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A RATE-INDEPENDENT MODEL FOR PERMANENT INELASTIC EFFECTS IN SHAPE MEMORY MATERIALS

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Abstract. This paper addresses a three-dimensional model for isothermal stress-induced transformation in shape memory polycrystalline materials in presence of permanent inelastic effects. The basic features of the model are recalled and the constitutive and the three-dimensional quasi-static evolution problem are proved to be well-posed. Finally, we discuss the convergence of the model to reduced/former ones by means of a rigorous Γ-convergence analysis.

1. **Introduction.** Shape-memory alloys (SMA) are active materials showing an amazing thermo-mechanical behavior. At high temperatures they are *super-elastic*, namely they fully recover comparably large strains up to 5-8% (note that ordinary steels plasticize around 1% strains). At lower temperatures, deformations are permanent but the material can be forced to recover its original shape by means of a thermal cycle. This is the so called *shape memory effect*. Additionally, some SMAs are ferromagnetic and large strains can be activated at a distance by controlling a magnetic field. At the microscopic level, SMAs experience an abrupt structural phase change at the metallic lattice level between a highly symmetric crystallographic phase called *austenite* (mostly cubic, predominant at higher temperatures) and less symmetric phases called *martensites* (different variants due to symmetry breaking, energetically favorable at lower temperatures). The different geometry of these crystallographic phases is responsible for the macroscopically observed inelastic strain.

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The amazing material behavior of SMAs is nowadays exploited in a variety of different technological contexts ranging from Aerospace, to Earthquake, to Biomechanical Engineering. New applications of SMAs are constantly emerging. This fact triggers an intense research in the direction of the efficient description of the corresponding material behavior. In fact, the Engineering and Materials literature on SMAs models is vast and SMA behavior has been investigated at all scales (microscopic, mesoscopic with volume fractions, macroscopic) and by means of a full menagerie of models. The reader should refer to [2, 10, 11, 21, 22, 24, 26, 28, 30, 35, 52, 48, 51, 53, 54, 56, 58, 59] for some references. On the other hand, the mathematical treatment of SMA model is comparably less developed. Some comprehensive results in this sense refer to either the original formulations or modifications of the Frémond [22] and the Falk, Falk & Konopka [20, 21] models. With no claim of completeness, the reader is referred to [1, 3, 14, 16, 17, 29, 50, 61] and the related references for a collection of mathematical results.

We shall here focus on a phenomenological model for polycrystalline materials originally advanced by Souza, Mamiya, & Zouain [57] and subsequently refined by Auricchio & Petrini [6, 7] (the *SA model* in the following). The SA model shows some distinctive advantage with respect to former contributions in terms both of simplicity (8 easily fitted material parameters are required for the full 3D thermomechanical description) and robustness with respect to discretizations. These desirable features are distinguishing the SA model with respect to competitors and have recently attracted a growing attention in the SMA Engineering community. As for the mathematical viewpoint, the isothermal SA model has been already addressed from the mathematical and numerical-theoretical viewpoints in [5] and [41, 42], respectively. As regards, the non-isothermal situation, one has to mention the papers [43, 40] where the temperature of the specimen is assumed to be changing in time, being however *given a-priori* and the more recent [31, 32] where a fully coupled thermo-mechanical in one dimension is addressed. Some extensions of the SA model to non-symmetric material behaviors and ferromagnetic SMAs have been also considered [9, 12].

Experimental evidence shows that SMAs present permanent inelasticity and degradation effects during iterated loading and unloading cycles. As an example, Figure 1 from [4] reports the experimental stress-strain response of a Ni-Ti wire subjected to a strain driven uniaxial cyclic tension test. The material shows an increasing level of permanent inelasticity that saturates on a stable value after a certain number of cycles. The same Figure highlights also the occurrence of degradation, namely the lowering of both activation stresses for the transformation (i.e. the top and the bottom branches of the hysteretic loop). The relevance of these permanent inelastic effects is crucial as most SMA devices works under cyclic actions. In this regard, some models taking into account permanent inelastic effects are available [13, 27, 36, 49] but, to our knowledge, the only mathematical results in this direction have been obtained by Chemetov [15] for the *training* effect in Frémond's model [23] and by Kružík & Zimmer [34] in a rate-independent context.

This paper is focused on a new model for SMAs including permanent inelastic effects. The model has been introduced in [8] as an extension of the original SA model in the direction of the description of training and degradation. This extension basically relies on the introduction of an extra (tensorial) internal variable in order to keep track of the accumulated plastic history. In particular, the good features of the original SA model (namely its variational structure, simplicity, robustness, and

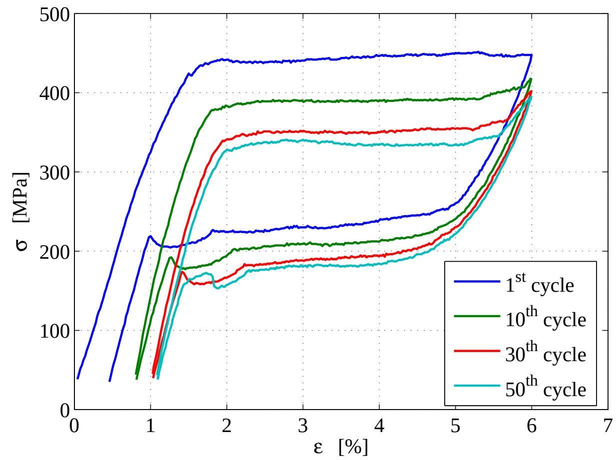

FIGURE 1. Experimental results on a SMA Ni-Ti wire. Cyclic tension test: stress versus strain up to 6% strain [4].

effectiveness) are here preserved. The model is recalled in Section 2 below whereas numerical experiments and validation are to be found in the original paper [8].

The main result of this paper is the well-posedness analysis of both the constitutive material relation (a tensorial nonlinear variational inequality) and the corresponding full quasi-static evolution problem (i.e., its coupling with the equilibrium system). In particular, we shall frame our analysis within the by-now classical theory of *energetic formulations* of rate-independent processes advanced by MIELKE & THEIL [46]. By-products of the existence argument are convergence results for time discretizations.

Eventually, we shed light on the connection of the current model with former/reduced ones by means of a rigorous analysis based on the variational concept of Γ-convergence. In particular, we present some convergence analysis with respect specific parameters asymptotics and, by letting the permanent plastic transformation radius to infinity, we show that the model reduces to the original SA model, with no permanent inelastic effects. On the other hand, some constrained plasticity model can be obtained as an asymptotic limit of the model. These convergences confirm once again the robustness of the proposed modeling perspective.

2. **The model.** We recall here the basic features of our SMA model with permanent inelastic effects. Further details are reported in the above-mentioned contributions where the reader can find a thorough discussion on motivation, numerical experiments, and validation.

Let us denote by $\mathbb{R}^{3\times3}_{\rm sym}$ the space of symmetric $3{\times}3$ tensors endowed with the usual scalar product $a{:}b = {\rm tr}(ab) := a_{ij}b_{ij}$ (summation convention) and the corresponding norm $|a| = \sqrt{a{:}a}$. Recall that the space $\mathbb{R}^{3\times3}_{\rm sym}$ can be orthogonally decomposed as $\mathbb{R}^{3\times3}_{\rm sym} = \mathbb{R}^{3\times3}_{\rm dev} \oplus \mathbb{R}1_2$, where $\mathbb{R}1_2$ is the subspace spanned by the identity 2-tensor 1_2, while $\mathbb{R}^{3\times3}_{\rm dev}$ is the subspace of all *deviatoric* symmetric 3×3 tensors. Given the displacement $u : \Omega \to \mathbb{R}^3$ from the fixed reference configuration $\Omega \subset \mathbb{R}^3$ we let

$$\varepsilon = (\varepsilon_{ij}) = \frac{1}{2}\big(u_{i,j} + u_{j,i}\big)$$

be the corresponding linearized strain ($u_{i,j} = \partial_j u_i$).

Moving within the classical theory of inelasticity at small strains (see [38]), we additively decompose $\varepsilon = \varepsilon^{\rm el} + \varepsilon^{\rm in}$ where $\varepsilon^{\rm el}$ represents the elastic part of the strain and $\varepsilon^{\rm in}$ is the inelastic part due to the martensitic transformation in the material. Further, we again decompose the latter as $\varepsilon^{\rm in} = \varepsilon^{\rm tr} + \varepsilon^{\rm pl}$ into a recoverable (or *transformation*) part $\varepsilon^{\rm tr}$ and a non-recoverable permanent (or *plastic*) part $\varepsilon^{\rm pl}$. Eventually, we have

$$\varepsilon = \varepsilon^{\rm el} + \varepsilon^{\rm tr} + \varepsilon^{\rm pl}.$$

We prescribe the stored energy (density) of the system $E = E(\varepsilon, \varepsilon^{\rm tr}, \varepsilon^{\rm pl})$ in the form

$$\begin{aligned} E(\varepsilon, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) &:= \frac{1}{2}(\varepsilon - \varepsilon^{\rm tr} - \varepsilon^{\rm pl}){:}\mathbb{C}{:}(\varepsilon - \varepsilon^{\rm tr} - \varepsilon^{\rm pl}) \\ &\quad + \alpha_T|\varepsilon^{\rm tr}| + \frac{1}{2}\varepsilon^{\rm tr}{:}\mathbb{H}^{\rm tr}{:}\varepsilon^{\rm tr} + \frac{1}{2}\varepsilon^{\rm pl}{:}\mathbb{H}^{\rm pl}{:}\varepsilon^{\rm pl} + \varepsilon^{\rm tr}{:}\mathbb{A}{:}\varepsilon^{\rm pl} + I(\varepsilon^{\rm tr} + \varepsilon^{\rm pl}), \end{aligned}$$

where $\mathbb{C}$ is the elasticity tensor, $\mathbb{H}^{\rm tr}$ and $\mathbb{H}^{\rm pl}$ are hardening tensors, $\mathbb{A}$ is a linear symmetric coupling tensor, and I is the indicator function of the ball $B := \{a \in \mathbb{R}^{3\times 3}_{\rm dev} \ : \ |a| \le \varepsilon_L\}$ for some $\varepsilon_L > 0$. In particular $I(a) = 0$ if $a \in B$ and $I(a) = \infty$ elsewhere. The material parameter α_T depends on temperature and behaves like $\beta(T - T_*)^+$ where $\beta > 0$ plays the role of the latent heat related to the first-order phase transition between austenite and martensite and T_* is the critical transition temperature at zero stress. As here we are interested in the isothermal situation, we shall fix the temperature from the very beginning to be higher than T_* so that the linear term $\alpha_T|\varepsilon^{\rm tr}|$ occurs yielding the classical superelestic behavior which is distinctive of shape memory alloys [6, 7, 57]. Note incidentally the latter behavior is not induced by the plastic evolution of $\varepsilon^{\rm pl}$. The first term in the definition of E is the fairly classical leading term in linearized (or small strain) plasticity whereas the quadratic terms are describing a combined hardening effect and the constraining term $I(\varepsilon^{\rm tr} + \varepsilon^{\rm pl})$ refers to the experimental evidence that the inelastic behavior of the material is confined to some bounded strain proportion. In particular, $\varepsilon_L > 0$ measures the maximal inelastic strain which can be obtained via reorientation of martensitic variants. Note that in the original formulation of [8] the constraining term $I(\varepsilon^{\rm tr})$ appears in the energy whereas here we have $I(\varepsilon^{\rm tr}+\varepsilon^{\rm pl})$ instead in order to bound the full inelastic strain (the experimental effectiveness of these two options being comparable). Other options such that considering two indicator functions $I(\varepsilon^{\rm tr}) + I(\varepsilon^{\rm pl})$ in the energy may also be considered with minor modifications.

The constitutive equations of the model read

$$\sigma = \frac{\partial E}{\partial \varepsilon}, \quad \xi^{\rm tr} = -\frac{\partial E}{\partial \varepsilon^{\rm tr}}, \quad \xi^{\rm pl} = -\frac{\partial E}{\partial \varepsilon^{\rm pl}}, \tag{1}$$

where σ is the stress and $\xi^{\rm tr}$ and $\xi^{\rm pl}$ are the thermodynamic forces associated with the internal variables $\varepsilon^{\rm tr}$ and $\varepsilon^{\rm pl}$, respectively.

