# DYNAMICAL BEHAVIOR OF NETWORKS OF NON-UNIFORM TIMOSHENKO BEAMS SYSTEM WITH BOUNDARY TIME-DELAY INPUTS

ZHONG-JIE HAN AND GEN-QI XU

Department of Mathematics, Tianjin University
Tianjin 300072, China

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ABSTRACT. The dynamical stability of planar networks of non-uniform Timoshenko beams system is considered. Suppose that the displacement and rotational angle is continuous at the common vertex of this network and the bending moment and shear force satisfies Kirchhoff's laws, respectively. Time-delay terms exist in control inputs at exterior vertices. The feedback control laws are designed to stabilize this kind of networks system. Then it is proved that the corresponding closed loop system is well-posed. Under certain conditions, the asymptotic stability of this system is shown. By a complete spectral analysis, the spectrum-determined-growth condition is proved to be satisfied for this system. Finally, the exponential stability of this system is discussed for a special case and some simulations are given to support these results.

1. **Introduction.** The dynamical behavior of flexible networks and their control problems (see [9], [10], [11] and [23]), are studied widely in engineering and applied mathematics. For such systems, due to high-precision requirements, we always need to take the time-delay effects into account, since time-delay is a universal phenomenon existing in almost every engineering practices (see [31], [38] and [39]). Normally, the presence of delays makes systems less stable (see [12], [13] and [33]). Especially for hyperbolic systems, a small delay always can make the energy of the controlled systems increase exponentially, which makes the control problem of them become very tough.

In engineering, time delays in control systems are roughly divided into three types: the internal delays, input delays and output delays. What we shall consider in this paper is a kind of planar networks of hyperbolic system with control input delays at the exterior vertices of networks. We shall discuss the stability of tree-shaped networks of non-uniform Timoshenko beams system, since non-uniform Timoshenko beam is the most realistic one of the 1-D distributed parameter models.

At present, there are many results about the dynamical behavior of flexible structure networks, for instance, Ammari *et al.* in [3], [4] and [5] discussed the stabilization problem of tree-shaped and star-shaped of elastic strings respectively, and asserted that the networks are asymptotically stable under certain conditions.

Similar method was used to consider the energy decay of elastic Euler-Bernoulli beams with star-shaped and tree-shaped network configuration (see [2]). Mercier in [28] gave a spectral analysis of a serially connected Euler-Bernoulli beams problems and proved the existence of the spectral gap for such systems. Mercier *et al.* in [29] studied the spectrum of a network of Euler-Bernoulli beams connected by $n$ vibrating point masses and discussed the controllability of this kind of network in [30]. Wang *et al.* in [44] obtained the Riesz basis and exponential stability of the flexible structure of a symmetric tree-shaped beam network, based on the asymptotic spectral distribution of the system. Han *et al.* in [17] studied the spectral distribution of a star-shaped networks of strings and obtained the asymptotic stability of the system. In [18], they discussed the dynamical behavior of tree-shaped networks of non-uniform strings with non-collocated feedbacks by Riesz basis approach. Zuazua *et al.* in [42] discussed the dynamical behavior of the wave equation on 1-d networks and gave a general method to study the stabilization problem of such systems. Moreover, by asymptotic spectral analysis, Xu *et al.* in [45] and [48] considered the stability of multi-connected wave systems. They in [19], [20], [21] and [46] studied the Riesz basis property of networks of Timoshenko beams and obtained that these systems satisfy spectrum-determined-growth condition, that is the growth order of the system is determined via its the spectral bound.

There also have been a few nice results on control problem of flexible systems with time delays. For example, Liang *et al.* in [24] and [25] introduced the Smith predictor and its variant to the boundary control of wave equation and Euler-Bernoulli beam equation with a delayed boundary measurement, and demonstrated the effectiveness of the proposed method by simulation. Morgul in [32] designed a class of dynamical controllers to robustly stabilize the wave equations against small time delays in the feedback loop. Guo *et al.* in [16] designed the observers and predictors to solve the stabilization problem of one-dimensional Schrödinger equation system. Han *et al.* in [22] discussed the exponential stability of Timoshenko beam with delay terms in boundary feedbacks. Xu *et al.* in [47] and [50] considered the Riesz basis properties, exact controllability and stability of string systems with time-delay boundary feedback controls. Nicaise *et al.* in [34] studied the stabilization of the wave equation with boundary or internal distributed delay and in [35] they applied the similar feedback controller to 1-D networks of wave systems and obtained the stability of the closed loop system under certain conditions.

However, there is few result about the dynamical behavior of networks of non-uniform hyperbolic system with time-delay feedbacks. We observe that for such networks, the operator determined by the closed loop systems are always non-normal and very complex. It is difficult to obtain some properties of the systems, such as distribution of spectrum, spectrum-determined-growth condition. Furthermore, time-delay problems for hyperbolic systems are always very tough due to its spectral property, even for single beam or string. Hence, such problems become very difficult to discuss. The aim of this paper is to study the dynamical behavior of a kind of planar networks of hyperbolic system—tree-shaped networks of non-uniform Timoshenko beams system with time-delay terms in the boundary control inputs (see Figure 1).

At the beginning, let us describe these tree-shaped networks of non-uniform Timoshenko beams on a planar graph. Let $G = (V, E)$ be a planar graph of tree shape with vertices $V = \{a_0, a_1, a_2, a_3, \cdots, a_n\}$ and edges $E = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$, where $a_0$ is its common vertex. Suppose that an elastic structure is defined on the
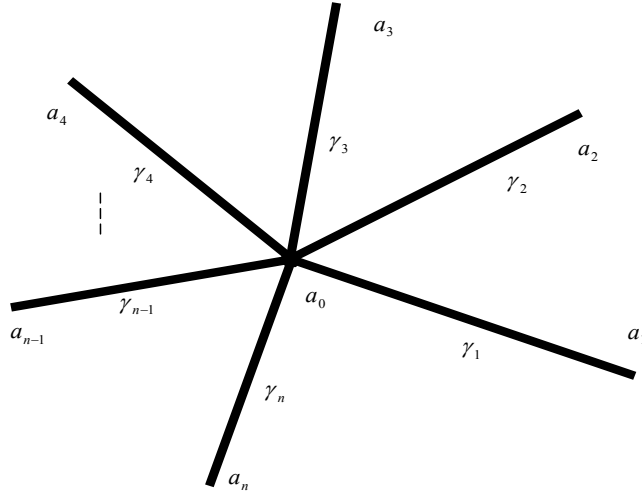
FIGURE 1. planar network

graph $G$, whose equilibrium position coincides with $G$. The function $w(s,t)$ and $\varphi(s,t)$ denotes the displacement and rotational angle depart from its equilibrium position in position $s \in G$ at time $t$, respectively. The notation $w_s(s,t)$, $\varphi_s(s,t)$ and $w_t(s,t)$, $\varphi_t(s,t)$ denotes the partial differential with respect to $s$ and $t$, respectively. We parameterize the edge $\gamma_j$ with start point $a_0$, end point $a_j$ and arc length $\ell_j$, such that,

$$w^j(x,t) = w(s,t)|_{\gamma_j}, \ x \in (0,\ell_j), \ s \in \gamma_j.$$

$$\varphi^j(x,t) = \varphi(s,t)|_{\gamma_j}, \ x \in (0,\ell_j), \ s \in \gamma_j.$$

For simplicity, we suppose that the length of each beam in the network is 1. Thus, the motion of the elastic structure on each $\gamma_j$ is governed as follows

$$\begin{cases} \rho_j(x)\frac{\partial^2 w^j(x,t)}{\partial t^2} - \frac{\partial}{\partial x}[k_j(x)(\frac{\partial w^j(x,t)}{\partial x} - \varphi^j(x,t))] = 0, \ t > 0, \ x \in (0,1), \\ I_{\rho_j}(x)\frac{\partial^2 \varphi^j(x,t)}{\partial t^2} - \frac{\partial}{\partial x}[EI_j(x)\frac{\partial \varphi^j(x,t)}{\partial x}] - k_j(x)(\frac{\partial w^j(x,t)}{\partial x} - \varphi^j(x,t)) = 0, \ t > 0, \ x \in (0,1), \end{cases}$$
(1)

where $j = 1, 2, \cdots, n$, and $\rho_j(x), I_{\rho_j}(x), EI_j(x), k_j(x)$ represents the density (i.e. the mass per unit length), moment of mass inertia, rigidity coefficient, shear modulus of elastic beam, respectively. More precise physical meaning of these quantities can be found in Timoshenko's book [41]. In this paper, we always assume

$$\rho_j(x), I_{\rho_j}(x), EI(x), k_j(x) \in C^1[0,1] \quad \text{and} \quad \rho_j(x), I_{\rho_j}(x), EI_j(x), k_j(x) > 0.$$

At the common vertex $a_0$, the beams satisfy the continuity conditions of displacements and angles, i.e.,

$$\begin{cases} w(a_0,t) = w^j(0,t), \ j = 1, 2, \cdots, n, \quad t > 0, \\ \varphi(a_0,t) = \varphi^j(0,t), \ j = 1, 2, \cdots, n, \quad t > 0. \end{cases}$$
(2)

Besides this, $w^j(x,t), \varphi^j(x,t)$, $j = 1, 2, \cdots, n$ at the common vertex $a_0$ satisfy the Kirchhoff's law, i.e.,

$$\sum_{j=1}^{n} k_j(0)(\frac{\partial w^j}{\partial x} - \varphi^j)(0,t) = 0, \quad \sum_{j=1}^{n} EI_j(0)\frac{\partial \varphi^j}{\partial x}(0,t) = 0. \tag{3}$$

In order to stabilize this kind of network, the controllers are designed at exterior vertices of the network to control the shear forces and bending moments of these beams, that is, at $a_j$, $j = 1, 2, \cdots, n$, $w^j(x,t), \varphi^j(x,t)$ satisfy the following conditions:

$$\begin{cases} k_j(1)(\frac{\partial w^j}{\partial x} - \varphi^j)(1,t) + w^j(1,t) = u_j, \ j = 1, 2, \cdots, n, \\ EI_j(1)\frac{\partial \varphi^j}{\partial x}(1,t) + \varphi^j(1,t) = v_j, \ j = 1, 2, \cdots, n, \end{cases} \tag{4}$$

where $u_j, v_j$ are control inputs, respectively. In fact, if these control inputs in (4) have damping or memory, which deduces time delays, then the actual effect of the control inputs are

$$\int_{-\tau_j}^{0} u_j(t+s)d\varrho_j(s), \quad \int_{-\tau_j}^{0} v_j(t+s)d\varrho_j(s), \ j = 1, 2, \cdots, n,$$

that is, $w^j(x,t), \varphi^j(x,t)$, $j = 1, 2, \cdots, n$ satisfy

$$\begin{cases} k_j(1)(\frac{\partial w^j}{\partial x} - \varphi^j)(1,t) + w^j(1,t) = \int_{-\tau_j}^{0} u_j(t+s)d\varrho_j(s), \ j = 1, 2, \cdots, n, \\ EI_j(1)\frac{\partial \varphi^j}{\partial x}(1,t) + \varphi^j(1,t) = \int_{-\tau_j}^{0} v_j(t+s)d\varrho_j(s), \ j = 1, 2, \cdots, n, \end{cases}$$

where $\tau_j$ is the maximal memory time and $\varrho_j(s)$ is a bounded variation function. In particular, if we choose $\varrho_j = \mu_j \text{sign}(s) + (1 - \mu_j)\text{sign}(s + \tau)$, where $\mu_j \in [0,1]$ are constants, then $d\varrho_j(s)$ are of point impulses at $s = 0, -\tau_j$. Hence,

$$\begin{cases} \int_{-\tau_j}^{0} u_j(t+s)d\varrho_j(s) = \mu_j u_j(t) + (1 - \mu_j)u_j(t - \tau_j), \\ \int_{-\tau_j}^{0} v_j(t+s)d\varrho_j(s) = \mu_j v_j(t) + (1 - \mu_j)v_j(t - \tau_j). \end{cases}$$

Thus, conditions (4) become the following conditions:

$$\begin{cases} k_j(1)(\frac{\partial w^j}{\partial x} - \varphi^j)(1,t) + w^j(1,t) = \mu_j u_j(t) + (1 - \mu_j)u_j(t - \tau_j), \ j = 1, 2, \cdots, n, \\ EI_j(1)\frac{\partial \varphi^j}{\partial x}(1,t) + \varphi^j(1,t) = \mu_j v_j(t) + (1 - \mu_j)v_j(t - \tau_j), \ j = 1, 2, \cdots, n. \end{cases} \tag{5}$$

Observing the velocities $\frac{\partial w^j}{\partial t}(1,t)$, $\frac{\partial \varphi^j}{\partial t}(1,t)$, $j = 1, 2, \cdots, n$, we adopt the following collocated velocity feedback control laws:

$$\begin{cases} u_j(t) = -\alpha_j \frac{\partial w^j}{\partial t}(1,t), \quad \alpha_j > 0, \ j = 1, 2, \cdots, n, \\ v_j(t) = -\beta_j \frac{\partial \varphi^j}{\partial t}(1,t), \quad \beta_j > 0, \ j = 1, 2, \cdots, n, \end{cases} \tag{6}$$

where $\alpha_j$, $\beta_j$ are feedback gains, $\frac{\partial w^j}{\partial t}(1, t - \tau_j) := f_j^0(t - \tau_j)$, $\frac{\partial \varphi^j}{\partial t}(1, t - \tau_j) := f_j^1(t - \tau_j)$, $t \in (0, \tau_j)$, $f_j^0$, $f_j^1$ are given functions. Thus, we have

$$\begin{cases} k_j(1)(\frac{\partial w^j}{\partial x} - \varphi^j)(1,t) + w^j(1,t) = -\alpha_j[\mu_j \frac{\partial w^j}{\partial t}(1,t) + (1 - \mu_j)\frac{\partial w^j}{\partial t}(1, t - \tau_j)], \ t > 0, \\ EI_j(1)\frac{\partial \varphi^j}{\partial x}(1,t) + \varphi^j(1,t) = -\beta_j[\mu_j \frac{\partial \varphi^j}{\partial t}(1,t) + (1 - \mu_j)\frac{\partial \varphi^j}{\partial t}(1, t - \tau_j)], \ t > 0, \end{cases} \tag{7}$$

In addition, we assume that the initial condition of system is given by

$$\begin{cases} w(s,0) = \bar{w}_0(s), \ w_t(s,0) = \bar{w}_1(s), \quad s \in G, \\ \varphi(s,0) = \bar{\varphi}_0(s), \ \varphi_t(s,0) = \bar{\varphi}_1(s), \quad s \in G. \end{cases} \tag{8}$$

Thus, under the feedback control laws (6), system (1)–(5) together with (7)-(8) becomes a closed loop system described by

$$\begin{cases} \rho_j(x)w_{tt}^j(x,t) - [k_j(x)(w_x^j(x,t) - \varphi^j(x,t))]_x = 0, \ t > 0, \ x \in (0,1), \\ I_{\rho_j}(x)\varphi_{tt}^j(x,t) - [EI_j(x)\varphi_x^j(x,t)]_x \\ \quad -k_j(x)(w_x^j(x,t) - \varphi^j(x,t)) = 0, \ t > 0, \ x \in (0,1), \\ w(a_0,t) = w^j(0,t), \quad \varphi(a_0,t) = \varphi^j(0,t), \ t > 0, \\ k_j(1)(w_x^j - \varphi^j)(1,t) + w^j(1,t) = -\alpha_j[\mu_j w_t^j(1,t) \\ \quad +(1-\mu_j)w_t^j(1,t-\tau_j)], \ t > 0, \\ EI_j(1)\varphi_x^j(1,t) + \varphi^j(1,t) = -\beta_j[\mu_j\varphi_t^j(1,t) \\ \quad +(1-\mu_j)\varphi_t^j(1,t-\tau_j)], \ t > 0, \\ \sum_{j=1}^{n} k_j(0)(w_x^j - \varphi^j)(0,t) = 0, \quad \sum_{j=1}^{n} EI_j(0)\varphi_x^j(0,t) = 0, \ t > 0, \\ w_t^j(1,t-\tau_j) = f_j^0(t-\tau_j), \quad \varphi_t^j(1,t-\tau_j) = f_j^1(t-\tau_j), \ t \in (0,\tau_j), \\ w(s,0) = \bar{w}_0(s), \ w_t(s,0) = \bar{w}_1(s), \quad s \in G, \\ \varphi(s,0) = \bar{\varphi}_0(s), \ \varphi_t(s,0) = \bar{\varphi}_1(s), \quad s \in G, \\ j = 1,2,\cdots,n. \end{cases} \tag{9}$$

Here and hereafter we shall use abbreviations $w_t = \frac{\partial w}{\partial t}$ and $w_x = \frac{\partial w}{\partial x}$.

