon} f_{**}$$

with

$$\begin{aligned} N_*(p, \epsilon) &= \langle \omega(p_0), p\rangle + O(\sqrt{\epsilon}), \\ \omega_*(p, \epsilon) &= \frac{\partial N_*}{\partial p}(p, \epsilon) = \omega(p_0) + O(\sqrt{\epsilon}), \\ \|f_{**}\| &\leq c_1\sqrt{\epsilon}\exp\left(-\frac{c_2}{\kappa}\right). \end{aligned}$$

2) For all $(p(0), q(0)) \in O_*(p_0, \epsilon) \times \mathbb{T}^n$*, there is a torus*

$$\hat{p}(t) = p(0),\ \hat{q}(t) = q(0) + \omega_{**}(p_0, p(0), q(0), \epsilon)t\,(\mathrm{mod} 2\pi),\ t \in \mathbb{R}$$

with

$$\omega_{**}(p_0, p(0), q(0), \epsilon) = \omega(p_0) + O(\sqrt{\epsilon}),$$

such that the orbit $(p(t), q(t))$ *starting from* $(p(0), q(0))$ *of (1) to satisfy the estimates*

$$|p(t) - \hat{p}(t)| \le c_3\kappa\sqrt{\epsilon},$$
$$|q(t) - \hat{q}(t)| \le c_4\kappa,$$

provided $|t| \le c_1 \exp\left(\frac{c_2}{4\kappa}\right)$.

To prove Proposition 1 we introduce a coordinate transformation $\Phi_* : (\mathcal{O}(0,1) \times \mathbb{T}^n) + \delta \to (O(p_0, \epsilon) \times \mathbb{T}^n) + \delta$,

$$p = p_0 + \sqrt{\epsilon}P,\ q = Q.$$

Under this transformation the Hamiltonian is reduced to the form

$$\begin{aligned}\hat{H}(P, Q) &= \frac{H(p, q)}{\sqrt{\epsilon}}\\ &= \frac{h(p_0)}{\sqrt{\epsilon}} + \langle \omega(p_0), P\rangle + O(\sqrt{\epsilon}P^2) + \sqrt{\epsilon} f_*(p_0 + \sqrt{\epsilon}P, Q, \epsilon).\end{aligned}$$

Without loss of generality, let $h(p_0) = 0$, and write $\varepsilon = \sqrt{\epsilon}$, $\omega_0 = \omega(p_0)$ and

$$f_0(P, Q, p_0, \varepsilon) = O(P^2) + f_*(p_0 + \varepsilon P, Q, \varepsilon^2).$$

Then

$$\hat{H}(P, Q) = \langle \omega_0, P\rangle + \varepsilon f_0(P, Q, p_0, \varepsilon), \tag{8}$$

and (P, Q) is defined on $(\mathcal{O}(0,1) \times \mathbb{T}^n) + \delta$. Obviously, by (3) we have

$$|f_0(P, Q, p_0, \varepsilon)| < c_5 \tag{9}$$

on $((\mathcal{O}(0,1) \times \mathbb{T}^n) + \delta) \times D \times [0,1]$.

Rewrite $\hat{H}, P$ and Q as H, p and q, and omit the parameters p_0 and ε in the arguments of f_0. Thus, Hamiltonian system (1) is changed into

$$\dot{p} = -H_q(p, q),\ \dot{q} = H_p(p, q)$$

with Hamiltonian

$$H(p, q) = \langle \omega_0, p\rangle + \varepsilon f_0(p, q). \tag{10}$$

Here ω_0 suits inequality (7).

For a real analytic function f, its Fourier's expansion is

$$f(p, q) = \sum_{k \in \mathbb{Z}^n} f_k e^{i\langle k, q\rangle}.$$

Let

$$\begin{aligned}\bar{f}(p) &= f_0(p),\\ [f]_L(p, q) &= \sum_{k \in \mathbb{Z}^n,\, 0<|k|\le L} f_k(p) e^{i\langle k, q\rangle},\\ R_L f(p, q) &= \sum_{k \in \mathbb{Z}^n,\, k|>L} f_k(p) e^{i\langle k, q\rangle}.\end{aligned}$$

We need the following lemmas.