The model is completed by prescribing a flow rule for the internal variables $\varepsilon^{\rm tr}$ and $\varepsilon^{\rm pl}$. This is achieved by introducing the positively 1-homogeneous dissipation (density) function $D : \mathbb{R}^{3\times 3}_{\rm dev} \times \mathbb{R}^{3\times 3}_{\rm dev} \to [0, \infty)$

$$D(\dot\varepsilon^{\rm tr}, \dot\varepsilon^{\rm pl}) = \left((R^{\rm tr})^p|\dot\varepsilon^{\rm tr}|^p + (R^{\rm pl})^p|\dot\varepsilon^{\rm pl}|^p\right)^{1/p}, \quad p \in [1, \infty]$$

where $R^{\rm tr}, R^{\rm pl}$ are representing positive transformation radii. As usual, in case $p = \infty$ the latter means

$$D(\dot\varepsilon^{\rm tr}, \dot\varepsilon^{\rm pl}) = \max\left\{R^{\rm tr}|\dot\varepsilon^{\rm tr}|, R^{\rm pl}|\dot\varepsilon^{\rm pl}|\right\}.$$

The generalized normality assumption [38] entail that the constitutive material relation reads

$$\begin{pmatrix} 0 \\ \partial_{\dot\varepsilon^{\rm tr}} D(\dot\varepsilon^{\rm tr}, \dot\varepsilon^{\rm pl}) \\ \partial_{\dot\varepsilon^{\rm pl}} D(\dot\varepsilon^{\rm tr}, \dot\varepsilon^{\rm pl}) \end{pmatrix} + \begin{pmatrix} \partial_{\varepsilon} E(\varepsilon, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) \\ \partial_{\varepsilon^{\rm tr}} E(\varepsilon, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) \\ \partial_{\varepsilon^{\rm pl}} E(\varepsilon, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) \end{pmatrix} \ni \begin{pmatrix} \sigma \\ 0 \\ 0 \end{pmatrix}. \tag{2}$$

Here, the symbol ∂ is systematically used for the subdifferential (with respect to the indicated variables) in the sense of Convex Analysis. Along with the above choices for E and D, by fixing for instance $p = 1$, the latter constitutive relations read

$$\begin{aligned} \mathbb{C}(\varepsilon - \varepsilon^{\rm tr} - \varepsilon^{\rm pl}) &= \sigma, \\ R^{\rm tr}\partial|\dot\varepsilon^{\rm tr}| + \alpha_T \partial|\varepsilon^{\rm tr}| + \mathbb{H}^{\rm tr}\varepsilon^{\rm tr} + \partial I(\varepsilon^{\rm tr} + \varepsilon^{\rm pl}) &\ni \sigma - \mathbb{A}\varepsilon^{\rm pl}, \\ R^{\rm pl}\partial|\dot\varepsilon^{\rm pl}| + \mathbb{H}^{\rm pl}\varepsilon^{\rm pl} + \partial I(\varepsilon^{\rm tr} + \varepsilon^{\rm pl}) &\ni \sigma - \mathbb{A}\varepsilon^{\rm tr}. \end{aligned}$$

Note that the dynamics of the internal parameters is here fully reversible. In particular, the residual plasticity *tensor* $\varepsilon^{\rm pl}$ is not subject to irrevesibility constraints, for simplicity. Let us however mention that it would be possible to augment the model by adding an extra internal *scalar* variable, the accumulated plastic hystory, say, in order to take irreversibility into account.

As for the full quasi-static evolution of the material we shall couple the constitutive relation (2) with the equilibrium equation

$$\operatorname{div} \sigma + f = 0 \quad \text{in} \quad \Omega, \tag{3}$$

where f is a given body force, along with the boundary conditions

$$\sigma n = g \quad \text{on} \quad \Gamma_{\rm tr}, \quad u = u^{\rm Dir} \quad \text{on} \quad \Gamma_{\rm Dir}. \tag{4}$$

Here n is the outer unit normal to the boundary $\partial\Omega$, g is a given traction on $\Gamma_{\rm tr} \subset \partial\Omega$, and $u^{\rm Dir}$ is a prescribed displacement on $\Gamma_{\rm Dir} = \partial\Omega \setminus \Gamma_{\rm tr}$, respectively.

The evolution problems (2) and (2)-(4) consist in a tensorial evolutionary variational inequality, possibly coupled with a linear elliptic PDE system. As inertia and viscosity effects are neglected, time plays here the role of a parameter and the whole problem is invariant under time rescalings. Namely, the model is *rate-independent* and we frame our analysis in the context of *energetic formulations* of rate-independent processes recently proposed by Mielke & Theil [46] (see also [37, 47]). This approach is based on *equivalently* reformulating the differential problems as the coupling of a *global stability* condition and an *energy conservation* relation. Relevant definitions and details are given below.

3. **Assumptions and preliminaries.**

We shall now prepare some notation and summarize our assumptions.

3.1. **Reference configuration and prescribed boundary displacement.**

For all $u \in H^1_{\rm loc}(\mathbb{R}^3;\mathbb{R}^3)$ the standard symmetric gradient $(\nabla u + \nabla u^\top)/2$ of u will be denoted by $\varepsilon(u) \in L^2_{\rm loc}(\mathbb{R}^3;\mathbb{R}^{3\times3}_{\rm sym})$.

Let Ω be a non-empty, connected, bounded, and open subset of $\mathbb{R}^3$ with Lipschitz boundary. Let $\Gamma_{\rm tr}, \Gamma_{\rm Dir} \subset \partial\Omega$ with $\Gamma_{\rm tr} \cup \Gamma_{\rm Dir} = \partial\Omega$, $\Gamma_{\rm tr} \cap \Gamma_{\rm Dir} = \emptyset$. We will assume that $\mathcal{H}^2(\Gamma_{\rm Dir}) > 0$. This implies that the well known Korn inequality (see, for instance, [19], Thm. 3.1) holds:

$$c_{\rm Korn}||u||^2_{H^1(\Omega;\mathbb{R}^3)} \le ||u||^2_{L^2(\Gamma_{\rm Dir};\mathbb{R}^3)} + ||\varepsilon(u)||^2_{L^2(\Omega;\mathbb{R}^{3\times3}_{\rm sym})} \tag{5}$$

for any $u \in H^1(\Omega;\mathbb{R}^3)$, and for some constant $c_{\rm Korn} > 0$. Finally we prescribe some non-homogeneous Dirichlet boundary condition $u^{\rm Dir}$ on $\Gamma_{\rm Dir}$ which we think as the trace of a (not renamed) function $u^{\rm Dir} \in W^{1,1}(0,T;H^1(\Omega;\mathbb{R}^3))$.

Given any A, $B \in \mathbb{R}^{3\times3\times3}$ (3-tensors), we define the *triple* contraction product $A \,\therefore\, B$ as the scalar $A \,\therefore\, B := A_{ijk}B_{ijk}$.

3.2. **Elastic energy.** Let $\mathbb{C}$ be the elastic tensor, i.e. a symmetric and positive definite 4-tensor $\mathbb{C} \in \mathbb{R}^{3\times3\times3\times3}_{\rm sym}$. The stored elastic energy functional $\mathcal{C} : L^2(\Omega;\mathbb{R}^{3\times3}_{\rm sym}) \to [0,+\infty)$ is given by

$$\mathcal{C}(a) := \frac{1}{2}\int_\Omega a{:}\mathbb{C}{:}a \,\mathrm{d}x.$$

3.3. **Inelastic energy.** As for the stored inelastic (transformation and plastic) energy, we shall prescribe the function $F_0 : \mathbb{R}^{3\times3}_{\rm dev} \times \mathbb{R}^{3\times3}_{\rm dev} \to [0,+\infty]$ as

$$F_0(a,b) := \alpha_T|a| + \frac{1}{2}a{:}\mathbb{H}^{\rm tr}{:}a + \frac{1}{2}b{:}\mathbb{H}^{\rm pl}{:}b + a{:}\mathbb{A}{:}b + I(a+b)$$

where $\mathbb{H}^{\rm tr}, \mathbb{H}^{\rm pl}, \mathbb{A} \in \mathbb{R}^{3\times3\times3\times3}_{\rm sym}$ are symmetric 4-tensors mapping $\mathbb{R}^{3\times3}_{\rm dev}$ into itself and such that

$$(a,b) \mapsto \frac{1}{2}a{:}\mathbb{H}^{\rm tr}{:}a + \frac{1}{2}b{:}\mathbb{H}^{\rm pl}{:}b + a{:}\mathbb{A}{:}b$$

is positive definite. The stored inelastic energy functional $\mathcal{F}_0 : L^2(\Omega;\mathbb{R}^{3\times3}_{\rm dev} \times \mathbb{R}^{3\times3}_{\rm dev}) \to [0,+\infty]$ is defined as

$$\mathcal{F}_0(a,b) := \begin{cases} \displaystyle\int_\Omega F_0(a,b)\,\mathrm{d}x, & \text{if } F_0(a,b) \in L^1(\Omega) \\ \infty & \text{else.} \end{cases}$$

In the following we will also deal with some regularization of F. More precisely we introduce an approximation parameter $\rho > 0$ and some functions $F_\rho \in \mathcal{C}^{2,1}(\mathbb{R}^{3\times3}_{\rm dev} \times \mathbb{R}^{3\times3}_{\rm dev})$ with F_ρ pointwise increasing in ρ, $\nabla^2 F_\rho$ bounded and uniformly positive definite, and $F_\rho(0) = 0$. For all $\rho \geq 0$, let $\mathcal{F}_\rho : L^2(\Omega;\mathbb{R}^{3\times3}_{\rm dev} \times \mathbb{R}^{3\times3}_{\rm dev}) \to [0,+\infty)$ be defined by

$$\mathcal{F}_\rho(a,b) := \int_\Omega F_\rho(a,b)\,\mathrm{d}x.$$

Note that the original modelling choice from [8] corresponds to the non-regularized case $\rho = 0$. Still, the smooth situation $\rho > 0$ bears some interest as it allows a continuous dependence result and is hence better suited for numerical implementation.

3.4. **State space and stored energy.** We specialize the definition of energy density functional, for all $\rho \geq 0$, as

$$E_\rho(\varepsilon,\varepsilon^{\rm tr},\varepsilon^{\rm pl}) := \frac{1}{2}\mathbb{C}(\varepsilon - \varepsilon^{\rm tr} - \varepsilon^{\rm pl}){:}(\varepsilon - \varepsilon^{\rm tr} - \varepsilon^{\rm pl}) + F_\rho(\varepsilon^{\rm tr},\varepsilon^{\rm pl}).$$

Let us now define the space

$$\mathcal{Y} := H^1(\Omega;\mathbb{R}^3) \times H^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev}) \times H^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev}).$$

For the sake of taking into account Dirichlet boundary conditions we shall define, for all $\bar u \in H^1(\Omega;\mathbb{R}^3)$,

$$\mathcal{Y}(\bar u) := \{(u,\varepsilon^{\rm tr},\varepsilon^{\rm pl}) \in \mathcal{Y} \;:\; u = \bar u \text{ on } \Gamma_{\rm Dir}\}.$$

We are now in the position of defining the total stored energy functional $\mathcal{E}_\rho : \mathcal{Y} \to [0,\infty]$ as

$$\mathcal{E}_\rho(u,\varepsilon^{\rm tr},\varepsilon^{\rm pl}) := \mathcal{C}(\varepsilon(u)-\varepsilon^{\rm tr}-\varepsilon^{\rm pl}) + \mathcal{F}_\rho(\varepsilon^{\rm tr},\varepsilon^{\rm pl}) + \frac{\nu}{2}\int_\Omega |\nabla\varepsilon^{\rm tr}|^2 {\rm d}x + \frac{\bar\nu}{2}\int_\Omega |\nabla\varepsilon^{\rm pl}|^2 {\rm d}x,$$

where $\nu, \bar\nu > 0$ are given. The last two terms above are expected to measure some non-local interaction effect for the internal variables. Indeed, gradients of inelastic strains have already been considered in the frame of shape-memory materials by Frémond [23] and the reader is referred also to Arndt et al. [3], Fried & Gurtin [25], Kružík et al. [33], Mielke & Roubíček [44], Roubíček [55, 56] for examples and discussions on nonlocal energy contributions.