Note that when $\mu_j = 1$, $j = 1,2,\cdots,n$, this closed loop system becomes a proportional velocity feedback controlled system, which can achieve exponential stability under certain conditions (see [44], [48] for tree-shaped network of Euler-Bernoulli beams and strings). In this paper, we shall prove that when $\mu_j \neq 1$, in which case the system has time-delay terms, this system also can be exponentially stable under certain conditions.

The structure of the paper is as follows. In Section 2, we formulate our problem (9) in a Hilbert state space setting and then obtain the well-posedness of this system. We show that the closed loop system is asymptotically stable under certain conditions. In Section 3, we give a complete asymptotic analysis for the spectrum of the system operator and prove that the operator has compact resolvent whose spectrum is located in a strip parallel to the imaginary axis. In Section 4, we prove the completeness and Riesz basis property of the eigenvectors and generalized eigenvectors of the system operator. Hence the spectrum-determined-growth condition holds. In Section 5, for a special simple case, we discuss the exponential stability of this kind of planar networks of Timoshenko beams system and finally give some simulations to support these results.

2. **Well-posedness.** In this section we shall study the well-posedness and asymptotic stability of the closed loop system (9). For this aim, we shall formulate this system in a Hilbert state space setting. Firstly, let us introduce auxiliary functions

$$\begin{aligned} p^j(x,t) &= \frac{\partial w^j}{\partial t}(1,t-\tau_j x), \ x \in [0,1], \ j = 1,2,\cdots,n, \\ q^j(x,t) &= \frac{\partial \varphi^j}{\partial t}(1,t-\tau_j x), \ x \in [0,1], \ j = 1,2,\cdots,n, \end{aligned}$$

which implies

$$\frac{\partial p^j}{\partial s}(s,t) = -\tau_j \frac{\partial p^j}{\partial t}(s,t), \quad \frac{\partial q^j}{\partial s}(s,t) = -\tau_j \frac{\partial q^j}{\partial t}(s,t),$$

and

$$p^j(x,0) = f_j^0(-x\tau_j), \quad q^j(x,0) = f_j^1(-x\tau_j), \ x \in (0,1),$$

where $f_j^0$ and $f_j^1$ are given in (9). Thus, system (9) becomes

$$\begin{cases} \rho_j(x)w_{tt}^j(x,t) - [k_j(x)(w_x^j(x,t) - \varphi^j(x,t))]_x = 0, \ t > 0, \ x \in (0,1), \\ I_{\rho_j}(x)\varphi_{tt}^j(x,t) - [EI_j(x)\varphi_x^j(x,t)]_x \\ \quad -k_j(x)(w_x^j(x,t) - \varphi^j(x,t)) = 0, \ t > 0, \ x \in (0,1), \\ p_x^j(x,t) = -\tau_j p_t^j(x,t), q_x^j(x,t) = -\tau_j q_t^j(x,t), \ t > 0, \ x \in (0,1), \\ w(a_0,t) = w^j(0,t), \quad \varphi(a_0,t) = \varphi^j(0,t), \ t > 0, \\ p^j(0,t) = w_t^j(1,t), \ q^j(0,t) = \varphi_t^j(1,t), \ t > 0, \\ k_j(1)(w_x^j - \varphi^j)(1,t) = -w^j(1,t) - \alpha_j[\mu_j p^j(0,t) \\ \quad +(1 - \mu_j)p^j(1,t)], \ t > 0, \\ EI_j(1)\varphi_x^j(1,t) = -\varphi^j(1,t) - \beta_j[\mu_j q^j(0,t) + (1 - \mu_j)q^j(1,t)], \ t > 0, \\ \sum\limits_{j=1}^{n} k_j(0)(w_x^j - \varphi^j)(0,t) = 0, \sum\limits_{j=1}^{n} EI_j(0)\varphi_x^j(0,t) = 0, \ t > 0, \\ p^j(x,0) = f_j^0(-x\tau_j), \quad q^j(x,0) = f_j^1(-x\tau_j), \ x \in (0,1), \\ w(s,0) = \bar{w}_0(s), \ w_t(s,0) = \bar{w}_1(s), \quad s \in G, \\ \varphi(s,0) = \bar{\varphi}_0(s), \ \varphi_t(s,0) = \bar{\varphi}_1(s), \quad s \in G, \\ j = 1, 2, \cdots, n. \end{cases} \quad (10)$$

Set

$$\begin{cases} W(x,t) := (w^1(x,t), w^2(x,t), \cdots, w^n(x,t))^T, \\ \Phi(x,t) := (\varphi^1(x,t), \varphi^2(x,t), \cdots, \varphi^n(x,t))^T, \\ P(x,t) := (p^1(x,t), p^2(x,t), \cdots, p^n(x,t))^T, \\ Q(x,t) := (q^1(x,t), q^2(x,t), \cdots, q^n(x,t))^T, \\ F^0(x) := (f_1^0(-x\tau_j), f_2^0(-x\tau_j), \cdots, f_n^0(-x\tau_j))^T, \\ F^1(x) := (f_1^1(-x\tau_j), f_2^1(-x\tau_j), \cdots, f_n^1(-x\tau_j))^T. \end{cases}$$

We define $n \times n$ matrices to be

$$\rho := \mathrm{diag}(\rho_1(x), \rho_2(x), \cdots, \rho_n(x)), \quad k := \mathrm{diag}(k_1(x), k_2(x), \cdots, k_n(x)),$$

$$I_\rho := \mathrm{diag}(I_{\rho_1}(x), I_{\rho_2}(x), \cdots, I_{\rho_n}(x)), \quad EI := \mathrm{diag}(EI_1, EI_2, \cdots, EI_n),$$

$$\tau := \mathrm{diag}(\tau_1, \tau_2, \cdots, \tau_n), \quad \mu := \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_n),$$

$$\beta := \mathrm{diag}(\beta_1, \beta_2, \cdots, \beta_n), \quad \alpha := \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_n).$$

Then equation (10) can be rewritten as the following equation in $\mathbb{C}^n$:

$$\begin{cases} \rho(x)W_{tt}(x,t) - [k(x)(W_x(x,t) - \Phi(x,t))]_x = 0, \quad t > 0, \ x \in (0,1), \\ I_\rho(x)\Phi_{tt}(x,t) - [EI(x)\Phi_x(x,t)]_x \\ \quad -k(x)(W_x(x,t) - \Phi(x,t)) = 0, \quad t > 0, \ x \in (0,1), \\ P_x(x,t) = -\tau P_t(x,t), \quad Q_x(x,t) = -\tau Q_t(x,t), \quad t > 0, \ x \in (0,1), \\ C_{(n-1)\times n}W(0,t) = 0, \quad C_{(n-1)\times n}\Phi(0,t) = 0, \quad t > 0, \\ P(0,t) = W_t(1,t), \ Q(0,t) = \Phi_t(1,t), \quad t > 0, \\ k(1)(W_x - \Phi)(1,t) = -W(1,t) - \alpha[\mu P(0,t) + (1 - \mu)P(1,t)], \quad t > 0, \\ EI(1)\Phi_x(1,t) = -\Phi(1,t) - \beta[\mu Q(0,t) + (1 - \mu)Q(1,t)], \quad t > 0, \\ I_{1\times n}k(0)(W_x - \Phi)(0,t) = 0, \quad I_{1\times n}EI(0)\Phi_x(0,t) = 0, \quad t > 0, \\ P(x,0) = F^0(x), \quad Q(x,0) = F^1(x), \quad x \in [0,1], \\ W(s,0) = \bar{W}_0(s), \ W_t(s,0) = \bar{W}_1(s), \quad s \in G, \\ \Phi(s,0) = \bar{\Phi}_0(s), \ \Phi_t(s,0) = \bar{\Phi}_1(s), \quad s \in G, \end{cases} \quad (11)$$

where

$$C_{(n-1)\times n} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \cdots & \cdots \\ 0 & \cdots & \cdots & -1 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{bmatrix}_{(n-1)\times n} \tag{12}$$

and

$$I_{1\times n} = (1, 1, \cdots, 1)_{1\times n}. \tag{13}$$

Now, let us formulate system (11) in an appropriate Hilbert state space setting. Let $H^k(0,1)(k=1,2)$ be the usual Sobolev space and $L^2(0,1)$ be the usual Hilbert space.

Set

$$V_E^k := \{u = (u^j)_{j=1}^n \in \Pi_{j=1}^n H^k(0,1) \mid u^i(0) = u^j(0) = u(a_0), \forall i, j = 1, 2, 3, \cdots, n\}.$$

Let the state space be $\mathcal{H} = V_E^1 \times \Pi_{i=1}^n L^2(0,1) \times V_E^1 \times \Pi_{i=1}^n L^2(0,1) \times \Pi_{i=1}^n L^2(0,1) \times \Pi_{i=1}^n L^2(0,1)$ equipped with an inner product, for

$$(W_i, Z_i, \Phi_i, \Psi_i, P_i, Q_i)^T = \left((w_i^j)_{j=1}^n, (z_i^j)_{j=1}^n, (\varphi_i^j)_{j=1}^n, (\psi_i^j)_{j=1}^n, (p_i^j)_{j=1}^n, (q_i^j)_{j=1}^n\right)^T \in \mathcal{H},$$

$i = 1, 2$, via

$$((W_1, Z_1, \Phi_1, \Psi_1, P_1, Q_1)^T, (W_2, Z_2, \Phi_2, \Psi_2, P_2, Q_2)^T)_{\mathcal{H}}$$

$$= \sum_{j=1}^n \int_0^1 k_j(x)(w_{1,x}^j(x) - \varphi_1^j(x))\overline{(w_{2,x}^j(x) - \varphi_2^j(x))}dx + \sum_{j=1}^n \int_0^1 \rho_j(x)z_1^j(x)\overline{z_2^j(x)}dx$$

$$+ \sum_{j=1}^n \int_0^1 EI_j(x)\varphi_{1,x}^j(x)\overline{\varphi_{2,x}^j(x)}dx + \sum_{j=1}^n \int_0^1 I_{\rho_j}(x)\psi_1^j(x)\overline{\psi_2^j(x)}dx$$

$$+ \sum_{j=1}^n \int_0^1 p_1^j(x)\overline{p_2^j(x)}dx + \sum_{j=1}^n \int_0^1 q_1^j(x)\overline{q_2^j(x)}dx + \sum_{j=1}^n w^j(1)\overline{w^j(1)} + \sum_{j=1}^n \varphi^j(1)\overline{\varphi^j(1)}.$$

We define an operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\mathcal{A}\begin{bmatrix} W \\ Z \\ \Phi \\ \Psi \\ P \\ Q \end{bmatrix} = \begin{pmatrix} Z \\ \rho^{-1}(x)\left[k(x)(W_x(x) - \Phi(x))\right]_x \\ \Psi \\ I_\rho^{-1}(x)\left([EI(x)\Phi_x(x)]_x + k(x)(W_x(x) - \Phi(x))\right) \\ -\tau^{-1}P_x \\ -\tau^{-1}Q_x \end{pmatrix}, \tag{14}$$

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ (W, Z, \Phi, \Psi, P, Q) \in \mathcal{H} \,\middle|\, \begin{array}{l} W \in V_E^2, \ Z \in V_E^1, \ \Phi \in V_E^2, \ \Psi \in V_E^1, \\ Z(1) = P(0), \ \Psi(1) = Q(0), \\ I_{1\times n}k(0)(W_x - \Phi)(0) = 0, \\ I_{1\times n}EI(0)\Phi_x(0) = 0, \\ k(1)(W_x - \Phi)(1) = -W(1) - \alpha[\mu P(0) \\ \quad + (1-\mu)P(1)], \\ EI(1)\Phi_x(1) = -\Phi(1) - \beta[\mu Q(0) \\ \quad + (1-\mu)Q(1)], \end{array} \right\}. \tag{15}$$

Then, we rewrite (9) as an evolutionary equation in $\mathcal{H}$

$$\begin{cases} \frac{dU(t)}{dt} = \mathcal{A}U(t), & t > 0, \\ U(0) = U_0, \end{cases} \tag{16}$$

where $U(t) = (W(\cdot, t), W_t(\cdot, t), \Phi(\cdot, t), \Phi_t(\cdot, t), P(\cdot, t), Q(\cdot, t))^T$ and

$$U(0) = (\bar{W}_0, \bar{W}_1, \bar{\Phi}_0, \bar{\Phi}_1, F^0, F^1)^T \in \mathcal{H}$$

is given. We have the following result.

**Lemma 2.1.** *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then $0 \in \rho(\mathcal{A})$ and $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$.*

*Proof.* Clearly, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Let us show that $\mathcal{A}^{-1}$ exists. Let $(W, Z, \Phi, \Psi, P, Q)^T \in \mathcal{D}(\mathcal{A})$ which satisfies

$$\mathcal{A}(W, Z, \Phi, \Psi, P, Q)^T = 0.$$

Then by (14)–(15), we have

$$Z = \Psi = P = Q = 0$$

and $w^j(x)$, $\varphi^j(x)$, $j = 1, 2, \cdots, n$ satisfy

$$k_j(x)[w_x^j(x) - \varphi^j(x)]_x = 0, \tag{17}$$

$$[EI_j(x)\varphi_x^j(x)]_x + k_j(x)(w_x^j - \varphi^j)(x) = 0, \tag{18}$$

$$\sum_{j=1}^n k_j(0)(w_x^j - \varphi^j)(0) = 0, \quad \sum_{j=1}^n EI_j(0)\varphi_x^j(0) = 0, \tag{19}$$

$$k_j(1)(w_x^j - \varphi)(1) = -w^j(1), \quad EI_j(1)\varphi_x^j(1) = -\varphi^j(1), \tag{20}$$

$$w^j(0) = w^i(0), \quad \varphi^j(0) = \varphi^i(0), \ i, j = 1, 2, \cdots, n. \tag{21}$$

Multiply (17), (18) by $w^j(x)$ and $\varphi^j(x)$ respectively. Then integrating the obtained identities from 0 to 1 yields

$$
\begin{aligned}
0 &= \int_0^1 [k_j(x)(w_x^j - \varphi^j)(x)]_x \overline{w^j(x)} dx \\
&= k_j(x)(w_x^j - \varphi^j)(x)\overline{w^j(x)}|_0^1 - \int_0^1 k_j(x)(w_x^j - \varphi^j)\overline{w_x^j(x)} dx \\
&= -w^j(1)\overline{w^j(1)} - \int_0^1 k_j(x)(w_x^j - \varphi^j)(x)\overline{w_x^j(x)} dx
\end{aligned}
\tag{22}
$$

and

$$
\begin{aligned}
0 &= \int_0^1 ([EI_j(x)\varphi_x^j(x)]_x \overline{\varphi^j(x)} + k_j(x)(w_x^j - \varphi^j)(x)\overline{\varphi^j(x)}) dx \\
&= EI_j(x)\varphi_x^j(x)\overline{\varphi^j(x)}|_0^1 - \int_0^1 EI_j(x)\varphi_x^j(x)\overline{\varphi_x^j(x)} dx + \int_0^1 k_j(x)(w_x^j - \varphi^j)(x)\overline{\varphi^j(x)} dx \\
&= -\varphi^j(1)\overline{\varphi^j(1)} - \int_0^1 EI_j(x)\varphi_x^j(x)\overline{\varphi_x^j(x)} dx + \int_0^1 k_j(x)(w_x^j - \varphi^j)(x)\overline{\varphi^j(x)} dx.
\end{aligned}
\tag{23}
$$

Then summing (22) and (23) leads to

$$-|w^j(1)|^2 - |\varphi^j(1)|^2 - \int_0^1 k_j(w_x^j - \varphi^j)(x)\overline{(w_x^j - \varphi^j)(x)} dx - \int_0^1 EI_j\varphi_x^j\overline{\varphi_x^j} dx = 0. \tag{24}$$

So we have

$$\sum_{j=1}^{n}[-|w^j(1)|^2 - |\varphi^j(1)|^2 - \int_0^1 k_j(x)(w_x^j - \varphi^j)(x)\overline{(w_x^j - \varphi^j)(x)}dx$$

$$- \int_0^1 EI_j(x)\varphi_x^j(x)\overline{\varphi_x^j(x)}dx]$$

$$= -\sum_{j=1}^{n}|w^j(1)|^2 - \sum_{j=1}^{n}|\varphi^j(1)|^2 - \sum_{j=1}^{n}\int_0^1 k_j(x)|(w_x^j - \varphi^j)(x)|^2 dx$$

$$- \sum_{j=1}^{n}\int_0^1 EI_j(x)|\varphi_x^j(x)|^2 dx$$

$$= 0.$$

Thus, $w^j(x) = \varphi^j(x) = 0$, $j = 1, 2, \cdots, n$ since $k_j(x)$, $EI_j(x) > 0$. Therefore, $(W, Z, \Phi, \Psi, P, Q) = 0$, which implies that $\mathcal{A}$ is injective.

