NETWORKS AND HETEROGENEOUS MEDIA
doi:10.3934/nhm.2011.6.61

Volume **6**, Number **1**, March **2011**
pp. **61–75**

ON THE CONVERGENCE RATE IN MULTISCALE HOMOGENIZATION OF FULLY NONLINEAR ELLIPTIC PROBLEMS

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(Communicated by Kenneth Karlsen)

Abstract. This paper concerns periodic multiscale homogenization for fully nonlinear equations of the form $u^\epsilon + H^\epsilon\left(x, \frac{x}{\epsilon}, \dots, \frac{x}{\epsilon^k}, Du^\epsilon, D^2u^\epsilon\right) = 0$. The operators H^ϵ are a regular perturbations of some uniformly elliptic, convex operator H. As $\epsilon \to 0^+$, the solutions u^ϵ converge locally uniformly to the solution u of a suitably defined effective problem. The purpose of this paper is to obtain an estimate of the corresponding rate of convergence. Finally, some examples are discussed.

1. **Introduction.** For $\epsilon > 0$ we consider the multiscale homogenization problem for equations of the form

$$u^\epsilon + H^\epsilon\left(x, \frac{x}{\epsilon}, \dots, \frac{x}{\epsilon^k}, Du^\epsilon, D^2u^\epsilon\right) = 0, \qquad x \in \mathbb{R}^n. \tag{1}$$

The operators H^ϵ are periodic, uniformly elliptic, regular perturbations of some convex operator H (namely, $H^\epsilon \to H$ locally uniformly as $\epsilon \to 0^+$; for the precise assumptions, see Section 2 below). It is well known that, as $\epsilon \to 0^+$, the solution u^ϵ of (1) converges locally uniformly to the solution of the *effective* problem (see Alvarez, Bardi and the second author [4])

$$u + \overline{H}\left(x, Du, D^2u\right) = 0, \qquad x \in \mathbb{R}^n \tag{2}$$

where the *effective Hamiltonian* $\overline{H}$ is defined via iterative homogenization. The purpose of this paper is to investigate the corresponding rate of convergence.

In the framework of viscosity solution theory (see the monographs by Braides and Defranceschi [8], Bensoussan, J.L.Lions and Papanicolaou [9], Jikov, Kozlov and Oleinik [19] for homogenization in the variational setting), the study of homogenization started with the seminal paper by P.L. Lions, Papanicolaou and Varadhan [22] concerning first order periodic Hamilton-Jacobi equations. A crucial advance was

2000 *Mathematics Subject Classification.* Primary: 35B27; Secondary: 35J60, 49L25.
Key words and phrases. Nonlinear elliptic equations, Bellman equations, multiscale homogenization, rate of convergence.

made by Evans [14, 15] with the introduction of the *perturbed test function method*. By means of this very adaptable technique he proved that the solutions u^ϵ of problem (1) with two scales, i.e. $k = 1$, converge locally uniformly to the solution u of (2) where the effective Hamiltonian $\overline{H}$ is defined by the following *cell problem*: for every $(\overline{x}, \overline{p}, \overline{X}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}^n$ find the unique value $\overline{H}(\overline{x}, \overline{p}, \overline{X})$ such that there exists a periodic solution $w = w(y)$ (the so-called *corrector*) of

$$H(\overline{x}, y, \overline{p}, \overline{X} + D^2_{yy} w) = \overline{H}(\overline{x}, \overline{p}, \overline{X}), \qquad y \in \mathbb{R}^n.$$

The effective Hamiltonian $\overline{H}$ can be also defined via the ergodic approximation: $\overline{H}(\overline{x}, \overline{p}, \overline{X})$ is the uniform limit of $-\lambda w_\lambda$ as $\lambda \to 0^+$, where the function $w_\lambda = w_\lambda(y; \overline{x}, \overline{p}, \overline{X})$ solves the approximated cell problem

$$\lambda w_\lambda + H(\overline{x}, y, \overline{p}, \overline{X} + D^2_{yy} w_\lambda) = 0, \qquad y \in \mathbb{R}^n. \tag{3}$$

The latter definition is more general than the former (see: [3, 6, 10] and references therein). The homogenization theory for fully nonlinear equation has been extended in several directions (see Alvarez and Bardi [2] for a general review) and also beyond the periodic setting (see [23, 24, 10]).

The multiscale homogenization problem for fully nonlinear equations was recently studied in [4, 5], respectively for second and first order equations. For problem (1), it was ascertained that u^ϵ converges locally uniformly to the solution u of the equation (2) with an effective operator $\overline{H}$ defined by an iterative homogenization process (see Section 2 for the detailed calculations).

An interesting problem connected with the homogenization theory is the estimation in terms of the scale parameter ϵ of the rate of convergence of the solutions of the perturbed problem to the solution of the homogenized one. This question has been tackled up for the first time by Capuzzo Dolcetta and Ishii [12] for first order equations. For $k = 1$, they proved that u^ϵ converges uniformly to u with a rate of order 1/3, namely $\|u^\epsilon - u\|_\infty \le C\epsilon^{1/3}$. In [25], the same rate of convergence has been obtained for the corresponding multiscale homogenization problem.

Concerning rates of convergence for second order problems, the two authors [11] considered the case of convex uniformly elliptic equations. For $k = 1$, $H^\epsilon \equiv H$ and H of the form

$$H(x, y, p, X) := \max_{\theta \in \Theta} \{-\mathrm{tr}\,(a(x, y, \theta)X) - f(x, y, \theta) \cdot p - l(x, y, \theta)\}$$

they proved that the solution u^ϵ to (1) converges uniformly to u and that there exists a positive α such that $\|u^\epsilon - u\|_\infty \le C\epsilon^\alpha$, with α depending on the regularity of u^ϵ and u.

The purpose of this paper is to obtain an estimate of the rate of convergence for the multiscale homogenization of fully nonlinear uniformly elliptic equations. In other words, we want to estimate $\|u^\epsilon - u\|_\infty$ where u^ϵ and u are respectively the solution to problems (1) and (2). As an important byproduct, we shall obtain that, in several cases, u^ϵ converges to u uniformly on the whole $\mathbb{R}^n$.

In this respect this paper extends the results of our previous one [11] in two directions: for $k = 1$ we consider Hamiltonian H^ϵ which in general are nonconvex (but they converge locally uniformly to a convex operator H) and, mainly, we address the multiscale homogenization problem.

Let us stress some features of our arguments. Following the approach in [12] we shall use the doubling of variables technique between the starting functions u^ϵ and the effective one u perturbed with an approximated corrector λw_λ. This latter term has the crucial role of linking the Hamiltonians H^ϵ with the effective Hamiltonian $\overline{H}$ (note that in general there is no estimate of the term $H^\epsilon - \overline{H}$). In order to deal with the dependance of w_λ on the slow variables, we shall invoke the regularity theory for convex uniformly elliptic equations (see the book by Gilbarg and Trudinger [18] and also Safonov [26]). The exponent α in the rate of convergence ϵ^α we obtain depends on the regularity of u^ϵ and u.

This paper is organized as follows: Section 2 is devoted to the homogenization framework (in particular, the definition of $\overline{H}$) and to state our main result. Since it is used in the proof of the main result, the case with discount a and $k = 1$ is studied in Section 3. Section 4 is devoted to the proof of the main result. In Section 5 we illustrate the problem with some examples.

2. **Mathematical framework and main result.**

We shall denote by $\mathbb{S}^n$ the space of symmetric $n \times n$ real matrices endowed with the usual norm. For any continuous function f, $J^+_x f$ and $J^-_x f$ stand respectively for the super and the subdifferential of f at the point x (we refer the reader to [13] for the precise definition and main properties).

We shall assume that the Hamiltonians H, $H^\epsilon : \mathbb{R}^n \times \mathbb{R}^{nk} \times \mathbb{R}^n \times \mathbb{S}^n \to \mathbb{R}$ fulfill the following hypotheses:

(A_1) H is convex in X.