Before moving on, let us explicitly note that both E_ρ and $\mathcal{E}_\rho$ are uniformly convex, independently of ρ, with respect to the metric in

$$Y := \mathbb{R}^{3\times3}_{\rm sym} \times \mathbb{R}^{3\times3}_{\rm dev} \times \mathbb{R}^{3\times3}_{\rm dev}$$

and that of $\mathcal{Y}$, respectively. We shall term the corresponding uniform convexity constant with $c_{\rm conv} > 0$ in the following.

3.5. **Load and traction.** We assume to be given the body force $f \in W^{1,1}(0,T;L^2(\Omega;\mathbb{R}^3))$ and a surface traction $g \in W^{1,1}(0,T;L^2(\Gamma_{\rm tr};\mathbb{R}^3))$. Then the total load $\ell \in W^{1,1}(0,T;(H^1(\Omega;\mathbb{R}^3))')$ for the system is given by

$$\langle \ell(t), u\rangle := \int_\Omega f\cdot u \,{\rm d}x + \int_{\Gamma_{\rm tr}} g\cdot u \,{\rm d}\mathcal{H}^2,$$

for all $u \in H^1(\Omega;\mathbb{R}^3)$ and $t\in[0,T]$, where, as usual, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(H^1(\Omega;\mathbb{R}^3))'$ and $H^1(\Omega;\mathbb{R}^3)$.

3.6. **Dissipation potential.** Recall that $D : \mathbb{R}^{3\times3}_{\rm dev}\times\mathbb{R}^{3\times3}_{\rm dev} \to [0,+\infty)$ is continuous, positively 1-homogeneous, and fulfills the *triangle inequality*

$$D(a_1+a_2, b_1+b_2) \le D(a_1,b_1) + D(a_2,b_2) \tag{6}$$

for all $a_1, a_2, b_1, b_2 \in \mathbb{R}^{3\times3}_{\rm dev}$.

Next, we define the corresponding dissipation functional $\mathcal{D} : L^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev}\times\mathbb{R}^{3\times3}_{\rm dev}) \to [0,+\infty)$ as

$$\mathcal{D}(a,b) := \int_\Omega D(a,b)\,{\rm d}x.$$

Finally, for any $(\varepsilon^{\rm tr},\varepsilon^{\rm pl}) : [0,T] \to \mathbb{R}^{3\times3}_{\rm dev}\times\mathbb{R}^{3\times3}_{\rm dev}$ and $[s,t]\subset[0,T]$ we let

$$\mathrm{Diss}_D(\varepsilon^{\rm tr},\varepsilon^{\rm pl};[s,t]) := \sup\Big\{ \sum_{i=1}^N D(\varepsilon^{\rm tr}(t_i)-\varepsilon^{\rm tr}(t_{i-1}), \varepsilon^{\rm pl}(t_i)-\varepsilon^{\rm pl}(t_{i-1})) : \{s=t_0<t_1<\dots<t_{N-1}<t_N=t\}\Big\}$$

where the supremum is taken over the set of all finite partitions. An analogous notion $\mathrm{Diss}_{\mathcal{D}}(\varepsilon^{\rm tr},\varepsilon^{\rm pl};[s,t])$ based on the functional $\mathcal{D}$ for functions of time taking values in $L^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev}\times\mathbb{R}^{3\times3}_{\rm dev})$ will also be considered.

4. **The constitutive relation.** This section focuses on the constitutive relation problem. For a fixed $\rho \geq 0$, we aim at determining conditions on the given stress $\sigma : [0,T] \to \mathbb{R}^{3\times 3}_{\rm sym}$ and initial values $(\varepsilon_0, \varepsilon_0^{\rm tr}, \varepsilon_0^{\rm pl})$ in order to possibly solve the constitutive relation (2) along with

$$(\varepsilon(0), \varepsilon^{\rm tr}(0), \varepsilon^{\rm pl}(0)) = (\varepsilon_0, \varepsilon_0^{\rm tr}, \varepsilon_0^{\rm pl}). \tag{7}$$

The interest for the constitutive relation problem is twofold. From the one hand, a detailed study of the latter is usually an important step toward the direction of the investigation of the full quasi-static evolution problem. This is especially true with respect to numerics where generally material updates are computed only locally. On the other hand, the full quasi-static evolution problem might reduce to the zero-dimensional constitutive relation problem under specific geometric restrictions or symmetries. This is particularly the case of the evolution of a shape memory wire which is clamped on one and subject to a specific time-dependent traction at the other (no distributed forces). By assuming that the material is homogeneous in space at the initial time, one gets that the same holds for all future times. In particular, the material evolves according solely to the constitutive relation.

Our first aim is to provide an equivalent version of (2), (7) in the frame of energetic formulations [39]. In particular, let us define the set of *stable states* at time $t \in [0,T]$ as

$$\begin{aligned} S_\rho(t) \ := \Big\{ &(\varepsilon, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) \in Y : E_\rho(\varepsilon, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) < \infty \ \text{ and, for all } \ (\bar\varepsilon, \bar\varepsilon^{\rm tr}, \bar\varepsilon^{\rm pl}) \in Y, \\ &E_\rho(\varepsilon, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) - \sigma(t){:}\varepsilon \leq \ E_\rho(\bar\varepsilon, \bar\varepsilon^{\rm tr}, \bar\varepsilon^{\rm pl}) - \sigma(t){:}\bar\varepsilon + D(\varepsilon^{\rm tr} - \bar\varepsilon^{\rm tr}, \varepsilon^{\rm pl} - \bar\varepsilon^{\rm pl}) \Big\}. \end{aligned} \tag{8}$$

For an *energetic solution* we mean an everywhere defined triplet $(\varepsilon, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) : [0,T] \to Y$ such that $(\varepsilon(0), \varepsilon^{\rm tr}(0), \varepsilon^{\rm pl}(0)) = (\varepsilon_0, \varepsilon_0^{\rm tr}, \varepsilon_0^{\rm pl})$, the function $t \mapsto \dot\sigma(t){:}\varepsilon(t)$ is integrable, and, for all $t \in [0,T]$, we have

Global stability:

$$(\varepsilon(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t)) \in S_\rho(t) \tag{9}$$

Energy conservation:

$$\begin{aligned} &E_\rho(\varepsilon(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t)) - \sigma(t){:}\varepsilon(t) + {\rm Diss}_D((\varepsilon^{\rm tr}, \varepsilon^{\rm pl}), [0,t]) \\ &= E_\rho(\varepsilon(0), \varepsilon^{\rm tr}(0), \varepsilon^{\rm pl}(0)) - \sigma(0){:}\varepsilon(0) - \int_0^t \dot\sigma(s){:}\varepsilon(s)\,{\rm d}s. \end{aligned} \tag{10}$$

As the energy E_ρ is uniformly convex, energetic solutions and classical strong solutions coincide [39] (σ being sufficiently smooth). We however focus here on the energetic formulation as is enlightens the variational structure of the problem and is somehow more suited for proving convergence results. In particular, energetic formulations are quite naturally linked to time discretizations.

4.1. **The incremental problem.** In order to construct an energetic solution to the constitutive relation problem, one considers an implicit time discretization procedure. Let us fix the partition $P := \{0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T\}$ with diameter $\tau := \max_{i=1,\ldots,N}(t_i - t_{i-1})$. Moreover, let $(\varepsilon_0, \varepsilon_0^{\rm tr}, \varepsilon_0^{\rm pl}) \in S_\rho(0)$ be a given initial datum. We solve iteratively the minimum problem

$$(\varepsilon_i, \varepsilon_i^{\rm tr}, \varepsilon_i^{\rm pl}) = \underset{(\varepsilon, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) \in Y}{\rm Arg\,Min} \Big(E_\rho(\varepsilon, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) - \sigma(t_i){:}\varepsilon + D(\varepsilon^{\rm tr} - \varepsilon_{i-1}^{\rm tr}, \varepsilon^{\rm pl} - \varepsilon_{i-1}^{\rm pl}) \Big) \tag{11}$$

for $i = 1, \ldots, N$. This can be uniquely done as, for all $(\bar{\varepsilon}^{\rm tr}, \bar{\varepsilon}^{\rm pl}) \in \mathbb{R}^{3\times3}_{\rm dev} \times \mathbb{R}^{3\times3}_{\rm dev}$ and $t \in [0,T]$, the function

$$(\varepsilon, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) \mapsto E_\rho(\varepsilon, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) - \sigma(t){:}\varepsilon + D(\varepsilon^{\rm tr} - \bar{\varepsilon}^{\rm tr}, \varepsilon^{\rm pl} - \bar{\varepsilon}^{\rm pl}),$$

is uniformly convex. The latter procedure is generally referred to as the *the incremental problem* associated to the constitutive relation.

By using the triangle inequality (6) we show that the minimization property (11) entails that $(\varepsilon_i, \varepsilon_i^{\rm tr}, \varepsilon_i^{\rm pl}) \in S_\rho(t_i)$ that is

$$(\varepsilon_i, \varepsilon_i^{\rm tr}, \varepsilon_i^{\rm pl}) = \mathop{\rm Arg\,Min}_{(\bar{\varepsilon}, \bar{\varepsilon}^{\rm tr}, \bar{\varepsilon}^{\rm pl}) \in Y} \Big(E_\rho(\bar{\varepsilon}, \bar{\varepsilon}^{\rm tr}, \bar{\varepsilon}^{\rm pl}) - \sigma(t_i){:}\bar{\varepsilon} + D(\bar{\varepsilon}^{\rm tr} - \varepsilon_i^{\rm tr}, \bar{\varepsilon}^{\rm pl} - \varepsilon_i^{\rm pl}) \Big) \qquad (12)$$

for all $i = 1, \ldots, N$. Indeed, for any $(\bar{\varepsilon}, \bar{\varepsilon}^{\rm tr}, \bar{\varepsilon}^{\rm pl}) \in Y$, we get

$$\begin{aligned} &E_\rho(\varepsilon_i, \varepsilon_i^{\rm tr}, \varepsilon_i^{\rm pl}) - \sigma(t_i){:}\varepsilon_i + D(\varepsilon_i^{\rm tr} - \varepsilon_{i-1}^{\rm tr}, \varepsilon_i^{\rm pl} - \varepsilon_{i-1}^{\rm pl}) \\ &\quad \le E_\rho(\bar{\varepsilon}, \bar{\varepsilon}^{\rm tr}, \bar{\varepsilon}^{\rm pl}) - \sigma(t_i){:}\bar{\varepsilon} + D(\bar{\varepsilon}^{\rm tr} - \varepsilon_{i-1}^{\rm tr}, \bar{\varepsilon}^{\rm pl} - \varepsilon_{i-1}^{\rm pl}) \\ &\quad \le E_\rho(\bar{\varepsilon}, \bar{\varepsilon}^{\rm tr}, \bar{\varepsilon}^{\rm pl}) - \sigma(t_i){:}\bar{\varepsilon} + D(\bar{\varepsilon}^{\rm tr} - \varepsilon_i^{\rm tr}, \bar{\varepsilon}^{\rm pl} - \varepsilon_i^{\rm pl}) + D(\varepsilon_i^{\rm tr} - \varepsilon_{i-1}^{\rm tr}, \varepsilon_i^{\rm pl} - \varepsilon_{i-1}^{\rm pl}) \end{aligned}$$

where in the last line we used (6). Now the term $D(\varepsilon_i^{\rm tr} - \varepsilon_{i-1}^{\rm tr}, \varepsilon_i^{\rm pl} - \varepsilon_{i-1}^{\rm pl})$ cancels out and we are done.