Next, let us prove that $\mathcal{A}$ is surjective. In fact, for any fixed

$$(F_1, F_2, F_3, F_4, F_5, F_6) \in \mathcal{H},$$

where $F_i = (\widehat{f}_i^1, \widehat{f}_i^2, \cdots, \widehat{f}_i^n)^T$, we consider the solvability of

$$\mathcal{A}(W, Z, \Phi, \Psi, P, Q)^T = (F_1, F_2, F_3, F_4, F_5, F_6),$$

that is

$$z^j = \widehat{f}_1^j, \tag{25}$$

$$\rho_j^{-1}(x)[k_j(x)(w_x^j - \varphi^j)(x)]_x = \widehat{f}_2^j, \tag{26}$$

$$\psi^j = \widehat{f}_3^j, \tag{27}$$

$$I_{\rho_j}^{-1}(x)([EI_j(x)\varphi_x^j(x)]_x + k_j(x)(w_x^j - \varphi^j)(x)) = \widehat{f}_4^j, \tag{28}$$

$$-\tau_j^{-1}p_x^j = \widehat{f}_5^j, \tag{29}$$

$$-\tau_j^{-1}q_x^j = \widehat{f}_6^j, \tag{30}$$

with conditions

$$\begin{cases} z^j(1) = p^j(0), \quad \psi^j(1) = q^j(0), \\ \sum_{j=1}^{n} k_j(0)(w_x^j - \varphi^j)(0) = 0, \\ \sum_{j=1}^{n} EI_j(0)\varphi_x^j(0) = 0, \\ k_j(1)(w_x^j - \varphi^j)(1) = -w^j(1) - \alpha_j[\mu_j\widehat{f}_1^j(0) + (1 - \mu_j)\widehat{f}_1^j(1)], \\ EI_j(1)\varphi_x^j(1) = -\varphi^j(1) - \beta_j[\mu_j\widehat{f}_3^j(0) + (1 - \mu_j)\widehat{f}_3^j(1)]. \end{cases} \tag{31}$$

Integrating (29) and (30) from 0 to 1 respectively, we have

$$p^j(x) - p^j(0) = \int_0^x \tau_j\widehat{f}_5^j(s)ds, \quad q^j(x) - q^j(0) = \int_0^x \tau_j\widehat{f}_6^j(s)ds,$$

which yields

$$p^j(x) = p^j(0) + \int_0^x \tau_j\widehat{f}_5^j(s)ds = \widehat{f}_1^j(1) + \tau_j\int_0^x \widehat{f}_5^j(s)ds, \tag{32}$$

$$q^j(x) = q^j(0) + \int_0^x \tau_j\widehat{f}_6^j(s)ds = \widehat{f}_3^j(1) + \tau_j\int_0^x \widehat{f}_6^j(s)ds. \tag{33}$$

Now, let us find a solution to (26) and (28) which satisfying (31). Multiplying (26) and (28) by the conjugates of test functions $g_j(x)$, $\widetilde{g}_j(x) \in V_E^1$, respectively and then integrating them from 0 to 1, we obtain

$$\int_0^1 [k_j(x)(w_x^j - \varphi^j)(x)]_x \overline{g_j(x)} dx = \int_0^1 \rho_j(x) \widehat{f}_2^j(x) \overline{g_j(x)} dx,$$

$$\int_0^1 ([EI_j(x)\varphi_x^j(x)]_x + k_j(x)(w_x^j(x) - \varphi^j(x))) \overline{\widetilde{g}_j(x)} = \int_0^1 I_{\rho_j}(x) \widehat{f}_4^j(x) \overline{\widetilde{g}_j(x)} dx.$$

Hence, we have

$$
\begin{aligned}
&\sum_{j=1}^n \int_0^1 \rho_j(x) \widehat{f}_2^j(x) \overline{g_j(x)} dx + \sum_{j=1}^n \int_0^1 I_{\rho_j}(x) \widehat{f}_4^j(x) \overline{\widetilde{g}_j(x)} dx \\
=\ & \sum_{j=1}^n \int_0^1 [k_j(x)(w_x^j - \varphi^j)(x)]_x \overline{g_j(x)} dx \\
& + \sum_{j=1}^n \int_0^1 ([EI_j(x)\varphi_x^j(x)]_x + k_j(x)(w_x^j(x) - \varphi^j(x))) \overline{\widetilde{g}_j(x)} dx \\
=\ & \sum_{j=1}^n k_j(x)(w_x^j - \varphi^j)(x) \overline{g_j(x)}|_0^1 - \sum_{j=1}^n \int_0^1 k_j(x)(w_x^j - \varphi^j)(x) \overline{g_{j,x}(x)} dx \\
& + \sum_{j=1}^n EI_j(x)\varphi_x^j(x) \overline{\widetilde{g}_j(x)}|_0^1 - \sum_{j=1}^n \int_0^1 EI_j(x)\varphi_x^j(x) \overline{\widetilde{g}_{j,x}(x)} dx \\
& + \sum_{j=1}^n \int_0^1 k_j(x)(w_x^j - \varphi^j)(x) \overline{\widetilde{g}_j(x)} dx \\
=\ & -\sum_{j=1}^n \int_0^1 k_j(x)(w_x^j - \varphi^j)(x) \overline{(g_{j,x} - \widetilde{g}_j)(x)} dx - \sum_{j=1}^n \int_0^1 EI_j(x)\varphi_x^j(x) \overline{\widetilde{g}_{j,x}(x)} dx \\
& + \sum_{j=1}^n k_j(x)(w_x^j - \varphi^j)(x) \overline{g_j(x)}|_0^1 + \sum_{j=1}^n EI_j(x)\varphi_x^j(x) \overline{\widetilde{g}_j(x)}|_0^1 \\
=\ & -\sum_{j=1}^n \int_0^1 k_j(x)(w_x^j - \varphi^j)(x) \overline{(g_{j,x} - \widetilde{g}_j)(x)} dx - \sum_{j=1}^n \int_0^1 EI_j(x)\varphi_x^j(x) \overline{\widetilde{g}_{j,x}(x)} dx \\
& + \sum_{j=1}^n k_j(1)(w_x^j - \varphi^j)(x) \overline{g_j(1)} - \left[ \sum_{j=1}^n k_j(0)(w_x^j - \varphi^j)(0) \right] \overline{g_1(0)} \\
& + \sum_{j=1}^n EI_j(1)\varphi_x^j(1) \overline{\widetilde{g}_j(1)} - \left[ \sum_{j=1}^n EI_j(0)\varphi_x^j(0) \right] \overline{\widetilde{g}_1(0)} \\
=\ & -\sum_{j=1}^n \int_0^1 k_j(x)(w_x^j - \varphi^j)(x) \overline{(g_{j,x} - \widetilde{g}_j)(x)} dx - \sum_{j=1}^n \int_0^1 EI_j(x)\varphi_x^j(x) \overline{\widetilde{g}_{j,x}(x)} dx \\
& + \sum_{j=1}^n [-w^j(1) - \alpha_j(\mu_j \widehat{f}_1^j(0) + (1 - \mu_j)\widehat{f}_1^j(1))] \overline{g_j(1)} \\
& + \sum_{j=1}^n [-\varphi^j(1) - \beta_j(\mu_j \widehat{f}_3^j(0) + (1 - \mu_j)\widehat{f}_3^j(1))] \overline{\widetilde{g}_j(1)}. \tag{34}
\end{aligned}
$$

Then we define the following bilinear function $B(\cdot, \cdot)$:

$$
\begin{aligned}
B(X,Y) &= \sum_{j=1}^{n} \int_0^1 k_j(x)(w_x^j(x) - \varphi^j(x))\overline{g_{j,x}(x) - \widetilde{g}_j(x)}dx \\
&\quad + \sum_{j=1}^{n} \int_0^1 EI_j(x)\varphi_x^j(x)\overline{\widetilde{g_{j,x}}(x)}dx \\
&\quad + \sum_{j=1}^{n} w^j(1)\overline{g_j(1)} + \sum_{j=1}^{n} \varphi^j(1)\overline{\widetilde{g_j}(1)},
\end{aligned}
$$

where

$$
X = (W, \Phi) = \left((w^j)_{j=1}^n, (\varphi^j)_{j=1}^n\right), \ Y = (G, \widetilde{G}) = \left((g_j)_{j=1}^n, (\widetilde{g}_j)_{j=1}^n\right) \in V_E^1 \times V_E^1.
$$

Thus, $B(\cdot, \cdot)$ is a bounded bilinear function, since

$$
\begin{aligned}
&|B(X,Y)| \\
&\leq M\Big(\sum_{j=1}^n \int_0^1 k_j(x)|(w_x^j - \varphi^j)(x)|^2 dx + \sum_{j=1}^n \int_0^1 EI_j(x)|\varphi_x^j(x)|^2 dx + \sum_{j=1}^n |w^j(1)|^2 + \sum_{j=1}^n |\varphi^j(1)|^2\Big)^{1/2} \\
&\quad \cdot \Big(\sum_{j=1}^n \int_0^1 k_j(x)|(g_{j,x} - \widetilde{g}_j)(x)|^2 dx + \sum_{j=1}^n \int_0^1 EI_j(x)|\widetilde{g_{j,x}}(x)|^2 dx + \sum_{j=1}^n |g_j(1)|^2 + \sum_{j=1}^n |\widetilde{g}_j(1)|^2\Big)^{1/2} \\
&= M\|X\|_{V_E^1 \times V_E^1} \cdot \|Y\|_{V_E^1 \times V_E^1},
\end{aligned}
$$

where $M$ is a positive constant.

Moreover, $B(\cdot, \cdot)$ satisfies that

$$
\begin{aligned}
|B(X,X)| &= \sum_{j=1}^{n} \int_0^1 k_j(x)(w_x^j - \varphi^j)(x)\overline{(w_x^j - \varphi^j)(x)}dx \\
&\quad + \sum_{j=1}^{n} \int_0^1 EI_j(x)\varphi_x^j(x)\overline{\varphi_x^j(x)}dx + \sum_{j=1}^{n} |w^j(1)|^2 + \sum_{j=1}^{n} |\varphi^j(1)|^2 \\
&= \|X\|_{V_E^1 \times V_E^1}^2.
\end{aligned}
$$

Choosing $G, \widetilde{G} \in (\Pi_{i=1}^n C_0^\infty(0,1)) \cap V_E^1 \subset V_E^1$, by Lax-Milgram's theorem, we obtain that there is a unique solution $X = (W, \Phi) = \left((w^j)_{j=1}^n, (\varphi^j)_{j=1}^n\right) \in V_E^1 \times V_E^1$ to the equation (34), where $\left((w^j)_{j=1}^n, (\varphi^j)_{j=1}^n\right)$ satisfies

$$
\begin{cases}
\rho_j^{-1}(x)[k(x)(w_x^j - \varphi^j)(x)]_x = \widehat{f_2^j}, \\
I_{\rho_j}^{-1}(x)([EI_j(x)\varphi_x^j(x)]_x + k_j(x)(w_x^j - \varphi^j)(x)) = \widehat{f_4^j},
\end{cases} \tag{35}
$$

which implies that $W(x), \Phi(x) \in (\Pi_{i=1}^n H^2(0,1)) \cap V_E^1$, that is $X \in V_E^2 \times V_E^2$. By selecting special test functions $g_j(x)$ and $\widetilde{g}_j(x)$, together with (35), we get that $W(x), \Phi(x)$ satisfy the following conditions

$$
\sum_{j=1}^n k_j(0)(w_x^j - \varphi^j)(0) = 0,
$$

$$
\sum_{j=1}^n EI_j(0)\varphi_x^j(0) = 0,
$$

$$
k_j(1)(w_x^j - \varphi^j)(1) = -w^j(1) - \alpha_j[\mu_j \widehat{f_1^j}(0) + (1 - \mu_j)\widehat{f_1^j}(1)],
$$

$$
EI_j(1)\varphi_x^j(1) = -\varphi^j(1) - \beta_j[\mu_j \widehat{f_3^j}(0) + (1 - \mu_j)\widehat{f_3^j}(1)].
$$

Summarizing the above calculation, we have obtained the unique vector

$$(W, Z, \Phi, \Psi, P, Q) \in \mathcal{D}(\mathcal{A})$$

such that $\mathcal{A}(W, Z, \Phi, \Psi, P, Q) = (F_1, F_2, F_3, F_4, F_5, F_6)$. Therefore, $\mathcal{A}$ is surjective in $\mathcal{H}$. Hence $0 \in \rho(\mathcal{A})$ by the Inverse operator theorem. Furthermore, according to the Sobolev embedding theorem (see [1]), we obtain that $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$. The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As a direct consequence of Lemma 2.1, we have the following result.

**Corollary 1.** *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then the spectrum of $\mathcal{A}$ consists of isolated eigenvalues of finite multiplicity only, i.e., $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$.*

Suppose that

$$\mu_j \geq \frac{1}{2}, \quad j = 1, 2, \cdots, n. \tag{36}$$

Under this condition, we shall show that $\mathcal{A}$ is dissipative in $\mathcal{H}$. For this aim, we choose positive real constants $\eta_j$, $\zeta_j$, $j = 1, 2, \cdots, n$ such that

$$\begin{aligned} \tau_j(1 - \mu_j)\alpha_j \leq \eta_j \leq \tau_j(3\mu_j - 1)\alpha_j, \ j = 1, 2, \cdots, n, \\ \tau_j(1 - \mu_j)\beta_j \leq \zeta_j \leq \tau_j(3\mu_j - 1)\beta_j, \ j = 1, 2, \cdots, n. \end{aligned} \tag{37}$$

These constants $\eta_j, \zeta_j, \ j = 1, 2, \cdots, n$ exist owing to condition (36). Then we introduce a new inner product in $\mathcal{H}$: for

$$(W_i, Z_i, \Phi_i, \Psi_i, P_i, Q_i)^T = ((w_i^j)_{j=1}^n, (z_i^j)_{j=1}^n, (\varphi_i^j)_{j=1}^n, (\psi_i^j)_{j=1}^n, (p_i^j)_{j=1}^n, (q_i^j)_{j=1}^n)^T \in \mathcal{H}, i = 1, 2,$$

$$\begin{aligned} &((W_1, Z_1, \Phi_1, \Psi_1, P_1, Q_1)^T, (W_2, Z_2, \Phi_2, \Psi_2, P_2, Q_2)^T)_1 \\ &= \sum_{j=1}^n \int_0^1 k_j(x)(w_{1,x}^j - \varphi_1^j)(x)\overline{(w_{2,x}^j - \varphi_2^j)(x)}dx + \sum_{j=1}^n \int_0^1 \rho_j(x)z_1^j(x)\overline{z_2^j(x)}dx \\ &\quad + \sum_{j=1}^n \int_0^1 EI_j(x)\varphi_{1,x}^j(x)\overline{\varphi_{2,x}^j(x)}dx + \sum_{j=1}^n \int_0^1 I_{\rho_j}(x)\psi_1^j(x)\overline{\psi_2^j(x)}dx \\ &\quad + \sum_{j=1}^n \int_0^1 \eta_j p_1^j(x)\overline{p_2^j(x)}dx + \sum_{j=1}^n \int_0^1 \zeta_j q_1^j(x)\overline{q_2^j(x)}dx + \sum_{j=1}^n w^j(1)\overline{w^j(1)} \\ &\quad + \sum_{j=1}^n \varphi^j(1)\overline{\varphi^j(1)}. \end{aligned}$$

It is easy to verify that $(\cdot, \cdot)_1$ is equivalent to the usual inner product $(\cdot, \cdot)_\mathcal{H}$ of $\mathcal{H}$. Then we have the following result.