(A_2) H^ϵ is periodic in $y_1, \dots, y_k$ and

$$|H^\epsilon(x, y_1, \dots, y_k, 0, 0)| \le C,$$
$$|H^\epsilon(x, y_1, \dots, y_k, p, X) - H^\epsilon(x, y_1, \dots, y_k, q, X)| \le C|p - q|,$$
$$|H^\epsilon(x_1, y_1, \dots, y_k, p, X) - H^\epsilon(x_2, z_1, \dots, z_k, p, X)| \le$$
$$\le C(1 + |p| + \|X\|)(|x_1 - x_2| + \sum_{i=1}^{k} |y_i - z_i|).$$

Moreover H^ϵ is uniformly elliptic: there exists a positive constant ν such that, for $X \ge Y$, it verifies

$$\nu^{-1}\|X - Y\| \le H^\epsilon(x, y_1, \dots, y_k, p, X) - H^\epsilon(x, y_1, \dots, y_k, p, Y) \le \nu\|X - Y\|.$$

(A_3) There exists a continuous function $\omega = \omega(\epsilon, x)$ such that, for every $x, y_i, p \in \mathbb{R}^n$ and $X \in \mathbb{S}^n$, there holds

$$|H^\epsilon (x, y_1, \dots, y_k, p, X) - H (x, y_1, \dots, y_k, p, X)| \le \omega(\epsilon, x) (1 + |p| + \|X\|) .$$

For the sake of simplicity, we shall consider in (A_3) only functions ω having the form

$$\omega(\epsilon, x) = \omega_1(\epsilon) + \omega_2(\epsilon)|x|^2 \tag{4}$$

where ω_i are modulus of continuity. Actually, one can easily adapt our arguments to the case of ω with different behavior as $|x| \to +\infty$ just modifying the penalization term in the proof of Theorem 2.1.

We observe that, with assumptions (A_1)–(A_3), (1) admits a unique viscosity solution (see [14, 15]).

The *effective Hamiltonian* $\overline{H}$ (see [4]) is defined via iterative homogenization as follows:

Set $H_0 = H$ and, for $i = 0, \dots, k-1$, fix $\overline{x}, \overline{y}_1, \dots, \overline{y}_{k-i-1}, \overline{p} \in \mathbb{R}^n$ and $\bar{X} \in \mathbb{S}^n$; for $\lambda > 0$ the problem

$$\lambda v + H_i(\overline{x}, \overline{y}_1, \dots, \overline{y}_{k-i-1}, z, \overline{p}, \overline{X} + D^2_{zz} v) = 0 \qquad \text{in } \mathbb{R}^n, \quad v \text{ periodic}$$

admits exactly one solution $v = v(z)$. As $\lambda \to 0^+$, it turns out that $\lambda v(z)$ converges uniformly to a constant that we denote by $-H_{i+1}(\overline{x}, \overline{y}_1, \dots, \overline{y}_{k-i-1}, \overline{p}, \overline{X})$. Finally, we define $\overline{H} := H_k$.

Let us state our main result

Theorem 2.1. *Under Assumptions* (A_1)-(A_3)*, there exist a positive constant* M *and* $\alpha \in (0,1)$ *such that*

$$|u^\epsilon(x) - u(x)| \le M\left[\epsilon^\alpha + \omega_1(\epsilon) + \omega_2(\epsilon)\left(1 + |x|^2\right)\right] \qquad \forall \epsilon \in (0,1),\, x \in \mathbb{R}^n. \tag{5}$$

The proof is deferred to Section 4.

Corollary 2.1. *Under Assumptions* (A_1)-(A_3) *with* $\omega_2 \equiv 0$ *in* (4)*, the function* u^ϵ *converges to* u *uniformly on the whole* $\mathbb{R}^n$ *with the rate*

$$\|u^\epsilon - u\|_\infty \le M[\epsilon^\alpha + \omega_1(\epsilon)].$$

3. **Two scale case with discount** a**.** This section is devoted to the case of two scales with a *discount* $a \in (0,1)$, namely to equations of the form

$$au^\epsilon + H^\epsilon\left(x, \frac{x}{\epsilon}, Du^\epsilon, D^2u^\epsilon\right) = 0, \qquad x \in \mathbb{R}^n. \tag{6}$$

A similar problem has been studied in [11] in the case $a = 1$. We will follow the argument used there, but we will pay a particular attention to the constants involved in the equation, especially to the influence of the parameter a on the rate of convergence. In the following section this estimate will be an essential step in the proof of Theorem 2.1.

It is well-known (see: [4] and also [2, 1, 14, 15] for the case $H^\epsilon \equiv H$) that, as $\epsilon \to 0^+$, the solution u^ϵ converges locally uniformly to u, solution to the effective equation

$$au + \overline{H}\left(x, Du, D^2u\right) = 0 \qquad x \in \mathbb{R}^n. \tag{7}$$

The effective $\overline{H}$ is defined as follows: for every positive λ, the *cell problem*

$$\lambda w^\lambda + H(\overline{x}, y, \overline{p}, \overline{X} + D^2_{yy}w^\lambda) = 0 \qquad y \in \mathbb{R}^n \tag{8}$$

admits exactly one periodic solution $w^\lambda = w^\lambda(y; \overline{x}, \overline{p}, \overline{X})$. As $\lambda \to 0^+$, the function λw^λ converges to a constant that we denote by $-\overline{H}(\overline{x}, \overline{p}, \overline{X})$. Let us now state the main result of this section.

Theorem 3.1. *Assume hypotheses* (A_1)-(A_3)*. Assume further*

(A_4) $H^\epsilon = H^\epsilon(x, y, p, X)$ *is periodic in* x *and* $\omega_2 \equiv 0$ *in* (4).

Then there exist two positive constants M *and* $\alpha \in (0,1)$ *(both independent of* a*) such that*

$$\sup_{x \in \mathbb{R}^n} |u^\epsilon(x) - u(x)| \le \frac{M}{a}[\epsilon^\alpha + \omega(\epsilon)] \qquad \forall \epsilon \in (0,1).$$

The proof is postponed at the end of this section. In the next two lemmas, we recall some properties of the approximated corrector and respectively of the effective Hamiltonian. We refer the reader to the papers [1, 2, 6, 14, 15] for the detailed proof.

Lemma 3.2. *Let* $w^\lambda = w^\lambda(y; \overline{x}, \overline{p}, \overline{X})$ *be the solution of* (8)*. There exists* $C_1 > 0$ *such that*

a) $\|\lambda w^\lambda(\cdot; x, p, X)\|_\infty \le C_1(1 + |p| + \|X\|)$, $\forall x, p, X$;

b) $\lambda|D_X w^\lambda|$, $\lambda|D_p w^\lambda| \le C_1$, $\lambda|D_x w^\lambda(y;x,p,X)| \le C_1(1+|p|+\|X\|)$ *(in viscosity sense);*

c) for some $\alpha \in (0,1)$, $\|w^\lambda(\cdot;x,p,X) - w^\lambda(0;x,p,X)\|_{C^{2,\alpha}(\mathbb{R}^N)} \le C_1(1+|p|+\|X\|)$, $\forall x,p,X,\lambda$;

d) $\left|\lambda w^\lambda(y;x,p,X) + \overline{H}(x,p,X)\right| \le \lambda C_1(1+|p|+\|X\|)$, $\forall y,x,p,X$.

Lemma 3.3. *There exists* $\tilde{C}_1 > 0$ *such that*

a) $|\overline{H}(x,p_1,X_1) - \overline{H}(x,p_2,X_2)| \le \tilde{C}_1\left(|p_1-p_2| + \|X_1-X_2\|\right)$;

b) $|\overline{H}(x_1,p,X) - \overline{H}(x_2,p,X)| \le \tilde{C}_1(1+|p|+\|X\|)|x_1-x_2|$;

c) $\overline{H}$ *is uniformly elliptic and convex with respect to* X.