Finally, note that, given $\varepsilon_i^{\rm tr}$ and $\varepsilon_i^{\rm pl}$, the tensor $\varepsilon_i \in \mathbb{R}^{3\times3}_{\rm sym}$ minimizing $\varepsilon \mapsto E_\rho(\varepsilon, \varepsilon_i^{\rm tr}, \varepsilon_i^{\rm pl}) - \sigma(t_i){:}\varepsilon$ is uniquely determined and depends *linearly* on $\varepsilon_i^{\rm tr}, \varepsilon_i^{\rm pl}$, and $\sigma(t_i)$. In particular, we have that

$$\varepsilon_i = L(\varepsilon_i^{\rm tr}, \varepsilon_i^{\rm pl}, \sigma(t_i))$$

for some given linear and continuous operator $L : \mathbb{R}^{3\times3}_{\rm dev} \times \mathbb{R}^{3\times3}_{\rm dev} \times \mathbb{R}^{3\times3}_{\rm sym} \to \mathbb{R}^{3\times3}_{\rm sym}$. Note that, if $(\varepsilon, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) \in S_\rho(t)$ then necessarily $\varepsilon = L(\varepsilon^{\rm tr}, \varepsilon^{\rm pl}, \sigma(t))$.

4.2. **Well-posedness result.** We have the following.

Theorem 4.1 (Well-posedness for the constitutive relation)**.** *Let $\rho \ge 0$. Given $\sigma \in W^{1,1}(0,T;\mathbb{R}^{3\times3}_{\rm sym})$ and $(\varepsilon_0, \varepsilon_0^{\rm tr}, \varepsilon_0^{\rm pl}) \in S_\rho(0)$ there exists an* energetic solution *$t \mapsto (\varepsilon(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t))$ to (9)-(10). Moreover, $t \mapsto (\varepsilon(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t)) \in W^{1,1}(0,T;Y)$.*

If $\rho > 0$, the solution depends continuously on data. In particular, there exists a positive constant $c_{\rm dep}$ depending just on parameters such that, given two solutions $t \mapsto (\varepsilon_j(t), \varepsilon_j^{\rm tr}(t), \varepsilon_j^{\rm pl}(t))$ corresponding to data $(\sigma_j, \varepsilon_{0,j}, \varepsilon_{0,j}^{\rm tr}, \varepsilon_{0,j}^{\rm pl})$ for $j = 1, 2$, one has

$$\begin{aligned} &|(\varepsilon_1 - \varepsilon_2)(t)|^2 + |(\varepsilon_1^{\rm tr} - \varepsilon_2^{\rm tr})(t)|^2 + |(\varepsilon_1^{\rm pl} - \varepsilon_2^{\rm pl})(t)|^2 \\ &\qquad \le c_{\rm dep}\big(|\varepsilon_{0,1} - \varepsilon_{0,2}|^2 + |\varepsilon_{0,1}^{\rm tr} - \varepsilon_{0,2}^{\rm tr}|^2 + |\varepsilon_{0,1}^{\rm pl} - \varepsilon_{0,2}^{\rm pl}|^2 \\ &\qquad + \|\sigma_1 - \sigma_2\|^2_{W^{1,1}(0,t;\mathbb{R}^{3\times3}_{\rm sym})}\big) \quad \forall t \in [0,T]. \end{aligned} \qquad (13)$$

In particular, if $\rho > 0$ the solution is unique.

We shall not provide here a full proof of this result. Indeed, in the smooth situation of $\rho > 0$, the result follows at once from the general theory from [39]. The non-smooth case of $\rho = 0$ the argument is just slightly more delicate and has been already detailed in the close situation in [5]. We provide here a sketch of the argument for the reader's convenience.

The construction of an energetic solution builds up on the passage to the limit in the time-discretization diameter in *incremental solutions*, namely solutions of incremental problems. Assume to be given a sequence of partitions $P^n = \{0 =$

$t_0^n < \cdots < t_{N^n}^n = T\}$ with diameters $\tau^n = \max_{i=1,\ldots,N^n}(t_i^n - t_{i-1}^n)$ going to 0 and solve the corresponding incremental problems (11). We denote by $(\varepsilon_n, \varepsilon_n^{\rm tr}, \varepsilon_n^{\rm pl})$ the incremental solutions, i.e. the right-continuous piecewise constant interpolants of $\{(\varepsilon_i^n, \varepsilon_i^{{\rm tr},n}, \varepsilon_i^{{\rm pl},n})\}_{i=0}^{N^n}$ on the partitions P^n. Moreover, let us denote by $\tau^n : [0,T] \to [0,T]$ the function $\tau^n(t) := t_n^i$ for $t \in [t_n^i, t_n^{i+1})$, $i = 0, \ldots, N^n - 1$.

By using the minimality from (11) one deduces that

$$\max_{t\in[0,T]} E_\rho(\varepsilon_n(t), \varepsilon_n^{\rm tr}(t), \varepsilon_n^{\rm pl}(t)) \ \text{ and } \ {\rm Diss}_D((\varepsilon_n^{\rm tr}, \varepsilon_n^{\rm pl}), [0,T]) \\ \text{are bounded independently of } n. \tag{14}$$

Now, Helly's selection principle, entails the possibility of finding a (not relabelled) subsequence of partitions and a non-decreasing function $\phi : [0,T] \to [0,+\infty)$ such that

$$(\varepsilon_n^{\rm tr}(t), \varepsilon_n^{\rm pl}(t)) \to (\varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t)), \quad {\rm Diss}_D((\varepsilon_n^{\rm tr}, \varepsilon_n^{\rm pl}), [0,t]) \to \phi(t) \quad \forall t \in [0,T], \tag{15}$$

$${\rm Diss}_D((\varepsilon^{\rm tr}, \varepsilon^{\rm pl}), [s,t]) \le \phi(t) - \phi(s) \quad \forall [s,t] \subset [0,T]. \tag{16}$$

Hence, for all $t \in [0,T]$, we finally obtain the unique limit

$$\varepsilon(t) = L(\varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t), \sigma(\tau_n(t)))$$

since

$$\varepsilon_n(t) = L(\varepsilon_n^{\rm tr}(t), \varepsilon_n^{\rm pl}(t), \sigma(\tau_n(t))) \to L(\varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t), \sigma(t)) = \varepsilon(t).$$

It is a standard matter to check the global stability of $t \mapsto (\varepsilon(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t))$ as the set of stable states is closed due to the continuity of σ, E_ρ, and D.

An upper estimate on the energy comes from choosing

$$(\varepsilon, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) = (\varepsilon_{i-1}, \varepsilon_{i-1}^{\rm tr}, \varepsilon_{i-1}^{\rm pl})$$

in (11) and summing on i as we have

$$E_\rho(\varepsilon_n(t), \varepsilon_n^{\rm tr}(t), \varepsilon_n^{\rm pl}(t)) - \sigma(\tau^n(t)){:}\varepsilon_n(t) + {\rm Diss}_D((\varepsilon_n^{\rm tr}(t), \varepsilon_n^{\rm pl}(t)), [0, \tau^n(t)])$$
$$\le E_\rho(\varepsilon_0, \varepsilon_0^{\rm tr}, \varepsilon_0^{\rm pl}) - \sigma(0){:}\varepsilon_0 - \int_0^{\tau^n(t)} \dot\sigma{:}\varepsilon_n \, {\rm d}s.$$

The lower energy estimate is instead a consequence of global stability (9) in the same spirit of [39, Prop. 5.7]. Hence, the absolute continuity of $t \mapsto (\varepsilon(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t))$ ensues uniformly with respect to n and ρ. As a consequence, $(\varepsilon_n^{\rm tr}, \varepsilon_n^{\rm pl}) \to (\varepsilon^{\rm tr}, \varepsilon^{\rm pl})$ uniformly by the Ascoli-Arzelà Theorem and

$$\varepsilon_n = L(\varepsilon_n^{\rm tr}, \varepsilon_n^{\rm pl}, \sigma_n) \to L(\varepsilon^{\rm tr}, \varepsilon^{\rm pl}, \sigma) = L\varepsilon \quad \text{uniformly in } \ [0,T]$$

by the continuity of L. Moreover, the convergence of energy and dissipation can also be achieved. We summarize these facts in the following.

Lemma 4.2 (Convergence of incremental solutions)**.** *Let $(\varepsilon_n, \varepsilon_n^{\rm tr}, \varepsilon_n^{\rm pl})$ denote the (unique) incremental solutions related to a sequence of partitions P^n with diameters $\tau^n = \max_{i=1,\ldots,N^n}(t_i^n - t_{i-1}^n)$ going to 0. Then, we have that, at least for a not relabeled subsequence (the whole sequence for $\rho > 0$), for all $t \in [0,T]$,*

$$(\varepsilon_n(t), \varepsilon_n^{\rm tr}(t), \varepsilon_n^{\rm pl}(t)) \to (\varepsilon(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t)), \tag{17}$$

$${\rm Diss}_D((\varepsilon_n^{\rm tr}, \varepsilon_n^{\rm pl}), [0,t]) \to {\rm Diss}_D((\varepsilon^{\rm tr}, \varepsilon^{\rm pl}), [0,t]), \tag{18}$$

$$E_\rho(\varepsilon_n(t), \varepsilon_n^{\rm tr}(t), \varepsilon_n^{\rm pl}(t)) \to E_\rho(\varepsilon(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t)) \tag{19}$$

where $(\varepsilon, \varepsilon^{\rm tr}, \varepsilon^{\rm pl})$ in an energetic solution.

Let us now provide a uniform bound on the continuity modulus of $t \mapsto (\varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t))$ by exploiting uniform convexity and global stability. Let $[s,t] \subset [0,T]$ be given. Since $(\varepsilon(s), \varepsilon^{\rm tr}(s), \varepsilon^{\rm pl}(s)) \in S_\rho(s)$, we get

$$\begin{aligned}
&{\rm c_{conv}}(|\varepsilon(t)-\varepsilon(s)|^2 + |\varepsilon^{\rm tr}(t)-\varepsilon^{\rm tr}(s)|^2 + |\varepsilon^{\rm pl}(t)-\varepsilon^{\rm pl}(s)|^2)\\
&\quad \le E_\rho(\varepsilon(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t)) - \sigma(s) : \varepsilon(t) + D(\varepsilon^{\rm tr}(t)-\varepsilon^{\rm tr}(s), \varepsilon^{\rm pl}(t)-\varepsilon^{\rm pl}(s))\\
&\quad - E_\rho(\varepsilon(s), \varepsilon^{\rm tr}(s), \varepsilon^{\rm pl}(s)) + \sigma(s) : \varepsilon(s)\\
&\quad \le E_\rho(\varepsilon(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t)) - \sigma(t) : \varepsilon(t) + {\rm Diss}_D((\varepsilon^{\rm tr}, \varepsilon^{\rm pl}), [s,t])\\
&\quad - E_\rho(\varepsilon(s), \varepsilon^{\rm tr}(s), \varepsilon^{\rm pl}(s)) + \sigma(s) : \varepsilon(s) - (\sigma(s)-\sigma(t)) : \varepsilon(t)\\
&\quad \le -\int_s^t \dot\sigma(r) : (\varepsilon(r)-\varepsilon(t))\,{\rm d}r
\end{aligned}$$

where $\rm c_{conv}$ is the uniform convexity constant of E_ρ. By applying the Gronwall Lemma we deduce that

$$|\varepsilon(t)-\varepsilon(s)| + |\varepsilon^{\rm tr}(t)-\varepsilon^{\rm tr}(s)| + |\varepsilon^{\rm pl}(t)-\varepsilon^{\rm pl}(s)| \le c_{\rm abs}\int_s^t |\dot\sigma(r)|\,{\rm d}r,$$

for some $c_{\rm abs} > 0$ depending only on $\rm c_{conv}$.

The continuous dependence proof follows at once by repeating the argument of [5, Thm. 3.4]. Moreover, in the very same spirit of [5, Lemma 3.6], in case $\rho > 0$ we are in the position of proving an a priori error bound on the discretization. In particular, we have the following.