**Lemma 2.2.** *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before, condition (36) be fulfilled. Then $\mathcal{A}$ is a dissipative operator in $\mathcal{H}$.*

*Proof.* For any real

$$\widetilde{U} = ((w^j(x))_{j=1}^n, (z^j(x))_{j=1}^n, (\varphi^j(x))_{j=1}^n, (\psi^j(x))_{j=1}^n, (p^j(x))_{j=1}^n, (q^j(x))_{j=1}^n)^T \in \mathcal{D}(\mathcal{A}),$$

we have

$$\begin{aligned} (\mathcal{A}\widetilde{U}, \widetilde{U})_1 &= \sum_{j=1}^n \int_0^1 k_j(x)(z_x^j - \psi^j)(w_x^j - \varphi^j)dx + \sum_{j=1}^n \int_0^1 EI_j(x)\psi_x^j \varphi_x^j dx \\ &\quad + \sum_{j=1}^n \int_0^1 \rho_j(x)[\rho_j^{-1}(x)k_j(x)(w_{xx}^j - \varphi_x^j)]z^j dx \end{aligned}$$

$$+\sum_{j=1}^{n}\int_{0}^{1}I_{\rho_j}(x)[I_{\rho_j}^{-1}(x)(EI_j(x)\varphi_x^j)_x+I_{\rho_j}^{-1}(x)k_j(x)(w_x^j-\varphi^j)]\psi^j\,dx$$

$$+\sum_{j=1}^{n}\int_{0}^{1}\eta_j[-\tau_j^{-1}p_x^j]p^j\,dx+\sum_{j=1}^{n}\int_{0}^{1}\zeta_j[-\tau_j^{-1}q_x^j]q^j\,dx+\sum_{j=1}^{n}z^j(1)w^j(1)$$

$$+\sum_{j=1}^{n}\psi^j(1)\varphi^j(1)$$

$$=\quad\sum_{j=1}^{n}k_j(w_x^j-\varphi^j)z^j|_0^1+\sum_{j=1}^{n}EI_j\varphi_x^j\psi^j|_0^1-\sum_{j=1}^{n}\frac{\tau_j^{-1}\eta_j}{2}(p^j)^2|_0^1$$

$$-\sum_{j=1}^{n}\frac{\tau_j^{-1}\zeta_j}{2}(q^j)^2|_0^1+\sum_{j=1}^{n}z^j(1)w^j(1)+\sum_{j=1}^{n}\psi^j(1)\varphi^j(1).$$

Substituting the conditions of (10) into the above equation leads to

$$(\mathcal{A}\widetilde{U},\widetilde{U})_1\quad=\quad-\sum_{j=1}^{n}\alpha_j\mu_j(p^j(0))^2-\sum_{j=1}^{n}\alpha_j(1-\mu_j)p^j(1)p^j(0)$$

$$-\sum_{j=1}^{n}\beta_j\mu_j(q^j(0))^2-\sum_{j=1}^{n}\beta_j(1-\mu_j)q^j(1)q^j(0)$$

$$-\sum_{j=1}^{n}\frac{\tau_j^{-1}\eta_j}{2}[(p^j(1))^2-(p^j(0))^2]-\frac{\tau_j^{-1}\zeta_j}{2}[(q^j(1))^2-(q^j(0))^2]$$

$$=\quad-\sum_{j=1}^{n}(\alpha_j\mu_j-\frac{\eta_j}{2\tau_j})(p^j(0))^2-\sum_{j=1}^{n}(\beta_j\mu_j-\frac{\zeta_j}{2\tau_j})(q^j(0))^2-\sum_{j=1}^{n}\frac{\eta_j}{2\tau_j}(p^j(1))^2$$

$$-\sum_{j=1}^{n}\frac{\zeta_j}{2\tau_j}(q^j(1))^2-\sum_{j=1}^{n}\alpha_j(1-\mu_j)p^j(0)p^j(1)-\sum_{j=1}^{n}\beta_j(1-\mu_j)q^j(0)q^j(1).$$

Using Cauchy-Schwarz's inequality, we have

$$-\alpha_j(1-\mu_j)p^j(0)p^j(1)\leq\frac{\alpha_j(1-\mu_j)}{2}[(p^j(0))^2+(p^j(1))^2],$$

$$-\beta_j(1-\mu_j)q^j(0)q^j(1)\leq\frac{\beta_j(1-\mu_j)}{2}[(q^j(0))^2+(q^j(1))^2].$$

Therefore, it follows that

$$(\mathcal{A}\widetilde{U},\widetilde{U})_1$$

$$\leq\quad-\sum_{j=1}^{n}[\alpha_j\mu_j-\frac{\eta_j}{2\tau_j}-\frac{\alpha_j(1-\mu_j)}{2}](p^j(0))^2-\sum_{j=1}^{n}[\beta_j\mu_j-\frac{\zeta_j}{2\tau_j}-\frac{\beta_j(1-\mu_j)}{2}](q^j(0))^2$$

$$-\sum_{j=1}^{n}[\frac{\eta_j}{2\tau_j}-\frac{\alpha_j(1-\mu_j)}{2}](p^j(1))^2-\sum_{j=1}^{n}[\frac{\zeta_j}{2\tau_j}-\frac{\beta_j(1-\mu_j)}{2}](q^j(1))^2.\qquad(38)$$

From (37), we obtain that

$$\alpha_j\mu_j-\frac{\eta_j}{2\tau_j}-\frac{\alpha_j(1-\mu_j)}{2}\geq0,\quad\frac{\eta_j}{2\tau_j}-\frac{\alpha_j(1-\mu_j)}{2}\geq0,$$

$$\beta_j\mu_j-\frac{\zeta_j}{2\tau_j}-\frac{\beta_j(1-\mu_j)}{2}\geq0,\quad\frac{\zeta_j}{2\tau_j}-\frac{\beta_j(1-\mu_j)}{2}\geq0.$$

Hence $(\mathcal{A}\widetilde{U},\widetilde{U})_1\leq0$ which implies $\mathcal{A}$ is dissipative in $\mathcal{H}$. $\qquad\square$

We have the following result.

**Theorem 2.3.** *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before, condition (36) be fulfilled. Then $\mathcal{A}$ generates a $C_0$ semigroup of contractions on $\mathcal{H}$. Hence, system (16) is well-posed.*

*Proof.* From Lemma 2.2, we obtain $\mathcal{A}$ is a dissipative operator in $\mathcal{H}$. Furthermore, we have proved in Lemma 2.1 that $\mathcal{A}^{-1}$ is compact, which implies that there always exists at least one $\lambda_0 \in \rho(\mathcal{A})$, $\lambda_0 > 0$. Hence $\mathcal{A} - \lambda_0 I$ is surjective for this $\lambda_0$. Then according to Lumer-Phillips theorem (see [36]), we obtain that $\mathcal{A}$ generates a $C_0$ semigroup of contractions on $\mathcal{H}$.                                                   □

Now, we proceed to discuss the asymptotic stability of the system (16). For any $\lambda \in \sigma(\mathcal{A})$, let $U = (W, Z, \Phi, \Psi, P, Q)$ be its corresponding eigenvector. Then

$$(\lambda I - \mathcal{A})U = 0, \tag{39}$$

which leads to

$$Z(x) = \lambda W(x), \ \Psi(x) = \lambda \Phi(x), \ j = 1, 2, \cdots, n$$

and $W, \Phi, P, Q$ satisfies

$$\begin{cases} \rho(x)\lambda^2 W(x) - [k(x)(W'(x) - \Phi(x))]' = 0, \ x \in (0,1), \\ I_\rho(x)\lambda^2 \Phi(x) - [EI(x)\Phi'(x)]' - k(x)(W'(x) - \Phi(x)) = 0, \ x \in (0,1), \\ P'(x) = -\tau\lambda P(x), Q'(x) = -\tau\lambda Q(x), \\ C_{(n-1)\times n}W(0) = 0, \quad C_{(n-1)\times n}\Phi(0) = 0, \\ P(0) = \lambda W(1), \ Q(0) = \lambda \Phi(1), \\ k(1)(W' - \Phi)(1) = -W(1) - \alpha[\mu P(0) + (1-\mu)P(1)], \\ EI(1)\Phi'(1) = -\Phi(1) - \beta[\mu Q(0) + (1-\mu)Q(1)], \\ I_{1\times n}k(0)(W' - \Phi)(0) = 0, \quad I_{1\times n}EI(0)\Phi'(0) = 0. \end{cases} \tag{40}$$

We have the following theorem.

**Theorem 2.4.** *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before. Then when $\mu_j > \frac{1}{2}$, $j = 1, 2, \cdots, n$, the closed loop system (16) is asymptotically stable.*

*Proof.* In order to get the asymptotic stability of the closed loop system given by (9), it only needs to check that there is no eigenvalue on imaginary axis by Lyubich and Phóng's theorem (see [26]). We have gotten that $0 \notin \sigma(\mathcal{A})$ and $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ by Lemma 2.1 and Corollary 1. So we shall confirm that for any $\xi \in \mathbb{R}$, $\xi \neq 0$, $\lambda = i\xi$ is not the eigenvalue of $\mathcal{A}$.

If $\lambda = i\xi$, $\xi \in \mathbb{R}$, $\xi \neq 0$ is an eigenvalue of $\mathcal{A}$ and

$$U = ((w^j)_{j=1}^n, (i\xi w^j)_{j=1}^n, (\varphi^j)_{j=1}^n, (i\xi \varphi^j)_{j=1}^n, (p^j)_{j=1}^n, (q^j)_{j=1}^n)^T$$

is its corresponding eigenvector, then we have

$$0 = \Re(\mathcal{A}U, U)_1 = \Re\lambda(U, U)_1,$$

which together with (38) implies that

$$p^j(0) = p^j(1) = q^j(0) = q^j(1) = 0.$$

Thus, $w^j(x)$ and $\varphi^j(x)$ satisfy the following equations

$$
\begin{cases}
\rho(x)\lambda^2 w^j(x) - [k_j(x)(w_x^j(x) - \varphi^j(x))]_x = 0, \; x \in (0,1), \\
I_{\rho_j}(x)\lambda^2 \varphi^j(x) - [EI_j(x)\varphi_x^j(x)]_x - k_j(x)(w_x^j(x) - \varphi^j(x)) = 0, \; x \in (0,1), \\
p_x^j(x) = -\tau_j \lambda p^j(x), \quad q_x^j(x) = -\tau_j \lambda q^j(x), \\
w(a_0) = w^j(0), \quad \varphi(a_0) = \varphi^j(0), \\
p^j(0) = \lambda w^j(1), \; q^j(0) = \lambda \varphi^j(1), \\
k_j(1)(w_x^j - \varphi^j)(1) = 0, \quad EI_j(1)\varphi_x^j(1) = 0, \\
\sum\limits_{j=1}^{n} k_j(0)(w_x^j - \varphi^j)(0) = 0, \quad \sum\limits_{j=1}^{n} EI_j(0)\varphi_x^j(0) = 0, \\
p^j(0) = p^j(1) = q^j(0) = q^j(1) = 0, \\
j = 1, 2, \cdots, n, \; x \in (0,1),
\end{cases}
\tag{41}
$$

which leads to

$$
w^j(1) = w_x^j(1) = \varphi^j(1) = \varphi_x^j(1) = 0, \; j = 1, 2, \cdots, n.
$$

Thus, according to the general theory of ordinary differential equations, we get

$$
w^j = \varphi^j = p^j = q^j = 0, \; j = 1, 2, \cdots, n.
$$

Hence $U = 0$, which is a contradiction that $U$ is an eigenvector of $\mathcal{A}$. Therefore, the system (16) is asymptotically stable due to Lyubich and Phóng's theorem. $\square$

**Remark 1.** If there are some $\mu_i$, $i \in \{1, 2, \cdots, n\}$ such that $\mu_i < \frac{1}{2}$, then there always exist some $\lambda \in \sigma(\mathcal{A})$ such that $\Re\lambda \geq 0$, even for single beam (see [22]), which implies that this kinds of networks systems are unstable.

3. **Asymptotic analysis of spectrum of the system operator.** In this section, we shall discuss the asymptotic distribution of the spectrum of the closed loop system (16). We have obtained from Corollary 1 that the spectrum of $\mathcal{A}$ consists only of isolated eigenvalues of finite multiplicity. Thus, it is sufficient to discuss the distribution of the eigenvalue of $\mathcal{A}$.

Let us consider the ordinary differential equations (40) to obtain the distribution of $\sigma(\mathcal{A})$. In order to calculate the eigenvalues of $\mathcal{A}$, we shall first get the fundamental matrix to the differential equations (40). Here we mainly employ the asymptotic technique in [27]. By (40), we obtain

$$
\begin{cases}
\lambda^2 \rho(x)W(x) - k'(x)(W'(x) - \Phi(x)) - k(x)(W''(x) - \Phi'(x)) = 0, \\
\lambda^2 I_\rho(x)\Phi(x) - EI'(x)\Phi'(x) - EI(x)\Phi''(x) - k(x)(W'(x) - \Phi(x)) = 0.
\end{cases}
\tag{42}
$$

From (42), it follows that

$$
\begin{cases}
W''(x) = k^{-1}(x)[\lambda^2 \rho(x)W(x) - k'(x)(W' - \Phi)(x)] + \Phi'(x), \\
\Phi''(x) = EI^{-1}(x)[\lambda^2 I_\rho(x)\Phi(x) - EI'(x)\Phi'(x) - k(x)(W' - \Phi)(x)].
\end{cases}
\tag{43}
$$

Set

$$
Y_1 := (W, \Phi)^T, \; Y_2 := \frac{Y_1'}{\lambda}, \; Y := \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}.
\tag{44}
$$

Then we rewrite (43) as follows

$$
\frac{dY}{dx} = (\lambda A_1 + A_0 + \lambda^{-1} A_{-1})Y
\tag{45}
$$

where

$$
A_1(x) := \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\rho(x)k^{-1}(x) & 0 & 0 & 0 \\
0 & I_\rho(x)(EI)^{-1}(x) & 0 & 0
\end{bmatrix},
$$

$$A_0(x) := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -k'(x)k^{-1}(x) & 1 \\ 0 & 0 & -k(x)(EI)^{-1}(x) & -EI'(x)(EI)^{-1}(x) \end{bmatrix}$$

and

$$A_{-1}(x) := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & k'(x)k^{-1}(x) & 0 & 0 \\ 0 & k(x)(EI)^{-1}(x) & 0 & 0 \end{bmatrix}.$$

Do the following transformation

$$Y(x) = T_0(x)Z(x) \tag{46}$$

where

$$T_0(x) := \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \sqrt{\rho(x)k^{-1}(x)} & 0 & -\sqrt{\rho(x)k^{-1}(x)} & 0 \\ 0 & \sqrt{I_\rho(x)EI^{-1}(x)} & 0 & -\sqrt{I_\rho(x)EI^{-1}(x)} \end{bmatrix}. \tag{47}$$

Substituting (46) into (45) yields

$$\begin{aligned} \frac{dZ(x)}{dx} &= T_0^{-1}(x)[\lambda A_1(x) + A_0(x) + \lambda^{-1}A_{-1}(x) - T_0'(x)T_0^{-1}(x)]T_0(x)Z(x) \\ &= [\lambda\widetilde{\Lambda}_1(x) - T_0^{-1}(x)T_0'(x) + T_0^{-1}(x)A_0(x)T_0(x) + \lambda^{-1}T_0^{-1}(x)A_{-1}(x)T_0]Z(x), (48) \end{aligned}$$

where

$$\widetilde{\Lambda}_1(x) := \begin{bmatrix} \sqrt{\rho(x)k^{-1}(x)} & 0 & 0 & 0 \\ 0 & \sqrt{I_\rho(x)EI^{-1}(x)} & 0 & 0 \\ 0 & 0 & -\sqrt{\rho(x)k^{-1}(x)} & 0 \\ 0 & 0 & 0 & -\sqrt{I_\rho(x)EI^{-1}(x)} \end{bmatrix}.$$

