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NON-STANDARD DYNAMICS OF ELASTIC COMPOSITES

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Abstract. An elastic medium with a large number of small axially symmetric solid particles is considered. It is assumed that the particles are identically oriented and under the influence of elastic medium they move translationally or rotate around symmetry axis but the direction of their symmetry axes does not change. The asymptotic behavior of small oscillations of the system is studied, when the diameters of particles and distances between the nearest particles are decreased. The equations, describing the homogenized model of the system, are derived. It is shown that the homogenized equations correspond to a non-standard dynamics of elastic medium. Namely, the homogenized stress tensor linearly depends not only on the strain tensor but also on the rotation tensor.

1. **Introduction.** One of the fundamental postulates of the elastic medium mechanics is the statement that under small deformations the stress tensor $\sigma[\underline{u}] = \{\sigma_{ij}[\underline{u}]\}_{i,j=1}^3$ in the medium linearly depends on the strain tensor $e[\underline{u}] = \{e_{ij}[\underline{u}] = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})\}_{i,j=1}^3$ of the medium. Such a dependence is represented with the aid of the elasticity tensor $A = \{a_{npqr}(\underline{x}, t)\}_{n,p,q,r=1}^3$. Namely, the following Hooke law holds:

$$\sigma[\underline{u}] = Ae[\underline{u}], \tag{1.1}$$

where $\underline{u} = \underline{u}(\underline{x}, t)$ is the displacement of the medium, and the fourth rank tensor $A = \{a_{npqr}(\underline{x}, t)\}_{n,p,q,r=1}^3$ is symmetrical with respect to permutation of pairs of subscripts and of subscripts in pairs themselves. This law is experimentally corroborated for the wide class of the homogeneous elastic materials. It appears that Hooke law also holds for many heterogeneous (composite) media with fine-dispersed inclusions. In [5], for example, it is shown that in the medium with perfectly rigid inclusions, the diameters of which tend to zero, the asymptotic behavior of solutions of the elasticity theory equation is expressed by the homogenized equation

$$\rho\frac{\partial^2 \underline{u}}{\partial t^2} - \operatorname{div}\sigma[\underline{u}] = \rho\underline{f}.$$

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It is a common equation of the elasticity theory for anisotropic medium in which the tensor $\sigma[\underline{u}]$ is determined by equality (1.1), where A is the effective elasticity tensor of the composite medium, possessing all known symmetry properties of the effective anisotropic medium. But the main thing of that equation is the fact that the stress tensor linearly depends only on the strain tensor. However, in the literature we have seen an increasing number of papers devoted to the study (both theoretical and experimental) of complex substances the properties of which are essentially different from the ones postulating in classical elasticity theory ([7], [9], [15], [17], [18], [12]).

Particularly, the stress tensor in those papers depends not only on the strain tensor but also on the rotation tensor $\omega[\underline{u}] = \{\omega_{ij}[\underline{u}] = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i})\}_{i,j=1}^3$. Namely, for such substances Hooke law holds in such a non-standard form:

$$\sigma[\underline{u}] = A^D e[\underline{u}] + A^R \omega[\underline{u}], \tag{1.2}$$

where the fourth-rank tensors A^D and A^R can be considered as the deformative and rotational parts of the elasticity tensor respectively. Moreover, those parts don't possess the symmetry properties postulating in classical continuum mechanics.

The examples of such substances are some kinds of liquid crystals, polymers, polycrystalline materials and so on ([12]). Analogous phenomena also occur in some fluid media, for example, in suspensions of magnetizable prolate particles subjected to the influence of strong magnetic fields. However, the characteristic of the microstructure of such substances is not clearly discussed in all papers known to us, thus the reasons of their mentioned behavior appear to be hidden.

In this paper we suggest the simplest example of an elastic composite material for the homogenized motion model of which Hooke law of the form (1.2) holds.

Namely, we consider an elastic medium with a large number of small perfectly rigid inclusions which are the prolate particles oriented along the fixed direction $\underline{l}$. Under the influence of the elastic medium the particles can move translationally or rotate around symmetry axis but the direction of their symmetry axes does not change. Such a motion of the composite can be realized, for example, if the particles are strongly magnetizable and subjected to the influence of the strong magnetic field, so that they are oriented along the field direction B (see Figure1).

We study the asymptotic behavior of such a composite when the diameters of inclusions tend to zero and the inclusions are distributed in the whole volume. As a result, we obtain the homogenized model of motion for which Hooke law of the form (1.2) holds.

2. **Statement of the problem.** Consider a bounded domain Ω in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Suppose that this domain is filled with composite substance consisting of elastic medium and a large number $N_\varepsilon = O(\varepsilon^{-3})$ of small solids Q_ε^i bounded by smooth surfaces ∂Q_ε^i. Further we will call them "the particles".

Let $\Omega_\varepsilon = \Omega \setminus \bigcup_{i=1}^{N} Q_\varepsilon^i$ be a domain filled with the elastic medium, ρ_e and ρ_s be the specific mass density of the elastic medium and of solid particles respectively, $\{a_{npqr}\}$ be the fourth-rank tensor (elasticity tensor) which is supposed to be positive definite and bounded, $\sigma[\underline{u}] = \big\{\sigma_{np}[\underline{u}] = \sum_{q,r=1}^{3} a_{npqr} e_{qr}[\underline{u}]\big\}_{n,p=1}^3$ be the stress tensor in elastic medium, $\underline{x}_\varepsilon^i$ be the position of the center of mass of Q_ε^i, $\underline{u}_\varepsilon^i$ be the displacement of the center of mass of Q_ε^i, $\underline{\theta}_\varepsilon^i$ be the rotation vector of Q_ε^i, m_ε^i be the

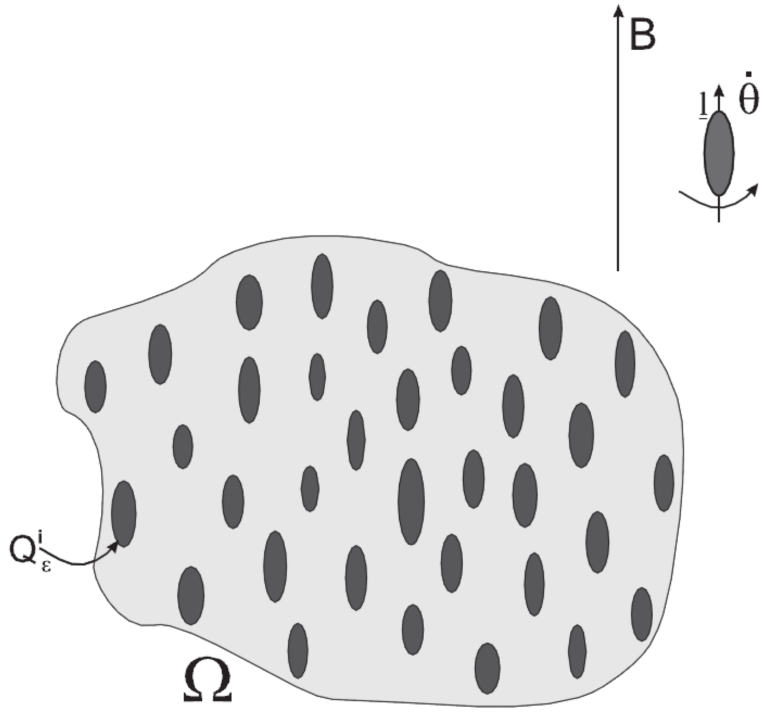

FIGURE 1. The elastic medium with oriented particles

mass of Q^i_ε, I^i_ε be the inertia tensor of Q^i_ε, and let $\underline{u}_\varepsilon = \underline{u}_\varepsilon(\underline{x}, t)$ be the displacement of the elastic medium.

Consider the following system of equations:

$$\rho_e \frac{\partial^2 \underline{u}_\varepsilon}{\partial t^2} - \sum_{n,p,q,r=1}^{3} \frac{\partial}{\partial x_p}\{a_{npqr} e_{qr}[\underline{u}_\varepsilon]\}\underline{e}_n = \rho_e \underline{f}_\varepsilon, \quad \underline{x} \in \Omega_\varepsilon; \tag{2.3}$$

$$\underline{u}_\varepsilon = \underline{u}^i_\varepsilon + \underline{\theta}^i_\varepsilon \times (\underline{x} - \underline{x}^i_\varepsilon), \quad \underline{\theta}^i_\varepsilon = P^d \underline{\theta}^i_\varepsilon, \quad \underline{x} \in \partial Q^i_\varepsilon; \tag{2.4}$$

$$m^i_\varepsilon \ddot{\underline{u}}^i_\varepsilon + \int\limits_{\partial Q^i_\varepsilon} \sigma[\underline{u}_\varepsilon]\nu \, ds = \int\limits_{Q^i_\varepsilon} \rho_s \underline{f}_\varepsilon \, d\underline{x}; \tag{2.5}$$

$$P^d \frac{d}{dt}\left[I^i_\varepsilon \dot{\underline{\theta}}^i_\varepsilon\right] + P^d \int\limits_{\partial Q^i_\varepsilon} (\underline{x} - \underline{x}^i_\varepsilon) \times \sigma[\underline{u}_\varepsilon]\underline{\nu} \, ds = P^d \int\limits_{Q^i_\varepsilon} (\underline{x} - \underline{x}^i_\varepsilon) \times \rho_s \underline{f}_\varepsilon \, d\underline{x}, \tag{2.6}$$

where $\underline{f}_\varepsilon = \underline{f}_\varepsilon(\underline{x}, t)$ is the external force acting on the composite, ν is the unit inner normal vector to the surface ∂Q^i_ε, and P^d is a projection operator onto some fixed d-dimensional subspace $S^d \subset \mathbb{R}^3$.

Depending on d, such a system describes non-stationary motions of elastic composite under various regimes of particles rotations. Namely, if $d = 3$ then the particles can rotate without any constraints. Such a situation was considered in

[5]. If $d = 0$ then the particles move translationally without any rotations. In this paper, we focuss on the non-standard cases where $d = 1$ or $d = 2$.

The case $d = 1$ can be realized, for example, if we consider strongly magnetizable prolate ellipsoidal particles in the strong magnetic field directed along a constant vector $\underline{B}$. Then all the particles are aligned along $\underline{B}$ ([11]), and under the influence of elastic forces they can move translationally or rotate only around their symmetry axis $\underline{l} = \underline{B}$, but the direction of their symmetry axis does not change (see Figure1). In this case, subspace S^1 is a linear subspace spanned by vector $\underline{l}$.

The case $d = 2$ can be realized, for example, if we consider strongly magnetizable oblate ellipsoidal particles in the strong magnetic field. Moreover, it is assumed that the particles are aligned in such a way that their symmetry axes are identically oriented along the direction $\underline{l}$ perpendicular to the field direction $\underline{B}$ and they can rotate both around their symmetry axis and around the field direction. In this case, subspace S^2 is a linear subspace spanned by vectors $\underline{l}$ and $\underline{B}$. The result both in case $d = 1$ and in case $d = 2$ is qualitatively the same: the stress tensor in the homogenized material is expressed via the strain tensor and the rotation tensor in accordance with (1.2).