Remark 3.1. The effective problem (7) satisfies the hypothesis required for the regularity result in Safonov [26]. It follows that there exist $N>0$ and $\bar{\alpha} \in (0,1)$ (both independent of a) such that:

$$\begin{aligned} &\|u\|_\infty,\ \|Du\|_\infty,\ \|D^2u\|_\infty \le N \\ &\|u\|_{C^{2,\bar{\alpha}}(B(x,1))} \le N \qquad \forall x \in \mathbb{R}^n. \end{aligned} \tag{9}$$

Indeed the first inequality (i.e. $\|u\|_\infty \le N$) is obtained following the arguments in [1, 6] (here, the periodicity assumption in (A_4) plays a crucial role) while the other inequalities are consequence of the first one and of the result by Gilbarg and Trudinger [18] and Safonov [26].

It is expedient for our purpose to study the approximated cell problem

$$\lambda w^\lambda_{\epsilon,r} + H^\epsilon_r(y, D_y w^\lambda_{\epsilon,r}, D^2_{yy} w^\lambda_{\epsilon,r}; \overline{x}, \overline{p}, \overline{X}) = 0 \qquad y \in \mathbb{R}^n, \quad w^\lambda_{\epsilon,r} \text{ periodic}, \tag{10}$$

where $\lambda > 0$ and

$$H^\epsilon_r(y,q,Y;\overline{x},\overline{p},\overline{X}) := \min_{|\xi_1|,|\xi_2|\le r} H^\epsilon(\overline{x}+\xi_1, y+\xi_2, \overline{p}+\epsilon q, \overline{X}+Y).$$

This definition of H^ϵ_r is in the same spirit of the approximated Hamiltonians introduced in [3] and in the *shaking of coefficients* method by Krylov (see [20] and [7]); we shall use these approximations in order to overcome the lack of uniform continuity of H^ϵ.

Let us observe that, owing to Assumptions (A_1)-(A_4), the operator H^ϵ_r is periodic in y and $\overline{x}$ and it is uniformly elliptic in Y. Furthermore, for some positive constant C_2, independent of ϵ and r, there holds

$$\begin{aligned} \left|H^\epsilon_r(y,q,Y;\overline{x},\overline{p},\overline{X}) - H^\epsilon_r(y',q',Y';\overline{x},\overline{p},\overline{X})\right| &\le C_2\left(\|Y-Y'\| + \epsilon|q-q'|\right) \\ &\quad + C_2|y-y'|\left(1+|\overline{p}|+\epsilon|q'|+\|\overline{X}\|+\|Y'\|\right), \end{aligned} \tag{11}$$

$$\begin{aligned} C_2\epsilon|q| + \omega(\epsilon)\left(1+|\overline{p}|+\|\overline{X}\|+\|Y\|\right) &\ge H^\epsilon_r(y,q,Y;\overline{x},\overline{p},\overline{X}) - H(\overline{x},y,\overline{p},\overline{X}+Y) \ge \\ &- C_2\epsilon|q| - (C_2 r + \omega(\epsilon))\left(1+|\overline{p}|+\|\overline{X}\|+\|Y\|\right) \end{aligned} \tag{12}$$

for every $\overline{x}, y, y', q, q', \overline{p} \in \mathbb{R}^n$ and $\overline{X}, Y, Y' \in \mathbb{S}^n$.

In the following Lemma, we collect some properties of $w^\lambda_{\epsilon,r}$.

Lemma 3.4. *There exists a unique bounded solution* $w^\lambda_{\epsilon,r}(\cdot;x,p,X)$ *to* (10). *Moreover there exists a positive constant* C_3, *depending only on the parameters entering in Assumptions* (A_1)-(A_4) *(i.e., independent of* λ, ϵ, r, x, p, X*) such that*

a) $\|\lambda w^\lambda_{\epsilon,r}(\cdot;x,p,X)\|_\infty \le C_3(1+|p|+\|X\|)$, $\forall x,p,X$;

b) $\left|\lambda w^\lambda_{\epsilon,r}(y;x,p,X) + \overline{H}(x,p,X)\right| \le C_3[\omega(\epsilon)+\epsilon+r+\lambda](1+|p|+\|X\|)$ $\forall y,x,p,X$.

Proof. We first establish that there exists a unique bounded solution $w^\lambda_{\epsilon,r}$ to (10). To this end, we observe that a Comparison Principle holds for problem (10). For $\tau := \tilde{C}_3[\omega(\epsilon)+\epsilon+r](1+|\overline{p}|+\|\overline{X}\|)$, the functions

$$w^{\pm}(y) := w^\lambda(y;\overline{x},\overline{p},\overline{X}) \pm \lambda^{-1}\tau \tag{13}$$

are respectively a super- and a subsolution to problem (10). Actually, for $\tilde{C}_3 := 2(1+C_2)(1+C_1)$, we have

$$\begin{aligned}\lambda w^+ + H^\epsilon_r(y, D_y w^+, D^2_{yy}w^+;\overline{x},\overline{p},\overline{X}) &= \lambda w^\lambda + H^\epsilon_r(y, D_y w^\lambda, D^2_{yy}w^\lambda;\overline{x},\overline{p},\overline{X}) + \tau \\ &\geq -[C_2C_1\epsilon + (C_2 r+\omega(\epsilon))(1+C_1)]\left(1+|\overline{p}|+\|\overline{X}\|\right)+\tau \geq 0\end{aligned}$$

(here, the rightmost inequality of (12) and Lemma 3.2-(c) have been used) so our claim for w^+ is completely proved. Being similar, the proof that w^- is a subsolution is omitted. Applying the Perron method, one can establish that problem (10) admits exactly one solution.

Let us now pass to the proof of estimates (a) and (b). The proof of point (a) relies on the same arguments of those of Lemma 3.2-(a) and we refer to [2, 6] for the proof.

(b). Let us first notice that, since $w^\pm$ in (13) are a super and a subsolution to problem (10), there holds

$$\lambda \sup_y \left|w^\lambda_{\epsilon,r}(y;\overline{x},\overline{p},\overline{X}) - w^\lambda(y;\overline{x},\overline{p},\overline{X})\right| \leq \tilde{C}_3[\omega(\epsilon)+\epsilon+r](1+|\overline{p}|+\|\overline{X}\|) \quad \forall\lambda,\epsilon,r \tag{14}$$

for every $(\overline{x},\overline{p},\overline{X})$. Hence Lemma 3.2-(d) and estimate (14) yield

$$\begin{aligned}\left|\lambda w^\lambda_{\epsilon,r}(y;\overline{x},\overline{p},\overline{X}) + \overline{H}(\overline{x},\overline{p},\overline{X})\right| &\leq \lambda\left|w^\lambda_{\epsilon,r}(y;\overline{x},\overline{p},\overline{X}) - w^\lambda(y;\overline{x},\overline{p},\overline{X})\right| + \\ &\left|\lambda w^\lambda(y;\overline{x},\overline{p},\overline{X}) + \overline{H}(\overline{x},\overline{p},\overline{X})\right| \leq C_3[\omega(\epsilon)+\epsilon+r+\lambda](1+|\overline{p}|+\|\overline{X}\|)\end{aligned}$$

for $C_3 = \max\{\tilde{C}_3, C_1\}$. □

Proof of Theorem 3.1. Fix $\epsilon \in (0,1)$. For every λ, $r \in (0,1)$, $\lambda \geq \epsilon^2$, let us introduce the function

$$\varphi(x) := u^\epsilon(x) - u(x) - \epsilon^2 w^\lambda_{\epsilon,r}\left(\frac{x}{\epsilon};[u](x)\right) \tag{15}$$

where

$$w^\lambda_{\epsilon,r}\left(y;[u](x)\right) := w^\lambda_{\epsilon,r}\left(y;x,Du(x),D^2u(x)\right).$$

The Comparison Principle for problems (6) and (7) ensures that u^ϵ and u are bounded, moreover it is easy to see that by (A_4) u^ϵ and u are periodic. By bounds in (3.1), Lemma 3.4-(a) and the periodicity of u and u^ϵ, the function $w^\lambda_{\epsilon,r}(\cdot/\epsilon;[u](\cdot))$ is bounded. Hence, there exists a point $\hat{x}$ where the function φ attains its maximum.