Set

$$E(x,\lambda) := \exp(\lambda \int_0^x \widetilde{\Lambda}_1(s)ds) \tag{49}$$

and the formal asymptotic series

$$Z(x,\lambda) := \sum_{k=0}^{\infty} \lambda^{-k} S_{-k}(x) E(x,\lambda),$$

where $S_{-k}(x)$, $k = 0, 1, 2, \cdots$ will be determined later. Substituting $Z(x,\lambda)$ into (48) yields

$$\begin{aligned} \frac{dZ(x)}{dx} &= \sum_{k=0}^{\infty} \lambda^{-k} S_{-k}'(x) E(x,\lambda) + \sum_{k=0}^{\infty} \lambda^{-k} S_{-k}(x) E'(x,\lambda) \\ &= [\lambda\widetilde{\Lambda}_1(x) - T_0^{-1}T_0' + T_0^{-1}A_0T_0 + \lambda^{-1}T_0^{-1}A_{-1}T_0] \sum_{k=0}^{\infty} \lambda^{-k} S_{-k}(x) E(x,\lambda). \end{aligned}$$

In order to identify $S_{-k}(x)$, $k = 0, 1, 2, \cdots$, following the asymptotic technique of Theorem 2.8.2 in [27], let each coefficient with the same power of $\lambda$ on each hand of above equation be equal, that is,

$$\lambda \quad : \quad S_0(x)\widetilde{\Lambda}_1(x) = \widetilde{\Lambda}_1(x)S_0(x), \tag{50}$$

$$1 \quad : \quad S_0'(x) + S_{-1}(x)\widetilde{\Lambda}_1(x) = \widetilde{\Lambda}_1(x)S_{-1}(x) + \left[-T_0^{-1}T_0' + T_0^{-1}A_0T_0\right]S_0(x), \tag{51}$$

$$\lambda^{-1} \quad : \quad S_{-1}'(x) + S_{-2}(x)\widetilde{\Lambda}_1(x) = \widetilde{\Lambda}_1(x)S_{-2}(x) + [-T_0^{-1}T_0' + T_0^{-1}A_0T_0]S_{-1}(x)$$

$$+T_0^{-1}A_{-1}T_0S_0(x), \tag{52}$$

$$\lambda^{-k} \quad : \quad S'_{-k}(x)+S_{-(k+1)}(x)\widetilde{\Lambda}_1(x)=\widetilde{\Lambda}_1(x)S_{-(k+1)}(x)+[-T_0^{-1}T_0'+T_0^{-1}A_0T_0]S_{-k}(x)$$
$$+T_0^{-1}A_{-1}T_0S_{-k+1}(x). \tag{53}$$

Then by (50)–(51), we obtain $S_0 = \mathrm{diag}(S_0^1(x), S_0^2(x), S_0^3(x), S_0^4(x))$, where

$$
\begin{aligned}
S_0^1(x) &= S_0^3(x) = e^{\int_0^x(-\left(\sqrt{\rho(x)k^{-1}(x)}\right)'[2\sqrt{\rho(x)k^{-1}(x)}]^{-1}-k'(x)[2k(x)]^{-1})ds} \\
&= \left(\sqrt{\rho(x)k^{-1}(x)}[\sqrt{\rho(0)k(0)^{-1}}]^{-1}\right)^{-\frac{1}{2}}\left(k(x)k(0)^{-1}\right)^{-\frac{1}{2}} \\
&= \left(\sqrt{\rho(x)k(x)}[\sqrt{\rho(0)k(0)}]^{-1}\right)^{-\frac{1}{2}}, \\
S_0^2(x) &= S_0^4(x) = e^{\int_0^x(-\left(\sqrt{I_\rho(x)EI^{-1}(x)}\right)'[2\sqrt{I_\rho(x)EI^{-1}(x)}]^{-1}-EI'(x)[2EI(x)]^{-1})ds} \\
&= \left(\sqrt{I_\rho(x)[EI(x)]^{-1}}[\sqrt{I_\rho(0)[EI(0)]^{-1}}]^{-1}\right)^{-\frac{1}{2}}\left(EI(x)[EI(0)]^{-1}\right)^{-\frac{1}{2}} \\
&= \left(\sqrt{I_\rho(x)EI(x)}[\sqrt{I_\rho(0)EI(0)}]^{-1}\right)^{-\frac{1}{2}}.
\end{aligned}
$$

Then we can calculate $S_{-k}$, $k = 1, 2, 3 \cdots$ by (53) similarly. Therefore, by Theorem 2.8.2 in [27], together with the above calculations, we have the following result.

**Lemma 3.1.** *Under the transformation $Y(x) = T_0Z(x)$, when conditions*

$$\det(\sqrt{I_\rho(x)EI^{-1}(x)} - \sqrt{\rho(x)k^{-1}(x)}) \neq 0 \tag{54}$$

*and*

$$\sqrt{\frac{\rho_j}{k_j}} \neq \sqrt{\frac{\rho_i}{k_i}}, \quad \sqrt{\frac{I_{\rho_j}}{EI_j}} \neq \sqrt{\frac{I_{\rho_i}}{EI_i}}, \quad i \neq j, \ i,j = 1, 2, \cdots, n \tag{55}$$

*are fulfilled, the expression of asymptotic fundamental matrix to (48) is given as follows*

$$\widetilde{E}(x, \lambda) = \sum_{k=0}^{\infty} \lambda^{-k}S_{-k}(x)E(x, \lambda), \tag{56}$$

*where $E(x, \lambda)$ is defined as (49), $S_0$ is given as before and $S_{-k, \, k=1,2,3,\cdots}$ are uniformly bounded and can be calculated similarly.*

**Remark 2.** It is guaranteed by Theorem 2.8.2 in [27] that $\widetilde{E}(x, \lambda)$ can be expanded by the formal asymptotic series (56). Here we only calculate $S_0$. That is because by calculating the spectrum below, we find that it is sufficient to use $S_0$ to estimate the asymptotic distribution of the spectrum of the system operator. In addition, conditions (54) and (55) are required in calculating $P_{-k}$, $k = 1, 2, \cdots$.

Thus, by inverse transformation, the fundamental matrix to (42) is $T_0\widetilde{E}(x, \lambda)$. Now, let us consider the spectrum of $\mathcal{A}$ and get the asymptotic distribution of them. For this aim, we translate the eigenvalue problem into boundary eigenvalue problem with asymptotic linearized parameters. According to (40), a direct calculation yields

$$P(x) = P(0)e^{-\tau\lambda x} = \lambda W(1)e^{-\tau\lambda x}, \quad Q(x) = Q(0)e^{-\tau\lambda x} = \lambda\Phi(1)e^{-\tau\lambda x},$$

and

$$P(1) = P(0)e^{-\tau\lambda} = \lambda W(1)e^{-\tau\lambda}, \quad Q(1) = Q(0)e^{-\tau\lambda} = \lambda\Phi(1)e^{-\tau\lambda}.$$

Inserting the above into the boundary conditions of (40), we have

$$\begin{cases} k(1)(W_x - \Phi)(1) = -W(1) - \alpha\mu\lambda W(1) - \alpha(1-\mu)\lambda W(1)e^{-\tau\lambda}, \\ EI(1)\Phi_x(1) = -\Phi(1) - \beta\mu\lambda\Phi(1) - \beta(1-\mu)\lambda\Phi(1)e^{-\tau\lambda}. \end{cases} \tag{57}$$

Let $Y$ be defined as (44). We can rewrite boundary conditions of (40) as follows

$$B_1 Y(0) + B_2 Y(1) = 0,$$

where

$$B_1 := \begin{bmatrix} C_{(n-1)\times n} & 0 & 0 & 0 \\ 0 & C_{(n-1)\times n} & 0 & 0 \\ 0 & I_{1\times n}k(0) & I_{1\times n}k(0)\lambda & 0 \\ 0 & 0 & 0 & I_{1\times n}EI(0)\lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4n\times 4n},$$

$$B_2 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ I + \alpha\mu\lambda + \alpha(1-\mu)\lambda e^{-\tau\lambda} & -k(1) & k(1)\lambda & 0 \\ 0 & I + \beta\mu\lambda + \beta(1-\mu)\lambda e^{-\tau\lambda} & 0 & EI(1)\lambda \end{bmatrix}_{4n\times 4n}.$$

Set

$$H(\lambda) := B_1 T_0(0) + B_2 T_0(1)\widetilde{E}(1,\lambda)$$

where $T_0(x)$ and $\widetilde{E}(1,\lambda)$ is given by (47), (56), respectively.

Thus, according to [46], we have

**Lemma 3.2.** *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then $\lambda \in \sigma(\mathcal{A})$ if and only if $\lambda$ satisfies*

$$\Delta(\lambda) := \det H(\lambda) = 0. \tag{58}$$

Since all coefficients in $\Delta(\lambda)$ are real constants, we have the following result.

**Corollary 2.** *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then the spectrum of $\mathcal{A}$ distributes in conjugate pairs on the complex plane, i.e., $\sigma(\mathcal{A}) = \overline{\sigma(\mathcal{A})}$.*

From Lemma 3.2, we only need to identify the zeros of $\Delta(\lambda)$. Firstly, let us get the asymptotic expression of $\Delta(\lambda)$. Set

$$[A]_1 := A + \mathcal{O}(\lambda^{-1}).$$

Then we can rewrite $\widetilde{E}(1,\lambda)$ as follows

$$\widetilde{E}(1,\lambda) = S_0(1)\begin{bmatrix} [1]_1 e^{\lambda\int_0^1\sqrt{\rho k^{-1}}dx} & [0]_1 e^{\lambda\int_0^1\sqrt{I_\rho EI^{-1}}dx} & [0]_1 e^{-\lambda\int_0^1\sqrt{\rho k^{-1}}dx} & [0]_1 e^{-\lambda\int_0^1\sqrt{I_\rho EI^{-1}}dx} \\ [0]_1 e^{\lambda\int_0^1\sqrt{\rho k^{-1}}dx} & [1]_1 e^{\lambda\int_0^1\sqrt{I_\rho EI^{-1}}dx} & [0]_1 e^{-\lambda\int_0^1\sqrt{\rho k^{-1}}dx} & [0]_1 e^{-\lambda\int_0^1\sqrt{I_\rho EI^{-1}}dx} \\ [0]_1 e^{\lambda\int_0^1\sqrt{\rho k^{-1}}dx} & [0]_1 e^{\lambda\int_0^1\sqrt{I_\rho EI^{-1}}dx} & [1]_1 e^{-\lambda\int_0^1\sqrt{\rho k^{-1}}dx} & [0]_1 e^{-\lambda\int_0^1\sqrt{I_\rho EI^{-1}}dx} \\ [0]_1 e^{\lambda\int_0^1\sqrt{\rho k^{-1}}dx} & [0]_1 e^{\lambda\int_0^1\sqrt{I_\rho EI^{-1}}dx} & [0]_1 e^{-\lambda\int_0^1\sqrt{\rho k^{-1}}dx} & [1]_1 e^{-\lambda\int_0^1\sqrt{I_\rho EI^{-1}}dx} \end{bmatrix}.$$

Hence, it follows that

$$B_1 T_0(0) = \begin{bmatrix} C_{(n-1)\times n} & 0 & C_{(n-1)\times n} & 0 \\ 0 & C_{(n-1)\times n} & 0 & C_{(n-1)\times n} \\ I_{1\times n}\sqrt{\rho(0)k(0)}\lambda & I_{1\times n}k(0) & -I_{1\times n}\sqrt{\rho(0)k(0)}\lambda & I_{1\times n}k(0) \\ 0 & I_{1\times n}\sqrt{I_\rho(0)EI(0)}\lambda & 0 & -I_{1\times n}\sqrt{I_\rho(0)EI(0)}\lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$B_2 T_0(1)$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I + \alpha\mu\lambda + \alpha(1-\mu)\lambda e^{-\tau\lambda} + \sqrt{\rho(1)k(1)}\lambda & -k(1) \\ 0 & I + \beta\mu\lambda + \beta(1-\mu)\lambda e^{-\tau\lambda} + \sqrt{I_\rho(1)EI(1)}\lambda \\ 0 & 0 \\ 0 & 0 \\ I + \alpha\mu\lambda + \alpha(1-\mu)\lambda e^{-\tau\lambda} - \sqrt{\rho(1)k(1)}\lambda & -k(1) \\ 0 & I + \beta\mu\lambda + \beta(1-\mu)\lambda e^{-\tau\lambda} - \sqrt{I_\rho(1)EI(1)}\lambda \end{bmatrix}_{4n \times 4n}.$$

Therefore, when $\Re\lambda \to +\infty$,

$$\left| \lim_{\Re\lambda \to +\infty} \frac{\Delta(\lambda)}{\lambda^{(2n+2)} e^{\sum_{j=1}^{n} \lambda \int_0^1 (\sqrt{\rho_j k_j^{-1}} + \sqrt{I_{\rho_j} EI_j^{-1}})dx}} \right|$$

$$= \sum_{j=1}^{n} \sqrt{\rho_j(0)k_j(0)} \cdot \sum_{j=1}^{n} \sqrt{I_{\rho_j}(0)EI_j(0)} \prod_{j=1}^{n} (\alpha_j\mu_j + \sqrt{\rho_j(1)k_j(1)}) \left( \sqrt{\frac{\rho_j(1)k_j(1)}{\rho_j(0)k_j(0)}} \right)^{-\frac{1}{2}}$$

$$\cdot \prod_{j=1}^{n} (\beta_j\mu_j + \sqrt{I_{\rho_j}(1)EI_j(1)}) \left( \sqrt{\frac{I_{\rho_j}(1)EI_j(1)}{I_{\rho_j}(0)EI_j(0)}} \right)^{-\frac{1}{2}}. \tag{59}$$

Set $\Upsilon := \{j | \mu_j \neq 1, \ j = 1, 2, \cdots, n\}$. Similarly, when $\Re\lambda \to -\infty$,

$$\left| \lim_{\Re\lambda \to -\infty} \frac{\Delta(\lambda)}{\lambda^{(2n+2)} e^{-\sum_{j=1}^{n} \lambda [\int_0^1 (\sqrt{\rho_j k_j^{-1}} + \sqrt{I_{\rho_j} EI_j^{-1}})dx] - \sum_{j\in\Upsilon} 2\lambda\tau_j}} \right|$$

$$= \sum_{j=1}^{n} \sqrt{\rho_j(0)k_j(0)} \cdot \sum_{j=1}^{n} \sqrt{I_{\rho_j}(0)EI_j(0)} \prod_{j\in\Upsilon} \alpha_j(1-\mu_j) \left( \sqrt{\frac{\rho_j(1)k_j(1)}{\rho_j(0)k_j(0)}} \right)^{-\frac{1}{2}}$$

$$\cdot \prod_{j\in\Upsilon} \beta_j(1-\mu_j) \left( \sqrt{\frac{I_{\rho_j}(1)EI_j(1)}{I_{\rho_j}(0)EI_j(0)}} \right)^{-\frac{1}{2}}. \tag{60}$$

From (59) and (60) together with the dissipativity of $\mathcal{A}$, we have the following result about the distribution of spectrum of $\mathcal{A}$.

**Theorem 3.3.** *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then the spectrum of $\mathcal{A}$ is located on the left hand side of complex plane and in a strip parallel to imaginary axis, i.e., there exists a sufficiently large positive constant h such that*

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} | \Delta(\lambda) = 0\} \subset \{\lambda \in \mathbb{C} | -h \leq \Re\lambda \leq 0\}.$$

By the definition of sine-type function (see [46]), we obtain that $\lambda^{-(2n+2)}\Delta(\lambda)$ is an entire function of sine type. Then by Levin's theorem (see [6]), we have the following corollary.