The system of equations (2.3)-(2.6) is supplemented by the initial conditions

$$\underline{u}_\varepsilon(\underline{x}, 0) = \underline{u}_{\varepsilon 0}(\underline{x}), \quad \frac{\partial \underline{u}_\varepsilon}{\partial t}(\underline{x}, t)\Big|_{t=0} = \underline{v}_{\varepsilon 0}(\underline{x}), \quad \underline{x} \in \Omega_\varepsilon; \tag{2.7}$$

$$\underline{u}^i_\varepsilon(0) = \underline{u}^i_{\varepsilon 0},\ \dot{\underline{u}}^i_\varepsilon(0) = \underline{v}^i_\varepsilon,\ \underline{\theta}^i_\varepsilon(0) = \underline{\theta}^i_{\varepsilon 0} = P^d \underline{\theta}^i_{\varepsilon 0},\ \dot{\underline{\theta}}^i_\varepsilon(0) = \underline{\omega}^i_\varepsilon = P^d \underline{\omega}^i_\varepsilon \tag{2.8}$$

($\underline{u}_{\varepsilon 0}(\underline{x}) = \underline{u}^i_{\varepsilon 0} + \underline{\theta}^i_{\varepsilon 0} \times (\underline{x} - \underline{x}^i_\varepsilon)$ and $\underline{v}_{\varepsilon 0}(\underline{x}) = \underline{v}^i_\varepsilon + \underline{\omega}^i_\varepsilon \times (\underline{x} - \underline{x}^i_\varepsilon)$ at $\underline{x} \in \partial Q^i_\varepsilon$) and the boundary condition on $\partial\Omega$

$$\underline{u}_\varepsilon(\underline{x}, t) = \underline{0}, \quad \underline{x} \in \partial\Omega. \tag{2.9}$$

Theorem 1. *There exists a unique solution of the problem* (2.3) − (2.9).

We do not give here the proof of the theorem. The main goal of the paper is to study the asymptotic behavior of the problem (2.3) − (2.9) solution as $\varepsilon \to 0$.

Before formulating the main result we introduce some definitions and assumptions.

3. **Additional assumptions and the main result.**

Let d^i_ε be the diameter of ellipsoidal particle Q^i_ε, $B(Q^i_\varepsilon)$ be a minimal ball containing Q^i_ε, and R^i_ε be the distance from the ball $B(Q^i_\varepsilon)$ to other minimal balls and to the boundary $\partial\Omega$. We suppose that both d^i_ε and R^i_ε satisfy the following inequalities:

$$C_1\varepsilon \le d^i_\varepsilon, R^i_\varepsilon \le C_2\varepsilon, \tag{3.1}$$

where constants C_1 and C_2 do not depend on ε ($0 < C_1 < C_2 < \infty$).

Suppose that rotation of the particle Q^i_ε is given by the vector $\underline{\theta}^i_\varepsilon$ such that $\underline{\theta}^i_\varepsilon = P^d \underline{\theta}^i_\varepsilon$, where P^d is the projection operator onto some fixed d-dimensional subspace $S^d \subset \mathbb{R}^3$. Consider a cube K^y_h with the side length h ($\varepsilon \ll h \ll 1$) centered at $\underline{y} \in \Omega$. We assume that the edges of this cube are parallel to the coordinate axes. Let $J^{\widehat{\theta}}_\varepsilon[K^y_h]$ be the following class of vector-functions:

$$J^{\widehat{\theta}}_\varepsilon[K^y_h] = \{\underline{w}_\varepsilon \in H^1(K^y_h);\ \underline{w}_\varepsilon(\underline{x}) = \underline{w}^i_\varepsilon + [P^d \underline{\theta}^i_\varepsilon + (1 - P^d)\widehat{\underline{\theta}}] \times (\underline{x} - \underline{x}^i_\varepsilon),\ \underline{x} \in Q^i_\varepsilon \cap K^y_h\},$$

where $\underline{w}_\varepsilon^i$ and $\underline{\theta}_\varepsilon^i$ are arbitrary vectors, and $\widehat{\theta}$ is a given vector. Consider a minimization problem in this class for the following functional (*mesocharacteristic*):

$$A_{\varepsilon h}^{\gamma}(\underline{w}_\varepsilon, \underline{y}, T) = E_{K_h^y}[\underline{w}_\varepsilon, \underline{w}_\varepsilon] + \\ + P_{K_h^y}^{\varepsilon h \gamma}\big[\underline{w}_\varepsilon(\underline{x}) - \sum_{n,p=1}^{3} T_{np}\underline{\varphi}^{np}(\underline{x}-\underline{y}), \underline{w}_\varepsilon(\underline{x}) - \sum_{q,r=1}^{3} T_{qr}\underline{\varphi}^{qr}(\underline{x}-\underline{y})\big], \tag{3.2}$$

where

$$E_G[\underline{u}_\varepsilon, \underline{v}_\varepsilon] = \int\limits_G \sum_{n,p,q,r=1}^{3} a_{npqr} e_{np}[\underline{u}_\varepsilon] e_{qr}[\underline{v}_\varepsilon] d\underline{x}, \tag{3.3}$$

$$P_G^{\varepsilon h \gamma}[\underline{u}_\varepsilon(\underline{x}), \underline{v}_\varepsilon(\underline{x})] = h^{-2-\gamma} \int\limits_G \langle \underline{u}_\varepsilon(\underline{x}), \underline{v}_\varepsilon(\underline{x}) \rangle \, dx, \tag{3.4}$$

$$\underline{\varphi}^{qr}(\underline{x}) = \frac{1}{2}(x_r \underline{e}^q + x_q \underline{e}^r), \tag{3.5}$$

$e_{kl}[\underline{u}] = \frac{1}{2}\big(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\big)$, $T = \{T_{qr}\}$ is an arbitrary symmetric second rank tensor, and $0 < \gamma < 2$ is a penalty parameter.

This mesocharacteristic plays the crucial role in our consideration. Roughly speaking, it allows us to compute the energy of the composite in some mesoscopic cube of size h ($\varepsilon \ll h \ll 1$), which is a so-called representative volume element. In other words, if a composite can be described within the effective single medium approach, then the elastic properties of the composite can be determined by calculation or measurements in some representative volume element of an intermediate mesoscale h, which is why we choose cube K_h^y.

Next, observe that the first term (3.3) in (3.2) represents the energy of the composite. The minimizer $\underline{w}_\varepsilon$ of (3.2) is "close", up to an additive constant, to the true global minimizer $\underline{u}_\varepsilon$ of the variational problem, which corresponds to (2.3)-(2.9) if the tensor T is chosen appropriately. Now one should choose T. If the single medium homogenized description is possible, then $\underline{u}_\varepsilon(\underline{x})$ is "close" to some smooth (homogenized) vector-function $\underline{u}(\underline{x})$, which depends only on macroscopic variable $\underline{x}$ and does not depend on ε, so that it does not vary on the microscale ε. We then minimize the energy of the composite, adding the constraint that the minimizer $\underline{w}_\varepsilon$ is "close" to the linear part (differential) of the global minimizer $\underline{u}$, so that $|\underline{w}_\varepsilon - \underline{u}| = o(h) \sim h^{1+\frac{\gamma}{2}}$ for some $\gamma > 0$. This condition is imposed by introducing the penalty term (3.4).

It can be proved that there exists the unique vector-function which minimizes the functional (3.2); the minimal value of this functional is given by

$$\min_{\underline{w}_\varepsilon \in J_\varepsilon^{\widehat{\theta}}[K_h^y]} A_{\varepsilon h}^{\gamma}(\underline{w}_\varepsilon, \underline{y}, T) = \sum_{n,p,q,r=1}^{3} a_{npqr}^{0,\gamma}(\underline{y}, S^d, \varepsilon, h) T_{np} T_{qr} + \\ + 2\sum_{n,p=1}^{3}\sum_{q=1}^{3} b_{npq}^{\gamma}(\underline{y}, S^d, \varepsilon, h) T_{np}\widehat{\theta}_q + \sum_{q,r=1}^{3} c_{qr}^{\gamma}(\underline{y}, S^d, \varepsilon, h)\widehat{\theta}_q\widehat{\theta}_r, \tag{3.6}$$

where $a_{npqr}^{0,\gamma}(\underline{y}, S^d, \varepsilon, h)$, $b_{npq}^{\gamma}(\underline{y}, S^d, \varepsilon, h)$ and $c_{qr}^{\gamma}(\underline{y}, S^d, \varepsilon, h)$ are the components of the fourth-, third- and second-rank tensors respectively, defined as follows

$$a^{0,\gamma}_{npqr}(\underline{y},S^d,\varepsilon,h)=E_{K^y_h}[\underline{w}^{np},\underline{w}^{qr}]+P^{\varepsilon h\gamma}_{K^y_h}\big[\underline{w}^{np}(\underline{x})-\underline{\varphi}^{np}(\underline{x}-\underline{y}),\underline{w}^{qr}(\underline{x})-\underline{\varphi}^{qr}(\underline{x}-\underline{y})\big], \tag{3.7}$$

$$b^{\gamma}_{npq}(\underline{y},S^d,\varepsilon,h)=E_{K^y_h}[\underline{w}^{np},\underline{v}^{q}]+P^{\varepsilon h\gamma}_{K^y_h}\big[\underline{w}^{np}(\underline{x})-\underline{\varphi}^{np}(\underline{x}-\underline{y}),\underline{v}^{q}(\underline{x})\big], \tag{3.8}$$

$$c^{\gamma}_{qr}(\underline{y},S^d,\varepsilon,h)=E_{K^y_h}[\underline{v}^{q},\underline{v}^{r}]+P^{\varepsilon h\gamma}_{K^y_h}\big[\underline{v}^{q}(\underline{x}),\underline{v}^{r}(\underline{x})\big]. \tag{3.9}$$

Here $\underline{w}^{np}(\underline{x})$ is the vector-function that minimizes the functional (3.2) in $J^{\underline{0}}_{\varepsilon}[K^y_h]$ as $T=T^{np}=\dfrac{1}{2}(\underline{e}^n\otimes\underline{e}^p+\underline{e}^p\otimes\underline{e}^n)$, $\underline{v}^q(\underline{x})$ is the vector-function minimizing the functional (3.2) in $J^{\underline{e}^q}_{\varepsilon}[K^y_h]$ as $T=0$, and $\underline{e}^n$ $(n=1,2,3)$ form an orthonormal basis in $\mathbb{R}^3$.

Starting from the solution $\{\underline{u}_\varepsilon(\underline{x},t),\ \underline{u}^i_\varepsilon,\ \underline{\theta}^i_\varepsilon=P^d\underline{\theta}^i_\varepsilon,\ i=\overline{1,N_\varepsilon}\}$ of the problem (2.3) – (2.5) we construct the vector function

$$\underline{\tilde{u}}_\varepsilon(\underline{x},t)=\chi_\varepsilon(\underline{x})\underline{u}_\varepsilon(\underline{x},t)+\sum_{i=1}^{N_\varepsilon}\chi^i_\varepsilon(\underline{x})[\underline{u}^i_\varepsilon+\underline{\theta}^i_\varepsilon\times(\underline{x}-\underline{x}^i_\varepsilon)], \tag{3.10}$$

where $\chi_\varepsilon(\underline{x})$ is the characteristic function of the domain Ω_ε, filled with the elastic media, and $\chi^i_\varepsilon(\underline{x})$ is the characteristic function of a particle Q^i_ε. We also denote by

$$\rho_\varepsilon(\underline{x})=\rho_e\chi_\varepsilon(\underline{x})+\rho_s\sum_{i=1}^{N_\varepsilon}\chi^i_\varepsilon(\underline{x})$$

the density of composite “the elastic media-the particles”.

We assume that the following conditions hold:

3.0) the sequence $\rho_\varepsilon(\underline{x})$ converges weakly* in $L^\infty(\Omega)$ to a function $\rho(\underline{x})>0$ and the sequence $\underline{f}_\varepsilon(\underline{x})$ converges weakly in $\mathbf{L_2(\Omega)}$ to a vector-function $\underline{f}(\underline{x})$, as $\varepsilon\to 0$.

3.1) the sequence of initial vector-functions $\underline{\tilde{u}}_{\varepsilon 0}(\underline{x})=\tilde{u}_\varepsilon(\underline{x},0)$ and $\underline{\tilde{v}}_{\varepsilon 0}(\underline{x})=\left.\dfrac{\partial\underline{\tilde{u}}_\varepsilon(\underline{x},t)}{\partial t}\right|_{t=0}$ converges weakly in $\mathbf{L_2(\Omega)}$ to vector-functions $\underline{u}_0(\underline{x})$ and $\underline{v}_0(\underline{x})$ respectively, as $\varepsilon\to 0$.