Lemma 1. ([16, 18]) *Assume that* ω_0 *satisfies the condition (7). Then the homological equation*

$$\langle \omega_0, S_q \rangle + [f]_L(p,q) = 0$$

has only one real analytic solution S *with* $\bar{S} = 0$. *Moreover, for any* σ *with* $0 < \sigma < \delta$,

$$\|S\|_{D_0 - \sigma} \le \frac{c_6}{\alpha \sigma^\tau} \|[f]_L\|_{D_0}.$$

Here $D_0 = (\mathcal{O}(0,1) \times T^n) + \delta$.

Lemma 2. ([1]) *Assume* $f(q)$ *to be real analytic in* $\mathbb{T}^n + \delta$. *Then, as* $0 < 2\sigma_0 < \nu$ *and* $\sigma_0 + \nu < \delta < 1$, *on* $\mathbb{T}^n + (\delta - \sigma_0 - \nu)$ *one has*

$$\|R_L l\| < \left(\frac{2n}{e}\right)^n \frac{\|l\|}{\sigma_0^{n+1}} e^{-L\nu}. \tag{11}$$

Proof of Proposition 1. Consider Hamiltonian (10). Let

$$D_k = (\mathcal{O}(0,1) \times \mathbb{T}^n) + (\delta - 4k\kappa),\ k = 0, 1, 2, \cdots, J(\kappa).$$

Simply write $\Phi_0 = \Phi_*$, $H_0 = H$ and $N_0(p,\varepsilon) = \langle \omega_0, p \rangle$. Assume that under jth step Hamiltonian (10) is changed into the form

$$\begin{aligned} H_j(p,q) &= N_j(p,\varepsilon) + \varepsilon f_j(p,q), && (12)\\ N_j(p,q) &= \langle \omega_0, p \rangle + \widetilde{N}_j(p,\varepsilon), && (13)\\ \widetilde{N}_j(p,\varepsilon) &= \sum_{i=0}^{j-1} \bar{f}_j(p,\varepsilon), && (14)\\ |f_j| &\le \frac{1}{2^{j+1}} M, && (15) \end{aligned}$$

defined on D_j.

We introduce a symplectic transformation $\Phi_{j+1} : D_{j+1} \to D_j$ by $\Phi_{j+1} = \phi_{j+1}^1$. Here ϕ_{j+1}^t is the flow of the Hamiltonian system

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi_{j+1}^t = \varepsilon \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix} \nabla S_j(\phi_{j+1}^t), \tag{16}$$

where S_j will later be determined by equation (24). By applying Taylor's formula, we have

$$\begin{aligned} H_{j+1}(p,q) &= H_j \circ \Phi_{j+1}(p,q) \\ &= N_j \circ \Phi_{j+1}(p,q) + \varepsilon f_j \circ \Phi_{j+1}(p,q) \\ &= N_j(p,q) + \varepsilon \{N_j, S_j\} + \varepsilon^2 \int_0^1 (1-t) \{\{N_j, S_j\}, S_j\} \circ \phi_{j+1}^t \mathrm{d}t \\ &\qquad + \varepsilon f_j(p,q) + \varepsilon^2 \int_0^1 \{f_j, S_j\} \circ \phi_{j+1}^t \mathrm{d}t \\ &= N_j(p,\varepsilon) + \varepsilon \bar{f}_j(p,\varepsilon) \\ &\qquad + \varepsilon^2 \int_0^1 (1-t) \{\{N_j, S_j\}, S_j\} \circ \phi_{j+1}^t \mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&+\varepsilon R_L f_j(p,q)\\
&+\varepsilon\{\widetilde{N}_j, S_j\}\\
&+\varepsilon^2\int_0^1 \{f_j, S_j\}\circ\phi_{j+1}^t \mathrm{d}t\\
&+\varepsilon(\{N_0, S_j\} + [f_j]_L).
\end{aligned}
$$