Lemma 4.3 (Error bound)**.** *Let $\rho > 0$. Then, there exists a positive constant $c_{\rm err}$ depending on data such that*

$$\max_{t\in[0,T]} \big(|(\varepsilon-\varepsilon_n)(t)| + |(\varepsilon^{\rm tr}-\varepsilon^{\rm tr}_n)(t)| + |(\varepsilon^{\rm pl}-\varepsilon^{\rm pl}_n)(t)|\big) \le c_{\rm err}\sqrt{\tau}.$$

Moreover, for $\rho > 0$, the convergence in (17) is uniform in time.

5. The quasi-static evolution problem.

The results of Section 4 can be reproduced at the level of the full three-dimensional quasi-static evolution problem ensuing from the combination of the constitutive relation (2) and the corresponding initial condition (7) with the quasi-static equilibrium equation (3) along with the boundary conditions (4).

By recalling the notation and assumptions of Section 3, we shall start by making precise the notion of energetic solution of the quasi-static evolution problem. Energetic solutions are everywhere defined functions $t \in [0,T] \mapsto (u(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t)) \in \mathcal{Y}(u^{\rm Dir}(t))$ such that $(u(0), \varepsilon^{\rm tr}(0), \varepsilon^{\rm pl}(0)) = (u_0, \varepsilon^{\rm tr}_0, \varepsilon^{\rm pl}_0)$ for some given initial datum $(u_0, \varepsilon^{\rm tr}_0, \varepsilon^{\rm pl}_0) \in \mathcal{Y}(u^{\rm Dir}(0))$, the function $t \mapsto \langle \dot\ell(t), u(t)\rangle$ is integrable and, for any $t \in [0,T]$, we have

Global stability:

$$\begin{aligned}
&(u(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t)) \in \Big\{ (u, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) \in \mathcal{Y}(u^{\rm Dir}(t)) \quad \text{such that}\\
&\quad \mathcal{E}_\rho(u, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) < \infty \ \text{ and, for all } \ (\bar u, \bar\varepsilon^{\rm tr}, \bar\varepsilon^{\rm pl}) \in \mathcal{Y}(u^{\rm Dir}(t)), \qquad (20)\\
&\quad \mathcal{E}_\rho(u, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) - \langle \ell(t), u\rangle \le \mathcal{E}_\rho(\bar u, \bar\varepsilon^{\rm tr}, \bar\varepsilon^{\rm pl}) - \langle \ell(t), \bar u\rangle + \mathcal{D}(\varepsilon^{\rm tr}-\bar\varepsilon^{\rm tr}, \varepsilon^{\rm pl}-\bar\varepsilon^{\rm pl}) \Big\}.
\end{aligned}$$

Energy conservation:

$$\mathcal{E}_\rho(u(t),\varepsilon^{\rm tr}(t),\varepsilon^{\rm pl}(t)) - \langle \ell(t), u(t)\rangle + {\rm Diss}_{\mathcal{D}}((\varepsilon^{\rm tr},\varepsilon^{\rm pl}),[0,t]) \\ = \mathcal{E}_\rho(u(0),\varepsilon^{\rm tr}(0),\varepsilon^{\rm pl}(0)) - \langle \ell(0), u(0)\rangle - \int_0^t \langle \dot\ell(s), u(s)\rangle\,{\rm d}s. \tag{21}$$

In case $\rho > 0$, the energy functional $\mathcal{E}_\rho$ is uniformly convex and smooth and the analysis of the latter energetic formulation follows from the general theory of [39]. Some extra care is needed in case $\rho = 0$ where smoothness is lost. In this case, the analysis of Section 4 can be adapted to the quasi-static evolution problem, possibly referring to [5] for analogous computations. Alternatively, one can rely on the asymptotic analysis of the forthcoming Subsection 6.4 and deduce the existence of an energetic solution for $\rho = 0$ from the forthcoming Theorem 6.3.

5.1. **Well-posedness result.** First of all, we perform a change of variables in (20)-(21) in order to reduce to the case of homogeneous Dirichlet boundary conditions. Indeed, we let $v = u - u^{\rm Dir} \in \mathcal{Y}_0 := \mathcal{Y}(0)$ and compute that

$$\mathcal{E}_\rho(u,\varepsilon^{\rm tr},\varepsilon^{\rm pl}) - \langle \ell, u\rangle \\ = \mathcal{E}_\rho(v,\varepsilon^{\rm tr},\varepsilon^{\rm pl}) + \int_\Omega \mathbb{C}(\varepsilon(v)-\varepsilon^{\rm tr}-\varepsilon^{\rm pl}){:}\varepsilon(u^{\rm Dir}) - \langle \ell, v\rangle + \mathcal{C}(\varepsilon(u^{\rm Dir})) - \langle \ell, u^{\rm Dir}\rangle,$$

we conclude that $(u,\varepsilon^{\rm tr},\varepsilon^{\rm pl})$ is an energetic solution of (20)-(21) if and only if $(v,\varepsilon^{\rm tr},\varepsilon^{\rm pl}) : t\in[0,T] \mapsto \mathcal{Y}_0$ is such that $(v(0),\varepsilon^{\rm tr}(0),\varepsilon^{\rm pl}(0)) = (v_0,\varepsilon^{\rm tr}_0,\varepsilon^{\rm pl}_0) = (u_0 - u^{\rm Dir}(0),\varepsilon^{\rm tr}_0,\varepsilon^{\rm pl}_0)$, the function $t\mapsto \langle \dot{\tilde\ell}(t),(v(t),\varepsilon^{\rm tr}(t),\varepsilon^{\rm pl}(t))\rangle$ is integrable, and, for all $t\in[0,T]$,

Global stability in the variable v:

$$(v(t),\varepsilon^{\rm tr}(t),\varepsilon^{\rm pl}(t)) \in \mathcal{S}_\rho(t) := \Big\{(v,\varepsilon^{\rm tr},\varepsilon^{\rm pl})\in\mathcal{Y}_0 : \mathcal{E}_\rho(v,\varepsilon^{\rm tr},\varepsilon^{\rm pl}) < \infty \text{ and} \\ \text{for all } (\bar v,\bar\varepsilon^{\rm tr},\bar\varepsilon^{\rm pl})\in\mathcal{Y}_0,\ \mathcal{E}_\rho(v,\varepsilon^{\rm tr},\varepsilon^{\rm pl}) - \langle\tilde\ell(t),(v,\varepsilon^{\rm tr},\varepsilon^{\rm pl})\rangle \le \mathcal{E}_\rho(\bar v,\bar\varepsilon^{\rm tr},\bar\varepsilon^{\rm pl}) \\ - \langle\tilde\ell(t),(\bar v,\bar\varepsilon^{\rm tr},\bar\varepsilon^{\rm pl})\rangle + \mathcal{D}(\varepsilon^{\rm tr}-\bar\varepsilon^{\rm tr},\varepsilon^{\rm pl}-\bar\varepsilon^{\rm pl})\Big\}, \tag{22}$$

Energy conservation in the variable v:

$$\mathcal{E}_\rho(v(t),\varepsilon^{\rm tr}(t),\varepsilon^{\rm pl}(t)) - \langle\tilde\ell(t),(v(t),\varepsilon^{\rm tr}(t),\varepsilon^{\rm pl}(t))\rangle + {\rm Diss}_{\mathcal{D}}((\varepsilon^{\rm tr},\varepsilon^{\rm pl}),[0,t]) \\ = \mathcal{E}_\rho(v(0),\varepsilon^{\rm tr}(0),\varepsilon^{\rm pl}(0)) - \langle\tilde\ell(0),(v(0),\varepsilon^{\rm tr}(0),\varepsilon^{\rm pl}(0))\rangle \\ - \int_0^t \langle\dot{\tilde\ell}(s),(v(s),\varepsilon^{\rm tr}(s),\varepsilon^{\rm pl}(s))\rangle\,{\rm d}s, \tag{23}$$

where $\tilde\ell : [0,T]\to\mathcal{Y}_0'$ is defined, for all $t\in[0,T]$, as

$$\langle\tilde\ell(t),(v,\varepsilon^{\rm tr},\varepsilon^{\rm pl})\rangle := -\int_\Omega \mathbb{C}(\varepsilon(v)-\varepsilon^{\rm tr}-\varepsilon^{\rm pl}) : \varepsilon(u^{\rm Dir}) + \langle\ell(t),v\rangle \\ \forall (v,\varepsilon^{\rm tr},\varepsilon^{\rm pl})\in\mathcal{Y}_0,\ t\in[0,T].$$

Notice that $u^{\rm Dir}\in W^{1,1}(0,T;H^1(\Omega;\mathbb{R}^3))$ and $\ell\in W^{1,1}(0,T;(H^1(\Omega;\mathbb{R}^3))')$ entail that $\tilde\ell\in W^{1,1}(0,T;\mathcal{Y}_0')$.

Now we are in position of stating our well-posedness result.

Theorem 5.1 (Well-posedness for the quasi-static evolution problem)**.** *Let $\rho\ge 0$. Given $\tilde\ell\in W^{1,1}(0,T;\mathcal{Y}_0')$ and $(v_0,\varepsilon^{\rm tr}_0,\varepsilon^{\rm pl}_0)\in\mathcal{S}_\rho(0)$, there exists an* energetic solution

$(v, \varepsilon^{\rm tr}, \varepsilon^{\rm pl})$ *of the quasi-static evolution problem* (22)-(23). *Moreover,* $(v, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) \in W^{1,1}(0, T; \mathcal{Y}_0)$.

If $\rho > 0$, *the solution depends continuously on data. In particular, there exists a positive constant* $c_{\rm dep,2}$ *depending just on parameters such that, given two solutions* $t \mapsto (v_j(t), \varepsilon^{\rm tr}_j(t), \varepsilon^{\rm pl}_j(t))$ *corresponding to data* $(\tilde{\ell}_j, v_{0,j}, \varepsilon^{\rm tr}_{0,j}, \varepsilon^{\rm pl}_{0,j})$ *for* $j = 1, 2$, *one has*

$$\begin{aligned}
&\|(v_1 - v_2)(t)\|^2_{H^1(\Omega;\mathbb{R}^3)} + |(\varepsilon^{\rm tr}_1 - \varepsilon^{\rm tr}_2)(t)|^2_{H^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev})} + |(\varepsilon^{\rm pl}_1 - \varepsilon^{\rm pl}_2)(t)|^2_{H^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev})} \\
&\quad \leq c_{\rm dep,2}\Big(|v_{0,1} - v_{0,2}|^2_{H^1(\Omega;\mathbb{R}^3)} + |\varepsilon^{\rm tr}_{0,1} - \varepsilon^{\rm tr}_{0,2}|^2_{H^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev})} \\
&\quad + |\varepsilon^{\rm pl}_{0,1} - \varepsilon^{\rm pl}_{0,2}|^2_{H^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev})} + \|\tilde{\ell}_1 - \tilde{\ell}_2\|^2_{W^{1,1}(0,t;\mathcal{Y}_0')}\Big) \quad \forall t \in [0, T].
\end{aligned} \tag{24}$$

In particular, if $\rho > 0$ *the solution is unique.*

As already mentioned, we shall not provide a full proof of the latter result. For the sake of definiteness, we however present here the corresponding incremental problems which read: given a sequence of partitions $P^n = \{0 = t^n_0 < t^n_1 < \cdots < t^n_{N^n-1} < t^n_{N^n} = T\}$ with diameters $\tau^n = \max_{i=1,\dots,N^n}(t^n_i - t^n_{i-1})$ going to 0, find

$$\begin{aligned}
(v^n_i, \varepsilon^{{\rm tr},n}_i, \varepsilon^{{\rm pl},n}_i) = \mathop{\rm Arg\,Min}_{(v,\varepsilon^{\rm tr},\varepsilon^{\rm pl})\in\mathcal{Y}_0} \Big(&\mathcal{E}_\rho(v, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}) - \langle \tilde{\ell}(t^n_i), (v, \varepsilon^{\rm tr}, \varepsilon^{\rm pl})\rangle \\
&+ \mathcal{D}(\varepsilon^{\rm tr} - \varepsilon^{{\rm tr},n}_{i-1}, \varepsilon^{\rm pl} - \varepsilon^{{\rm pl},n}_{i-1})\Big)
\end{aligned} \tag{25}$$

for $i = 1, \dots, N^n$ along with $(v^n_0, \varepsilon^{{\rm tr},n}_0, \varepsilon^{{\rm pl},n}_0) = (v_0, \varepsilon^{\rm tr}_0, \varepsilon^{\rm pl}_0)$. The latter minimum problems are uniquely solvable as the underlying functionals are uniformly convex.