**Corollary 3.** *Let $\mathcal{A}$ be defined as before. Then $\sigma(\mathcal{A})$ is a finite union of separable sets.*

Moreover, when $|\Re\lambda| \leq h$ and $|\lambda|$ is large enough, a direct calculation yields

$$\begin{aligned} \lambda^{-(2n+2)}\Delta(\lambda) &= \lambda^{-(2n+2)} \det(B_1 T_0(0) + B_2 T_0(1)\widetilde{E}(1,\lambda)) \\ &= \Delta_1(\lambda)\Delta_2(\lambda) + \mathcal{O}(\lambda^{-1}), \end{aligned}$$

where

$$\Delta_1(\lambda) \quad := \quad \det \begin{bmatrix} C_{(n-1)\times n} \\ I_{1\times n}\sqrt{\rho(0)k(0)} \\ [\alpha\mu+\alpha(1-\mu)e^{-\tau\lambda}+\sqrt{\rho(1)k(1)}]S_0^1(1)e^{\lambda\int_0^1\sqrt{\rho k^{-1}}dx} \\ \\ C_{(n-1)\times n} \\ -I_{1\times n}\sqrt{\rho(0)k(0)} \\ [\alpha\mu+\alpha(1-\mu)e^{-\tau\lambda}-\sqrt{\rho(1)k(1)}]S_0^3(1)e^{-\lambda\int_0^1\sqrt{\rho k^{-1}}dx} \end{bmatrix}_{2n\times 2n} \tag{61}$$

and

$$\Delta_2(\lambda) \quad := \quad \det \begin{bmatrix} C_{(n-1)\times n} \\ I_{1\times n}\sqrt{I_\rho(0)EI(0)} \\ [\beta\mu+\beta(1-\mu)e^{-\tau\lambda}+\sqrt{I_\rho(1)EI(1)}]S_0^2(1)e^{\lambda\int_0^1\sqrt{I_\rho EI^{-1}}dx} \\ \\ C_{(n-1)\times n} \\ -I_{1\times n}\sqrt{I_\rho(0)EI(0)} \\ [\beta\mu+\beta(1-\mu)e^{-\tau\lambda}-\sqrt{I_\rho(1)EI(1)}]S_0^4(1)e^{-\lambda\int_0^1\sqrt{I_\rho EI^{-1}}dx} \end{bmatrix}_{2n\times 2n} \tag{62}$$

Clearly, when $|\Re\lambda| \le h$ and $|\lambda| \to +\infty$, $\Delta(\lambda)$ has asymptotic zeros

$$\Delta_1(\lambda) = 0, \quad \Delta_2(\lambda) = 0. \tag{63}$$

Thus we have

**Theorem 3.4.** *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then the asymptotic values of the spectrum of $\mathcal{A}$ can be determined by the zeros of $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$.*

4. **Spectrum-determined-growth condition.** In this section, we shall discuss the completeness and Riesz basis property of (generalized) eigenvectors of $\mathcal{A}$ so as to obtain the spectrum-determined-growth condition for this system (16), which is a very special property for the infinite-dimensional systems. Then the stability of the system (16) can be determined based on the distribution of the spectrum of $\mathcal{A}$.

Firstly, let us establish the completeness of the (generalized) eigenvectors of $\mathcal{A}$. To do this, we define an auxiliary operator $\mathcal{A}_0$ in $\mathcal{H}$ as follows.

$$\mathcal{A}_0 \begin{bmatrix} W \\ Z \\ \Phi \\ \Psi \\ P \\ Q \end{bmatrix} = \begin{pmatrix} Z \\ \rho^{-1}(x)\left[[k(x)(W_x(x)-\Phi(x))]_x\right] \\ \Psi \\ I_\rho^{-1}(x)\left[[EI(x)\Phi_x(x)]_x+k(x)(W_x(x)-\Phi(x))\right] \\ -\tau^{-1}P_x \\ -\tau^{-1}Q_x \end{pmatrix} \tag{64}$$

where

$$\mathcal{D}(\mathcal{A}_0) = \left\{ (W,Z,\Phi,\Psi,P,Q)\in\mathcal{H} \left| \begin{array}{l} W\in V_E^2,\ Z\in V_E^1,\ \Phi\in V_E^2,\ \Psi\in V_E^1, \\ Z(1)=P(0)=P(1),\ \Psi(1)=Q(0)=Q(1), \\ I_{1\times n}k(0)(W_x-\Phi)(0)=0, \\ I_{1\times n}EI(0)\Phi_x(0)=0, \\ k(1)(W_x-\Phi)(1)=-W(1), \\ EI(1)\Phi_x(1)=-\Phi(1), \end{array} \right. \right\}. \tag{65}$$

It is easy to check that the following result holds.

**Lemma 4.1.** *$\mathcal{A}_0$ is a skew-adjoint operator in $\mathcal{H}$ and hence*

$$\|R(\lambda,\mathcal{A}_0)\| \le \frac{1}{|\Re\lambda|}, \quad \forall\lambda\in\rho(\mathcal{A}_0)\backslash\{0\}.$$

Then we have the following result.

**Theorem 4.2.** *Suppose that the conditions (36), (54) and (55) are fulfilled. Then the system of the (generalized) eigenvectors of $\mathcal{A}$ is of completeness in $\mathcal{H}$.*

*Proof.* We mainly use the technique of the proof in [20] to show the completeness of the system of the (generalized) eigenvectors of $\mathcal{A}$ in $\mathcal{H}$.

Define

$$\mathrm{Span}(\mathcal{A}) := \overline{\{\sum_k y_k, y_k \in E(\lambda_k, \mathcal{A})\mathcal{H}, \lambda_k \in \sigma(\mathcal{A})\}},$$

where $E(\lambda_k, \mathcal{A})$ is the Riesz projector corresponding to $\lambda_k$. Then the completeness of the (generalized) eigenvectors of $\mathcal{A}$ is $\mathrm{Span}(\mathcal{A}) = \mathcal{H}$.

Assume $\widetilde{U} = (\widetilde{u}, \widetilde{v}, \widetilde{g}, \widetilde{h}, \widetilde{p}, \widetilde{q}) \in \mathcal{H}$ and $\widetilde{U} \perp \mathrm{Span}(\mathcal{A})$. Denote by $R^*(\lambda, \mathcal{A})$ the conjugate operator of the resolvent operator of $\mathcal{A}$. $R^*(\lambda, \mathcal{A})\widetilde{U}$ is an entire function on $\mathbb{C}$ valued in $\mathcal{H}$ since $\widetilde{U} \perp \mathrm{Span}(\mathcal{A})$. Thus for any $\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_3, \widetilde{F}_4, \widetilde{F}_5, \widetilde{F}_6) \in \mathcal{H}$,

$$\widetilde{G}(\lambda) := (\widetilde{F}, R^*(\lambda, \mathcal{A})\widetilde{U})_{\mathcal{H}}, \quad \forall \lambda \in \mathbb{C} \tag{66}$$

is an entire function which satisfies $\lim\limits_{\Re\lambda \to +\infty} G(\lambda) = 0$, since $\mathcal{A}$ generates a $C_0$ semigroup of contractions. In particular, for $\lambda \in \rho(\mathcal{A})$, it holds that $G(\lambda) = (R(\lambda, \mathcal{A})\widetilde{F}, \widetilde{U})_{\mathcal{H}}$.

We consider the resolvent problem

$$(\lambda I - \mathcal{A})\widetilde{Y}_1 = \widetilde{F}, \quad (\lambda I - \mathcal{A}_0)\widetilde{Y}_2 = \widetilde{F}, \quad \lambda \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}_0) \cap \mathbb{R}_-, \tag{67}$$

where $\widetilde{Y}_1 := (\widetilde{W}_1, \widetilde{Z}_1, \widetilde{\Phi}_1, \widetilde{\Psi}_1, \widetilde{P}_1, \widetilde{Q}_1) \in \mathcal{D}(\mathcal{A})$, $\widetilde{Y}_2 = (\widetilde{W}_2, \widetilde{Z}_2, \widetilde{\Phi}_2, \widetilde{\Psi}_2, \widetilde{P}_2, \widetilde{Q}_2) \in \mathcal{D}(\mathcal{A}_0)$. Set $\widetilde{Y}_3(\lambda) := \widetilde{Y}_1 - \widetilde{Y}_2 = (\widetilde{W}_3, \widetilde{Z}_3, \widetilde{\Phi}_3, \widetilde{\Psi}_3, \widetilde{P}_3, \widetilde{Q}_3)$. Thus $\widetilde{Y}_3$ satisfies the following equations:

$$\begin{cases} \rho(x)\lambda^2\widetilde{W}_3(x) - [k(x)(\widetilde{W}_{3,x}(x) - \widetilde{\Phi}_3(x))]_x = 0, \ x \in (0,1), \\ I_\rho(x)\lambda^2\widetilde{\Phi}_3(x) - [EI(x)\widetilde{\Phi}_{3,x}(x)]_x - k(x)(\widetilde{W}_{3,x}(x) - \widetilde{\Phi}_3(x)) = 0, \ x \in (0,1), \\ \widetilde{P}_{3,x}(x) = -\tau\lambda\widetilde{P}_3(x), \widetilde{Q}_{3,x}(x) = -\tau\lambda\widetilde{Q}_3(x), \\ \widetilde{Z}_3 = \lambda\widetilde{W}_3, \quad \widetilde{\Psi}_3 = \lambda\widetilde{\Phi}_3, \\ C_{(n-1)\times n}\widetilde{W}_3(0) = 0, \quad C_{(n-1)\times n}\widetilde{\Phi}_3(0) = 0, \\ \widetilde{P}_3(0) = \lambda\widetilde{W}_3(1), \ \widetilde{Q}_3(0) = \lambda\widetilde{\Phi}_3(1), \\ k(1)(\widetilde{W}_{3,x} - \widetilde{\Phi}_3)(1) + \widetilde{W}_3(1) + \alpha[\mu\widetilde{P}_3(0) + (1-\mu)\widetilde{P}_3(1)] = -\alpha[\mu\widetilde{P}_2(0) + (1-\mu)\widetilde{P}_2(1)], \\ EI(1)\widetilde{\Phi}_{3,x}(1) + \widetilde{\Phi}_3(1) + \beta[\mu\widetilde{Q}_3(0) + (1-\mu)\widetilde{Q}_3(1)] = -\beta[\mu\widetilde{Q}_2(0) + (1-\mu)\widetilde{Q}_2(1)], \\ I_{1\times n}k(0)(\widetilde{W}_{3,x} - \widetilde{\Phi}_3)(0) = 0, \quad I_{1\times n}EI(0)\widetilde{\Phi}_{3,x}(0) = 0, \\ \widetilde{Z}_2(1) = \widetilde{P}_2(0) = \widetilde{P}_2(1), \ \widetilde{\Psi}_2(1) = \widetilde{Q}_2(0) = \widetilde{Q}_2(1). \end{cases} \tag{68}$$

Similar to [20], we shall prove $\widetilde{U} = 0$ via three steps below, which implies $\mathrm{Span}(\mathcal{A}) = \mathcal{H}$.

*Step 1).* For $\lambda \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}_0) \cap \mathbb{R}_-$, there exists a constant $\widetilde{M}_1 > 0$, such that

$$\|\widetilde{Z}_3(1)\| \le \widetilde{M}_1\|\widetilde{Z}_2(1)\|, \quad \|\widetilde{\Psi}_3(1)\| \le \widetilde{M}_1\|\widetilde{\Psi}_2(1)\|,$$

$$\|\widetilde{P}_3(1)\| \le \widetilde{M}_1\|\widetilde{P}_2(1)\|, \quad \|\widetilde{Q}_3(1)\| \le \widetilde{M}_1\|\widetilde{Q}_2(1)\|.$$

*Step 2).* For $\lambda \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}_0) \cap \mathbb{R}_-$, $\|\widetilde{Y}_3(\lambda)\|$ satisfies

$$|\lambda|\|\widetilde{Y}_3(\lambda)\|^2 \le \widetilde{M}_2(|\lambda|\|\widetilde{Y}_2\| + \|\widetilde{F}\|)^2, \tag{69}$$

where $\widetilde{M}_2$ is a positive constant.

*Step 3).* For $\lambda \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}_0) \cap \mathbb{R}_-$,

$$\|R(\lambda, \mathcal{A})\widetilde{F}\| = \|\widetilde{Y}_1\| \le 2\sqrt{\widetilde{M_2}}|\lambda|^{-\frac{1}{2}}\|\widetilde{F}\| + |\lambda|^{-1}\|\widetilde{F}\|.$$

In fact, from Lemma 4.1, we have

$$\|\widetilde{Y}_2\| = \|(\lambda - \mathcal{A}_0)^{-1}\widetilde{F}\| \le \|(\lambda - \mathcal{A}_0)^{-1}\|\|\widetilde{F}\| \le \frac{1}{|\Re\lambda|}\|\widetilde{F}\|, \quad \forall \lambda \in \rho(\mathcal{A}_0)\backslash\{0\}.$$

Then for $\lambda \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}_0) \cap \mathbb{R}_-$,

$$\|R(\lambda, \mathcal{A})\widetilde{F}\| = \|\widetilde{Y}_1\| \le \|\widetilde{Y}_2\| + \|\widetilde{Y}_3\| \le \|\widetilde{F}\|\left[\frac{1}{|\lambda|} + 2\sqrt{\widetilde{M_2}}|\lambda|^{-\frac{1}{2}}\right].$$

Therefore, when $\lambda \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}_0) \cap \mathbb{R}_-$, $\lim\limits_{\lambda \to -\infty} \|R(\lambda, \mathcal{A})\widetilde{F}\| = \|\widetilde{Y}_2 + \widetilde{Y}_3\| = 0$, which implies

$$\lim_{\lambda \to -\infty} |\widetilde{G}(\lambda)| = 0. \tag{70}$$

When condition (36) is fulfilled, we have shown in last section that $\mathcal{A}$ is dissipative in $\mathcal{H}$ and $0 \in \rho(\mathcal{A})$, which implies $|\widetilde{G}(\lambda)|$ is bounded on the domain $\Re\lambda \ge \hat{d}, \hat{d} > 0$. By Phragmén-Lindelöf theorem (see [51]), together with (70), we have that $|\widetilde{G}(\lambda)|$ is bounded on the sector with boundary rays $\Re\lambda = \hat{d}, \Im\lambda \ge 0$ and $\Re\lambda \le \hat{d}, \Im\lambda = 0$, since $\widetilde{G}(\lambda)$ is an entire function of finite exponential type. Similarly, we obtain $|\widetilde{G}(\lambda)|$ is bounded on the sector with boundary rays $\Re\lambda = \hat{d}, \Im\lambda \le 0$ and $\Re\lambda \le \hat{d}, \Im\lambda = 0$. Therefore, $|\widetilde{G}(\lambda)|$ is uniformly bounded on $\mathbb{C}$. Hence,

$$|\widetilde{G}(\lambda)| \le M, \quad \forall \lambda \in \mathbb{C}.$$

Furthermore, the Liouville's theorem deduces that $G(\lambda)$ is constant, since $\widetilde{G}(\lambda)$ is an entire function. Then $\lim\limits_{\lambda \to \infty} \widetilde{G}(\lambda) = 0$ yields $\widetilde{G}(\lambda) \equiv 0$. Note that $\widetilde{G}(\lambda) = (\widetilde{F}, R^*(\lambda, \mathcal{A})\widetilde{U})_{\mathcal{H}}$ holds for any $\widetilde{F} \in \mathcal{H}$. It must be $R^*(\lambda, \mathcal{A})\widetilde{U} = 0$, which means $\widetilde{U} = 0$. Therefore, $\text{Span}(\mathcal{A}) = \mathcal{H}$. $\square$

To study the Riesz basis generation of the (generalized) eigenvectors of $\mathcal{A}$, we give some definitions about Riesz basis, Riesz basis with parentheses and Riesz basis of subspaces (see [8], [14], [15] and [37]).