3.2) for some real number $\gamma>0$ the following limits exist heterogeneously at $\underline{x}\in\Omega$:

$$a)\ \lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\frac{a^{0,\gamma}_{npqr}(\underline{x},S^d,\varepsilon,h)}{h^3}=\lim_{h\to 0}\underline{\lim_{\varepsilon\to 0}}\frac{a^{0,\gamma}_{npqr}(\underline{x},S^d,\varepsilon,h)}{h^3}=a^0_{npqr}(\underline{x},S^d),$$

$$b)\ \lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\frac{b^{\gamma}_{npq}(\underline{x},S^d,\varepsilon,h)}{h^3}=\lim_{h\to 0}\underline{\lim_{\varepsilon\to 0}}\frac{b^{\gamma}_{npq}(\underline{x},S^d,\varepsilon,h)}{h^3}=b_{npq}(\underline{x},S^d),$$

$$c)\ \lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\frac{c^{\gamma}_{qr}(\underline{x},S^d,\varepsilon,h)}{h^3}=\lim_{h\to 0}\underline{\lim_{\varepsilon\to 0}}\frac{c^{\gamma}_{qr}(\underline{x},S^d,\varepsilon,h)}{h^3}=c_{qr}(\underline{x},S^d),$$

where $\{a^0_{npqr}(\underline{x},S^d)\}$, $\{b_{npq}(\underline{x},S^d)\}$, $\{c_{qr}(\underline{x},S^d)\}$ are continuous tensors (at $\underline{x}\in\Omega$).

Note, that the existence of limits 3.2) is a general restriction on the spatial distributions of the particles. Since we do not require any spatial periodicity, we have to impose some conditions on these distributions. In section 7, we provide an example where limits 3.2) are calculated explicitly.

Remark. If the limits in 3.2) exist for some $\gamma > 0$, then they exist for any $\gamma > 0$ and the limiting tensors do not depend on γ; moreover, $\{a^0_{npqr}(\underline{x}, S^d)\}$ and $\{c_{qr}(\underline{x}, S^d)\}$ are positive definite tensors (these facts can be proved analogously to [13]).

Now we are in a position to formulate the main mathematical result of this paper.

Theorem 2. *Let conditions 3.0)-3.2) hold. Then the sequence of vector-functions* $\underline{\tilde{u}}_\varepsilon(\underline{x}, t)$*, defined by* (3.10)*, converges weakly in* $\mathbf{L_2}(\mathbf{\Omega} \times [\mathbf{0}, \mathbf{T}])$ *(for any* $T > 0$*) to a vector-function* $\underline{u}(\underline{x}, t)$*, which is a solution of the following homogenized problem:*

$$\rho \frac{\partial^2 \underline{u}}{\partial t^2} - \sum_{n,p,q,r=1}^{3} \frac{\partial}{\partial x_p} \Big[a^D_{npqr}(\underline{x}, S^d) e_{qr}[u] + a^R_{npqr}(\underline{x}, S^d) \omega_{qr}[u] \Big] \underline{e}^n = \rho \underline{f}, \quad \underline{x} \in \Omega,\ t > 0; \tag{3.11}$$

$$\underline{u}(\underline{x}, t) = \underline{0}, \quad \underline{x} \in \partial\Omega, \quad t > 0; \tag{3.12}$$

$$\underline{u}(\underline{x}, 0) = \underline{u}_0(\underline{x}), \quad \frac{\partial \underline{u}}{\partial t}(\underline{x}, t)\Big|_{t=0} = \underline{v}_0(\underline{x}), \quad \underline{x} \in \Omega. \tag{3.13}$$

Here

$$a^D_{npqr} = a^0_{npqr} + \frac{1}{2} \sum_{l=1}^{3} b_{qrl} \epsilon_{lnp}, \qquad a^R_{npqr} = \frac{1}{4} \sum_{l,m=1}^{3} c_{lm} \epsilon_{lnp} \epsilon_{mqr} + \frac{1}{2} \sum_{l=1}^{3} b_{npl} \epsilon_{lqr}, \tag{3.14}$$

$$\omega_{qr}[u] = \frac{1}{2} \Big(\frac{\partial u_q}{\partial x_r} - \frac{\partial u_r}{\partial x_q} \Big), \tag{3.15}$$

where $\{\epsilon_{lnp}\}$ *is Levi-Civita permutation tensor.*

The problem (3.11) $-$ (3.13) *has the unique solution.*

The proof of this theorem is given in sections 4-6. First, in section 4, using Laplace transform, we formulate a stationary version of the problem (2.3)–(2.9) with the spectral parameter λ . Then we reduce it to a variational form for $\lambda > 0$. In section 5, we study the asymptotic behavior of the solution of the variational problem as $\varepsilon \to 0$ by using a method close to the computation of a Γ-limit. We find the homogenized variational functional and the system of Euler equations corresponding to this functional. Finally, in section 6 we study the analytic properties of the solutions of these equations in the parameter λ, and, applying the inverse Laplace transform, obtain the homogenized non-stationary problem (3.11) –(3.13).

4. **Variational formulation of the stationary problem.** Use the Laplace transform of the functions to be found: $\underline{u}_\varepsilon(\underline{x}, t) \to \underline{v}_\varepsilon(\underline{x}, \lambda)$, $\underline{u}^i_\varepsilon(t) \to \underline{u}^i_\varepsilon(\lambda)$, $\underline{\theta}^i_\varepsilon(t) \to \underline{\theta}^i_\varepsilon(\lambda)$. Taking into account the properties of the Laplace transform, we rewrite the problem (2.3)-(2.5) in the form

$$\lambda^2\rho_e\underline{u}_\varepsilon-\sum_{n,p,q,r=1}^{3}\frac{\partial}{\partial x_p}\{a_{npqr}e_{qr}[\underline{u}_\varepsilon]\}\underline{e}_n=\rho_e\underline{f}_\varepsilon+\lambda\rho_e\underline{u}_{\varepsilon 0}(\underline{x})+\rho_e\underline{v}_{\varepsilon 0}(\underline{x}),\quad \underline{x}\in\Omega_\varepsilon, \tag{4.1}$$

$$\underline{u}_\varepsilon=\underline{u}^i_\varepsilon+\underline{\theta}^i_\varepsilon\times(\underline{x}-\underline{x}^i_\varepsilon),\quad \underline{\theta}^i_\varepsilon=P^d\underline{\theta}^i_\varepsilon,\quad \underline{x}\in\partial Q^i_\varepsilon, \tag{4.2}$$

$$\lambda^2 m^i_\varepsilon\underline{u}^i_\varepsilon+\int\limits_{\partial Q^i_\varepsilon}\sigma[\underline{u}_\varepsilon]\nu\,ds=\lambda m^i_\varepsilon\underline{u}^i_{\varepsilon 0}+m^i_\varepsilon\underline{v}^i_\varepsilon+\int\limits_{Q^i_\varepsilon}\rho_s\underline{f}_\varepsilon\,d\underline{x}, \tag{4.3}$$

$$\begin{aligned}&\lambda^2P^d[I^i_\varepsilon\underline{\theta}^i_\varepsilon]+P^d\int\limits_{\partial Q^i_\varepsilon}(\underline{x}-\underline{x}^i_\varepsilon)\times\sigma[\underline{u}_\varepsilon]\nu\,ds\\ =&\lambda P^d[I^i_\varepsilon\underline{\theta}^i_{\varepsilon 0}]+P^d[I^i_\varepsilon\underline{\omega}^i_\varepsilon]+P^d\int\limits_{Q^i_\varepsilon}(\underline{x}-\underline{x}^i_\varepsilon)\times\rho_s\underline{f}_\varepsilon\,d\underline{x},\end{aligned} \tag{4.4}$$

$$\underline{u}_\varepsilon(\underline{x})=0,\quad \underline{x}\in\partial\Omega, \tag{4.5}$$

where $\mathrm{Re}\lambda>0$. We extend the displacement function $\underline{u}_\varepsilon(\underline{x},\lambda)$ onto the particles Q^i_ε according to (4.2) and keep the same notations for the extended function.

Fix now $\lambda>0$. Then the problem (4.1)- (4.5) is equivalent to the variational problem

$$\Phi_\varepsilon(\underline{u}_\varepsilon)=\min_{\underline{u}'_\varepsilon\in\overset{\circ}{J}_\varepsilon(\Omega)}\Phi_\varepsilon(\underline{u}'_\varepsilon), \tag{4.6}$$

where $\overset{\circ}{J}_\varepsilon(\Omega)$ is the class of vector-functions from $\overset{\circ}{H}{}^1(\Omega)$ which are equal to $\underline{a}^i_\varepsilon+\underline{\theta}^i_\varepsilon\times(\underline{x}-\underline{x}^i_\varepsilon)$ on the particles Q^i_ε ($\underline{a}^i_\varepsilon$ and $\underline{\theta}^i_\varepsilon=P^d\underline{\theta}^i_\varepsilon$ are arbitrary vectors), and

$$\Phi_\varepsilon(\underline{u}_\varepsilon)=\int\limits_\Omega\big\{\lambda^2\rho_\varepsilon\langle\underline{u}_\varepsilon,\underline{u}_\varepsilon\rangle+\sum_{n,p,q,r=1}^{3}a_{npqr}e_{np}[\underline{u}_\varepsilon]e_{qr}[\underline{u}_\varepsilon]-2\rho_\varepsilon\langle\lambda\underline{u}_{\varepsilon 0}+\underline{v}_{\varepsilon 0}+\underline{f}_\varepsilon,\underline{u}_\varepsilon\rangle\big\}\,d\underline{x}, \tag{4.7}$$

where $\lambda>0$.

The main goal is to investigate the asymptotic behavior of the solution $\underline{u}_\varepsilon(\underline{x})$ of minimization problem (4.6), as $\varepsilon\to 0$. To formulate the homogenization result, we consider the minimization problem

$$\Phi_0(\underline{u})=\min_{\underline{u}'\in\overset{\circ}{H}{}^1(\Omega)}\Phi_0(\underline{u}'), \tag{4.8}$$

where

$$\begin{aligned}&\Phi_0(\underline{u})\\ =&\int\limits_\Omega\Big\{\lambda^2\rho\langle\underline{u},\underline{u}\rangle+\sum_{n,p,q,r=1}^{3}a^0_{npqr}(\underline{x})e_{np}[\underline{u}]e_{qr}[\underline{u}]-2\sum_{n,p=1}^{3}\sum_{q=1}^{3}b_{npq}(\underline{x})e_{np}[\underline{u}]\Big[\frac{1}{2}\mathrm{rot}\,\underline{u}\Big]_q\end{aligned}$$

$$+\sum_{q,r=1}^{3} c_{qr}(\underline{x})\Big[\frac{1}{2}\mathrm{rot}\,\underline{u}\Big]_q\Big[\frac{1}{2}\mathrm{rot}\,\underline{u}\Big]_r - 2\rho\langle \underline{f}+\lambda\underline{u}_0+\underline{v}_0,\underline{u}\rangle\Big\}\,d\underline{x}. \tag{4.9}$$

The minimizer of this problem is the solution of the following boundary value problem:

$$\lambda^2\rho\underline{u}-\sum_{n,p,q,r=1}^{3}\frac{\partial}{\partial x_p}\Big[a^D_{npqr}(\underline{x})e_{qr}[u]+a^R_{npqr}(\underline{x})\omega_{qr}[u]\Big]\underline{e}^n=\rho\underline{f}+\lambda\rho\underline{u}_0+\rho\underline{v}_0,\quad \underline{x}\in\Omega, \tag{4.10}$$

$$\underline{u}(\underline{x},\lambda)=0,\quad \underline{x}\in\partial\Omega. \tag{4.11}$$

The asymptotic behavior as $\varepsilon\to 0$ of the solution of problem (4.6) is given by the following theorem.

Theorem 3. *Let conditions 3.0)-3.2) hold. Then the solution $\underline{u}_\varepsilon(\underline{x},\lambda)$ of the problem (4.6) for any $\lambda>0$ converges strongly in $\mathbf{L_2(\Omega)}$ to the solution $\underline{u}(\underline{x},\lambda)$ of the problem (4.8), as $\varepsilon\to 0$:*

$$\underline{u}_\varepsilon(\underline{x},\lambda)\xrightarrow[\varepsilon\to 0]{}\underline{u}(\underline{x},\lambda)\quad \textit{strongly}\quad \textit{in}\quad \mathbf{L_2(\Omega)}.$$

The proof of this theorem is given in section 5.