Denote

$$
\begin{aligned}
\widetilde{N}_{j+1} &= \widetilde{N}_j + \varepsilon \bar{f}_j, && (17)\\
N_{j+1} &= N_j + \varepsilon \bar{f}_j, && (18)\\
f_{j+1}^1 &= \varepsilon\int_0^1 (1-t)\{\{N_j, S_j\}, S_j\}\circ\phi_{j+1}^t \mathrm{d}t, && (19)\\
f_{j+1}^2 &= R_L f_j, && (20)\\
f_{j+1}^3 &= \{\widetilde{N}_j, S_j\}, && (21)\\
f_{j+1}^4 &= \varepsilon\int_0^1 \{f_j, S_j\}\circ\phi_{j+1}^t \mathrm{d}t, && (22)\\
f_{j+1} &= f_{j+1}^1 + f_{j+1}^2 + f_{j+1}^3 + f_{j+1}^4. && (23)
\end{aligned}
$$

To determine transformation Φ_{j+1} we choose S_j satisfying

$$\{N_0, S_j\} + [f_j]_L = 0. \tag{24}$$

It follows that under transformation Φ_{j+1} Hamiltonian H_j is reduced to the form

$$H_{j+1} = H_j\circ\Phi_{j+1} = N_{j+1} + \varepsilon f_{j+1}. \tag{25}$$

Inductively, by (3), (9), (15) and (17), one has

$$\|\widetilde{N}_j\| \le \varepsilon\sum_{i=0}^{j-1}\frac{1}{2^{i+1}}M \le M\varepsilon \tag{26}$$

on D_j.

Take

$$\sigma_0 = \frac{1}{8}\kappa,\ \ \nu = \frac{3}{8}\kappa.$$

Thus, by Lemma 2 and the definition of $L(\kappa)$, on $D_j - \frac{1}{2}\kappa$,

$$\|f_{j+1}^2\|_{D_j-\frac{1}{2}\kappa} = \|R_L f_j\|_{D_j-\frac{1}{2}\kappa} \le \left(\frac{2n}{e}\right)^n \frac{\|f_j\|}{\sigma_0^{n+1}} e^{-L\nu} \le \frac{1}{8}\|f_j\|_{D_j}.$$

Hence,

$$
\begin{aligned}
\|f_{j+1}^2\|_{D_{j+1}} \le \|f_{j+1}^2\|_{D_j-\frac{1}{2}\kappa} &\le \frac{1}{8}\|f_j\|_{D_j}, && (27)\\
\|\nabla R_L f_j\|_{D_j-\kappa} &\le \frac{1}{4\kappa}\|f_j\|_{D_j}. && (28)
\end{aligned}
$$

On the basis of (27) and Cauchy's formula, we derive

$$\|[f_j]_L\|_{D_j-\frac{1}{2}\kappa} \le \|\bar{f}_j\|_{D_j} + \|R_L f_j\|_{D_j-\frac{1}{2}\kappa} + \|f_j\|_{D_j} \le 3\|f_j\|_{D_j}. \tag{29}$$

Let $(P_t, Q_t) = \phi_{j+1}^t(p, q)$. By applying (24), (29), Lemma 1 and Cauchy's formula, for all $(P_t, Q_t) \in D_j - 2\kappa$ with $0 \le t \le 1$, we have

$$\begin{aligned} |(p,q) - (P_t, Q_t)| &\le \varepsilon \|\nabla S_j(P_t, Q_t)\|_{D_j - 2\kappa} \\ &\le \frac{2\varepsilon}{\kappa} \|S_j(P_t, Q_t)\|_{D_j - \frac{3}{2}\kappa} \\ &\le \frac{2c_6\varepsilon}{\alpha\kappa^{\tau+1}} \|[f_j]\|_{D_j - \frac{1}{2}\kappa} \\ &\le \frac{6c_6\varepsilon}{\alpha\kappa^{\tau+1}} \|f_j\|_{D_j} \\ &\le \frac{6c_6\varepsilon}{\alpha\kappa^{\tau+1}} \frac{1}{2^{j+1}} M \\ &\le \frac{1}{2^{j+1}} \kappa \\ &< \kappa, \end{aligned} \tag{30}$$

provided ε satisfies

$$\frac{6c_6\varepsilon}{\alpha\kappa^{\tau+1}} \le \kappa. \tag{A}$$

By the geometric lemma in [1], $\phi_{j+1}^{-t}(D_j - 2\kappa) \supset D_j - 3\kappa$, and ϕ_{j+1}^{-t} is a diffeomorphism defined on D_{j+1}. This shows that $\phi_{j+1}^t(D_{j+1}) \subset D_j$.