By denoting by $(v_n, \varepsilon^{\rm tr}_n, \varepsilon^{\rm pl}_n)$ the incremental solution (see Section 4) and along the lines of Lemmas 4.2-4.3 above we also have the following.

Lemma 5.2 (Convergence of incremental solutions)**.** *Let* $(v_n, \varepsilon^{\rm tr}_n, \varepsilon^{\rm pl}_n)$ *denote the (unique) incremental solutions of* (25) *related to a sequence of partitions* P^n *with diameters* $\tau^n = \max_{i=1,\dots,N^n}(t^n_i - t^n_{i-1})$ *going to* 0. *Then we have that, at least for a not relabeled subsequence (the whole sequence for* $\rho > 0$*), for all* $t \in [0, T]$,

$$(v_n(t), \varepsilon^{\rm tr}_n(t), \varepsilon^{\rm pl}_n(t)) \to (v(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t)), \tag{26}$$

$$\mathrm{Diss}_{\mathcal{D}}((\varepsilon^{\rm tr}_n, \varepsilon^{\rm pl}_n), [0, t]) \to \mathrm{Diss}_{\mathcal{D}}((\varepsilon^{\rm tr}, \varepsilon^{\rm pl}), [0, t]), \tag{27}$$

$$\mathcal{E}_\rho(u_n(t), \varepsilon^{\rm tr}_n(t), \varepsilon^{\rm pl}_n(t)) \to \mathcal{E}_\rho(u(t), \varepsilon^{\rm tr}(t), \varepsilon^{\rm pl}(t)) \tag{28}$$

where $(v, \varepsilon^{\rm tr}, \varepsilon^{\rm pl})$ *in an energetic solution.*

The main difference here from the proofs of Lemmas 4.2-4.3 is that the strong convergence in (26) cannot be inferred by compactness (which would indeed yield weak convergence only) but is recovered from the convergence of energies (28) as the functional $\mathcal{E}_\rho$ is uniformly convex [60, Thm. 2.2, p. 252].

6. **Asymptotic analysis.** In this last section we shall prove some asymptotic results connecting the present model with former ones. In particular, we are mainly concerned with the limits $R^{\rm tr} \to \infty$ and $R^{\rm pl} \to \infty$ which correspond to the pure plastic and pure SMA limits, respectively, and the regularization limit $\rho \to 0$.

By formally taking $R^{\rm tr} = \infty$, we have that the energetic solution of the constitutive material relation (2) and (7) with $\varepsilon^{\rm tr}_0 = 0$ is indeed solving the *constrained* *linearized plasticity* problem

$$\begin{pmatrix} 0 \\ \partial_{\dot\varepsilon^{\rm pl}} D(0,\dot\varepsilon^{\rm pl}) \end{pmatrix} + \begin{pmatrix} \partial_\varepsilon E_\rho(\varepsilon,0,\varepsilon^{\rm pl}) \\ \partial_{\varepsilon^{\rm pl}} E_\rho(\varepsilon,0,\varepsilon^{\rm pl}) \end{pmatrix} \ni \begin{pmatrix} \sigma \\ 0 \end{pmatrix}, \quad (\varepsilon(0),\varepsilon^{\rm pl}(0)) = (\varepsilon_0,\varepsilon_0^{\rm pl}). \tag{29}$$

On the other hand, the formal choice $R^{\rm pl} = \infty$ with $\varepsilon_0^{\rm pl} = 0$ consists in solving the *original SA model* without permanent inelastic effects [5]. Namely,

$$\begin{pmatrix} 0 \\ \partial_{\dot\varepsilon^{\rm tr}} D(\dot\varepsilon^{\rm tr},0) \end{pmatrix} + \begin{pmatrix} \partial_\varepsilon E_\rho(\varepsilon,\varepsilon^{\rm tr},0) \\ \partial_{\varepsilon^{\rm tr}} E_\rho(\varepsilon,\varepsilon^{\rm tr},0) \end{pmatrix} \ni \begin{pmatrix} \sigma \\ 0 \end{pmatrix}, \quad (\varepsilon(0),\varepsilon^{\rm tr}(0)) = (\varepsilon_0,\varepsilon_0^{\rm tr}). \tag{30}$$

The aim of this section is to provide a rigorous analysis of the latter limits as well as the discussion of the limit $\rho \to 0$ in the regularization parameter.

6.1. **The general strategy.** In the following, we systematically exploit the theory of [45] where sufficient conditions in order to possibly pass to the limit within a sequence of energetic formulations are discussed. By referring specifically to the notation of the quasi-static evolution problem, assume to be given a sequence of functionals $(\mathcal{E}^k,\mathcal{D}^k)$ for $k \in \mathbb{N}\cup\{\infty\}$ and assume, for simplicity, that $(v_0,\varepsilon_0^{\rm tr},\varepsilon_0^{\rm pl}) = (0,0,0)$ and that the load ℓ and the boundary datum $u^{\rm Dir}$ are fixed independently of k (more elaborated situations may be discussed with little additional intricacy).

Let now $(v_k,\varepsilon_k^{\rm tr},\varepsilon_k^{\rm pl})$ be an energetic solution associated to the pair $(\mathcal{E}^k,\mathcal{D}^k)$ for $k\in\mathbb{N}$ and assume that $\mathcal{E}^k$ are uniformly convex in $\mathcal{Y}_0$, independently of k, and

$$\mathcal{E}^\infty \le \Gamma - \liminf_{k\to\infty} \mathcal{E}^k \quad \text{w.r.t. the weak topology in } \ \mathcal{Y}_0, \tag{31}$$

$$\mathcal{D}^\infty \le \Gamma - \liminf_{k\to\infty} \mathcal{D}^k \quad \text{w.r.t. the strong topology in } \ (L^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev}))^2, \tag{32}$$

where we have used a standard notation for the lim inf of a sequence of functionals with respect to Γ-convergence. The reader is referred to [18] for relevant materials and a collection of results.

The corresponding sets of stable states $\mathcal{S}^k(t)$ depending on $k \in \mathbb{N}\cup\{\infty\}$ and $t\in[0,T]$ are defined via

$$\begin{aligned} \mathcal{S}^k(t) := \Big\{ &(v,\varepsilon^{\rm tr},\varepsilon^{\rm pl}) \in \mathcal{Y}_0 : \mathcal{E}^k(v,\varepsilon^{\rm tr},\varepsilon^{\rm pl}) < \infty \ \text{ and, for all } \ (\bar v,\bar\varepsilon^{\rm tr},\bar\varepsilon^{\rm pl}) \in \mathcal{Y}_0, \\ &\text{we have } \mathcal{E}^k(v,\varepsilon^{\rm tr},\varepsilon^{\rm pl}) - \langle \tilde\ell(t),(v,\varepsilon^{\rm tr},\varepsilon^{\rm pl})\rangle \le \mathcal{E}^k(\bar v,\bar\varepsilon^{\rm tr},\bar\varepsilon^{\rm pl}) \\ &\quad - \langle \tilde\ell(t),(\bar v,\bar\varepsilon^{\rm tr},\bar\varepsilon^{\rm pl})\rangle + \mathcal{D}^k(\varepsilon^{\rm tr}-\bar\varepsilon^{\rm tr},\varepsilon^{\rm pl}-\bar\varepsilon^{\rm pl}) \Big\}. \end{aligned}$$

Given any $m \mapsto k_m \in \mathbb{N}$ increasing and unbounded, the sequence

$$(t_m, v_{k_m}, \varepsilon_{k_m}^{\rm tr}, \varepsilon_{k_m}^{\rm pl})_{m\in\mathbb{N}}$$

is called a *stable sequence* if

$$(v_{k_m}, \varepsilon_{k_m}^{\rm tr}, \varepsilon_{k_m}^{\rm pl}) \in \mathcal{S}^{k_m}(t_m) \ \text{ and } \ \sup_{m\in\mathbb{N}} \mathcal{E}^{k_m}(v_{k_m}, \varepsilon_{k_m}^{\rm tr}, \varepsilon_{k_m}^{\rm pl}) < \infty.$$

For the sake of notational simplicity we shall let

$$\mathcal{W}^k(t,v,\varepsilon^{\rm tr},\varepsilon^{\rm pl}) := \mathcal{E}^k(v,\varepsilon^{\rm tr},\varepsilon^{\rm pl}) - \langle \tilde\ell(t),(v,\varepsilon^{\rm tr},\varepsilon^{\rm pl})\rangle \quad \text{for } \ k\in\mathbb{N}\cup\{\infty\}.$$

We shall assume that the set of stable states shows some specific *upper semicontinuity* property [45, (2.11)]. In particular, we ask that for each stable sequence

$(t_m, v_{k_m}, \varepsilon^{\rm tr}_{k_m}, \varepsilon^{\rm pl}_{k_m}) \to (t, v, \varepsilon^{\rm tr}, \varepsilon^{\rm pl})$ weakly in $[0,T]\times\mathcal{Y}_0$ and for each $(\tilde v, \tilde\varepsilon^{\rm tr}, \tilde\varepsilon^{\rm pl}) \in \mathcal{Y}_0$ there exist $(\tilde v_{k_m}, \tilde\varepsilon^{\rm tr}_{k_m}, \tilde\varepsilon^{\rm pl}_{k_m}) \in \mathcal{Y}_0$ (not necessarily converging as $m\to\infty$) such that

$$\begin{aligned}\limsup_{m\to\infty} \Big(&\mathcal{W}^{k_m}(t_m, \tilde v_{k_m}, \tilde\varepsilon^{\rm tr}_{k_m}, \tilde\varepsilon^{\rm pl}_{k_m}) + \mathcal{D}^k(\varepsilon^{\rm tr}_{k_m} - \tilde\varepsilon^{\rm tr}_{k_m}, \varepsilon^{\rm pl}_{k_m} - \tilde\varepsilon^{\rm pl}_{k_m}) \\ &- \mathcal{W}^{k_m}(t_m, v_{k_m}, \varepsilon^{\rm tr}_{k_m}, \varepsilon^{\rm pl}_{k_m})\Big) \leq \mathcal{W}^\infty(t, \tilde v, \tilde\varepsilon^{\rm tr}, \tilde\varepsilon^{\rm pl}) \\ &\quad + \mathcal{D}^\infty(\varepsilon^{\rm tr} - \tilde\varepsilon^{\rm tr}, \varepsilon^{\rm pl} - \tilde\varepsilon^{\rm pl}) - \mathcal{W}^\infty(t, v, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}).\end{aligned} \tag{33}$$

By assuming the Γ-lim inf relations (31)-(32) and the upper semicontinuity condition (33), the result [45, Theorem 3.1] ensures that, at least for some relabeled subsequence,

$$(v_k, \varepsilon^{\rm tr}_k, \varepsilon^{\rm pl}_k) \to (v_\infty, \varepsilon^{\rm tr}_\infty, \varepsilon^{\rm pl}_\infty) \quad \text{at least pointwise in time and weakly in } \mathcal{Y}_0$$

where $(v_\infty, \varepsilon^{\rm tr}_\infty, \varepsilon^{\rm pl}_\infty)$ is an energetic solution associated to $(\mathcal{E}^\infty, \mathcal{D}^\infty)$. We shall specifically use this result in the following.

6.2. **The limit $R^{\rm tr} \to \infty$.** Let us firstly concentrate on the pure plastic limit by letting $R^{\rm tr} \to \infty$. In this case, the convergence result reads as follows.