5. **Proof of Theorem 3.** Let $\underline{u}_\varepsilon(\underline{x},\lambda)$ be the solution of the problem (4.6). Since $0\in\overset{\circ}{J}_\varepsilon(\Omega)$, we have:

$$\Phi_\varepsilon(\underline{u}_\varepsilon)\le\Phi_\varepsilon(0)=0. \tag{5.1}$$

Due to conditions 3.0)-3.1) and the first Korn's inequality (see [16])

$$\|\underline{u}_\varepsilon\|^2_{\overset{\circ}{H}{}^1(\Omega)}\le 2\int_\Omega\sum_{n,p=1}^{3}e^2_{np}[\underline{u}_\varepsilon]\,d\underline{x}, \tag{5.2}$$

(4.7) and (5.1) give:

$$\|\underline{u}_\varepsilon\|^2_{H^1(\Omega)}\le C. \tag{5.3}$$

Therefore the set of vector-functions $\{\underline{u}_\varepsilon(\underline{x},\lambda),\,\varepsilon>0\}$ is weakly compact in $H^1(\Omega)$. Due to the Embedding Theorem, this set is compact in $L_2(\Omega)$. Hence, there exists a subsequence $\{\underline{u}_{\varepsilon_k}(\underline{x},\lambda),\,\varepsilon>0\}$ which converges (weakly in $H^1(\Omega)$ and strongly in $L_2(\Omega)$) to some vector-function $\underline{u}(\underline{x},\lambda)$. As it is shown below, the limiting vector-function $\underline{u}(\underline{x},\lambda)$ is a solution of the problem (4.8). It can be proved (see Lemma 1) that

$$\int_\Omega\big[a^D_{npqr}+a^R_{npqr}\big]\frac{\partial u_n}{\partial x_p}\frac{\partial u_q}{\partial x_r}\,d\underline{x}\ge\|\underline{u}\|^2_{H^1(\Omega)},$$

and hence, problem (4.8) has a unique solution. From this it follows that the sequence $\{\underline{u}_\varepsilon(\underline{x},\lambda),\,\varepsilon>0\}$ is also convergent:

$$\underline{u}_\varepsilon\rightharpoonup\underline{u}\,\text{weakly in}\,H^1(\Omega),\ \underline{u}_\varepsilon\to\underline{u}\,\text{strongly in}\,L_2(\Omega). \tag{5.4}$$

Clearly, $\underline{u}(\underline{x}) \in \overset{\circ}{H}{}^1(\Omega)$. Show that for any vector-function $\underline{w} \in \overset{\circ}{H}{}^1(\Omega)$ the following inequality holds:

$$\Phi_0(\underline{u}) \leq \Phi_0(\underline{w}). \tag{5.5}$$

1. For any vector-function $\underline{w} \in \overset{\circ}{H}{}^1(\Omega) \bigcap C_0^2(\Omega)$ we construct a special vector-function $\underline{w}_{\varepsilon h} \in \overset{\circ}{J}_\varepsilon(\Omega)$, such that

$$\lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}}\, \Phi_\varepsilon(\underline{w}_{\varepsilon h}) \leq \Phi_0(\underline{w}). \tag{5.6}$$

Now we describe this construction. Cover the domain Ω with cubes $K_h^{\underline{x}_\alpha}$ centered at points $\underline{x}_\alpha \in \Omega$ with the edges of length h, which are parallel to the coordinate axis: $\overline{\Omega} \subset \bigcup_{\alpha\in\Lambda} K_h^{\underline{x}_\alpha}$. Let the centers $\underline{x}_\alpha \in \Omega$ of these cubes form a cubic lattice of period $h - h^{1+\frac{\gamma}{2}}$ $(0 < \gamma < 2)$, so that the cubes overlap. Due to the overlap of the cubes, we can further select smaller cubes $K_{h'}^{\underline{x}_\alpha}$ (with the edges of length $h' = h - 2h^{1+\frac{\gamma}{2}}$) which are concentric to $K_h^{\underline{x}_\alpha}$. It is well known (see [6]) that there exists a set of functions $\{\phi_\alpha^{\varepsilon h}(\underline{x}) \in C_0^\infty(\Omega)\}_{\alpha\in\Lambda}$ (called *a special partition of unity*) such that

$$1)\ \phi_\alpha^{\varepsilon h}(\underline{x}) = \begin{cases} 1, \underline{x} \in K_{h'}^{\underline{x}_\alpha} \\ 0, \underline{x} \notin K_h^{\underline{x}_\alpha} \end{cases}, \quad 2)\ 0 \leq \phi_\alpha^{\varepsilon h}(\underline{x}) \leq 1, \quad 3)\ |\nabla \phi_\alpha^{\varepsilon h}(\underline{x})| \leq \frac{c}{h^{1+\frac{\gamma}{2}}},$$
$$4)\ \sum_{\alpha\in\Lambda} \phi_\alpha^{\varepsilon h}(\underline{x}) \equiv 1,\ \underline{x} \in \overline{\Omega}, \quad 5)\ \phi_\alpha^{\varepsilon h}(\underline{x}) = C_\varepsilon^i,\ \underline{x} \in B(Q_\varepsilon^i), \tag{5.7}$$

where C_ε^i are the constants $(0 \leq C_\varepsilon^i \leq 1)$, and $B(Q_\varepsilon^i)$ are the balls centered at points $\underline{x}_\varepsilon^i$ with the radii d_ε^i (see (3.1)), which contain the particles Q_ε^i. For the sake of simplicity, we will omit the superscripts ε and h where it will not cause any confusion: $\phi_\alpha^{\varepsilon h}(\underline{x}) = \phi_\alpha(\underline{x})$.

For any vector-function $\underline{w}(\underline{x}) \in C_0^2(\Omega)$ we construct the vector-function $\underline{w}_{\varepsilon h}(\underline{x}) \in \overset{\circ}{J}_\varepsilon(\Omega)$ possessing the following properties. First, it approximates (in $L_2(\Omega)$) a given vector-function $\underline{w}(\underline{x}) \in \overset{\circ}{H}{}^1(\Omega)$ for small ε and h. Second, it "almost" minimizes the functional (3.2).

Note that any vector-function $\underline{w}(\underline{x}) \in C^2(K_h^{\underline{x}_\alpha})$ can be written in the form

$$\underline{w}(\underline{x}) = \underline{w}(\underline{x}^\alpha) + \sum_{n,p=1}^{3} \big(e_{np}[\underline{w}(\underline{x}^\alpha)]\underline{\varphi}^{np}(\underline{x} - \underline{x}^\alpha) +$$
$$+ w_{np}[\underline{w}(\underline{x}^\alpha)]\underline{\psi}^{np}(\underline{x} - \underline{x}^\alpha)\big) + \underline{g}_\alpha(\underline{x}), \quad \underline{x} \in K_h^{\underline{x}_\alpha}, \tag{5.8}$$

where

$$e_{np}[\underline{w}(\underline{x}_\alpha)] = \frac{1}{2}\Big(\frac{\partial \underline{w}_n}{\partial x_p}(\underline{x}_\alpha) + \frac{\partial \underline{w}_p}{\partial x_n}(\underline{x}_\alpha)\Big), \quad w_{np}[\underline{w}(\underline{x}_\alpha)] = \frac{1}{2}\Big(\frac{\partial \underline{w}_n}{\partial x_p}(\underline{x}_\alpha) - \frac{\partial \underline{w}_p}{\partial x_n}(\underline{x}_\alpha)\Big),$$

the vector-function $\underline{\varphi}^{np}(\underline{x})$ is defined in (3.5),

$$\underline{\psi}^{np}(\underline{x}) = \frac{1}{2}(x_p \underline{e}^n - x_n \underline{e}^p), \tag{5.9}$$

and $D^k \underline{g}_\alpha(\underline{x}) = \underline{O}(h^{2-k})$, $k = \overline{0,2}$. Define the quasi-minimizer $\underline{w}_{\varepsilon h}(\underline{x})$ as follows:

$$\underline{w}_{\varepsilon h}(\underline{x}) = \sum_{\alpha\in\Lambda}\Big\{\underline{w}(\underline{x}_\alpha) + \sum_{n,p=1}^{3} e_{np}[\underline{w}(\underline{x}_\alpha)]\underline{v}^{np,\underline{0}}_{\alpha,\varepsilon h}(\underline{x}) + \\ + \sum_{n,p=1}^{3} w_{np}[\underline{w}(\underline{x}_\alpha)]\underline{\psi}^{np}(\underline{x}-\underline{x}_\alpha) - \sum_{k=1}^{3}\big[\frac{1}{2}\mathrm{rot}\,\underline{w}(\underline{x}_\alpha)\big]_k \underline{v}^k_{\alpha,\varepsilon h}\Big\}\cdot\phi_\alpha(\underline{x}), \tag{5.10}$$

where the vector-functions $\underline{v}^{np,\underline{0}}_{\alpha,\varepsilon h}(\underline{x})$ are the minimizers of the functional (3.2) in $J^{\underline{0}}_\varepsilon[K^y_h]$ as $T = T^{np} = \frac{1}{2}(\underline{e}^n\otimes\underline{e}^p + \underline{e}^p\otimes\underline{e}^n)$, and $\underline{v}^k_{\alpha,\varepsilon h}(\underline{x})$ are the minimizers of the functional (3.2) in $J^{\underline{e}^k}_\varepsilon[K^y_h]$ as $T = 0$.

It is obvious that $\underline{w}_{\varepsilon h}(\underline{x}) \in \overset{\circ}{J}_\varepsilon(\Omega)$. Let us calculate the functional (4.7) on the vector-function $\underline{w}_{\varepsilon h}(\underline{x})$. To this end, we distinguish the leading term in $e_{kl}[\underline{w}_{\varepsilon h}]$:

$$e_{kl}[\underline{w}_{\varepsilon h}(\underline{x})] = \sum_{\alpha\in\Lambda}\Big\{\sum_{n,p=1}^{3} e_{np}[\underline{w}(\underline{x}_\alpha)]e_{kl}[\underline{v}^{np,\underline{0}}_{\alpha,\varepsilon h}(\underline{x})] + \\ - \sum_{m=1}^{3}\big[\frac{1}{2}\mathrm{rot}\,\underline{w}(\underline{x}_\alpha)\big]_m e_{kl}[\underline{v}^m_{\alpha,\varepsilon h}]\Big\}\phi_\alpha(\underline{x}) + \delta_{\varepsilon h}(\underline{x}), \tag{5.11}$$

where $\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\|\delta_{\varepsilon h}\|_{L_2(\Omega)} = 0$ (for more details see, for example, [3] and [6]). Then, using (5.7) and (5.11), similarly to [1], [2], [3], [4] and [6] we can show that

$$E_\Omega[\underline{w}_{\varepsilon h}, \underline{w}_{\varepsilon h}] = \sum_{\alpha\in\Lambda}\sum_{n,p,q,r=1}^{3} e_{np}[\underline{w}(\underline{x}_\alpha)]e_{qr}[\underline{w}(\underline{x}_\alpha)]E_{K^{\underline{x}_\alpha}_{h'}}[\underline{v}^{np,\underline{0}}_{\alpha,\varepsilon h}, \underline{v}^{qr,\underline{0}}_{\alpha,\varepsilon h}] - \\ - 2\sum_{\alpha\in\Lambda}\sum_{n,p=1}^{3}\sum_{q=1}^{3} e_{np}[\underline{w}(\underline{x}_\alpha)]\big[\frac{1}{2}\mathrm{rot}\,\underline{w}(\underline{x}_\alpha)\big]_q E_{K^{\underline{x}_\alpha}_{h'}}[\underline{v}^{np,\underline{0}}_{\alpha,\varepsilon h}, \underline{v}^q_{\alpha,\varepsilon h}] + \\ + \sum_{\alpha\in\Lambda}\sum_{q,r=1}^{3}\big[\frac{1}{2}\mathrm{rot}\,\underline{w}(\underline{x}_\alpha)\big]_q\big[\frac{1}{2}\mathrm{rot}\,\underline{w}(\underline{x}_\alpha)\big]_r E_{K^{\underline{x}_\alpha}_{h'}}[\underline{v}^q_{\alpha,\varepsilon h}, \underline{v}^r_{\alpha,\varepsilon h}] + L(\varepsilon, h), \tag{5.12}$$

where $\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}} L(\varepsilon, h) = 0$.