If ε satisfies

$$\frac{3c_6M\varepsilon}{\alpha\kappa^{\tau+2}} \le \frac{1}{8}, \tag{B}$$

then, by Lemma 1, Cauchy's formula, (16) and (26), we obtain

$$\begin{aligned} \|f_{j+1}^3\|_{D_j - 2\kappa} &= \|\{\widetilde{N}_j, S_j\}\|_{D_j - 2\kappa} \\ &\le \left\|\frac{\partial \widetilde{N}_j}{\partial p}\right\|_{D_j - 2\kappa} \left\|\frac{\partial S_j}{\partial q}\right\|_{D_j - 2\kappa} \\ &\le \frac{1}{\kappa^2} \|\widetilde{N}_j\|_{D_j} \|S_j\|_{D_j - \frac{3}{2}\kappa} \\ &\le \frac{M\varepsilon}{\kappa^2} \cdot \frac{c_6}{\alpha\kappa^\tau} \|[f_j]_L\|_{D_j - \frac{1}{2}\kappa} \\ &\le \frac{3c_6M\varepsilon}{\alpha\kappa^{\tau+2}} \|f_j\|_{D_j} \\ &\le \frac{1}{8} \|f_j\|_{D_j}. \end{aligned}$$

Hence,

$$\|f_{j+1}^3\|_{D_{j+1}} \le \|f_{j+1}^3\|_{D_j - 2\kappa} \le \frac{1}{8} \|f_j\|_{D_j}, \tag{31}$$

$$\|\nabla f_{j+1}^3\|_{D_j - 3\kappa} \le \frac{1}{\kappa} \|f_{j+1}^3\|_{D_j - 2\kappa} \le \frac{1}{8\kappa} \|f_j\|_{D_j}. \tag{32}$$

Similarly, as ε satisfies the inequality

$$\frac{3c_6M\varepsilon}{\alpha\kappa^{\tau+2}} \le \frac{1}{8}, \tag{C}$$

we derive

$$\begin{aligned}\|f_{j+1}^4\|_{D_{j+1}} &\le \varepsilon\|\{f_j,S_j\}\phi_{j+1}^t\|_{D_{j+1}}\\ &\le \varepsilon\|\{f_j,S_j\}\|_{D_j-3\kappa}\\ &\le \frac{\varepsilon}{\kappa^2}\|f_j\|_{D_j-2\kappa}\|S_j\|_{D_j-2\kappa}\\ &\le \frac{c_6\varepsilon}{\alpha\kappa^{\tau+2}}\|f_j\|_{D_j-2\kappa}\|[f_j]\|_{D_j-\kappa}^2\\ &\le \frac{1}{2^{j+1}}\frac{3c_6\varepsilon}{\alpha\kappa^{\tau+2}}M\|f_j\|_{D_j}\\ &\le \frac{1}{8}\|f_j\|_{D_j-3\kappa}\end{aligned}\tag{33}$$

for all $t\in[0,1]$.

Now we estimate f_{j+1}^1. Note that

$$\{N_j,S_j\}=\{\widetilde{N}_j,S_j\}+\{N_0,S_j\}=f_{j+1}^3-[f_j]_L,$$

which implies that

$$\{f_j+(1-t)\{N_j,S_j\},S_j\}=\{(1-t)f_{j+1}^3+t[f_j]_L+\bar{f}_j+R_Lf_j,S_j\}\tag{34}$$

from (24). Hence, by employing the conclusion $\phi_{j+1}^tD_{j+1}\subset D_j-3\kappa$, Cauchy's formula, (31), (33), (29), (27), (28), (32) and Lemma 1, we derive