For each $\tau \in (0,1)$, set $c := 3C_3(1+N)\frac{\epsilon^2}{\lambda\tau^2}$ and introduce the function

$$\tilde{\varphi}(x) := u^\epsilon(x) - u(x) - \epsilon^2 w\left(\frac{x}{\epsilon}\right) - c|x-\hat{x}|^2 \tag{16}$$

with $w := w^\lambda_{\epsilon,r}(\cdot;[u](\hat{x}))$. We notice that there holds: $\tilde{\varphi}(\hat{x}) = \varphi(\hat{x})$ and, for $x \in \partial B(\hat{x},\tau)$,

$$\begin{aligned}\tilde{\varphi}(\hat{x}) - \tilde{\varphi}(x) &= [\varphi(\hat{x})-\varphi(x)] - \epsilon^2\left[w^\lambda_{\epsilon,r}(x/\epsilon;[u](x)) - w^\lambda_{\epsilon,r}(x/\epsilon;[u](\hat{x}))\right] + c\tau^2 \\ &\geq -\epsilon^2\left[w^\lambda_{\epsilon,r}(x/\epsilon;[u](x)) - w^\lambda_{\epsilon,r}(x/\epsilon;[u](\hat{x}))\right] + c\tau^2 \\ &\geq -2C_3(1+\|Du\|_\infty + \|D^2u\|_\infty)\frac{\epsilon^2}{\lambda} + 3C_3(1+N)\frac{\epsilon^2}{\lambda} > 0\end{aligned}$$

(here, Lemma 3.4-(a) and relations (9) have been used). Whence, the function $\tilde\varphi$ has a maximum at some point $\tilde x \in B(\hat x, \tau)$, that we can assume to be strict by adding to u^ϵ a smooth function vanishing with its first and second derivatives at $\tilde x$. Hence, by standard arguments, we infer that, for every positive parameter σ, the function

$$\Phi(x,\xi) := u^\epsilon(x) - u(x) - \epsilon^2 w\left(\frac{\xi}{\epsilon}\right) - c|x-\hat x|^2 - \frac{\sigma}{2}|x-\xi|^2 \tag{17}$$

attains a maximum value in some point (x_σ, ξ_σ), with

$$x_\sigma, \xi_\sigma \to \tilde x \qquad \text{as} \quad \sigma \to +\infty. \tag{18}$$

Let us now claim that there exists a positive constant C_4 such that, for every $\eta > 0$, there exists two matrices X_1, $X_2 \in \mathbb{S}^n$ such that there holds

$$(Du(x_\sigma) + 2c(x_\sigma - \hat x) + \sigma(x_\sigma - \xi_\sigma), X_1) \in J^+_{x_\sigma} u^\epsilon, \tag{19}$$

$$(\frac{\sigma}{\epsilon}(x_\sigma - \xi_\sigma), X_2) \in J^-_{\xi_\sigma/\epsilon} w, \tag{20}$$

$$X_1 - X_2 \le D^2 u(x_\sigma) + (2c + \eta C_4) I. \tag{21}$$

In fact, applying [13, Thm 3.2] to u^ϵ and $W(\xi) := \epsilon^2 w(\xi/\epsilon)$ with the penalization term $\psi(x,\xi) := u(x) + c|x-\hat x|^2 + \frac{\sigma}{2}|x-\xi|^2$, we deduce that, for each $\eta > 0$, there exist two matrices X_1 and X_2 such that

$$(D_x\psi(x_\sigma,\xi_\sigma), X_1) \in J^+_{x_\sigma} u^\epsilon, \qquad (-D_\xi\psi(x_\sigma,\xi_\sigma), X_2) \in J^-_{\xi_\sigma} W$$

$$\begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} \le D^2\psi(x_\sigma,\xi_\sigma) + \eta \left(D^2\psi(x_\sigma,\xi_\sigma)\right)^2.$$

By the first two relations, properties (19) and (20) follow; indeed, (p, X) belongs to $J^-_{\xi_\sigma} W$ if, and only if, $(\epsilon^{-1}p, X)$ belongs to $J^-_{\xi_\sigma/\epsilon} w$. Furthermore, applying the last inequality to the vector (v, v), we infer

$$X_1 - X_2 \le D^2 u(x_\sigma) + 2cI + \eta \|(D^2\psi(x_\sigma,\xi_\sigma))^2\| I;$$

in particular, for $C_4 := \|(D^2\psi(x_\sigma,\xi_\sigma))^2\|$, inequality (21) is established.

Taking into account that u^ϵ is a subsolution to (6) and relation (19), we can write

$$\begin{aligned} 0 \ &\ge \ au^\epsilon(x_\sigma) + H^\epsilon\left(x_\sigma, x_\sigma/\epsilon, Du(x_\sigma) + 2c(x_\sigma - \hat x) + \sigma(x_\sigma - \xi_\sigma), X_1\right) \\ &\ge \ au^\epsilon(x_\sigma) + H^\epsilon\big(x_\sigma, x_\sigma/\epsilon, Du(x_\sigma) + 2c(x_\sigma - \hat x) + \sigma(x_\sigma - \xi_\sigma), \\ &\qquad\qquad X_2 + D^2u(x_\sigma) + (2c + \eta C_4)I\big) \\ &\ge \ au^\epsilon(x_\sigma) + H^\epsilon\left(x_\sigma, x_\sigma/\epsilon, Du(x_\sigma) + \sigma(x_\sigma - \xi_\sigma), X_2 + D^2u(x_\sigma)\right) \\ &\qquad\qquad -C\left[c|x_\sigma - \hat x| + c + \eta C_4\right] \end{aligned}$$

where the last two inequalities are due to relation (21) and Assumption (A_2). Moreover, by relations (9), for σ sufficiently large, we deduce

$$\begin{aligned} 0 \ge au^\epsilon(x_\sigma) + H^\epsilon&\left(x_\sigma, x_\sigma/\epsilon, Du(\hat x) + \sigma(x_\sigma - \xi_\sigma), X_2 + D^2u(\hat x)\right) \\ &\qquad - C\left[N|\hat x - x_\sigma|^{\bar\alpha} + c|x_\sigma - \hat x| + c + \eta C_4\right] \end{aligned}$$

On the other hand, being a solution to the (λ, ϵ, r)-cell problem (10) centered in $(\hat x, Du(\hat x), D^2u(\hat x))$, by relation (20), the function w verifies

$$0 \le \lambda w\left(\frac{\xi_\sigma}{\epsilon}\right) + H^\epsilon_r\left(\frac{\xi_\sigma}{\epsilon}, \frac{\sigma}{\epsilon}(x_\sigma - \xi_\sigma), X_2; \hat x, Du(\hat x), D^2u(\hat x)\right).$$

We choose $r=2\tau$ and we notice that, by (18) for σ sufficiently large, there holds

$$H_r^\epsilon\left(\frac{\xi_\sigma}{\epsilon},\frac{\sigma}{\epsilon}(x_\sigma-\xi_\sigma),X_2;\hat{x},Du(\hat{x}),D^2u(\hat{x})\right)\leq H^\epsilon\left(x_\sigma,\frac{x_\sigma}{\epsilon},Du(\hat{x})+\sigma(x_\sigma-\xi_\sigma),D^2u(\hat{x})+X_2\right).$$

The last three inequalities guarantee the following one:

$$\begin{aligned}0 &\geq au^\epsilon(x_\sigma)-\lambda w(\xi_\sigma/\epsilon)-C\left[N|\hat{x}-x_\sigma|^{\bar{\alpha}}+c|x_\sigma-\hat{x}|+c+\eta C_4\right]\\ &\geq au^\epsilon(x_\sigma)+\overline{H}(\hat{x},Du(\hat{x}),D^2u(\hat{x}))-C_3(1+2N)[\omega(\epsilon)+\epsilon+2\tau+\lambda]\\ &\qquad -C\left[N|\hat{x}-x_\sigma|^{\bar{\alpha}}+c|x_\sigma-\hat{x}|+c+\eta C_4\right]\end{aligned}$$

(in the last relation Lemma 3.4-(b) and estimates (9) have been applied). Since u is a classical solution to the effective problem (7), we infer

$$a[u^\epsilon(x_\sigma)-u(\hat{x})]\leq C_3(1+2N)[\omega(\epsilon)+\epsilon+2\tau+\lambda]+C\left[N|\hat{x}-x_\sigma|^{\bar{\alpha}}+c|x_\sigma-\hat{x}|+c+\eta C_4\right].$$

Letting $\eta\to 0$ and $\sigma\to+\infty$, by the limits (18), we obtain

$$a[u^\epsilon(\tilde{x})-u(\hat{x})]\leq C_5[\omega(\epsilon)+\epsilon+\tau^{\bar{\alpha}}+\lambda+c]$$

where the constant C_5 is independent of a, λ, ϵ and τ.