From (5.12), taking into account (3.7)-(3.9), we obtain

$$E_\Omega[\underline{w}_{\varepsilon h}, \underline{w}_{\varepsilon h}] \le \sum_{\alpha\in\Lambda}\Big\{\sum_{n,p,q,r=1}^{3} a^{0,\gamma}_{npqr}(\underline{x}_\alpha, \varepsilon, h)e_{np}[\underline{w}(\underline{x}_\alpha)]e_{qr}[\underline{w}(\underline{x}_\alpha)] - \\ -2\sum_{n,p=1}^{3}\sum_{q=1}^{3} b^\gamma_{npq}(\underline{x}_\alpha, \varepsilon, h)e_{np}[\underline{w}(\underline{x}_\alpha)]\big[\frac{1}{2}\mathrm{rot}\,\underline{w}(\underline{x}_\alpha)\big]_q + \\ + \sum_{q,r=1}^{3} c^\gamma_{qr}(\underline{x}_\alpha, \varepsilon, h)\big[\frac{1}{2}\mathrm{rot}\,\underline{w}(\underline{x}_\alpha)\big]_q\big[\frac{1}{2}\mathrm{rot}\,\underline{w}(\underline{x}_\alpha)\big]_r\Big\} + \bar{o}(1) \quad (\varepsilon \ll h \ll 1). \tag{5.13}$$

Here we add in the RHS of (5.12) the positive term

$$\sum_{\alpha\in\Lambda} P^{\varepsilon h\gamma}_{K_h^{\underline{x}_\alpha}}\Bigg[\sum_{n,p=1}^{3}\Big(\underline{v}^{np,\underline{0}}_{\alpha,\varepsilon h}-\underline{\varphi}^{np}(\underline{x}-\underline{x}_\alpha)\Big)e_{np}[\underline{w}(\underline{x}_\alpha)]-\sum_{n=1}^{3}\underline{v}^{n}_{\alpha,\varepsilon h}\Big[\frac{1}{2}\mathrm{rot}\,\underline{w}(\underline{x}_\alpha)\Big]_n,$$
$$\sum_{q,r=1}^{3}\Big(\underline{v}^{qr,\underline{0}}_{\alpha,\varepsilon h}-\underline{\varphi}^{qr}(\underline{x}-\underline{x}_\alpha)\Big)e_{qr}[\underline{w}(\underline{x}_\alpha)]-\sum_{q=1}^{3}\underline{v}^{q}_{\alpha,\varepsilon h}\Big[\frac{1}{2}\mathrm{rot}\,\underline{w}(\underline{x}_\alpha)\Big]_q\Bigg]$$

corresponding to the penalty terms in (3.7)-(3.9). Now we make use of inequality (5.13) to estimate the functional (4.7):

$$\Phi_\varepsilon(\underline{w}_{\varepsilon h})\le\sum_{\alpha\in\Lambda}h^3\Big\{\sum_{n,p,q,r=1}^{3}\frac{a^{0,\gamma}_{npqr}(\underline{x}_\alpha,\varepsilon,h)}{h^3}e_{np}[\underline{w}(\underline{x}_\alpha)]e_{qr}[\underline{w}(\underline{x}_\alpha)]-$$
$$-2\sum_{n,p=1}^{3}\sum_{q=1}^{3}\frac{b^{\gamma}_{npq}(\underline{x}_\alpha,\varepsilon,h)}{h^3}e_{np}[\underline{w}(\underline{x}_\alpha)]\Big[\frac{1}{2}\mathrm{rot}\,\underline{w}(\underline{x}_\alpha)\Big]_q+$$
$$+\sum_{q,r=1}^{3}\frac{c^{\gamma}_{qr}(\underline{x}_\alpha,\varepsilon,h)}{h^3}\Big[\frac{1}{2}\mathrm{rot}\,\underline{w}(\underline{x}_\alpha)\Big]_q\Big[\frac{1}{2}\mathrm{rot}\,\underline{w}(\underline{x}_\alpha)\Big]_r\Big\}+$$
$$+\lambda^2\int_\Omega\langle\rho_\varepsilon\underline{w}_{\varepsilon h},\underline{w}_{\varepsilon h}\rangle d\underline{x}-2\int_\Omega\rho_\varepsilon\langle\lambda\underline{u}_{\varepsilon 0}+\underline{v}_{\varepsilon 0}+\underline{f}_\varepsilon,\underline{w}_{\varepsilon h}\rangle\,dx+\bar{o}(1)\quad(\varepsilon\ll h\ll 1).\tag{5.14}$$

Using (5.8), (5.10) and taking into account the fact that the minimizers $\underline{v}^{np,\underline{0}}_{\alpha,\varepsilon h}(\underline{x})$ and $\underline{v}^{k}_{\alpha,\varepsilon h}(\underline{x})$ are close, in some sense, to $\underline{\varphi}^{np}(\underline{x}-\underline{x}_\alpha)$ and $\underline{0}$ respectively, we can show that (for more details see, for example, [1] and [3])

$$\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\,\|\underline{w}_{\varepsilon h}-\underline{w}\|_{L_2(\Omega)}=0.\tag{5.15}$$

Then, passing to the limit in (5.14) as $\varepsilon\to 0$ and $h\to 0$ and taking into consideration 3.0)-3.2) and the fact that $\underline{w}(\underline{x})\in C^2(\overline{\Omega})$, we obtain

$$\lim_{h\to 0}\overline{\lim_{\varepsilon\to 0}}\,\Phi_\varepsilon(\underline{w}_{\varepsilon h})\le\Phi_0(\underline{w}).$$

Thus, inequality (5.6) is proved. Next, from (5.6) and an obvious inequality $\Phi_\varepsilon(\underline{u}_\varepsilon)\le\Phi_\varepsilon(\underline{w}_{\varepsilon h})$ there follows the upper bound:

$$\overline{\lim_{\varepsilon\to 0}}\,\Phi_\varepsilon(\underline{u}_\varepsilon)\le\Phi_0(\underline{w}),\quad\forall\underline{w}\in\overset{\circ}{H}{}^1(\Omega).\tag{5.16}$$

2. Prove now the lower bound

$$\Phi_0(\underline{u})\le\underline{\lim}_{\varepsilon\to 0}\,\Phi_\varepsilon(\underline{u}_\varepsilon),\tag{5.17}$$

where the vector-function $\underline{u}(\underline{x})$ is defined in (5.4). For the sake of simplicity we first assume that the limiting vector-function is smooth enough: $\underline{u}(\underline{x})\in\overset{\circ}{H}{}^1(\Omega)\bigcap C^2_0(\Omega)$.

Consider a partition of the domain Ω by non-intersecting cubes $K_h^{\underline{x}_\alpha}$ aligned along the coordinate axes and centered at the points $\underline{x}_\alpha$ forming a cubic lattice of period h. In each cube the vector-function $\underline{u}(\underline{x})$ can be written in the form

$$\underline{u}(\underline{x}) = \underline{u}(\underline{x}^\alpha) + \sum_{n,p=1}^{3} \big(e_{np}[\underline{u}(\underline{x}^\alpha)]\underline{\varphi}^{np}(\underline{x} - \underline{x}^\alpha) + \\ + w_{np}[\underline{u}(\underline{x}^\alpha)]\underline{\psi}^{np}(\underline{x} - \underline{x}^\alpha)\big) + O(h^2), \quad \underline{x} \in K_h^{\underline{x}_\alpha}. \tag{5.18}$$

Then, in every internal with respect to Ω cube $K_h^{\underline{x}_\alpha}$ (which does not intersect the boundary $\partial\Omega$) consider a vector-function

$$\underline{u}_\varepsilon^\alpha(\underline{x}) = \underline{u}_\varepsilon(\underline{x}) - \underline{u}(\underline{x}^\alpha) - \sum_{n,p=1}^{3} w_{np}[\underline{u}(\underline{x}^\alpha)]\underline{\psi}^{np}(\underline{x} - \underline{x}^\alpha). \tag{5.19}$$

It is clear that $\underline{u}_\varepsilon^\alpha(\underline{x}) \in J_\varepsilon^{\widehat{\underline{\theta}}^\alpha}[K_h^{\underline{x}_\alpha}]$, where $\widehat{\underline{\theta}}^\alpha = -\frac{1}{2}\mathrm{rot}\,\underline{u}(\underline{x}_\alpha)$, and $e_{np}[\underline{u}_\varepsilon^\alpha] = e_{np}[\underline{u}_\varepsilon]$ in $K_h^{\underline{x}_\alpha}$. Therefore, from (3.2) and (3.6) for $T_{np} = e_{np}[\underline{u}(\underline{x}_\alpha)]$ we obtain

$$E_{K_h^{\underline{x}_\alpha}}[\underline{u}_\varepsilon, \underline{u}_\varepsilon] + \\ + P_{K_h^{y}}^{\varepsilon h\gamma}\big[\underline{u}_\varepsilon^\alpha(\underline{x}) - \sum_{n,p=1}^{3} e_{np}[\underline{u}(\underline{x}_\alpha)]\underline{\varphi}^{np}(\underline{x} - \underline{x}_\alpha), \underline{u}_\varepsilon^\alpha(\underline{x}) - \sum_{n,p=1}^{3} e_{np}[\underline{u}(\underline{x}_\alpha)]\underline{\varphi}^{np}(\underline{x} - \underline{x}_\alpha)\big] \geq \\ \geq \sum_{n,p,q,r=1}^{3} a_{npqr}^{0,\gamma}(\underline{x}_\alpha, \varepsilon, h) e_{np}[\underline{u}(\underline{x}_\alpha)] \cdot e_{qr}[\underline{u}(\underline{x}_\alpha)] - \\ -2\sum_{n,p=1}^{3}\sum_{q=1}^{3} b_{npq}^{\gamma}(\underline{x}_\alpha, \varepsilon, h) e_{np}[\underline{u}(\underline{x}_\alpha)]\big[\tfrac{1}{2}\mathrm{rot}\,\underline{u}(\underline{x}_\alpha)\big]_q + \\ + \sum_{q,r=1}^{3} c_{qr}^{\gamma}(\underline{x}_\alpha, \varepsilon, h)\big[\tfrac{1}{2}\mathrm{rot}\,\underline{u}(\underline{x}_\alpha)\big]_q\big[\tfrac{1}{2}\mathrm{rot}\,\underline{u}(\underline{x}_\alpha)\big]_r. \tag{5.20}$$

Estimate now the second term in the LHS of inequality (5.20). Taking into account (3.4), (5.4), (5.18) and (5.19), we have

$$\int_{K_h^{\underline{x}_\alpha}} \Big| \underline{u}_\varepsilon^\alpha(\underline{x}) - \sum_{n,p=1}^{3} e_{np}[\underline{u}(\underline{x}_\alpha)]\underline{\varphi}^{np}(\underline{x} - \underline{x}_\alpha) \Big|^2 dx = O(h^7). \tag{5.21}$$