$$\begin{aligned}\|f_{j+1}^1\|_{D_{j+1}} &\le \varepsilon(\|\nabla f_{j+1}^3\|_{D_j-3\kappa}+\|\nabla[f_j]_L\|_{D_j-3\kappa}+\|\nabla\bar{f}_j\|_{D_j-3\kappa}\\ &\quad+\|\nabla R_Lf_j\|_{D_j-3\kappa})\|\nabla S_j\|_{D_j-3\kappa}\\ &\le \frac{\varepsilon}{\kappa^2}(\frac{1}{8}\|f_j\|_{D_j}+\|[f_j]_L\|_{D_j-2\kappa}+\|\bar{f}_j\|_{D_j-2\kappa}+\frac{1}{4}\|f_j\|_{D_j})\|S_j\|_{D_j-2\kappa}\\ &\le \frac{\varepsilon}{\kappa^2}\left(\frac{3}{8}\|f_j\|_{D_j}+3\|f_j\|_{D_j}+\|f_j\|_{D_j}\right)\cdot\frac{3c_6}{\alpha\kappa^\tau}\|[f_j]\|_{D_j-\kappa}\\ &\le \frac{105c_6\varepsilon M}{8\alpha\kappa^{\tau+2}}\|f_j\|_{D_j}\\ &\le \frac{1}{8}\|f_j\|_{D_j},\end{aligned}\tag{35}$$

provided

$$\frac{105c_6\varepsilon M}{8\alpha\kappa^{\tau+2}}\le\frac{1}{8}.\tag{D}$$

Hence, from (23), (27), (31), (33) and (35), it follows that

$$\begin{aligned}\|f_{j+1}\|_{D_{j+1}} &\le \|f_{j+1}^1\|_{D_{j+1}}+\|f_{j+1}^2\|_{D_{j+1}}+\|f_{j+1}^3\|_{D_{j+1}}+\|f_{j+1}^4\|_{D_{j+1}}\\ &\le \frac{1}{2}\|f_j\|_{D_j}\\ &\le \frac{1}{2^{j+2}}M.\end{aligned}\tag{36}$$

Put $\Psi=\Phi_1\circ\cdots\circ\Phi_J$. Then $\Psi:D_J\to D_0$ and $D_j\supset D_{**}=(\mathcal{O}(0,1)\times\mathbb{T}^n)+\frac{1}{2}\delta$. Denote $\Psi(r,\theta)=(p,q)$. It leads

$$H_J(r,\theta)=\hat{H}\circ\Psi(r,\theta)=N_J(r,\varepsilon)+\varepsilon f_J(r,\theta)\tag{37}$$

with

$$\begin{aligned} \|\widetilde{N}_J\|_{D_{**}} &\leq M\varepsilon, &(38)\\ \|f_J\|_{D_{**}} &\leq \frac{1}{2^{J+1}}M = c_1 \exp\left(-\frac{c_2}{\kappa}\right). &(39)\end{aligned}$$

where $c_1 = \frac{1}{2}M$ and $c_2 = \frac{\log 2}{12}\delta$.

The Hamiltonian system with (37) is the following

$$\dot{r} = -\varepsilon\frac{\partial f_J}{\partial\theta}, \tag{40}$$

$$\dot{\theta} = \omega_0 + \frac{\partial \widetilde{N}_J}{\partial r} + \varepsilon\frac{\partial f_J}{\partial r}. \tag{41}$$

Take $D_* = (\mathcal{O}(0,1)\times\mathbb{T}^n) + \frac{1}{4}\delta$. By Cauchy's formula one has

$$\max\left\{\left\|\frac{\partial f_L}{\partial r}\right\|_{D_*}, \left\|\frac{\partial f_L}{\partial\theta}\right\|_{D_*}\right\} \leq \frac{4}{\delta}\|f_L\|_{D_{**}}. \tag{42}$$

Thus, for any $(r(0),\theta(0))\in\mathcal{O}(0,1)\times\mathbb{T}^n$, as $|t|\leq\exp\left(\frac{c_2}{2\kappa}\right)$, it follows that

$$|r(t)-r(0)|\leq c_7\varepsilon\exp\left(-\frac{c_2}{2\kappa}\right). \tag{43}$$