We choose $\lambda=\epsilon^{\theta_1}$, $\tau=\epsilon^{\theta_2}$. Recalling the definition of c ($c=3C_3(1+N)\epsilon^2\lambda^{-1}\tau^{-2}$) we infer

$$a[u^\epsilon(\tilde{x})-u(\hat{x})]\leq C_5[\omega(\epsilon)+\epsilon+\epsilon^{\theta_2\bar{\alpha}}+\epsilon^{\theta_1}+3C_3(1+N)\epsilon^{2-\theta_1-2\theta_2}].$$

Finally, relation $a\tilde{\varphi}(\tilde{x})\geq a\tilde{\varphi}(\hat{x})=a\varphi(\hat{x})\geq a\varphi(x)$ entails

$$a[u^\epsilon(x)-u(x)]\leq a[u^\epsilon(\tilde{x})-u(\hat{x})]+a[u(\hat{x})-u(\tilde{x})]+\epsilon^2 a\left[w^\lambda_{\epsilon,r}\left(\frac{x}{\epsilon};[u](x)\right)-w^\lambda_{\epsilon,r}\left(\frac{\tilde{x}}{\epsilon};[u](\hat{x})\right)\right].$$

Combining the previous two inequalities, estimates (9), Lemma 3.4-(a), for some constant C_6 with the same properties of C_5 (namely, it is independent of $a,\epsilon,\theta_1,\theta_2$) there holds

$$a[u^\epsilon(x)-u(x)]\leq C_6[\omega(\epsilon)+\epsilon+\epsilon^{\theta_2\bar{\alpha}}+\epsilon^{\theta_1}+\epsilon^{2-\theta_1-2\theta_2}].$$

By the arbitrariness of x, taking $\theta_1=\frac{\bar{\alpha}}{\bar{\alpha}+1}$ and $\theta_2=\frac{1}{\bar{\alpha}+1}$, we get the bound

$$a[u^\epsilon(x)-u(x)]\leq C_6[\omega(\epsilon)+\epsilon^{\frac{\bar{\alpha}}{\bar{\alpha}+1}}].$$

The proof of the bound for $u-u^\epsilon$ is similar and we shall omit it. □

Remark 3.2. Let us observe that by the above calculation $\alpha=\frac{\bar{\alpha}}{\bar{\alpha}+1}$ where $\bar{\alpha}$ is the Hölder exponent of u, see (9).

4. **Proof of Theorem 2.1.** This section is devoted to the proof of our main result stated in Theorem 2.1. For simplicity, we shall consider only the case $k=2$ since the general case can be dealt in a similar manner. In this case the construction of the effective Hamiltonian $\overline{H}$ requires two steps:

i) Fix $(\overline{x},\overline{y},\overline{p},\overline{X})$ and, for every positive λ, consider the *microscopic cell problem*

$$\lambda w^\lambda+H(\overline{x},\overline{y},z,\overline{p},\overline{X}+D^2_{zz}w^\lambda)=0\qquad y\in\mathbb{R}^n,\quad w^\lambda \text{ periodic.}\tag{22}$$

This problem admits exactly one periodic solution $w^\lambda = w^\lambda(z;\overline{x},\overline{y},\overline{p},\overline{X})$. As $\lambda \to 0^+$, the function λw^λ converges (uniformly in z) to some constant $-H_1(\overline{x},\overline{y},\overline{p},\overline{X})$.
ii) Fixed $(\overline{x},\overline{p},\overline{X})$, for each positive λ, let $W^\lambda = W^\lambda(y;\overline{x},\overline{p},\overline{X})$ be the solution of the *mesoscopic cell problem*

$$\lambda W^\lambda + H_1(\overline{x},y,\overline{p},\overline{X}+D^2_{yy}W^\lambda) = 0 \qquad y\in\mathbb{R}^n, \quad W^\lambda \text{ periodic.} \tag{23}$$

As before (since the operator H_1 enjoys the same properties of H, see [6] and also Lemma 3.3), as $\lambda \to 0^+$, the function λW^λ converges (uniformly in y) to $-\overline{H}(\overline{x},\overline{p},\overline{X})$.

The function W^λ satisfies regularity result similar to Lemma 3.2-(c):

Lemma 4.1. *There exist a positive constant C_1, depending only on the Assumptions (A_1)-(A_3), and a parameter $\tilde\alpha\in(0,1)$, depending continuously on $(\overline{p},\overline{X})$, such that*

$$\|W^\lambda(\cdot;\overline{x},\overline{p},\overline{X}) - W^\lambda(0;\overline{x},\overline{p},\overline{X})\|_{C^{2,\tilde\alpha}} \le C_1\left(1+|\overline{p}|+\|\overline{X}\|\right) \qquad \forall\lambda,(\overline{x},\overline{p},\overline{X}).$$

For our purpose, it is expedient to introduce the operators

$$H^\epsilon_r(y,z,q,Y;\overline{x},\overline{p},\overline{X}) := \min_{|\xi_1|,|\xi_2|,|\xi_3|\le r} H^\epsilon(\overline{x}+\xi_1, y+\xi_2, z+\xi_3, \overline{p}+\epsilon q, \overline{X}+Y)$$

and, for $\lambda>0$, the approximated multiscale cell problem

$$\lambda w^\lambda_{\epsilon,r} + H^\epsilon_r\left(y,\frac{y}{\epsilon},D_y w^\lambda_{\epsilon,r},D^2_{yy}w^\lambda_{\epsilon,r};\overline{x},\overline{p},\overline{X}\right) = 0 \qquad y\in\mathbb{R}^n, \quad w^\lambda_{\epsilon,r} \text{ periodic.} \tag{24}$$

We shall denote a solution of (24) by $w^\lambda_{\epsilon,r}(y;\overline{x},\overline{p},\overline{X})$ in order to display its dependence on the (fixed) parameters $(\overline{x},\overline{p},\overline{X})$. Some properties of $w^\lambda_{\epsilon,r}$ are collected in the following statements

Lemma 4.2. *Assume (A_1)-(A_3). There exists a unique periodic solution of (24). Moreover, there exists a positive constant C_2, independent of λ, ϵ, r, $\overline{x}$, $\overline{p}$ and $\overline{X}$, such that*

$$\|\lambda w^\lambda_{\epsilon,r}(\cdot;\overline{x},\overline{p},\overline{X})\|_\infty \le C_2(1+|\overline{p}|+\|\overline{X}\|), \qquad \forall\lambda,\epsilon,r,\overline{x},\overline{p},\overline{X}.$$

Since the proof of the previous lemma follows the same arguments of those of Lemma 3.4, we shall omit it.