Sum up inequality (5.20) over all cubes of our partition. Using (5.20)-(5.21) we obtain

$$\Phi_\varepsilon(\underline{u}_\varepsilon) \geq \sum_{\alpha\in\Lambda} h^3 \Big\{ \sum_{n,p,q,r=1}^{3} \frac{a_{npqr}^{0,\gamma}(\underline{x}_\alpha, \varepsilon, h)}{h^3} e_{np}[\underline{u}(\underline{x}_\alpha)] e_{qr}[\underline{u}(\underline{x}_\alpha)] - \\ -2\sum_{n,p=1}^{3}\sum_{q=1}^{3} \frac{b_{npq}^{\gamma}(\underline{x}_\alpha, \varepsilon, h)}{h^3} e_{np}[\underline{u}(\underline{x}_\alpha)]\big[\tfrac{1}{2}\mathrm{rot}\,\underline{u}(\underline{x}_\alpha)\big]_q + \\ + \sum_{q,r=1}^{3} \frac{c_{qr}^{\gamma}(\underline{x}_\alpha, \varepsilon, h)}{h^3}\big[\tfrac{1}{2}\mathrm{rot}\,\underline{u}(\underline{x}_\alpha)\big]_q\big[\tfrac{1}{2}\mathrm{rot}\,\underline{u}(\underline{x}_\alpha)\big]_r \Big\} + \\ + \lambda^2 \int_\Omega \langle \rho_\varepsilon \underline{u}_\varepsilon, \underline{u}_\varepsilon \rangle d\underline{x} - 2\int_\Omega \rho_\varepsilon \langle \lambda \underline{u}_{\varepsilon 0} + \underline{v}_{\varepsilon 0} + \underline{f}_\varepsilon, \underline{u}_\varepsilon \rangle\, d\underline{x} + O(h^{2-\gamma}) \quad (\varepsilon \ll h \ll 1). \tag{5.22}$$

Then, passing to the limit as $\varepsilon \to 0$ and $h \to 0$ in (5.22), and taking into account 3.0)-3.2), the fact that $\underline{u}(\underline{x}) \in C^2(\Omega)$ and $\gamma < 2$, we obtain

$$\underline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(\underline{u}_\varepsilon) \geq$$

$$\geq \int_\Omega \Big\{ \sum_{n,p,q,r=1}^{3} a^0_{npqr}(\underline{x}) e_{np}[\underline{u}(\underline{x})] e_{qr}[\underline{u}(\underline{x})] - 2 \sum_{n,p=1}^{3}\sum_{q=1}^{3} b_{npq}(\underline{x}) e_{np}[\underline{u}] \big[\frac{1}{2}\mathrm{rot}\,\underline{u}\big]_q +$$

$$+ \sum_{q,r=1}^{3} c_{qr}(\underline{x}) \big[\frac{1}{2}\mathrm{rot}\,\underline{u}\big]_q \big[\frac{1}{2}\mathrm{rot}\,\underline{u}\big]_r + \lambda^2 \langle \rho \underline{u}, \underline{u}\rangle - 2\rho\langle \lambda \underline{u}_0 + \underline{v}_0 + \underline{f}, \underline{u}\rangle \Big\} d\underline{x} = \Phi_0(\underline{u}).$$

Thus, the required inequality (5.17) is obtained under the assumption that the limiting vector-function $\underline{u}(\underline{x})$ is smooth. The proof for a non-smooth case $\underline{u}(\underline{x}) \in \overset{\circ}{H}{}^1(\Omega)$ is more technical, though its scheme is the same. Namely, it is necessary to construct smooth approximations $\underline{u}_\sigma(\underline{x})$ of the limiting vector-function, then to obtain for these approximations inequality, which is analogous to (5.17), and to pass to the limit as $\sigma \to 0$. The details of this construction are presented in [4].

The inequality (5.5) follows from (5.16) and (5.17). Theorem 3 is proved. □

6. **Proof of Theorem 2.** Note, that the convergence in Theorem 3 was proved for $\lambda > 0$ only. To prove the main Theorem 2, we need to apply the inverse Laplace transform to get the convergence of $\underline{u}_\varepsilon(\underline{x}, t)$ to $\underline{u}(\underline{x}, t)$. To this end, we need to extend these vector-functions analytically into the complex right half-plane and to establish their behavior as $\lambda \to \infty$.

Lemma 1. *For any vector-function $\underline{u} \in \overset{\circ}{H}{}^1(\Omega)$ the following inequality holds:*

$$\int_\Omega \big[a^D_{npqr} + a^R_{npqr}\big] \frac{\partial u_n}{\partial x_p}\frac{\partial u_q}{\partial x_r}\, d\underline{x} \geq \|\underline{u}\|^2_{H^1(\Omega)}, \tag{6.1}$$

where a^D_{npqr} and a^R_{npqr} are defined by (3.14).

Proof. For a given vector-function $\underline{u} \in \overset{\circ}{H}{}^1(\Omega)$ we construct a sequence $\underline{u}_{\varepsilon h}(\underline{x}) \in \overset{\circ}{J}_\varepsilon(\Omega)$ in accordance with (5.10). Using (5.13) and Korn's inequality (5.2), it is easy to see that

$$\lim_{h\to 0} \overline{\lim_{\varepsilon\to 0}} \|\underline{u}_{\varepsilon h}\|^2_{H^1(\Omega)} \leq \int_\Omega \big[a^D_{npqr} + a^R_{npqr}\big] \frac{\partial u_n}{\partial x_p}\frac{\partial u_q}{\partial x_r}\, d\underline{x} \leq C\|\underline{u}\|^2_{H^1(\Omega)}. \tag{6.2}$$

Taking into account (6.2) and (5.15), we conclude that the sequence $\underline{u}_{\varepsilon h}(\underline{x})$ (up to subsequence) converges weakly in $H^1(\Omega)$ to $\underline{u}(\underline{x})$. Therefore

$$\|\underline{u}\|^2_{H^1(\Omega)} \leq \underline{\lim}_{h\to 0}\, \underline{\lim}_{\varepsilon\to 0} \|\underline{u}_{\varepsilon h}\|^2_{H^1(\Omega)} \leq \int_\Omega \big[a^D_{npqr} + a^R_{npqr}\big] \frac{\partial u_n}{\partial x_p}\frac{\partial u_q}{\partial x_r}\, d\underline{x}.$$

Lemma is proved. □

Analogously to [1], [2], [3], [4] and [6], it may be shown that the family of solutions $\underline{u}_\varepsilon(\underline{x}, \lambda)$ of the problem (4.1)-(4.5) is analytic in the domain $\{\mathrm{Re}\lambda > 0\}$. Moreover, in this domain the following estimate holds:

$$\|\underline{u}_\varepsilon(\underline{x},\lambda)\|_{L_2(\Omega)} \le \frac{C}{|\lambda|}, \tag{6.3}$$

where the constant C does not depend on ε.

Thus, taking into account Theorem 3, the analyticity of $\underline{u}_\varepsilon(x,\lambda)$ in $\{\text{Re}\lambda > 0\}$ and the uniform bound (6.3), with the help of Vitali's theorem (see [14]) we conclude that $\underline{u}_\varepsilon(x,\lambda)$ converges in $L_2(\Omega)$ to some vector-function $\underline{w}(\underline{x},\lambda)$, uniformly with respect to λ in any compact subset of the domain $\{\text{Re}\lambda > 0\}$. Moreover, this vector-function is a solution of the problem (4.10)-(4.11) for $\lambda > 0$, analytic in the domain $\{\text{Re}\lambda > 0\}$, and in this domain

$$\|\underline{w}(\underline{x},\lambda)\|_{L_2(\Omega)} \le \frac{C}{|\lambda|}. \tag{6.4}$$

Show that problem (4.10)-(4.11) has a unique analytic solution for all $\text{Re}\lambda > 0$. This problem can be written in the following weak form:

$$L_\lambda[\underline{u},\underline{v}] = F_\lambda[\underline{v}], \forall \underline{v} \in \overset{\circ}{H}{}^1(\Omega),$$

where

$$L_\lambda[\underline{u},\underline{v}] = \lambda \int_\Omega \rho \underline{u}\cdot\overline{\underline{v}}\, d\underline{x} + \frac{1}{\lambda}\int_\Omega \left[a^D_{npqr} + a^R_{npqr}\right]\frac{\partial u_n}{\partial x_p}\frac{\partial \overline{v}_q}{\partial x_r}\, d\underline{x}$$

and

$$F_\lambda[\underline{v}] = \frac{1}{\lambda}\int_\Omega \left\langle \rho \underline{f} + \lambda\rho\underline{u}_0 + \rho\underline{v}_0, \overline{\underline{v}}\right\rangle d\underline{x}.$$

It is easy to see that

$$|L_\lambda[\underline{u},\underline{v}]| \le C\|\underline{u}\|_{H^1(\Omega)}\|\underline{v}\|_{H^1(\Omega)},\ F_\lambda[\underline{v}] \le C\|\underline{v}\|_{H^1(\Omega)},\ \text{Re}\lambda > 0. \tag{6.5}$$

Moreover, taking into account (6.1) and identity $a^D_{npqr} + a^R_{npqr} = a^D_{qrnp} + a^R_{qrnp}$, we obtain that

$$|L_\lambda[\underline{u},\underline{v}]| \ge C\|\underline{u}\|^2_{H^1(\Omega)},\ \text{Re}\lambda > 0. \tag{6.6}$$

Combining now (6.5)-(6.6) and using the Lax-Milgram Theorem, we conclude that there exists a unique solution $\underline{u}(\underline{x},\lambda)$ of problem (4.10)-(4.11) for any $\text{Re}\lambda > 0$. Moreover, this solution is analytic in right half-plane $\{\text{Re}\lambda > 0\}$, since the form $L_\lambda[\underline{u},\underline{v}]$ is analytic (see [10]). From this it follows that $\underline{w}(\underline{x},\lambda) = \underline{u}(\underline{x},\lambda)$ in $\{\text{Re}\lambda > 0\}$.

Due to the estimates (6.3) and (6.4), we can apply the inverse Laplace transform (see, for example, [14] and [8]) and prove, thereby, the statement of Theorem 2 (see details in [1], [2], [3], [4] and [6]). □

7. **Explicit formulas for the elastic modules for periodic array of particles.**

We now show the existence of the limits in condition 3.2) for a particular example of a periodic cubic lattice. Namely, let the particles Q^i_ε be the ellipsoids of revolution with the same semi-axes $a^i_\varepsilon = b^i_\varepsilon = a\varepsilon$ and $d^i_\varepsilon = d\varepsilon$ respectively ($a \ll d < \frac{1}{8}$). We suppose that all the particles Q^i_ε are aligned along the direction $\underline{l}$ and their centers $\underline{x}^i_\varepsilon$ form a cubic lattice of period ε.

Let K^i_ε be a cube of side length ε centered at the point $\underline{x}^i_\varepsilon$ and containing a particle Q^i_ε. Then $D^i_\varepsilon = K^i_\varepsilon \setminus Q^i_\varepsilon$ is a periodicity cell filled with the elastic medium. To obtain the standard unit cell we rescale D^i_ε by the factor ε^{-1} and shift its center to the origin. Then the domain $D = K \setminus Q$ is a unit periodicity cell where K is a cube of side length 1 centered at the origin and Q is an ellipsoid of revolution in K with the semi-axes $a = b$ and d respectively ($a \ll d < \frac{1}{8}$).

We prove the following.