**Definition 4.3.** The subspace $\{W_k\}_{k=1}^{\infty}$ is called a Riesz basis of subspaces for space $H$, if for every $f \in H$, there is a unique $f_k \in W_k$, $k = 1, 2, \cdots$ such that $f = \sum\limits_{k=1}^{\infty} f_k$ and there exists positive constants $C$ and $D$ such that for any $f \in \mathcal{H}$, $f = \sum\limits_{k=1}^{\infty} f_k$,

$$C\sum_{k=1}^{\infty} \|f_k\|^2 \le \|\sum_{k=1}^{\infty} f_k\|^2 \le D\sum_{k=1}^{\infty} \|f_k\|^2. \tag{71}$$

A sequence $\{y_i\}_{i=1}^{\infty}$ is called a Riesz basis with parentheses for $H$, if there is a method in parentheses; for instance, there is a sequence of integers $n_1 = 1 \le n_2 \le \cdots \le n_k \le \cdots$ such that

$$\overline{\text{Span}\{y_{n_k}, y_{n_k+1}, \cdots, y_{n_{k+1}-1}\}} = W_k, \quad k = 1, 2 \cdots. \tag{72}$$

form a Riesz basis of subspaces.

If the sequence $\{y_i\}_{i=1}^{\infty}$ satisfies condition

$$C \sum_{k=1}^{\infty} \|y_k\|^2 \leq \|\sum_{k=1}^{\infty} y_k\|^2 \leq D \sum_{k=1}^{\infty} \|y_k\|^2. \tag{73}$$

then $\{y_i\}_{i=1}^{\infty}$ is called a Riesz basis for $H$.

In order to deduce the Riesz basis property of the (generalized) eigenvectors of $\mathcal{A}$, we need the following result, which comes from [46] and is an extension of the result in [49].

**Theorem 4.4.** *Let $\mathcal{A}$ be the generator of a $C_0$ semigroup $T(t)$ on a separable Hilbert space $\mathcal{H}$. Suppose that the following conditions are satisfied:*

*1) The spectrum of $\mathcal{A}$ has a decomposition $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A})$, where $\sigma_2(\mathcal{A})$ consists of the isolated eigenvalues of $\mathcal{A}$ of finite multiplicity (repeated many times according to its algebraic multiplicity).*

*2) There exists a real number $\alpha \in \mathbb{R}$ such that $\sup\{\Re\lambda, \lambda \in \sigma_1(\mathcal{A})\} \leq \alpha \leq \inf\{\Re\lambda, \lambda \in \sigma_2(\mathcal{A})\}$.*

*3) The set $\sigma_2(\mathcal{A})$ is a finite unification of separable sets.*

*Then the following statements are true:*

*i) There exist two $T(t)$-invariant closed subspaces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$ such that $\sigma(\mathcal{A}|_{\mathcal{H}_1}) = \sigma_1(\mathcal{A})$ and $\sigma(\mathcal{A}|_{\mathcal{H}_2}) = \sigma_2(\mathcal{A})$; and there exists a finite combination $E(\Omega_k, \mathcal{A})$ of some $\{E(\lambda_k, \mathcal{A})\}_{k=1}^{\infty}$:*

$$E(\Omega_k, \mathcal{A}) = \sum_{\lambda \in \Omega_k \cap \sigma_2(\mathcal{A})} E(\lambda, \mathcal{A})$$

*such that $\{E(\Omega_k, \mathcal{A})\mathcal{H}_2\}_{k \in \mathbb{N}}$ forms a Riesz basis of subspaces for $\mathcal{H}_2$. Furthermore,*

$$\mathcal{H} = \overline{\mathcal{H}_1 \oplus \mathcal{H}_2} \quad \text{(algebraic direct sum)}.$$

*ii) If $\sup_{k \geq 1} \|E(\lambda_k, \mathcal{A})\| < \infty$, then $\mathcal{D}(\mathcal{A}) \subset \mathcal{H}_1 \oplus \mathcal{H}_2 \subset \mathcal{H}$ (algebraic direct sum).*

*iii) $\mathcal{H}$ has a decomposition of the topological direct sum, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, if and only if*

$$\sup_{n \geq 1} \left\| \sum_{k=1}^{n} E(\Omega_k, \mathcal{A}) \right\| < \infty.$$

According to Corollary 1, 3 and Theorem 3.3, 4.2, 4.4, we have the following theorem.

**Theorem 4.5.** *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before. If the conditions (36), (54) and (55) are fulfilled, then there is a sequence of (generalized) eigenvectors of $\mathcal{A}$ that forms a Riesz basis with parentheses for $\mathcal{H}$.*

*Proof.* Set $\sigma_1(\mathcal{A}) = \emptyset$, $\sigma_2(\mathcal{A}) = \sigma(\mathcal{A})$. From Corollary 1, 3 and Theorem 3.3, we obtain that all hypotheses in Theorem 4.4 are fulfilled. So it leads to the results of Theorem 4.4. Therefore, there is a sequence of (generalized) eigenvectors of $\mathcal{A}$ that forms a Riesz basis with parentheses for $\mathcal{H}_2$. According to Theorem 4.2 we have that the system of the (generalized) eigenvectors is complete in $\mathcal{H}$, i.e., $\mathcal{H}_2 = \mathcal{H}$. Hence this sequence is also a Riesz basis with parentheses for $\mathcal{H}$. $\qquad\square$

By the Riesz basis property together with the uniform boundedness of the multiplicities of the eigenvalues of $\mathcal{A}$, we have the following result.

**Corollary 4.** *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before. If the conditions (36), (54) and (55) are fulfilled, then the system associated with $\mathcal{A}$ satisfies the spectrum-determined-growth condition, i.e., $\omega(\mathcal{A}) = S(\mathcal{A})$, where $\omega(\mathcal{A}) = \lim\limits_{t\to\infty} \frac{1}{t} \ln \|e^{\mathcal{A}t}\|$ is the growth order of $e^{\mathcal{A}t}$ and $S(\mathcal{A}) = \sup\{\Re\lambda | \lambda \in \sigma(\mathcal{A})\}$ is the spectral bound of $\mathcal{A}$.*

5. **Exponential stability.** In this section, we shall consider a simple case for this kind of networks. Assume that all these $n$ Timoshenko beams in networks are the same, i.e., $\rho_j(x) = \widehat{\rho}(x)$, $k_j(x) = \widehat{k}(x)$, $I_{\rho_j}(x) = \widehat{I}_\rho(x)$, $EI_j(x) = \widehat{EI}(x)$, $j = 1, 2, \cdots, n$. Suppose that $\mu_j = \widehat{\mu}$, $\tau_j = \widehat{\tau}$, $\alpha_j = \widehat{\alpha}$, $\beta_j = \widehat{\beta}$, $j = 1, 2, \cdots, n$ in (10). We shall consider the exponential stability of networks for this simple case.

Obviously, all the results in sections above also hold for this simple case. Thus, this case satisfies the spectrum-determined-growth condition. We have obtained from Theorem 2.4 that this kind of system is asymptotically stable when $\mu_j > \frac{1}{2}$, $j = 1, 2, \cdots, n$. In order to discuss the exponential stability of the system, we have the following lemma.

**Lemma 5.1.** *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before and conditions (36), (54) and (55) be fulfilled. Then the system is exponentially stable if and only if*

$$\inf_{\lambda \in i\mathbb{R}} |\Delta(\lambda)| > 0.$$

*Proof.* The necessity is obvious. Under conditions (36), (54) and (55), we know from Corollary 4 that the system satisfies the spectrum-determined-growth condition. $|\Delta(\lambda)| \neq 0$ for all $\lambda \in i\mathbb{R}$ implies that there is no eigenvalue on the imaginary axis, which deduces the asymptotic stability of the system (see Theorem 2.4). Since $\inf\limits_{\lambda \in i\mathbb{R}} |\Delta(\lambda)| > 0$ asserts that the imaginary axis is not an asymptote of $\sigma(\mathcal{A})$, together with the spectrum-determined-growth condition, the system is exponentially stable. □

When $\widehat{\mu} = 1$, we can obtain the following asymptotic expression of the spectrum of $\mathcal{A}$.

**Lemma 5.2.** *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before and conditions (54) and (55) be fulfilled. When $\widehat{\mu} = 1$, the asymptotic expression of the spectrum of $\mathcal{A}$ is given as follows.*

$$\lambda_{1,n} = \begin{cases} \lambda_{1,n}^1 := \frac{1}{2 \int_0^1 \sqrt{\widehat{\rho}\widehat{k}^{-1}}dx} \ln\left|\frac{\widehat{\alpha} - \sqrt{\widehat{\rho}(1)\widehat{k}(1)}}{\widehat{\alpha} + \sqrt{\widehat{\rho}(1)\widehat{k}(1)}}\right| + 2n\pi i + \mathcal{O}(\frac{1}{n}), & \widehat{\alpha} > \sqrt{\widehat{\rho}(1)\widehat{k}(1)}, \\[4mm] \lambda_{1,n}^2 := \frac{1}{2 \int_0^1 \sqrt{\widehat{\rho}\widehat{k}^{-1}}dx} \ln\left|\frac{\widehat{\alpha} - \sqrt{\widehat{\rho}(1)\widehat{k}(1)}}{\widehat{\alpha} + \sqrt{\widehat{\rho}(1)\widehat{k}(1)}}\right| + (2n+1)\pi i + \mathcal{O}(\frac{1}{n}), & 0 < \widehat{\alpha} < \sqrt{\widehat{\rho}(1)\widehat{k}(1)} \end{cases} \tag{74}$$

*and*

$$\lambda_{2,n} = \begin{cases} \lambda_{2,n}^1 := \frac{1}{2 \int_0^1 \sqrt{\widehat{I_\rho}\widehat{EI}^{-1}}dx} \ln\left|\frac{\widehat{\beta} - \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)}}{\widehat{\beta} + \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)}}\right| + 2n\pi i + \mathcal{O}(\frac{1}{n}), & \widehat{\beta} > \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)}, \\[4mm] \lambda_{2,n}^2 := \frac{1}{2 \int_0^1 \sqrt{\widehat{I_\rho}\widehat{EI}^{-1}}dx} \ln\left|\frac{\widehat{\beta} - \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)}}{\widehat{\beta} + \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)}}\right| + (2n+1)\pi i + \mathcal{O}(\frac{1}{n}), & 0 < \widehat{\beta} < \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)}. \end{cases} \tag{75}$$

*Furthermore,*

*1). when $\widehat{\alpha} - \sqrt{\widehat{\rho}(1)\widehat{k}(1)} > 0$, the multiplicity of $\lambda_{1,n}^1$ is 1 and the multiplicity $\lambda_{1,n}^2$ is $n-1$,*

*when $\widehat{\alpha} - \sqrt{\widehat{\rho}(1)\widehat{k}(1)} < 0$, the multiplicity of $\lambda_{1,n}^1$ is $n-1$ and the multiplicity of $\lambda_{1,n}^2$ is 1;*

2). when $\widehat{\beta} - \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)} > 0$, the multiplicity of $\lambda_{2,n}^1$ is 1 and the multiplicity of $\lambda_{1,n}^1$ is $n-1$,

when $\widehat{\beta} - \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)} < 0$, the multiplicity of $\lambda_{2,n}^1$ is $n-1$ and the multiplicity of $\lambda_{2,n}^2$ is 1.

*Proof.* According to (61) and (62), we obtain

$$
\begin{aligned}
&\Delta_1(\lambda) \\
=& \sqrt{\widehat{\rho}(0)\widehat{k}(0)} \left| \begin{array}{c} \dfrac{-1}{[\widehat{\alpha}\widehat{\mu} + \widehat{\alpha}(1-\widehat{\mu})e^{-\widehat{\tau}\lambda} + \sqrt{\widehat{\rho}(1)\widehat{k}(1)}]S_0^1(1)e^{\lambda \int_0^1 \sqrt{\widehat{\rho k}^{-1}}dx}} \\[4mm] \dfrac{-1}{[\widehat{\alpha}\widehat{\mu} + \widehat{\alpha}(1-\widehat{\mu})e^{-\widehat{\tau}\lambda} - \sqrt{\widehat{\rho}(1)\widehat{k}(1)}]S_0^3(1)e^{-\lambda \int_0^1 \sqrt{\widehat{\rho k}^{-1}}dx}} \end{array} \right|^{n-1} \\[4mm]
&\cdot \left| \begin{array}{c} \dfrac{n}{[\widehat{\alpha}\widehat{\mu} + \widehat{\alpha}(1-\widehat{\mu})e^{-\widehat{\tau}\lambda} + \sqrt{\widehat{\rho}(1)\widehat{k}(1)}]S_0^1(1)e^{\lambda \int_0^1 \sqrt{\widehat{\rho k}^{-1}}dx}} \\[4mm] \dfrac{-n}{[\widehat{\alpha}\widehat{\mu} + \widehat{\alpha}(1-\widehat{\mu})e^{-\widehat{\tau}\lambda} - \sqrt{\widehat{\rho}(1)\widehat{k}(1)}]S_0^3(1)e^{-\lambda \int_0^1 \sqrt{\widehat{\rho k}^{-1}}dx}} \end{array} \right| \quad (76)
\end{aligned}
$$

and

$$
\begin{aligned}
&\Delta_2(\lambda) \\
=& \sqrt{\widehat{I_\rho}(0)\widehat{EI}(0)} \left| \begin{array}{c} \dfrac{-1}{[\widehat{\beta}\widehat{\mu} + \widehat{\beta}(1-\widehat{\mu})e^{-\widehat{\tau}\lambda} + \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)}]S_0^2(1)e^{\lambda \int_0^1 \sqrt{\widehat{I_\rho EI}^{-1}}dx}} \\[4mm] \dfrac{-1}{[\widehat{\alpha}\widehat{\mu} + \widehat{\alpha}(1-\widehat{\mu})e^{-\widehat{\tau}\lambda} - \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)}]S_0^4(1)e^{-\lambda \int_0^1 \sqrt{\widehat{I_\rho EI}^{-1}}dx}} \end{array} \right|^{n-1} \\[4mm]
&\cdot \left| \begin{array}{c} \dfrac{n}{[\widehat{\beta}\widehat{\mu} + \widehat{\beta}(1-\widehat{\mu})e^{-\widehat{\tau}\lambda} + \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)}]S_0^2(1)e^{\lambda \int_0^1 \sqrt{\widehat{I_\rho EI}^{-1}}dx}} \\[4mm] \dfrac{-n}{[\widehat{\beta}\widehat{\mu} + \widehat{\beta}(1-\widehat{\mu})e^{-\widehat{\tau}\lambda} - \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)}]S_0^4(1)e^{-\lambda \int_0^1 \sqrt{\widehat{I_\rho EI}^{-1}}dx}} \end{array} \right| . \quad (77)
\end{aligned}
$$

When $\widehat{\mu} = 1$, a direct calculation leads to

$$
\begin{aligned}
\Delta_1(\lambda) =& \sqrt{\widehat{\rho}(0)\widehat{k}(0)} \\
&\cdot \left| \begin{array}{cc} \dfrac{-1}{[\widehat{\alpha} + \sqrt{\widehat{\rho}(1)\widehat{k}(1)}]S_0^1(1)e^{\lambda \int_0^1 \sqrt{\widehat{\rho k}^{-1}}dx}} & \dfrac{-1}{[\widehat{\alpha}\widehat{\mu} - \sqrt{\widehat{\rho}(1)\widehat{k}(1)}]S_0^3(1)e^{-\lambda \int_0^1 \sqrt{\widehat{\rho k}^{-1}}dx}} \end{array} \right|^{n-1} \\
&\cdot \left| \begin{array}{cc} \dfrac{n}{[\widehat{\alpha}\widehat{\mu} + \sqrt{\widehat{\rho}(1)\widehat{k}(1)}]S_0^1(1)e^{\lambda \int_0^1 \sqrt{\widehat{\rho k}^{-1}}dx}} & \dfrac{-n}{[\widehat{\alpha} - \sqrt{\widehat{\rho}(1)\widehat{k}(1)}]S_0^3(1)e^{-\lambda \int_0^1 \sqrt{\widehat{\rho k}^{-1}}dx}} \end{array} \right| , \\
\Delta_2(\lambda) =& \sqrt{\widehat{I_\rho}(0)\widehat{EI}(0)} \\
&\cdot \left| \begin{array}{cc} \dfrac{-1}{[\widehat{\beta} + \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)}]S_0^2(1)e^{\lambda \int_0^1 \sqrt{\widehat{I_\rho EI}^{-1}}dx}} & \dfrac{-1}{[\widehat{\alpha} - \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)}]S_0^4(1)e^{-\lambda \int_0^1 \sqrt{\widehat{I_\rho EI}^{-1}}dx}} \end{array} \right|^{n-1} \\
&\cdot \left| \begin{array}{cc} \dfrac{n}{[\widehat{\beta} + \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)}]S_0^2(1)e^{\lambda \int_0^1 \sqrt{\widehat{I_\rho EI}^{-1}}dx}} & \dfrac{-n}{[\widehat{\beta} - \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)}]S_0^4(1)e^{-\lambda \int_0^1 \sqrt{\widehat{I_\rho EI}^{-1}}dx}} \end{array} \right| .
\end{aligned}
$$

In order to get the asymptotic expression of the spectrum of $\mathcal{A}$, it suffices to calculate the zeros of $\Delta_i(\lambda)$, $i = 1, 2$. By Rouché theorem (see [7]), we obtain (74) and (75) by a direct calculation. $\qquad\square$

Based on Lemma 5.1 and 5.2, we get the following result about the exponential stability of the closed loop system (10).