Proposition 4.1. *Under assumptions (A_1)-(A_3), there exist two positive constants M_1 and $\alpha_1 = \frac{\tilde\alpha}{\tilde\alpha+1}\in(0,\frac12)$ (where $\tilde\alpha$ as in Lemma 4.1) depending continuously and only on $|\overline{p}|$, $\|\overline{X}\|$ and on the parameters entering in Assumption (A_1)-(A_3) (in particular, independent of λ, ϵ, r, $\overline{x}$) such that*

$$|\lambda w^\lambda_{\epsilon,r}(y;\overline{x},\overline{p},\overline{X}) + \overline{H}(\overline{x},\overline{p},\overline{X})| \le M_1[\epsilon^{\alpha_1}+r+\omega(\epsilon,\overline{x})+\lambda], \qquad \forall y\in\mathbb{R}^n.$$

Proof. We claim that there exist a positive constant $\tilde M$ and $\alpha_1\in(0,1)$, depending continuously and only on $|\overline{p}|$, $\|\overline{X}\|$ and on Assumption (A_1)-(A_3) (in particular, independent of λ, ϵ, r, $\overline{x}$) such that

$$\lambda|w^\lambda_{\epsilon,r}(y;\overline{x},\overline{p},\overline{X}) - W^\lambda(y;\overline{x},\overline{p},\overline{X})| \le \tilde M[\epsilon^{\alpha_1}+r+\omega(\epsilon,\overline{x})], \qquad \forall y\in\mathbb{R}^n,\ \lambda\in(0,1) \tag{25}$$

where W^λ is the solution to the mesoscopic cell problem (23) centered in $(\overline{x},\overline{p},\overline{X})$. Actually, one can easily check that there exists a positive constant $\tilde C$, independent of ϵ, r and $(\overline{x},\overline{p},\overline{X})$, such that

$$\left|H^\epsilon_r(y,z,q,Y;\overline{x},\overline{p},\overline{X}) - H(\overline{x},y,z,\overline{p},\overline{X}+Y)\right| \le \tilde C(\epsilon+r+\omega(\epsilon,\overline{x}))[C_0+|q|+\|Y\|]$$

with $C_0 := 1+|\overline{p}|+\|\overline{X}\|$. It follows that, since $H^\epsilon_r(y,z,q,Y;\overline{x},\overline{p},\overline{X})$ converges locally uniformly to $H(\overline{x},y,z,\overline{p},\overline{X}+Y)$ as $(\epsilon,r)\to(0,0)$, the homogenized Hamiltonian of H^ϵ_r with respect to z coincides with the one of H with respect to the same variable and therefore is given by $H_1(\overline{x},y,\overline{p},\overline{X}+Y)$. By applying Theorem 3.1 with $\omega(\epsilon)$, a, u^ϵ, u replaced respectively by $\tilde{\omega} := \tilde{C}C_0[\epsilon+r+\omega(\epsilon,\bar{x})]$, λ, $w^\lambda_{\epsilon,r}$, W^λ and taking into account Remark 3.2 we infer our claim (25).

On the other hand, following the same arguments as in the proof of Lemma 3.4-(b) with H_1 in place of H^ϵ_r (hence the corresponding estimate does not depend on ϵ and r) and using Lemma 4.1, we notice that there exists a positive constant M_1, independent of λ and $(\overline{x},\overline{p},\overline{X})$, such that

$$\left|\lambda W^\lambda(y;\overline{x},\overline{p},\overline{X}) + \overline{H}(\overline{x},\overline{p},\overline{X})\right| \le M_1\lambda(1+|\overline{p}|+\|\overline{X}\|) \qquad \forall\lambda,\ y,\ (\overline{x},\overline{p},\overline{X}). \quad (26)$$

Finally, let us observe that there holds

$$|\lambda w^\lambda_{\epsilon,r}(y;\overline{x},\overline{p},\overline{X}) + \overline{H}(\overline{x},\overline{p},\overline{X})| \le \lambda|w^\lambda_{\epsilon,r}(y;\overline{x},\overline{p},\overline{X}) - W^\lambda(y;\overline{x},\overline{p},\overline{X})| + \\ |\lambda W^\lambda(y;\overline{x},\overline{p},\overline{X}) + \overline{H}(\overline{x},\overline{p},\overline{X})|;$$

substituting inequalities (25) and (26) in the previous one, we accomplish the proof of our statement. □

Proof of Theorem 2.1. We shall argue as in the proof of Theorem 3.1. Fix $\epsilon\in(0,1)$. For every λ, γ, $r\in(0,1)$, $\lambda\ge\epsilon^2$, let us introduce the function

$$\varphi(x) := u^\epsilon(x) - u(x) - \epsilon^2 w^\lambda_{\epsilon,r}\left(\frac{x}{\epsilon};[u](x)\right) - \frac{\gamma}{2}|x|^2 \quad (27)$$

where

$$w^\lambda_{\epsilon,r}\left(y;[u](x)\right) := w^\lambda_{\epsilon,r}\left(y;x,Du(x),D^2u(x)\right).$$

The Comparison Principle ensures that u^ϵ and u are bounded. In fact, invoking the result by Safonov [26], one can prove that there exist $N>0$ and $\bar{\alpha}\in(0,1)$ such that:

$$\|u\|_\infty,\ \|Du\|_\infty,\ \|D^2u\|_\infty \le N, \quad \|u\|_{C^{2,\bar{\alpha}}(B(x,1))} \le N \qquad \forall x\in\mathbb{R}^n. \quad (28)$$

By these estimates and Lemma 4.2, the function $w^\lambda_{\epsilon,r}(\cdot/\epsilon;[u](\cdot))$ is bounded. Hence, there exists a point $\hat{x}$ where the function φ attains its maximum.

Set $\tau := r/2$ and $c := 3C_2(1+N)\frac{\epsilon^2}{\lambda\tau^2}$, and introduce the function

$$\tilde{\varphi}(x) := u^\epsilon(x) - u(x) - \epsilon^2 w\left(\frac{x}{\epsilon}\right) - \frac{\gamma}{2}|x|^2 - c|x-\hat{x}|^2 \quad (29)$$

with $w := w^\lambda_{\epsilon,r}(\cdot;[u](\hat{x}))$. Arguing as before, by Lemma 4.2, one can easily check that the function $\tilde{\varphi}$ has a maximum in some point $\tilde{x}\in B(\hat{x},\tau)$, that we can assume to be strict by adding to u^ϵ a smooth function vanishing with its first and second derivatives at $\tilde{x}$. By standard arguments in viscosity solution theory, we infer that, for every positive parameter σ, the function

$$\Phi(x,\xi) := u^\epsilon(x) - u(x) - \epsilon^2 w\left(\frac{\xi}{\epsilon}\right) - \frac{\gamma}{2}|x|^2 - c|x-\hat{x}|^2 - \frac{\sigma}{2}|x-\xi|^2 \quad (30)$$

attains a maximum value in some point (x_σ,ξ_σ), with

$$x_\sigma,\xi_\sigma\to\tilde{x} \qquad \text{as} \quad \sigma\to+\infty. \quad (31)$$

Applying again [13, Thm 3.2] (now, the penalization term is $\psi(x,\xi) := u(x) + \frac{\gamma}{2}|x|^2 + c|x-\hat{x}|^2 + \frac{\sigma}{2}|x-\xi|^2$), we infer that there exists a positive constant $\tilde{C}$ such that, for every $\eta > 0$, there exists two matrices X_1, $X_2 \in \mathbb{S}^n$ such that there holds

$$(Du(x_\sigma) + \gamma x_\sigma + 2c(x_\sigma - \hat{x}) + \sigma(x_\sigma - \xi_\sigma), X_1) \in J^+_{x_\sigma} u^\epsilon, \tag{32}$$

$$(\frac{\sigma}{\epsilon}(x_\sigma - \xi_\sigma), X_2) \in J^-_{\xi_\sigma/\epsilon} w, \tag{33}$$

$$X_1 - X_2 \le D^2 u(x_\sigma) + (\gamma + 2c + \eta\tilde{C})I. \tag{34}$$

From now on the letter $\bar{M}$ stands for a positive constant, dependent only on the parameters entering in Assumptions (A$_1$)-(A$_3$) (i.e., independent on λ, ϵ, r, σ and τ) which may change from line to line.