Theorem 4. *For the cubic lattice described above the limits in condition 3.2) exist, the functions $a^0_{npqr}(\underline{y})$, $b_{npq}(\underline{y})$ and $c_{qr}(\underline{y})$ are constants and are given by the following formulas:*

$$a^0_{npqr} = a_{npqr} + \int\limits_K \sum_{k,l,s,t=1}^{3} a_{klst} e_{kl}\left[\underline{w}^{np}(\underline{z})\right] e_{st}\left[\underline{w}^{qr}(\underline{z})\right] d\underline{z},$$

$$b_{npq} = \int\limits_K \sum_{k,l,s,t=1}^{3} a_{klst} e_{kl}\left[\underline{w}^{np}(\underline{z})\right] e_{st}\left[\underline{u}^{q}(\underline{z})\right] d\underline{z},$$

$$c_{qr} = \int\limits_K \sum_{k,l,s,t=1}^{3} a_{klst} e_{kl}\left[\underline{u}^{q}(\underline{z})\right] e_{st}\left[\underline{u}^{r}(\underline{z})\right] d\underline{z},$$

where $\underline{w}^{np}(\underline{z})$ and $\underline{u}^q(\underline{z})$ are the solutions of the following problems, respectively:

$$\begin{cases} -\sum\limits_{k,l,s,t=1}^{3} \dfrac{\partial}{\partial z_s}\left\{a_{klst} e_{kl}[\underline{w}^{np}(\underline{z})]\right\}\underline{e}^t = 0, \quad \underline{z} \in K \setminus Q, \\ \underline{w}^{np}(\underline{z}) = -\underline{\varphi}^{np}(\underline{z}) + \theta^{np}\underline{l} \times \underline{z}, \quad \underline{z} \in Q, \\ P^{\underline{l}} \int\limits_{\partial Q} \underline{z} \times \sigma[\underline{w}^{np}]\nu \, d\underline{z} = \underline{0}, \\ \underline{w}^{np}\Big|_{F_i^+} = \underline{w}^{np}\Big|_{F_i^-}, \quad \sigma[\underline{w}^{np}]\Big|_{F_i^+} = \sigma[\underline{w}^{np}]\Big|_{F_i^-}, \end{cases} \tag{7.1}$$

and

$$\begin{cases} -\sum\limits_{k,l,s,t=1}^{3} \dfrac{\partial}{\partial z_s}\left\{a_{klst} e_{kl}[\underline{u}^{q}(\underline{z})]\right\}\underline{e}^t = \underline{0}, \quad \underline{z} \in K \setminus Q, \\ \underline{u}^{q}(\underline{z}) = [\theta^q\underline{l} + (1 - P^{\underline{l}})\underline{e}^q] \times \underline{z}, \quad \underline{z} \in Q, \\ P^{\underline{l}} \int\limits_{\partial Q} \underline{z} \times \sigma[\underline{u}^{q}]\nu \, d\underline{z} = \underline{0}, \\ \underline{u}^{q}\Big|_{F_i^+} = \underline{u}^{q}\Big|_{F_i^-}, \quad \sigma[\underline{u}^{q}]\Big|_{F_i^+} = \sigma[\underline{u}^{q}]\Big|_{F_i^-}. \end{cases} \tag{7.2}$$

Here $P^{\underline{l}}$ is a projection operator onto $\underline{l}$, F_i^+ and F_i^- are opposite faces of the cube K $(i = \overline{1,3})$.

Proof. Let $K^{\underline{y}}_h$ be a cube of side length h $(h \gg \varepsilon)$ centered at the point $\underline{y} \in \Omega$. We seek a function $\underline{u}^{np}_\varepsilon(\underline{x})$ minimizing functional (3.2) in $J^0_\varepsilon[K^{\underline{y}}_h]$ as $T = T^{np} = \frac{1}{2}(\underline{e}^n \otimes \underline{e}^p + \underline{e}^p \otimes \underline{e}^n)$ in the form

$$\underline{u}^{np}_\varepsilon(\underline{x}) = \underline{U}^{np}_\varepsilon(\underline{x}) + \underline{v}^{np}_\varepsilon(\underline{x}), \tag{7.3}$$

where

$$\underline{U}_\varepsilon^{np}(\underline{x}) = \underline{\varphi}^{np}(\underline{x} - \underline{y}_\varepsilon) + \varepsilon \underline{\tilde{w}}_\varepsilon^{np}\big(\frac{\underline{x} - \underline{y}_\varepsilon}{\varepsilon}\big). \tag{7.4}$$

Here $\underline{\tilde{w}}^{np}(\underline{x})$ is a periodic extension of the function $\underline{w}^{np}(\underline{x})$ and $\underline{y}_\varepsilon = \underline{x}_\varepsilon^i$ is the nearest to $\underline{y}$ center of particles Q_ε^i (for the sake of simplicity we assume that $\underline{y}_\varepsilon = \underline{y}$). Using the properties of the functions $\underline{\varphi}^{np}(\underline{x})$ and $\underline{w}_\varepsilon^{np}(\underline{x})$, we have

$$\underline{U}_\varepsilon^{np}(\underline{x}) = \underline{\varphi}^{np}(\underline{x}_\varepsilon^j - \underline{y}_\varepsilon) + \theta^{np}\underline{l} \times (\underline{x} - \underline{x}_\varepsilon^j),\ \underline{x} \in Q_\varepsilon^j. \tag{7.5}$$

Analogously, we seek a vector-function $\underline{w}_\varepsilon^q(\underline{x})$ minimizing functional (3.2) in $J_\varepsilon^{e^q}[K_h^y]$ as $T = 0$ in the form

$$\underline{w}_\varepsilon^q(\underline{x}) = \underline{W}_\varepsilon^q(\underline{x}) + \underline{h}_\varepsilon^q(\underline{x}), \tag{7.6}$$

where $\underline{W}_\varepsilon^q(\underline{x}) = \varepsilon \underline{\tilde{u}}^q\big(\frac{\underline{x} - \underline{y}_\varepsilon}{\varepsilon}\big)$, and $\underline{\tilde{u}}^q(\underline{x})$ is a periodic extension of the function $\underline{u}^q(\underline{x})$.

Next we obtain variational problems for the correctors $\underline{v}_\varepsilon^{np}(\underline{x})$ and $\underline{h}_\varepsilon^q(\underline{x})$. Analysis of those problems and substitution of (7.3)-(7.6) into (3.7)-(3.9), together with a periodicity of the structure, give

$$\frac{1}{h^3} a_{npqr}^{0,\gamma}(\underline{y}, \varepsilon, h) = \frac{1}{h^3} E_{K_h^y}[\underline{U}_\varepsilon^{np}, \underline{U}_\varepsilon^{qr}] + \bar{o}(1)\ (\varepsilon \ll h \ll 1),$$

$$\frac{1}{h^3} b_{npq}^{\gamma}(\underline{y}, \varepsilon, h) = \frac{1}{h^3} E_{K_h^y}[\underline{U}_\varepsilon^{np}, \underline{W}_\varepsilon^q] + \bar{o}(1)\ (\varepsilon \ll h \ll 1),$$

$$\frac{1}{h^3} c_{qr}^{\gamma}(\underline{y}, \varepsilon, h) = \frac{1}{h^3} E_{K_h^y}[\underline{W}_\varepsilon^q, \underline{W}_\varepsilon^r] + \bar{o}(1)\ (\varepsilon \ll h \ll 1).$$

The statement of Theorem 4 follows from the above representation. □

8. Asymptotic formulas for the elastic modules for small particles.

Note, that even in periodic case the obtained formulas for coefficients defined in *condition 3.2)* do not clarify their dependence on the particles orientation. Meanwhile, if the volume fraction of the particles are rather small then $a_{npqr}^0(\underline{x}, \underline{l}) = a_{npqr}$, $b_{npq}(\underline{x}, \underline{l}) = 0$, $c_{qr}(\underline{x}, \underline{l}) = 0$. Therefore, it would be interesting to obtain asymptotic expressions of smaller order for those coefficients which ascertain their dependence on the particles orientation.

To do this, we consider ellipsoidal particles oriented along vector $\underline{l}$ and suppose that their diameters are of order ε^α ($\alpha > \frac{5}{3}$). We prove the following theorem.

Theorem 5. *For small volume fraction of the particles the following asymptotic formulas hold:*

$$\tilde{a}_{npqr}(\underline{y}, \underline{l}, \varepsilon, h) \overset{def}{=} \frac{a_{npqr}^{0,\gamma}(\underline{y}, \underline{l}, \varepsilon, h)}{h^3} = a_{npqr} +$$
$$+\tau_\varepsilon \Big\{ \big[A_1(\delta_{nq}\delta_{pr} + \delta_{nr}\delta_{pq}) + A_2\delta_{np}\delta_{qr}\big] + B\big[l_n l_q \delta_{pr} + l_p l_q \delta_{nr} + l_n l_r \delta_{pq} + l_p l_r \delta_{nq}\big] +$$
$$+ C\big[l_n l_p \delta_{qr} + l_q l_r \delta_{np}\big] + D l_n l_p l_q l_r \Big\} + \bar{o}(\tau_\varepsilon),$$

$$\tilde{b}_{npq}(\underline{y}, \underline{l}, \varepsilon, h) \overset{def}{=} \frac{b_{npq}^{\gamma}(\underline{y}, \underline{l}, \varepsilon, h)}{h^3} = \tau_\varepsilon b \sum_{k=1}^{3} (l_n l_k \epsilon_{kpq} + l_p l_k \epsilon_{knq}) + \bar{o}(\tau_\varepsilon),$$

$$\tilde{c}_{qr}(\underline{y}, \underline{l}, \varepsilon, h) \stackrel{def}{=} \frac{c^{\gamma}_{qr}(\underline{y}, \underline{l}, \varepsilon, h)}{h^3} = \tau_\varepsilon c(\delta_{np} - l_n l_p) + \overline{o}(\tau_\varepsilon),$$

where $\tau_\varepsilon = \sum_{i \; K_h^y} \frac{|Q^i_\varepsilon|}{h^3}$ *is a volume fraction of the particles located in* K_h^y *(which is supposed to be of order* ε^β, $\beta > 2$*), and constants* A_1, A_2, B, C, D, b *and* c *depend on the form of solid* Q *(of unit diameter) oriented along axis* $\underline{e}^1$.

Sketch of the proof. For a single fixed ellipsoidal solid Q oriented along axis $\underline{e}^1$ we consider the following boundary-value problems:

$$\begin{cases} -\sum\limits_{k,l,s,t=1}^{3} \dfrac{\partial}{\partial z_s}\big\{a_{klst} e_{kl}[\underline{w}^{np}(\underline{z})]\big\}\underline{e}^t = 0, \quad \underline{z} \in \mathbb{R}^3 \setminus Q, \\ \underline{w}^{np}(\underline{z}) = \underline{\varphi}^{np}(\underline{z}) + \underline{a}^{np} + \theta^{np}\underline{e}^1 \times \underline{z}, \quad \underline{z} \in \partial Q, \\ \int\limits_{\partial Q} \sigma[\underline{w}^{np}]\nu \, d\underline{z} = \Big\langle \int\limits_{\partial Q} \underline{z} \times \sigma[\underline{w}^{np}]\nu \, d\underline{z}, \underline{e}^1 \Big\rangle = \underline{0}, \\ \underline{w}^{np} = O\Big(\dfrac{1}{|\underline{x}|^2}\Big), \; |\underline{x}| \to \infty, \end{cases} \tag{8.1}$$

and

$$\begin{cases} -\sum\limits_{k,l,s,t=1}^{3} \dfrac{\partial}{\partial z_s}\big\{a_{klst} e_{kl}[\underline{u}^{q}(\underline{z})]\big\}\underline{e}^t = 0, \quad \underline{z} \in \mathbb{R}^3 \setminus Q, \\ \underline{u}^{q}(\underline{z}) = \underline{a}^{q} + [\underline{e}^q(1-\epsilon_{q23}) + \theta^{q}\underline{e}^1] \times \underline{z}, \quad \underline{z} \in \partial Q, \\ \int\limits_{\partial Q} \sigma[\underline{u}^{q}]\nu \, d\underline{z} = \Big\langle \int\limits_{\partial Q} \underline{z} \times \sigma[\underline{u}^{q}]\nu \, d\underline{z}, \underline{e}^1 \Big\rangle = \underline{0}, \\ \underline{u}^{q} = O\Big(\dfrac{1}{|\underline{x}|^2}\Big), \; |\underline{x}| \to \infty. \end{cases} \tag{8.2}$$

There exist a unique solution $(\underline{w}^{np}, \underline{a}^{np}, \theta^{np})$ of problem (8.1) and a unique solution $(\underline{u}^{q}, \underline{a}^{q}, \theta^{q})$ of problem (8.2). Moreover, from the symmetry of problems (8.1)-(8.2) it follows that $\underline{a}^{np} = \underline{0}$, $\theta^{np} = 0$, $\underline{a}^{q} = \underline{0}$, $\theta^{q} = 0$ for all $n, p, q = \overline{1,3}$, and $\underline{u}^1 \equiv \underline{0}$.