**Theorem 5.3.** *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before and conditions (54) and (55) be fulfilled. If $\widehat{\mu} > \frac{1}{2}$, then the closed loop system (10) is exponentially stable.*

*Proof.* From Theorem 2.4, under the conditions $\widehat{\mu} > \frac{1}{2}$, system (10) is asymptotically stable. According to Lemma 5.1, in order to obtain the exponential stability of the system, it is sufficient to check

$$\inf_{\lambda \in i\mathbb{R}} |\Delta(\lambda)| > 0.$$

For this aim, set $\xi \in \mathbb{R}$. We only need to show that

$$\Delta_j(i\xi) \nrightarrow 0, \quad \xi \to \infty, \; j = 1, 2,$$

where $\Delta_j(\lambda), j = 1, 2$ are given by (76) and (77).

A direct calculation yields that $\Delta_1(i\xi) = 0$ is equivalent to

$$\begin{aligned}
\widehat{\alpha}\widehat{\mu}\sin(\xi\int_0^1\sqrt{\widehat{\rho k^{-1}}}dx) + \widehat{\alpha}(1-\widehat{\mu})\cos\tau\xi\sin(\xi\int_0^1\sqrt{\widehat{\rho k^{-1}}}dx) = 0, \\
\widehat{\alpha}(1-\widehat{\mu})\sin\tau\xi\sin(\xi\int_0^1\sqrt{\widehat{\rho k^{-1}}}dx) + \sqrt{\widehat{\rho}(1)\widehat{k}(1)}\cos(\xi\int_0^1\sqrt{\widehat{\rho k^{-1}}}dx) = 0
\end{aligned} \tag{78}$$

or

$$\begin{aligned}
\widehat{\alpha}\widehat{\mu}\cos(\xi\int_0^1\sqrt{\widehat{\rho k^{-1}}}dx) + \widehat{\alpha}(1-\widehat{\mu})\cos\tau\xi\cos(\xi\int_0^1\sqrt{\widehat{\rho k^{-1}}}dx) = 0, \\
\widehat{\alpha}(1-\widehat{\mu})\sin\tau\xi\cos(\xi\int_0^1\sqrt{\widehat{\rho k^{-1}}}dx) - \sqrt{\widehat{\rho}(1)\widehat{k}(1)}\sin(\xi\int_0^1\sqrt{\widehat{\rho k^{-1}}}dx) = 0.
\end{aligned} \tag{79}$$

Similarly, $\Delta_2(i\xi) = 0$ is equivalent to

$$\begin{aligned}
\widehat{\beta}\widehat{\mu}\sin(\xi\int_0^1\sqrt{\widehat{I_\rho \widehat{EI}^{-1}}}dx) + \widehat{\beta}(1-\widehat{\mu})\cos\tau\xi\sin(\xi\int_0^1\sqrt{\widehat{I_\rho \widehat{EI}^{-1}}}dx) = 0, \\
\widehat{\beta}(1-\widehat{\mu})\sin\tau\xi\sin(\xi\int_0^1\sqrt{\widehat{I_\rho \widehat{EI}^{-1}}}dx) + \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)}\cos(\xi\int_0^1\sqrt{\widehat{I_\rho \widehat{EI}^{-1}}}dx) = 0
\end{aligned} \tag{80}$$

or

$$\begin{aligned}
\widehat{\beta}\widehat{\mu}\cos(\xi\int_0^1\sqrt{\widehat{I_\rho \widehat{EI}^{-1}}}dx) + \widehat{\beta}(1-\widehat{\mu})\cos\tau\xi\cos(\xi\int_0^1\sqrt{\widehat{I_\rho \widehat{EI}^{-1}}}dx) = 0, \\
\widehat{\beta}(1-\widehat{\mu})\sin\tau\xi\cos(\xi\int_0^1\sqrt{\widehat{I_\rho \widehat{EI}^{-1}}}dx) - \sqrt{\widehat{I_\rho}(1)\widehat{EI}(1)}\sin(\xi\int_0^1\sqrt{\widehat{I_\rho \widehat{EI}^{-1}}}dx) = 0.
\end{aligned} \tag{81}$$

Now, we shall discuss the exponential stability of the system in the following two cases.

When $\widehat{\mu} = 1$, according to Lemma 5.2 together with Corollary 4, we get that system (10) is exponentially stable.

When $\frac{1}{2} < \widehat{\mu} < 1$, from the first equation in (78), we get $\widehat{\alpha}\widehat{\mu} + \widehat{\alpha}(1-\widehat{\mu})\cos\widehat{\tau}\xi = 0$ or $\sin(\xi\int_0^1\sqrt{\widehat{\rho k^{-1}}}dx) = 0$. Since it is easy to check that $\sin(\xi\int_0^1\sqrt{\widehat{\rho k^{-1}}}dx) \neq 0$, we can get

$$\cos\widehat{\tau}\xi = \frac{-\widehat{\mu}}{1-\widehat{\mu}}.$$

Putting it into the second equation in (78) leads to

$$\pm\widehat{\alpha}(1-\widehat{\mu})\sqrt{1-(\frac{-\widehat{\mu}}{1-\widehat{\mu}})^2}\sin(\xi\int_0^1\sqrt{\widehat{\rho k^{-1}}}dx) - \sqrt{\widehat{\rho}(1)\widehat{k}(1)}\cos(\xi\int_0^1\sqrt{\widehat{\rho k^{-1}}}dx) = 0.$$

Thus it is easy to get $\cos(\xi \int_0^1 \sqrt{\widehat{\rho}\widehat{k}^{-1}}dx) \neq 0$. Therefore,

$$\tan(\xi \int_0^1 \sqrt{\widehat{\rho}\widehat{k}^{-1}}dx) = \pm \frac{\sqrt{\widehat{\rho}(1)\widehat{k}(1)}}{\widehat{\alpha}(1-\widehat{\mu})\sqrt{1-(\frac{\widehat{\mu}}{1-\widehat{\mu}})^2}} = \pm \frac{\sqrt{\widehat{\rho}(1)\widehat{k}(1)}}{\widehat{\alpha}\sqrt{1-2\widehat{\mu}}}.$$

When $\frac{1}{2} < \widehat{\mu} < 1$, $\frac{\sqrt{\widehat{\rho}(1)\widehat{k}(1)}}{\widehat{\alpha}\sqrt{1-2\widehat{\mu}}}$ is an imaginary number. Since $\tan(\xi \int_0^1 \sqrt{\widehat{\rho}\widehat{k}^{-1}}dx) \in \mathbb{R}$
for $\forall \xi \in \mathbb{R}$, so equations (78) not hold. Similarly, equations (79) also not hold which
leads to $\Delta_1(i\xi) \nrightarrow 0$ for any $\xi \in \mathbb{R}$, $\xi \to \infty$ when $\frac{1}{2} < \widehat{\mu} < 1$. By similar method,
we can get $\Delta_2(i\xi) \nrightarrow 0$ for any $\xi \in \mathbb{R}$, $\xi \to \infty$ if $\frac{1}{2} < \widehat{\mu} < 1$.

Therefore, $\Delta(i\xi) \nrightarrow 0$ for any $\xi \in \mathbb{R}$, $\xi \to \infty$ when $\widehat{\mu} > \frac{1}{2}$. By Lemma 5.1, system
(10) is exponentially stable. The proof is complete. $\square$

6. **Simulations.** We give some simple numerical simulations for this kinds of net-
works. Assume that the network composes of three Timoshenko beams which are
the same, and the parameters in the closed loop system are positive constants. Set
$\widehat{\rho}(x) = 5$, $\widehat{I_\rho}(x) = 8$, $\widehat{k}(x) = 5$, $\widehat{EI}(x) = 2$, $\widehat{\tau} = 0.1$, $\widehat{\alpha} = 3$, $\widehat{\beta} = 3$. Then we get the
following system:

$$\begin{cases} 5w_{tt}^j(x,t) - 5(w_{xx}^j(x,t) - \varphi_x^j(x,t)) = 0, \; t > 0, \; x \in (0,1), \\ 8\varphi_{tt}^j(x,t) - 2\varphi_{xx}^j(x,t) - 5(w_x^j(x,t) - \varphi^j(x,t)) = 0, \; t > 0, \; x \in (0,1), \\ w(a_0,t) = w^j(0,t), \quad \varphi(a_0,t) = \varphi^j(0,t), \; t > 0, \\ 5(w_x^j - \varphi^j)(1,t) + w^j(1,t) = -3[\mu w_t^j(1,t) + (1-\mu)w_t^j(1,t-0.1)], \; t > 0, \\ EI_j(1)\varphi_x^j(1,t) + \varphi^j(1,t) = -3[\mu\varphi_t^j(1,t) + (1-\mu)\varphi_t^j(1,t-0.1)], \; t > 0, \\ \sum\limits_{j=1}^n 5(w_x^j - \varphi^j)(0,t) = 0, \quad \sum\limits_{j=1}^n 2\varphi_x^j(0,t) = 0, \; t > 0, \\ w_t^j(1,t-\tau_j) = f_j^0(t-\tau_j), \quad \varphi_t^j(1,t-\tau_j) = f_j^1(t-\tau_j), \; t \in (0,\tau_j), \\ w(s,0) = \bar{w}_0(s), \; w_t(s,0) = \bar{w}_1(s), \quad s \in G, \\ \varphi(s,0) = \bar{\varphi}_0(s), \; \varphi_t(s,0) = \bar{\varphi}_1(s), \quad s \in G, \\ j = 1,2,\cdots,n. \end{cases}$$

(82)

In order to discuss the stability of this kind of system, let us discuss the spectral
distributions of the corresponding system operator by the following cases.

**Case 1.** $\mu > 0.5$.

According to the fundamental solutions for Timoshenko beams (see [43]), using
Matlab Scientific Calculation, we can get many simulations about the spectral dis-
tribution of $\mathcal{A}$ by changing $\mu$. We find that when $\mu > 0.5$, the spectrum are always
located in the left hand of complex plane and away from the imaginary axis. Here
we show two figures of the simulations when $\mu > 0.5$. Fig.2 denotes the spectral
distribution of the system operator $\mathcal{A}$ when $\mu = 0.7$ and $\mu = 0.9$, where " * "
denotes the spectrum of $\mathcal{A}$.

From Fig.2, we find that when $\mu > 0.5$, all of the eigenvalues are always located
in the left hand of complex plane and away from the imaginary axis, which possibly
implies the exponential stability of the system due to the spectrum-determined-
growth condition. Furthermore, if choosing bigger $\mu > 0.5$, the maximum of the
real part of the eigenvalues $\max_{\lambda \in \sigma(\mathcal{A})}\{\Re\lambda\}$ becomes smaller. Moreover, in Fig.2,
we can see that all eigenvalues of $\mathcal{A}$ satisfy that $\Re\lambda < -0.1$ and $\Re\lambda < -0.3$ when
$\mu = 0.7$ and $\mu = 0.9$, respectively. Since the system satisfies spectrum-determined-
growth condition, it possibly indicates that the system is exponentially stable when
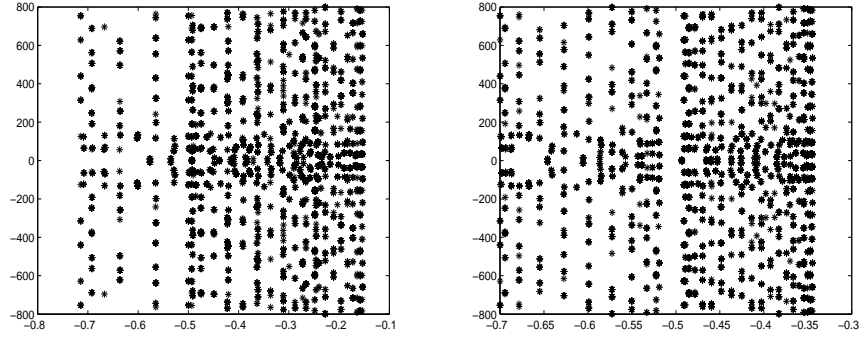
FIGURE 2.  Distribution of spectrum (Left: $\mu = 0.7$;   Right: $\mu = 0.9$)

$\mu = 0.7$ and $\mu = 0.9$. Furthermore, the exponential decay rate is less than $-0.1$ and $-0.3$, respectively.

**Case 2.**   $\mu = 0.5$.

Choosing $\mu = 0.5$, we get the following Fig. 3 about the spectral distribution of the eigenvalues of the system operator.
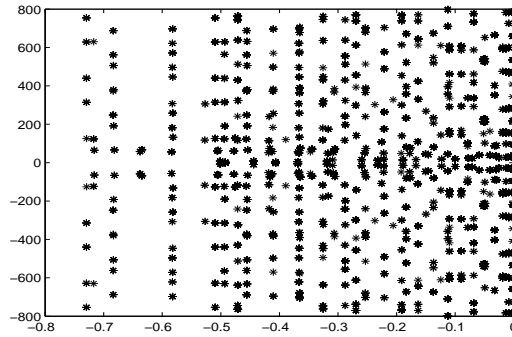


FIGURE 3.  Distribution of spectrum ($\mu = 0.5$)

From this figure, we find that when $\mu \to 0.5$, the distance between the maximum of the real part of the eigenvalues and the imaginary axis becomes very small. We see that the imaginary axis may be the asymptote of the eigenvalues of $\mathcal{A}$ when $\mu = 0.5$. Therefore, the system (82) is possibly asymptotically stable but not exponentially stable when $\mu = 0.5$.

**Case 3.**   $\mu < 0.5$.

When $\mu < 0.5$, we find that there always exists at least one eigenvalue of the system operator such that the real part of it is positive (see Fig.4), which implies that the system is unstable under this condition. Moreover, when $0.5 - \mu > 0$ becomes very small, the maximum of the real part of the eigenvalues is small but still positive (see the right figure in Fig.4 for $\mu = 0.49$). Thus, this kind of network system is still unstable.
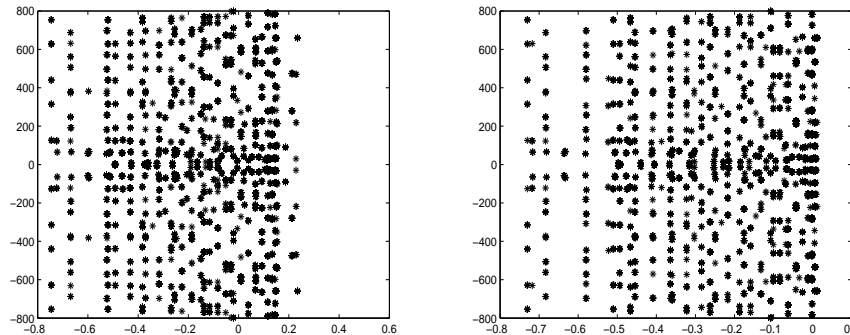
FIGURE 4. Distribution of spectrum (Left: $\mu = 0.3$;   Right: $\mu = 0.49$)

**Remark 3.** By the above simulations, we see that "$\mu = \frac{1}{2}$" is a critical for the stability of the system (10). Under this condition, the system (10) is always at most asymptotically stable.

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*E-mail address*: zjhan@tju.edu.cn (Z.J. Han)
*E-mail address*: gqxu@tju.edu.cn (G.Q. Xu)