Denote

$$\omega_*(r,\varepsilon) = \omega_0 + \frac{\partial\widetilde{N}_J}{\partial p}(r,\varepsilon).$$

Thus,

$$\begin{aligned} |\omega_*(r(t),\varepsilon)-\omega_*(r(0),\varepsilon)| &\leq c_8\varepsilon|r(t)-r(0)|\\ &\leq c_9\varepsilon\exp\left(-\frac{c_2}{2\kappa}\right) &(44)\end{aligned}$$

on $\mathcal{O}(0,1)\times\mathbb{T}^n$. It follows from (41), (39), (44) and Cauchy's formula that

$$|\theta(t)-\omega_*(r(0),\varepsilon)t-\theta(0)|\leq c_{10}\varepsilon\exp\left(-\frac{c_2}{4\kappa}\right), \tag{45}$$

provided $|t|\leq\exp\left(\frac{c_2}{4\kappa}\right)$.

Obviously, inequality (D) implies (A), (B) and (C). Let

$$\varepsilon_0(\alpha,\kappa) = \min\left\{\max\{\varepsilon:\varepsilon\geq 0 \text{ and } \varepsilon \text{ satisfies } (D)\}, \frac{1}{2}\right\},$$

that is,

$$\varepsilon_0(\alpha,\kappa) = \min\left\{\frac{\alpha\kappa^{\tau+2}}{105c_6M}, \frac{1}{2}\right\}.$$

Without loss of generality, take

$$\varepsilon_0(\alpha,\kappa) = \frac{\alpha\kappa^{\tau+2}}{105c_6M}. \tag{46}$$

Write $\epsilon_0 = \varepsilon_0^2$. Let $(p(t),q(t))$ be a solution starting from $(p(0),q(0))\in O(p_0,\epsilon)\times\mathbb{T}^n$ of system (1). Then $(P(t),Q(t))$ is a solution with $P(0)\in\mathcal{O}(0,1)$, of the system with Hamiltonian (8). By (r,θ) we denote a new coordinate variables under the change Ψ. By (30),

$$|(P,Q)-(r,\theta)| = |\Psi(r,\theta)-(r,\theta)| \le \sum_{j=0}^{J} \frac{1}{2^{j+1}}\kappa < \kappa, \tag{47}$$

which and (43) imply that, as $|t| \le \exp\left(\frac{c_2}{4\kappa}\right)$,

$$\begin{aligned} |P(t)-P(0)| &\le |P(t)-r(t)|+|P(0)-r(0)|+|r(t)-r(0)| \\ &\le 2\kappa + c_7\varepsilon \exp\left(-\frac{c_2}{2\kappa}\right) \\ &\le c_{11}\kappa; \qquad (48) \\ |Q(t)-\omega_*(r(0),\varepsilon)t-Q(0)| &\le |Q(t)-\theta(t)|+|\theta(t)-\omega_*(r(0),\varepsilon)t-\theta(0)| \\ &\quad +|Q(0)-\theta(0)| \\ &\le 2\kappa + c_{10}\varepsilon \exp\left(-\frac{c_2}{4\kappa}\right) \\ &\le c_{12}\kappa. \qquad (49) \end{aligned}$$

Note that $\Phi_*(P,Q)=(p,q)$, that is,

$$p = p_0 + \sqrt{\epsilon}P, \;\; q = Q.$$

Hence,

$$(r(0),\theta(0)) = \Psi^{-1}(P(0),Q(0)) = \Psi^{-1}\left(\frac{p(0)-p_0}{\sqrt{\epsilon}}, q(0)\right). \tag{50}$$

We use $\mathfrak{P}$ to denote the operator which projects the phase on the space of action variables. From (50), it follows that

$$r(0) = \mathfrak{P}\circ\Psi^{-1}\left(\frac{p(0)-p_0}{\sqrt{\epsilon}}, q(0)\right).$$