Being a solution to the starting problem (1) with $k = 2$, by relation (32), the function u^ϵ verifies

$$\begin{aligned} 0 \quad &\ge \quad u^\epsilon(x_\sigma) + H^\epsilon\left(x_\sigma, \frac{x_\sigma}{\epsilon}, \frac{x_\sigma}{\epsilon^2}, Du(x_\sigma) + \gamma x_\sigma + 2c(x_\sigma - \hat{x}) + \sigma(x_\sigma - \xi_\sigma), X_1\right) \\ &\ge \quad u^\epsilon(x_\sigma) + H^\epsilon\left(x_\sigma, \frac{x_\sigma}{\epsilon}, \frac{x_\sigma}{\epsilon^2}, Du(x_\sigma) + \sigma(x_\sigma - \xi_\sigma), X_2 + D^2u(x_\sigma)\right) \\ &\qquad -\bar{M}\left[\gamma|x_\sigma| + c|x_\sigma - \hat{x}| + \gamma + c + \eta\tilde{C}\right] \end{aligned}$$

where the last inequality is a consequence of relations (34) and the uniform ellipticity of H^ϵ. Moreover, for σ sufficiently large, relations (28) entail

$$\begin{aligned} 0 \ge u^\epsilon(x_\sigma) + H^\epsilon\left(x_\sigma, \frac{x_\sigma}{\epsilon}, \frac{x_\sigma}{\epsilon^2}, Du(\hat{x}) + \sigma(x_\sigma - \xi_\sigma), X_2 + D^2u(\hat{x})\right) \\ - \bar{M}\left[|\hat{x} - x_\sigma|^{\bar{\alpha}} + \gamma|x_\sigma| + c|x_\sigma - \hat{x}| + \gamma + c + \eta\tilde{C}\right] \end{aligned}$$

On the other hand, problem (24) centered in $(\hat{x}, Du(\hat{x}), D^2u(\hat{x}))$ and relation (33), imply that the function w verifies for σ sufficiently large

$$\begin{aligned} 0 \quad &\le \quad \lambda w(\xi_\sigma/\epsilon) + H^\epsilon_r\left(\frac{\xi_\sigma}{\epsilon}, \frac{\xi_\sigma}{\epsilon^2}, \frac{\sigma}{\epsilon}(x_\sigma - \xi_\sigma), X_2; \hat{x}, Du(\hat{x}), D^2u(\hat{x})\right) \\ &\le \quad \lambda w(\xi_\sigma/\epsilon) + H^\epsilon\left(x_\sigma, \frac{x_\sigma}{\epsilon}, \frac{x_\sigma}{\epsilon^2}, Du(\hat{x}) + \sigma(x_\sigma - \xi_\sigma), X_2 + D^2u(\hat{x})\right), \end{aligned}$$

where the latter inequality is due to our choice of r (and τ) and to the limits (31).

The last two inequalities ensure the following one:

$$0 \ge u^\epsilon(x_\sigma) - \lambda w(\xi_\sigma/\epsilon) - \bar{M}\left[|\hat{x} - x_\sigma|^{\bar{\alpha}} + \gamma|x_\sigma| + c|x_\sigma - \hat{x}| + \gamma + c + \eta\tilde{C}\right].$$

Moreover, owing to Proposition 4.1 and to estimates (28), we have

$$\begin{aligned} -\lambda w(\xi_\sigma/\epsilon) \quad &\ge \quad \overline{H}(\hat{x}, Du(\hat{x}), D^2u(\hat{x})) - \bar{M}[\epsilon^{\alpha_1} + r + \omega(\epsilon, \hat{x}) + \lambda] \\ &\ge \quad -u(\hat{x}) - \bar{M}[\epsilon^{\alpha_1} + r + \omega(\epsilon, \hat{x}) + \lambda] \end{aligned}$$

(in the last inequality, equation (2) has been used) where $\alpha_1 \in (0, 1)$ is a constant depending only on the parameters entering in the starting Assumptions (A$_1$)-(A$_3$) (i.e., independent on λ, ϵ, r, σ and τ).

Substituting the last inequality in the previous one, we obtain

$$\begin{aligned} u^\epsilon(x_\sigma) - u(\hat{x}) \le &\bar{M}\big[\epsilon^{\alpha_1} + r + \omega(\epsilon, \hat{x}) + \lambda + |\hat{x} - x_\sigma|^{\bar{\alpha}} \\ &+\gamma|x_\sigma| + c|x_\sigma - \hat{x}| + c + \gamma + \eta\tilde{C}\big]. \end{aligned}$$

Letting $\eta \to 0$, we deduce

$$u^\epsilon(x_\sigma) - u(\hat{x}) \le \bar{M}\left[\epsilon^{\alpha_1} + r + \omega(\epsilon, \hat{x}) + \lambda + |\hat{x} - x_\sigma|^{\bar{\alpha}} + \gamma|x_\sigma| + c|x_\sigma - \hat{x}| + c + \gamma\right];$$

as $\sigma \to +\infty$, taking into account the definition of r, by (31) we obtain

$$u^\epsilon(\tilde{x}) - u(\hat{x}) \le \bar{M}\left[\epsilon^{\alpha_1} + \omega(\epsilon,\hat{x}) + \lambda + \tau^{\bar{\alpha}} + \gamma|\tilde{x}| + c\tau + c + \gamma\right].$$

Choose $\lambda = \epsilon^{\theta_1}$, $\tau = \epsilon^{\theta_2}$ for some positive parameters θ_1 and θ_2. By the definition of c, we have

$$u^\epsilon(\tilde{x}) - u(\hat{x}) \le \bar{M}\left[\epsilon^{\alpha_1} + \omega(\epsilon,\hat{x}) + \epsilon^{\theta_1} + \epsilon^{\theta_2\bar{\alpha}} + \gamma|\tilde{x}| + \gamma + \epsilon^{2-\theta_1-2\theta_2}\right].$$

In conclusion, relation $\tilde{\varphi}(\tilde{x}) \ge \tilde{\varphi}(\hat{x}) = \varphi(\hat{x}) \ge \varphi(x)$ yields

$$u^\epsilon(x) - u(x) \le [u^\epsilon(\tilde{x}) - u(\hat{x})] + [u(\hat{x}) - u(\tilde{x})] + \epsilon^2\left[w^\lambda_{\epsilon,r}\left(\frac{x}{\epsilon};[u](x)\right) - w^\lambda_{\epsilon,r}\left(\frac{\tilde{x}}{\epsilon};[u](\hat{x})\right)\right] + \frac{\gamma}{2}\left(|x|^2 - |\tilde{x}|^2\right).$$

Taking into account the previous two inequalities, estimates (28) and Lemma 4.2, we obtain

$$u^\epsilon(x) - u(x) \le \bar{M}\left[\epsilon^{\alpha_1} + \omega(\epsilon,\hat{x}) + \epsilon^{\theta_1} + \epsilon^{\theta_2\bar{\alpha}} + \gamma|\tilde{x}| + \gamma + \epsilon^{2-\theta_1-2\theta_2}\right] + \frac{\gamma}{2}\left(|x|^2 - |\tilde{x}|^2\right).$$

Recall that the function ω has the form given in (4) and choose $\gamma = 8\bar{M}\omega_2(\epsilon)$. Hence, our choice of τ and a simple calculation give

$$\begin{aligned}
\bar{M}\left[\omega(\epsilon,\hat{x}) + \gamma|\tilde{x}|\right] - \frac{\gamma}{2}|\tilde{x}|^2 &= \bar{M}\omega_1(\epsilon) + \bar{M}\omega_2(\epsilon)\left[|\hat{x}|^2 + 8\bar{M}|\tilde{x}| - 4|\tilde{x}|^2\right] \\
&\le \bar{M}[\omega_1(\epsilon) + 2|\hat{x} - \tilde{x}|^2] + \bar{M}\omega_2(\epsilon)\left[-2|\tilde{x}|^2 + 8\bar{M}|\tilde{x}|\right] \\
&\le \bar{M}[\omega_1(\epsilon) + 2\epsilon^{2\theta_2}] + 8\bar{M}^3\omega_2(\epsilon).
\end{aligned}$$

Substituting this inequality in the previous one, we obtain

$$u^\epsilon(x) - u(x) \le \bar{M}\left[\epsilon^{\alpha_1} + \omega_1(\epsilon) + \omega_2(\epsilon) + \epsilon^{\theta_1} + \epsilon^{\theta_2\bar{\alpha}} + \omega_2(\epsilon)|x|^2 + \epsilon^{2-\theta_1-2\theta_2}\right] \quad \forall x.$$

In conclusion, for θ_1 and θ_2 sufficiently small, the proof of one side of the inequality is accomplished. The other part can be proved in parallel and we shall omit it. □

Remark 4.1. Choosing $\theta_1 = \bar{\alpha}/(1+\bar{\alpha})$ and $\theta_2 = 1/(1+\bar{\alpha})$, in equation (5) we obtain $\alpha = \min\left\{\frac{\tilde{\alpha}}{\tilde{\alpha}+1}, \frac{\bar{\alpha}}{\bar{\alpha}+1}\right\}$, where $\bar{\alpha}$ and $\tilde{\alpha}$ are the Hölder regularity exponents for the effective problem (2) (see Remark 3.1) and respectively for the mesoscopic cell problem (23) (see Lemma 4.1 and Prop. 4.1).