Theorem 6.1 (Plastic limit)**.** *Let $(v_k, \varepsilon^{\rm tr}_k, \varepsilon^{\rm pl}_k)$ be energetic solutions of the quasi-static evolution problem (22)-(23) for given $\ell \in W^{1,1}(0,T;\mathcal{Y}')$ and $(v_0, \varepsilon^{\rm tr}_0, \varepsilon^{\rm pl}_0) = (0,0,0)$ along with the choice*

$$\mathcal{E}^k = \mathcal{E}_\rho, \qquad \mathcal{D}^k(a,b) = \mathcal{D}(ka, b) \quad \forall a,\, b \in L^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev}).$$

Then, we have that, for all $t\in[0,T]$,

$$(v_k(t), \varepsilon^{\rm pl}_k(t)) \to (v_\infty(t), \varepsilon^{\rm pl}_\infty(t)) \quad \textit{weakly in } H^1(\Omega;\mathbb{R}^3)\times H^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev})$$

where $(v_\infty, \varepsilon^{\rm pl}_\infty)$ is an energetic solution of the constrained linearized plasticity *problem (29). In case $\rho > 0$ the whole sequence converges.*

Proof. Let us observe that $\mathcal{E}^k \stackrel{\Gamma}{\to} \mathcal{E}_\rho$ w.r.t. the weak topology of $\mathcal{Y}_0$, and

$$\mathcal{D}^k(a,b) \stackrel{\Gamma}{\to} \mathcal{D}^\infty(a,b) := \begin{cases} R^{\rm pl}|b| & a = 0 \\ \infty & a \neq 0 \end{cases}$$

w.r.t. the strong topology of $(L^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev}))^2$. The first convergence obviously follows from the lower semicontinuity of $\mathcal{E}_\rho$. As for the second, for all $(a_k, b_k) \to (a,b)$ strongly in $(L^1(\Omega;\mathbb{R}^{3\times3}_{\rm dev}))^2$ we have that

$$\liminf_{k\to\infty} \mathcal{D}^k(a_k, b_k) < \infty \quad \Longrightarrow \quad a = 0.$$

Hence, we have that

$$\mathcal{D}^\infty(a,b) \leq \liminf_{k\to\infty} \mathcal{D}^k(a_k, b_k).$$

On the other hand, $\mathcal{D}^k \to \mathcal{D}^\infty$ pointwise and the above mentioned Γ-convergence follows.

In the spirit of Subsection 6.1, in order to possibly pass to the limit in the sequence of energetic solutions $(v_k, \varepsilon^{\rm pl}_k, \varepsilon^{\rm tr}_k)$ we shall now check for the upper semicontinuity condition (33). Let us start by letting

$$Q := \{(a,b) \in \mathbb{R}^{3\times3}_{\rm dev} \times \mathbb{R}^{3\times3}_{\rm dev} \ : \ |a+b| \leq \varepsilon_L\}.$$

The set Q is non-empty, convex, and closed. We let $\pi : \mathbb{R}^{3\times3}_{\rm dev} \times \mathbb{R}^{3\times3}_{\rm dev} \to Q$ be the standard projection and assume $\nu = \bar\nu$ for simplicity (the case $\nu \neq \bar\nu$ would require to project with respect to a different metric). Moreover, let $\pi_1 : \mathbb{R}^{3\times3}_{\rm dev} \times \mathbb{R}^{3\times3}_{\rm dev} \to \mathbb{R}^{3\times3}_{\rm dev}$ and $\pi_2 : \mathbb{R}^{3\times3}_{\rm dev} \times \mathbb{R}^{3\times3}_{\rm dev} \to \mathbb{R}^{3\times3}_{\rm dev}$ be the projections on the first and the second component, respectively. Let the stable sequence $(t_m, v_{k_m}, \varepsilon^{\rm tr}_{k_m}, \varepsilon^{\rm pl}_{k_m})$ converges to $(t, v, \varepsilon^{\rm tr}, \varepsilon^{\rm pl})$ weakly in $[0,T] \times \mathcal{Y}_0$ and fix $(\tilde v, \tilde\varepsilon^{\rm tr}, \tilde\varepsilon^{\rm pl}) \in \mathcal{Y}_0$. We shall define

$$(\tilde v_{k_m}, \tilde\varepsilon^{\rm tr}_{k_m}, \tilde\varepsilon^{\rm pl}_{k_m}) := (\tilde v, \pi(\varepsilon^{\rm tr}_{k_m} - \varepsilon^{\rm tr} + \tilde\varepsilon^{\rm tr}, \varepsilon^{\rm pl}_{k_m} - \varepsilon^{\rm pl} + \tilde\varepsilon^{\rm pl})).$$

Let us fix, for notational simplicity

$$a_m := \pi_1(\pi(\varepsilon^{\rm tr}_{k_m} - \varepsilon^{\rm tr} + \tilde\varepsilon^{\rm tr}, \varepsilon^{\rm pl}_{k_m} - \varepsilon^{\rm pl} + \tilde\varepsilon^{\rm pl})),\ b_m := \pi_2(\pi(\varepsilon^{\rm tr}_{k_m} - \varepsilon^{\rm tr} + \tilde\varepsilon^{\rm tr}, \varepsilon^{\rm pl}_{k_m} - \varepsilon^{\rm pl} + \tilde\varepsilon^{\rm pl})).$$

Note that clearly $(\tilde\varepsilon^{\rm tr}_{k_m}, \tilde\varepsilon^{\rm pl}_{k_m}) \in Q$ and $(\tilde v_{k_m}, \tilde\varepsilon^{\rm tr}_{k_m}, \tilde\varepsilon^{\rm pl}_{k_m}) \to (\tilde v, \tilde\varepsilon^{\rm tr}, \tilde\varepsilon^{\rm pl})$ weakly in $\mathcal{Y}_0$ owing to the strong continuity of π on $(L^2(\Omega; \mathbb{R}^{3\times3}_{\rm dev}))^2$ and the fact that

$$\begin{aligned} &|\nabla \pi(\varepsilon^{\rm tr}_{k_m} - \varepsilon^{\rm tr} + \tilde\varepsilon^{\rm tr}, \varepsilon^{\rm pl}_{k_m} - \varepsilon^{\rm pl} + \tilde\varepsilon^{\rm pl})| \\ &\qquad \le |\nabla(\varepsilon^{\rm tr}_{k_m} - \varepsilon^{\rm tr} + \tilde\varepsilon^{\rm tr}, \varepsilon^{\rm pl}_{k_m} - \varepsilon^{\rm pl} + \tilde\varepsilon^{\rm pl})| \quad \text{a.e. in } \Omega \end{aligned} \tag{34}$$

as the projection is contractive.

We shall now check that the choice for $(\tilde v_{k_m}, \tilde\varepsilon^{\rm tr}_{k_m}, \tilde\varepsilon^{\rm pl}_{k_m})$ fulfills (33). Indeed, we are just interested in the situation when $\varepsilon^{\rm tr} = \tilde\varepsilon^{\rm tr}$ almost everywhere as, if this was not the case, the right hand side of (33) is ∞. Let us observe that

$$\begin{aligned} &\mathcal{D}^{k_m}(\varepsilon^{\rm tr}_{k_m} - \tilde\varepsilon^{\rm tr}_{k_m}, \varepsilon^{\rm pl}_{k_m} - \tilde\varepsilon^{\rm pl}_{k_m}) = \mathcal{D}^{k_m}(\varepsilon^{\rm tr}_{k_m} - a_m, \varepsilon^{\rm pl}_{k_m} - b_m) \\ &\overset{(\varepsilon^{\rm tr}_{k_m}, \varepsilon^{\rm pl}_{k_m}) \in Q}{\le} \mathcal{D}^{k_m}(\varepsilon^{\rm tr} - \tilde\varepsilon^{\rm tr}, \varepsilon^{\rm pl} - \tilde\varepsilon^{\rm pl}) = \mathcal{D}^{\infty}(0, \varepsilon^{\rm pl} - \tilde\varepsilon^{\rm pl}). \end{aligned} \tag{35}$$

As for the autonomous part of the energy we compute

$$\begin{aligned} &\mathcal{E}^{k_m}(\tilde v_{k_m}, \tilde\varepsilon^{\rm tr}_{k_m}, \tilde\varepsilon^{\rm pl}_{k_m}) - \mathcal{E}^{k_m}(v_{k_m}, \varepsilon^{\rm tr}_{k_m}, \varepsilon^{\rm pl}_{k_m}) = \mathcal{E}_\rho(\tilde v, a_m, b_m) - \mathcal{E}_\rho(v_{k_m}, \varepsilon^{\rm tr}_{k_m}, \varepsilon^{\rm pl}_{k_m}) \\ &= \mathcal{C}(\varepsilon(\tilde v) - a_m - b_m) + \mathcal{F}_\rho(a_m, b_m) + \frac{\nu}{2}\int_\Omega |\nabla a_m|^2 {\rm d}x + \frac{\nu}{2}\int_\Omega |\nabla b_m|^2 {\rm d}x \\ &- \mathcal{C}(\varepsilon(v_{k_m}) - \varepsilon^{\rm tr}_{k_m} - \varepsilon^{\rm pl}_{k_m}) - \mathcal{F}_\rho(\varepsilon^{\rm tr}_{k_m}, \varepsilon^{\rm pl}_{k_m}) - \frac{\nu}{2}\int_\Omega |\nabla \varepsilon^{\rm tr}_{k_m}|^2 {\rm d}x - \frac{\nu}{2}\int_\Omega |\nabla \varepsilon^{\rm pl}_{k_m}|^2 {\rm d}x \end{aligned} \tag{36}$$

By the strong convergence $(a_m, b_m) \to (\tilde\varepsilon^{\rm tr}, \tilde\varepsilon^{\rm pl})$ in $(L^2(\Omega; \mathbb{R}^{3\times3}_{\rm dev}))^2$ we get

$$\begin{aligned} &\limsup_{m\to+\infty} \big(\mathcal{C}(\varepsilon(\tilde v) - a_m - b_m) - \mathcal{C}(\varepsilon(v_{k_m}) - \varepsilon^{\rm tr}_{k_m} - \varepsilon^{\rm pl}_{k_m})\big) \\ &\qquad\qquad \le \mathcal{C}(\varepsilon(\tilde v) - \tilde\varepsilon^{\rm tr} - \tilde\varepsilon^{\rm pl}) - \mathcal{C}(\varepsilon(v) - \varepsilon^{\rm tr} - \varepsilon^{\rm pl}). \end{aligned}$$

Moreover, also exploiting the lower semicontinuity of $\mathcal{F}_\rho$ with respect to the weak topology of $L^2(\Omega; \mathbb{R}^{3\times3}_{\rm dev} \times \mathbb{R}^{3\times3}_{\rm dev})$ and its continuity with respect to the strong one, we obtain

$$\begin{aligned} &\limsup_{m\to\infty} \Big(\mathcal{F}_\rho(a_m, b_m) - \mathcal{F}_\rho(\varepsilon^{\rm tr}_{k_m}, \varepsilon^{\rm pl}_{k_m})\Big) \\ &= \limsup_{m\to\infty} \Big(\mathcal{F}_\rho(\varepsilon^{\rm tr}_{k_m} - \varepsilon^{\rm tr} + \tilde\varepsilon^{\rm tr}, \varepsilon^{\rm pl}_{k_m} - \varepsilon^{\rm pl} + \tilde\varepsilon^{\rm pl}) - \mathcal{F}_\rho(\varepsilon^{\rm tr}_{k_m}, \varepsilon^{\rm pl}_{k_m})\Big) \\ &\le \mathcal{F}_\rho(\tilde\varepsilon^{\rm tr}, \tilde\varepsilon^{\rm pl}) - \mathcal{F}_\rho(\varepsilon^{\rm tr}, \varepsilon^{\rm pl}). \end{aligned}$$