Let

$$\omega_{**}(p_0,p(0),q(0),\epsilon) = \omega_*\left(\mathfrak{P}\circ\Psi^{-1}\left(\frac{p(0)-p_0}{\sqrt{\epsilon}}, q(0)\right), \sqrt{\epsilon}\right). \tag{51}$$

Choose the torus as follows,

$$\hat{p}(t) = p(0), \;\; \hat{q}(t) = q(0) + \omega_{**}(p_0,p(0),q(0),\epsilon)t \,(\mathrm{mod} 2\pi), \; t \in \mathbb{R}.$$

Combining (48), (49) and the transformation Φ_*, the proof of Proposition 1 is finished. $\square$

4. **Proof of Theorem B.** In order to prove Theorem B, we regard κ as a function in ϵ. Choose

$$\kappa = \varepsilon^{\frac{1}{\tau+3}} = \epsilon^{\frac{1}{2(\tau+3)}}.$$

By applying (46) and the definition of ϵ_0, we obtain

$$\epsilon_0(\alpha) = \left(\frac{\alpha}{105Mc_6}\right)^{2(\tau+3)}.$$

Take

$$O_\epsilon = O(p_0,\epsilon).$$

According to Proposition 1 and its proof, as $0 < \epsilon \le \epsilon_0(\alpha)$, for all $(p(0),q(0)) \in O_\epsilon \times \mathbb{T}^n$, there exists a torus $\{(\hat{p}(t),\hat{q}(t))\}$ with frequency $\omega_{**}(p_0,p(0),q(0),\epsilon)$ such that the orbit $(p(t), q(t))$ starting from $(p(0), q(0))$ of (1) to satisfy the following inequalities

$$|p(t) - \hat{p}(t)| \leq c_3 \epsilon^{\frac{1}{2}+\frac{1}{2(\tau+3)}},$$
$$|q(t) - \hat{q}(t)| \leq c_4 \epsilon^{\frac{1}{2(\tau+3)}},$$

provided $|t| \leq c_1 \exp\left(\frac{c_2}{4}\epsilon^{-\frac{1}{2(\tau+3)}}\right)$. The proof of Theorem B is completed. □

5. **Proof of Theorem A.** Now we prove Theorem A. To this end we regard κ and α as a function in ϵ, respectively, in this section. Simply, let

$$\alpha = \kappa = \epsilon^{\frac{1}{2(\tau+4)}}. \tag{52}$$

By using (46) we determine

$$\epsilon_0 = \left(\frac{1}{105Mc_6}\right)^{2(\tau+4)}.$$

Define

$$D_\epsilon = \{p \in D : |\langle k, \omega(p)\rangle| \geq \alpha |k|^{-\tau}, \text{ for all } 0 \neq k \in \mathbb{Z}^n\}.$$

Here $\tau > n(n-1)$. This condition is a requirement of the measure estimate. Let

$$D^\tau_{\alpha,k} = \{y \in D : |\langle k, \omega(y)\rangle| \leq \alpha |k|^{-\tau}\},$$
$$D^\tau_\alpha = \bigcup_{0\neq k\in\mathbb{Z}^n} D^\tau_{\alpha,k}.$$

By assumption (H1) and Lemma 2.1 in [20], one has

$$\text{meas}(D^\tau_{\alpha,k}) = O\left(\alpha^{\frac{1}{n}}|k|^{-\frac{\tau+1}{n}}\right),$$
$$\text{meas}(D^\tau_\alpha) = O(\alpha^{\frac{1}{n}}).$$

Define

$$D_\epsilon = D - D^\tau_\alpha,$$
$$D_* = \bigcup_{\epsilon>0} D_\epsilon.$$

Then, by KAM theory, $\text{meas}D_\epsilon = \text{meas}D - O(\alpha^{\frac{1}{n}})$, and D_* is a set of full measure in $\mathbb{R}^n$, for any $\tau > n(n-1)$.