5. **Examples.** This Section is devoted to illustrate two examples; in the first one, an explicit estimate for the exponent α in (5) is exhibited. In the second we apply our results to an *unfair* stochastic differential game and in particular to stochastic optimal control problems.

Example 1 Let us consider the following problems with three scales

$$u^\epsilon - \mathrm{tr}\left[a(x)D^2u^\epsilon\right] + F_1\left(x, \frac{x}{\epsilon}, Du^\epsilon\right) + F_2\left(x, \frac{x}{\epsilon^2}, Du^\epsilon\right) = 0$$

where $a \in C^{1,\alpha^*}$ with $\alpha^* \in (0,1)$, $a \ge \nu I$ and $F_i = F_i(x,y,p)$ fulfill assumptions (A_1) and (A_2) (for $i = 1, 2$). In this case, the microscopic cell problem (22) centered in $(\bar{x}, \bar{y}, \bar{p}, \bar{X})$ reads

$$\lambda w^\lambda - \mathrm{tr}\left[a(\bar{x})D^2w^\lambda\right] + F_2(\bar{x}, z, \bar{p}) + F_1(\bar{x}, \bar{y}, \bar{p}) - \mathrm{tr}\left[a(\bar{x})\bar{X}\right] = 0.$$

Then the mesoscopic Hamiltonian (see [3]) H_1 has the form

$$H_1(x,y,p,X) = -\mathrm{tr}\left[a(x)X\right] + F_1(x,y,p) + \int_{[0,1)^n} F_2(x,z,p)\,dz;$$

furthermore, the mesoscopic cell problem (23) centered in $(\bar{x}, \bar{p}, \bar{X})$ reads

$$\lambda W^{\lambda} - \mathrm{tr}\left[a(\bar{x})D^2W^{\lambda}\right] + F_1(\bar{x}, y, \bar{p}) - \mathrm{tr}\left[a(\bar{x})\bar{X}\right] + \int_{[0,1)^n} F_2\left(\bar{x}, z, \bar{p}\right) dz = 0.$$

Since the coefficient of the second order term is constant, the regularity theory for elliptic equations (see [21, Chap. IV, Thm 6.3]) ensures that the solution W^{λ} belongs to $C^{2,\tilde{\alpha}}$ for every $\tilde{\alpha} \in (0,1)$ (namely, Lemma 4.1 holds for every $\tilde{\alpha} \in (0,1)$); moreover, the effective problem is

$$u - \mathrm{tr}\left[a(x)D^2u\right] + \int_{[0,1)^n} \left[F_1(x, z, Du) + F_2\left(x, z, Du\right)\right] dz = 0.$$

Invoking again the regularity theory for elliptic equations, we infer that the effective solution u belongs to $C^{2,\bar{\alpha}}$. Hence, Theorem 2.1 and Remark 4.1 guarantee that, for some positive M, there holds

$$\sup_{x\in\mathbb{R}^n} |u^{\epsilon}(x) - u(x)| \leq M\epsilon^{\frac{\bar{\alpha}}{\bar{\alpha}+1}}.$$

Furthermore, let us notice that, for $a \in C^{2,\beta}$ and $F_i \in C^{1,\beta}$ $(i = 1, 2)$ with $\beta \in (0,1)$, the solution u belongs to $C^{3,\beta}$; hence, for every $\alpha \in (0, \frac{1}{2})$, there exists a constant M_{α} such that

$$\sup_{x\in\mathbb{R}^n} |u^{\epsilon}(x) - u(x)| \leq M_{\alpha}\epsilon^{\alpha}.$$

EXAMPLE 2 Let us consider a stochastic differential game whose state variable evolves in a medium displaying heterogeneities of different scales and where a player may only "disturb" the other one. The dynamics are given by the stochastic differential equation

$$dx_s = f^{\epsilon}\left(x_s, \frac{x_s}{\epsilon}, \dots, \frac{x_s}{\epsilon^k}, \theta_s, \beta_s\right) ds + \sigma^{\epsilon}\left(x_s, \frac{x_s}{\epsilon}, \dots, \frac{x_s}{\epsilon^k}, \theta_s, \beta_s\right) dW_s, \qquad x_0 = x$$

where $(\Omega, \mathcal{F}, P)$ is a probability space, endowed with a continuous right filtration $(\mathcal{F}_t)_{0\leq t<+\infty}$ and a p-adapted Brownian motion W_t. The control law θ (respectively, β) belongs to the set $\mathcal{T}$ (resp., $\mathcal{B}$) of progressively measurable processes which take values in the compact set Θ (resp., B). The two controls θ and β are chosen respectively by the first and the second player whose purpose are opposite. The former wants to minimize the following cost function

$$P(x, \theta, \tau) := \mathbb{E}_x \int_0^{+\infty} \ell^{\epsilon}\left(x_s, \frac{x_s}{\epsilon}, \dots, \frac{x_s}{\epsilon^k}, \theta_s, \beta_s\right) e^{-s}\, ds$$

while the latter's aim is to maximize it. For $\varphi = f, \sigma, \ell$, we shall assume

$$\varphi^{\epsilon}(x, y_1, \dots, y_k, \theta, \beta) = \varphi_1(x, y_1, \dots, y_k, \theta) + \omega(\epsilon)\varphi_2(x, y_1, \dots, y_k, \theta, \beta)$$

(note that φ_1 is independent of the control β) where ω is a modulus of continuity. It is well known (see: [16, 17]) that the value function

$$u^{\epsilon}(x) := \inf_{\theta\in\Gamma} \sup_{\beta\in\mathcal{B}} P\left(x, \theta[\beta], \beta\right)$$

is a viscosity solution to problem (1) with

$$H^{\epsilon}(x, y_1, \dots, y_k, p, X) := \min_{\beta\in B} \max_{\theta\in\Theta} \{-\mathrm{tr}\left(a^{\epsilon}(x, y_1, \dots, y_k, \theta, \beta)X\right)$$
$$-f^{\epsilon}(x, y_1, \dots, y_k, \theta, \beta) \cdot p - \ell^{\epsilon}(x, y_1, \dots, y_k, \theta, \beta)\},$$

here $a^\epsilon = \sigma^\epsilon(\sigma^\epsilon)^T/2$ while Γ stands for the set of admissible *strategies* of the first player (namely, *nonanticipating* maps $\theta : \mathcal{B} \to \mathcal{T}$; for the precise definition and main properties, see [17]). We observe that, as $\epsilon \to 0$, H^ϵ converges locally uniformly to the operator

$$H(x, y_1, \dots, y_k, p, X) = \max_\theta \{-\mathrm{tr}\,(a_1 X) - f_1 \cdot p - \ell_1\} \qquad \text{with } a_1 = \sigma_1(\sigma_1)^T/2.$$

Invoking Corollary 2.1, we deduce that the value function u^ϵ converges uniformly in $\mathbb{R}^n$ to the solution u to the effective problem (2) with the rate

$$\sup_{x\in\mathbb{R}^n} |u^\epsilon(x) - u(x)| \le M\left[\epsilon^\alpha + \omega(\epsilon)\right].$$

Remark 5.1. Let us emphasize that the latter example encompasses stochastic optimal control problems. Indeed, in these cases, the second player is missing (that is, the set B reduces to a singleton). Moreover, in this context, the regular perturbation of the Hamiltonians (namely the fact that $H^\epsilon \to H$ locally uniformly) can be interpreted as a lack of information on the features of the problem.

Acknowledgments. The two authors are grateful to the referees for several comments and suggestions.

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Received October 2009; revised May 2010.

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