Finally, the quadratic terms in (36) can be handled as follows

$$\frac{\nu}{2}\int_\Omega |\nabla a_m|^2 \mathrm{d}x + \frac{\nu}{2}\int_\Omega |\nabla b_m|^2 \mathrm{d}x - \frac{\nu}{2}\int_\Omega |\nabla \varepsilon^{\rm tr}_{k_m}|^2 \mathrm{d}x - \frac{\nu}{2}\int_\Omega |\nabla \varepsilon^{\rm pl}_{k_m}|^2 \mathrm{d}x$$
$$\overset{(34)}{\le} \frac{\nu}{2}\int_\Omega |\nabla(\varepsilon^{\rm tr}_{k_m} - \varepsilon^{\rm tr} + \tilde\varepsilon^{\rm tr})|^2 \mathrm{d}x + \frac{\nu}{2}\int_\Omega |\nabla(\varepsilon^{\rm pl}_{k_m} - \varepsilon^{\rm pl} + \tilde\varepsilon^{\rm pl})|^2 \mathrm{d}x$$
$$- \frac{\nu}{2}\int_\Omega |\nabla \varepsilon^{\rm tr}_{k_m}|^2 \mathrm{d}x - \frac{\nu}{2}\int_\Omega |\nabla \varepsilon^{\rm pl}_{k_m}|^2 \mathrm{d}x$$
$$= \frac{\nu}{2}\int_\Omega |\nabla \tilde\varepsilon^{\rm tr}|^2 \mathrm{d}x + \frac{\nu}{2}\int_\Omega |\nabla \varepsilon^{\rm tr}|^2 \mathrm{d}x + \nu\int_\Omega \nabla\tilde\varepsilon^{\rm tr} \because \nabla(\varepsilon^{\rm tr}_{k_m} - \varepsilon^{\rm tr}) \mathrm{d}x$$
$$- \nu\int_\Omega \nabla\varepsilon^{\rm tr}_{k_m} \because \nabla\varepsilon^{\rm tr} \mathrm{d}x + \frac{\nu}{2}\int_\Omega |\nabla \tilde\varepsilon^{\rm pl}|^2 \mathrm{d}x + \frac{\nu}{2}\int_\Omega |\nabla \varepsilon^{\rm pl}|^2 \mathrm{d}x$$
$$+ \nu\int_\Omega \nabla\tilde\varepsilon^{\rm pl} \because \nabla(\varepsilon^{\rm pl}_{k_m} - \varepsilon^{\rm pl}) \mathrm{d}x - \nu\int_\Omega \nabla\varepsilon^{\rm pl}_{k_m} \because \nabla\varepsilon^{\rm pl} \mathrm{d}x$$

so that, by passing to the lim sup, we have

$$\limsup_{m\to\infty}\left(\frac{\nu}{2}\int_\Omega |\nabla a_m|^2 \mathrm{d}x + \frac{\nu}{2}\int_\Omega |\nabla b_m|^2 \mathrm{d}x - \frac{\nu}{2}\int_\Omega |\nabla \varepsilon^{\rm tr}_{k_m}|^2 \mathrm{d}x - \frac{\nu}{2}\int_\Omega |\nabla \varepsilon^{\rm pl}_{k_m}|^2 \mathrm{d}x\right)$$
$$\le \frac{\nu}{2}\int_\Omega |\nabla \tilde\varepsilon^{\rm tr}|^2 \mathrm{d}x + \frac{\nu}{2}\int_\Omega |\nabla \tilde\varepsilon^{\rm pl}|^2 \mathrm{d}x - \frac{\nu}{2}\int_\Omega |\nabla \varepsilon^{\rm tr}|^2 \mathrm{d}x - \frac{\nu}{2}\int_\Omega |\nabla \varepsilon^{\rm pl}|^2 \mathrm{d}x.$$

Eventually, we can pass to the lim sup in (36) and, using also (35), conclude that

$$\limsup_{m\to\infty}\Big(\mathcal{E}^{k_m}(\tilde v_{k_m}, \tilde\varepsilon^{\rm tr}_{k_m}, \tilde\varepsilon^{\rm pl}_{k_m}) + \mathcal{D}^k(\varepsilon^{\rm tr}_{k_m} - \tilde\varepsilon^{\rm tr}_{k_m}, \varepsilon^{\rm pl}_{k_m} - \tilde\varepsilon^{\rm pl}_{k_m}) - \mathcal{E}^{k_m}(v_{k_m}, \varepsilon^{\rm tr}_{k_m}, \varepsilon^{\rm pl}_{k_m})\Big)$$
$$\le \mathcal{E}^\infty(\tilde v, \tilde\varepsilon^{\rm tr}, \tilde\varepsilon^{\rm pl}) + \mathcal{D}^\infty(\varepsilon^{\rm tr} - \tilde\varepsilon^{\rm tr}, \varepsilon^{\rm pl} - \tilde\varepsilon^{\rm pl}) - \mathcal{E}^\infty(v, \varepsilon^{\rm tr}, \varepsilon^{\rm pl}).$$

As the treatment of the time-dependent terms is immediate due to the continuity of $\tilde\ell$, we readily conclude for the limsup condition (33) and the assertion follows from the general theory in [45]. □

6.3. **The limit $R^{\rm pl} \to \infty$.** By passing to the limit as $R^{\rm pl} \to 0$ starting from $\varepsilon^{\rm pl} = 0$ no permanent inelastic evolution takes place and the model reduces to the original SA one. More precisely, we have the following.

Theorem 6.2 (Shape memory limit). *Let $(v_k, \varepsilon^{\rm tr}_k, \varepsilon^{\rm pl}_k)$ be energetic solutions of the quasi-static evolution problem (22)-(23) for given $\ell \in W^{1,1}(0,T;\mathcal{Y}')$ and $(v_0, \varepsilon^{\rm tr}_0, \varepsilon^{\rm pl}_0) = (0,0,0)$ along with the choice*

$$\mathcal{E}^k = \mathcal{E}_\rho, \qquad \mathcal{D}^k(a,b) = \mathcal{D}(a, kb) \quad \forall a,\, b \in L^1(\Omega;\mathbb{R}^{3\times 3}_{\rm dev}).$$

Then, we have that

$$(v_k(t), \varepsilon^{\rm tr}_k(t)) \to (v_\infty(t), \varepsilon^{\rm tr}_\infty(t)) \quad \text{weakly in } H^1(\Omega;\mathbb{R}^3)\times H^1(\Omega;\mathbb{R}^{3\times 3}_{\rm dev})$$

where $(v_\infty, \varepsilon^{\rm tr}_\infty)$ is an energetic solution of the original SA model (30). In case $\rho > 0$ the whole sequence converges.

We report here no proof of the latter as it may be easily obtained by suitably modifying the argument for Theorem 6.1.

6.4. **The limit $\rho \to 0$.** Let us now comment on the possibility of passing to the limit as the regularization parameter $\rho \to 0$. One shall recall that the original modeling choice is $\rho = 0$ whereas the interest in considering the smooth situation $\rho > 0$ is related to uniqueness and discretizations. We prove the following.

Theorem 6.3 (Regularization limit)**.** *Let $(v_k, \varepsilon_k^{\rm tr}, \varepsilon_k^{\rm pl})$ be energetic solutions of the quasi-static evolution problem (22)-(23) for given $\ell \in W^{1,1}(0,T;\mathcal{Y}')$ and $(v_0, \varepsilon_0^{\rm tr}, \varepsilon_0^{\rm pl}) = (0,0,0)$ along with the choice*

$$\mathcal{E}^k = \mathcal{E}_{1/k}, \qquad \mathcal{D}^k = \mathcal{D}.$$

Then, we have that

$$(v_k(t), \varepsilon_k^{\rm tr}(t), \varepsilon_k^{\rm pl}(t)) \to (v_\infty(t), \varepsilon_\infty^{\rm tr}(t), \varepsilon_\infty^{\rm pl}(t)) \quad \textit{weakly in } \mathcal{Y}_0$$

where $(v_\infty, \varepsilon_\infty^{\rm tr}, \varepsilon_\infty^{\rm pl})$ is an energetic solution associated with the pair $(\mathcal{E}_0, \mathcal{D})$.

Proof. This argument is very close to that of Theorem 6.1. Note that $\mathcal{D}^k \stackrel{\Gamma}{\to} \mathcal{D}$ w.r.t. the strong topology in $(L^1(\Omega;\mathbb{R}^{3\times 3}_{\rm dev}))^2$, and $\mathcal{E}^k \stackrel{\Gamma}{\to} \mathcal{E}_0$ w.r.t. the weak topology in $\mathcal{Y}_0$, the first convergence being ensured by lower semicontinuity and the second by the monotone pointwise convergence of F_ρ to F_0.

Let now the stable sequence $(t_m, v_{k_m}, \varepsilon_{k_m}^{\rm tr}, \varepsilon_{k_m}^{\rm pl})$ converging to $(t, v, \varepsilon^{\rm tr}, \varepsilon^{\rm pl})$ weakly in $[0,T] \times \mathcal{Y}_0$ be given and fix $(\tilde{v}, \tilde{\varepsilon}^{\rm tr}, \tilde{\varepsilon}^{\rm pl}) \in \mathcal{Y}_0$. Exactly as in the proof of Theorem 6.1, the recovery sequence

$$(\tilde{v}_{k_m}, \tilde{\varepsilon}_{k_m}^{\rm tr}, \tilde{\varepsilon}_{k_m}^{\rm pl}) := (\tilde{v}, \pi(\varepsilon_{k_m}^{\rm tr} - \varepsilon^{\rm tr} + \tilde{\varepsilon}^{\rm tr}, \varepsilon_{k_m}^{\rm pl} - \varepsilon^{\rm pl} + \tilde{\varepsilon}^{\rm pl}))$$

turns out to be well-suited for the sake of proving (33). Indeed,

$$\begin{aligned}
&\mathcal{D}^{k_m}(\varepsilon_{k_m}^{\rm tr} - \tilde{\varepsilon}_{k_m}^{\rm tr}, \varepsilon_{k_m}^{\rm pl} - \tilde{\varepsilon}_{k_m}^{\rm pl}) = \mathcal{D}(\varepsilon_{k_m}^{\rm tr} - a_m, \varepsilon_{k_m}^{\rm pl} - b_m) \\
&\stackrel{(\varepsilon_{k_m}^{\rm tr}, \varepsilon_{k_m}^{\rm pl}) \in Q}{\leq} \mathcal{D}(\tilde{\varepsilon}^{\rm tr} - \varepsilon^{\rm tr}, \tilde{\varepsilon}^{\rm pl} - \varepsilon^{\rm pl}).
\end{aligned}$$

As for the energy, we have

$$\begin{aligned}
&\limsup_{m\to\infty} \Big(\mathcal{E}^{k_m}(\tilde{v}_{k_m}, \tilde{\varepsilon}_{k_m}^{\rm tr}, \tilde{\varepsilon}_{k_m}^{\rm pl}) - \mathcal{E}^{k_m}(v_{k_m}, \varepsilon_{k_m}^{\rm tr}, \varepsilon_{k_m}^{\rm pl}) \Big) \\
&\leq \limsup_{m\to\infty} \mathcal{E}_0(\tilde{v}_{k_m}, \tilde{\varepsilon}_{k_m}^{\rm tr}, \tilde{\varepsilon}_{k_m}^{\rm pl}) - \liminf_{m\to\infty} \mathcal{E}^{k_m}(v_{k_m}, \varepsilon_{k_m}^{\rm tr}, \varepsilon_{k_m}^{\rm pl}) \Big) \\
&\leq \mathcal{E}_0(\tilde{v}, \tilde{\varepsilon}^{\rm tr}, \tilde{\varepsilon}^{\rm pl}) - \mathcal{E}_0(v, \varepsilon^{\rm tr}, \varepsilon^{\rm pl})
\end{aligned}$$

where we used the monotone convergence of $\mathcal{F}_\rho$ to $\mathcal{F}_0$, the lower semicontinuity of $\mathcal{F}_0$ with respect to the weak topology of $L^2(\Omega;\mathbb{R}^{3\times 3}_{\rm dev} \times \mathbb{R}^{3\times 3}_{\rm dev})$ and its continuity with respect to the strong one whenever restricted to its effective domain, and the argument on the gradient terms in the energy from the proof of Theorem 6.1. Once again, the time-dependent linear terms make no trouble and we have (33). Eventually, the convergence statement follows from the general theory of [45]. □

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