Introduce the following fourth-, third- and second-rank tensors corresponding to Q:

$$a_{npqr}(Q) = \int\limits_{\mathbb{R}^3 \setminus Q} \sum_{i,k=1}^{3} e_{ik}[\underline{w}^{np}, \underline{w}^{qr}] \, d\underline{x},$$

$$b_{npq}(Q) = \int\limits_{\mathbb{R}^3 \setminus Q} \sum_{i,k=1}^{3} e_{ik}[\underline{w}^{np}, \underline{u}^{q}] \, d\underline{x}, \quad c_{np}(Q) = \int\limits_{\mathbb{R}^3 \setminus Q} \sum_{i,k=1}^{3} e_{ik}[\underline{u}^{n}, \underline{u}^{p}] \, d\underline{x}. \tag{8.3}$$

It is clear that $a_{npqr}(Q) = a_{pnqr}(Q) = a_{qrnp}(Q)$, $b_{npq}(Q) = b_{pnq}(Q)$ and $c_{np}(Q) = c_{pn}(Q)$. Moreover, taking into account the symmetry of problems (8.1)-(8.2) and orientation of Q, it can be proved that

$$\begin{gathered} a_{2222}(Q) = a_{3333}(Q), \; a_{1122}(Q) = a_{1133}(Q), \\ a_{1212}(Q) = a_{1313}(Q), \; a_{2222}(Q) = a_{2233}(Q) + 2a_{2323}(Q), \\ b_{123}(Q) = -b_{132}(Q), \; b_{npq}(Q) = 0 \quad \text{in all other cases}, \\ c_{22}(Q) = c_{33}(Q), \; c_{np}(Q) = 0 \quad \text{in all other cases}. \end{gathered} \tag{8.4}$$

Let Q^i_ε be a particle oriented along $\underline{l}$ with diameter d^i_ε. Consider the following boundary-value problems:

$$\begin{cases} -\sum\limits_{k,l,s,t=1}^{3} \dfrac{\partial}{\partial z_s}\big\{a_{klst}e_{kl}[\underline{w}_\varepsilon^{i,np}(\underline{z})]\big\}\underline{e}^t = 0, \quad \underline{z} \in \mathbb{R}^3 \setminus Q_\varepsilon^i, \\ \underline{w}_\varepsilon^{i,np}(\underline{z}) = \underline{\varphi}^{np}(\underline{z}) + \underline{a}_\varepsilon^{i,np} + \theta_\varepsilon^{i,np}\underline{l} \times \underline{z}, \quad \underline{z} \in \partial Q_\varepsilon^i, \\ \int\limits_{\partial Q_\varepsilon^i} \sigma[\underline{w}_\varepsilon^{i,np}]\nu\, d\underline{z} = P^{\underline{l}} \int\limits_{\partial Q_\varepsilon^i} \underline{z} \times \sigma[\underline{w}_\varepsilon^{i,np}]\nu\, d\underline{z} = \underline{0}, \\ \underline{w}_\varepsilon^{i,np} = O\Big(\dfrac{1}{|\underline{x}|^2}\Big),\ |\underline{x}| \to \infty, \end{cases} \tag{8.5}$$

and

$$\begin{cases} -\sum\limits_{k,l,s,t=1}^{3} \dfrac{\partial}{\partial z_s}\big\{a_{klst}e_{kl}[\underline{u}_\varepsilon^{i,q}(\underline{z})]\big\}\underline{e}^t = 0, \quad \underline{z} \in \mathbb{R}^3 \setminus Q_\varepsilon^i, \\ \underline{u}_\varepsilon^{i,q}(\underline{z}) = \underline{a}_\varepsilon^{i,q} + [\theta_\varepsilon^{i,q}\underline{l} + (1 - P^{\underline{l}})\underline{e}^q] \times \underline{z}, \quad \underline{z} \in \partial Q_\varepsilon^i, \\ \int\limits_{\partial Q_\varepsilon^i} \sigma[\underline{u}_\varepsilon^{i,q}]\nu\, d\underline{z} = P^{\underline{l}} \int\limits_{\partial Q_\varepsilon^i} \underline{z} \times \sigma[\underline{u}_\varepsilon^{i,q}]\nu\, d\underline{z} = \underline{0}, \\ \underline{u}_\varepsilon^{i,q} = O\Big(\dfrac{1}{|\underline{x}|^2}\Big),\ |\underline{x}| \to \infty. \end{cases} \tag{8.6}$$

Introduce the following fourth-, third- and second-rank tensors corresponding to Q_ε^i:

$$a_{npqr}(Q_\varepsilon^i) = \int\limits_{\mathbb{R}^3 \setminus Q_\varepsilon^i} \sum_{i,k=1}^{3} e_{ik}[\underline{w}_\varepsilon^{i,np}, \underline{w}_\varepsilon^{i,qr}]\, d\underline{x},$$

$$b_{npq}(Q_\varepsilon^i) = \int\limits_{\mathbb{R}^3 \setminus Q_\varepsilon^i} \sum_{i,k=1}^{3} e_{ik}[\underline{w}_\varepsilon^{i,np}, \underline{u}_\varepsilon^{i,q}]\, d\underline{x}, \quad c_{np}(Q_\varepsilon^i) = \int\limits_{\mathbb{R}^3 \setminus Q_\varepsilon^i} \sum_{i,k=1}^{3} e_{ik}[\underline{u}_\varepsilon^{i,n}, \underline{u}_\varepsilon^{i,p}]\, d\underline{x}. \tag{8.7}$$

It is clear, that up to rescaling and rotations problems (8.5) and (8.6) coincide with problems (8.1) and (8.2), respectively. Using this fact and (8.4), it can be proved that

$$a_{npqr}(Q_\varepsilon^i) = (d_\varepsilon^i)^3 \Big\{ \big[\acute{A}_1(\delta_{nq}\delta_{pr} + \delta_{nr}\delta_{pq}) + \acute{A}_2\delta_{np}\delta_{qr}\big] +$$
$$+ \acute{B}\big[l_n l_q \delta_{pr} + l_p l_q \delta_{nr} + l_n l_r \delta_{pq} + l_p l_r \delta_{nq}\big] + \acute{C}\big[l_n l_p \delta_{qr} + l_q l_r \delta_{np}\big] + \acute{D} l_n l_p l_q l_r \Big\},$$

$$b_{npq}(Q_\varepsilon^i) = (d_\varepsilon^i)^3 \acute{b} \sum_{k=1}^{3} (l_n l_k \epsilon_{kpq} + l_p l_k \epsilon_{knq}), \quad c_{qr}(Q_\varepsilon^i) = (d_\varepsilon^i)^3 \acute{c}(\delta_{np} - l_n l_p),$$

where $\acute{A}_1 = a_{2323}(Q)$, $\acute{A}_2 = a_{2233}(Q)$, $\acute{B} = a_{1212}(Q) - a_{2323}(Q)$, $\acute{C} = a_{1122}(Q) - a_{2233}(Q)$, $\acute{D} = a_{1111}(Q) + a_{2222}(Q) - 2a_{1122}(Q) - 4a_{1212}(Q)$, $\acute{b} = b_{123}(Q)$ and $\acute{c} = c_{22}(Q)$.

Let $K_h^{\underline{y}}$ be a cube of side length h ($h \gg \varepsilon$) centered at the point $\underline{y} \in \Omega$. Suppose that the diameters d_ε^i of the particles Q_ε^i are of order $O(\varepsilon^\alpha)$, $\alpha > \dfrac{5}{3}$. We seek a vector-function $\underline{v}_{\varepsilon h}^{np}(\underline{x})$ minimizing functional (3.2) in $J_\varepsilon^{\underline{0}}[K_h^{\underline{y}}]$ as $T = T^{np} = \dfrac{1}{2}(\underline{e}^n \otimes \underline{e}^p + \underline{e}^p \otimes \underline{e}^n)$ in the form

$$\underline{v}_{\varepsilon h}^{np}(\underline{x}) = \underline{V}_\varepsilon^{np}(\underline{x}) + \underline{w}_{\varepsilon h}^{np}(\underline{x}), \tag{8.8}$$

where

$$\underline{V}_\varepsilon^{np}(\underline{x}) = \underline{\varphi}^{np}(\underline{x}-\underline{y}) - {\sum_i}'_{K_h^y} \underline{w}_\varepsilon^{i,np}(\underline{x})\phi_\varepsilon^i(\underline{x}). \tag{8.9}$$

Here $\underline{w}_\varepsilon^{i,np}(\underline{x})$ is a solution of problem (8.5), ${\sum_i}'_{K_h^y}$ stands for summation over all particles Q_ε^i located in $K_{h'}^y$ ($h' = h - 2h^{1+\frac{\gamma}{2}}$, $\gamma > 0$) and

$$\phi_\varepsilon(\underline{x}) = \phi\Big(\frac{|\underline{x}-\underline{x}_\varepsilon^i|}{R_\varepsilon^i}\Big),\ \phi(t) = \begin{cases} 1,\ t \le \frac{1}{4}, \\ 0,\ t > \frac{1}{2} \end{cases} \in C^2(\mathbb{R}_+).$$

Using the properties of the functions $\underline{\varphi}^{np}(\underline{x})$, $\underline{w}_\varepsilon^{i,np}(\underline{x})$ and $\phi(t)$, we conclude that $\underline{V}_\varepsilon^{np}(\underline{x}) \in J_\varepsilon^{\underline{0}}[K_h^y]$.

Analogously, we seek a vector-function $\underline{v}_\varepsilon^q(\underline{x})$ minimizing functional (3.2) in $J_\varepsilon^{\underline{e}^q}[K_h^y]$ as $T = 0$ in the form

$$\underline{v}_\varepsilon^q(\underline{x}) = \underline{V}_\varepsilon^q(\underline{x}) + \underline{h}_\varepsilon^q(\underline{x}), \tag{8.10}$$

where

$$\underline{V}_\varepsilon^q(\underline{x}) = {\sum_i}'_{K_h^y} \underline{u}_\varepsilon^{i,q}(\underline{x})\phi_\varepsilon^i(\underline{x}). \tag{8.11}$$

Here $\underline{u}_\varepsilon^{i,q}(\underline{x})$ is a solution of problem (8.6). Using the properties of the functions $\underline{u}_\varepsilon^{i,q}(\underline{x})$ and $\phi(t)$, we conclude that $\underline{V}_\varepsilon^q(\underline{x}) \in J_\varepsilon^{\underline{e}^q}[K_h^y]$.

Next we obtain variational problems for the correctors $\underline{v}_\varepsilon^{np}(\underline{x})$ and $\underline{h}_\varepsilon^q(\underline{x})$. Analysis of those problems and substitution of (8.8)-(8.11) into (3.7)-(3.9) give

$$\tilde{a}_{npqr}(\underline{y},\underline{l},\varepsilon,h) = \frac{1}{h^3}E_{K_h^y}[\underline{V}_\varepsilon^{np},\underline{V}_\varepsilon^{qr}] + \overline{o}(\tau_\varepsilon) \quad (\varepsilon \ll h \ll 1),$$

$$\tilde{b}_{npq}(\underline{y},\underline{l},\varepsilon,h) = \frac{1}{h^3}E_{K_h^y}[\underline{V}_\varepsilon^{np},\underline{V}_\varepsilon^{q}] + \overline{o}(\tau_\varepsilon) \quad (\varepsilon \ll h \ll 1),$$

$$\tilde{c}_{np}(\underline{y},\underline{l},\varepsilon,h) = \frac{1}{h^3}E_{K_h^y}[\underline{V}_\varepsilon^{n},\underline{V}_\varepsilon^{p}] + \overline{o}(\tau_\varepsilon) \quad (\varepsilon \ll h \ll 1).$$

The statement of Theorem 5 with

$$A_1 = \frac{\acute{A}_1}{|Q|},\ A_2 = \frac{\acute{A}_2}{|Q|},\ B = \frac{\acute{B}}{|Q|},\ C = \frac{\acute{C}}{|Q|},\ D = \frac{\acute{D}}{|Q|},\ b = \frac{\acute{b}}{|Q|},\ c = \frac{\acute{c}}{|Q|}$$

follows from the above representation. □

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