For any $p_0 \in D_*$, let

$$\alpha(p_0) = \max\{\alpha : 0 < \alpha \leq 1 \text{ and } |\langle k, \omega(p_0)\rangle| \geq \alpha|k|^{-\tau} \text{ for all } 0 \neq k \in \mathbb{Z}^n\},$$
$$\epsilon(p_0) = \min\{\alpha(p_0)^{2(\tau+4)}, \epsilon_0\}.$$

Write $O_\epsilon(p_0) = O(p_0, \epsilon)$. If $0 < \epsilon \leq \epsilon(p_0)$, from Proposition 1, it follows that for all $(p(0), q(0)) \in O_\epsilon(p_0) \times \mathbb{T}^n$, as $|t| \leq c_1 \exp\left(\frac{c_2}{4}\epsilon^{-\frac{1}{2(\tau+4)}}\right)$,

$$|p(t) - p(0)| \leq c_3 \epsilon^{\frac{1}{2}+\frac{1}{2(\tau+4)}}.$$

For any $\epsilon \in (0, \epsilon_0]$, define

$$E_\epsilon = \bigcup_{p_0\in\{p\in D_*:\epsilon(p)\geq\epsilon\}} O_\epsilon(p_0). \tag{53}$$

This is an open subset of D, and $\text{meas}E_\epsilon = \text{meas}D - O\left(\epsilon^{\frac{1}{2n(\tau+4)}}\right)$ and $\lim_{\epsilon\to 0+} E_\epsilon = D_*$.

For any given $(p(0), q(0)) \in E_\epsilon \times \mathbb{T}^n$, by (53), there are $p_0 \in D_*$ and a constant $\epsilon(p_0) \geq \epsilon$ satisfying $p(0) \in O_\epsilon(p_0)$. By applying Proposition 1 to p_0 and $O_\epsilon(p_0)$, we derive that the orbit $(p(t), q(t))$ starting from $(p(0), q(0))$ satisfying the estimate

$$|p(t) - p(0)| \leq c_3 \epsilon^{\frac{1}{2} + \frac{1}{2(\tau+4)}},$$

provided $|t| \leq c_1 \exp\left(\frac{c_2}{4}\epsilon^{-\frac{1}{2(\tau+4)}}\right)$. The proof of Theorem A is completed. □

In the proof of Theorem A, the choices of κ and α may be an another form. More precisely, let

$$\alpha = \epsilon^\eta, \ \ \kappa = \epsilon^\iota \text{ and } \eta + 2\iota(\tau + 2) + \chi = \frac{1}{2} \tag{54}$$

for given positive constants η, ι and χ. By using (46), we obtain

$$\epsilon_0(\eta, \iota, \chi) = \left(\frac{1}{105 M c_6}\right)^{\frac{1}{\chi}}. \tag{55}$$

Similar to the proof of Theorem A, we could obtain that, as $0 < \epsilon \leq \min\{\epsilon(y_0), \epsilon_0(\eta, \iota, \chi)\}$, for all $(p(0), q(0)) \in O_\epsilon \times \mathbb{T}^n$, the orbit $(p(t), q(t))$ starting from $(p(0), q(0))$ is of near-invariant torus, and $p(t)$ and t satisfy the estimates with exponents $\frac{1}{2} + \iota$ and ι. Hence, from (54), the stable exponents of system (1) can be chose as $\frac{1}{2} + \frac{1}{2(\tau+2)} - \epsilon_{*1}$ and $\frac{1}{2(\tau+2)} - \epsilon_{*1}$, where ϵ_{*1} is an arbitrary small positive number.

Now consider Diophantine exponent τ. Under Rüssmann's non-degenerate condition, τ can be took as $\tau > n(n-1)$ due to the estimate of the measure. Combining the above analysis the stable exponents of system (1) can be chose as $\frac{1}{2} + \frac{1}{2n^2-2n+4} - \epsilon_{*2}$ and $\frac{1}{2n^2-2n+4} - \epsilon_{*2}$, where ϵ_{*2} is an arbitrary small positive number.

In a similar way, if system (1) satisfies the classical non-degeneracy, that is,

$$h_{pp} \neq 0,$$

then the stable exponents can be chose as $\frac{1}{2} + \frac{1}{2n+2} - \epsilon_{*3}$ and $\frac{1}{2n+2} - \epsilon_{*3}$, where ϵ_{*3} is an arbitrary small positive number.

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