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ON THE LOCATION OF THE 1-PARTICLE BRANCH OF THE SPECTRUM OF THE DISORDERED STOCHASTIC ISING MODEL

Michele Gianfelice

Dipartimento di Matematica
Università della Calabria
Ponte Pietro Bucci, cubo 30B, 87036 Arcavacata di Rende, Italy

Marco Isopi

Dipartimento di Matematica
"Sapienza" Università di Roma
Piazzale Aldo Moro 5, 00185 Roma, Italy

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Abstract. We analyse the lower non trivial part of the spectrum of the generator of the Glauber dynamics for a d-dimensional nearest neighbour Ising model with a bounded random potential. We prove conjecture 1 in [1]: for sufficently large values of the temperature, the first band of the spectrum of the generator of the process coincides with a closed non random segment of the real line.

1. Introduction.

In [1] the authors study the generator of the Glauber dynamics for a one dimension Ising model with random bounded potential. They prove that, for any realization of the potential and any value of the inverse temperature $\beta > 0$, the spectrum of the generator is the union of disjoint closed subsets of the real line (k-particle branches, $k \in \mathbb{N}^+$) and that, with probability one with respect to the distribution of the potential, is a non random set. In particular it is proved there that there exists a spectral gap and thus the model exhibits exponential relaxation to equilibrium. As is to be expected, and proved in [1], a relaxation rate which is valid for every realization is the same as that of the non-disordered model with a coupling constant that coincides with the maximum value of the coupling in the disordered model. For the average over the disorder of the single spin autocorrelation function, the speed of relaxation is somewhat larger as was proved in [14].

Boundedness of the potential is essential for all these results of fast convergence to equilibrium. In this case fairly detailed information on parts of the spectrum of the generator is available ([1], [14]). Also in more than one dimension convergence slightly slower than exponential on average can be proved at high temperature [2].

When the interactions are not bounded the situation is considerably different. Even in one dimension there is no spectral gap (see [13]) and relaxation rate is subexponential (see [12]).

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In [1] it is conjectured (conjecture 1, page 657) that results similar to those proved there for one dimension should hold for β small enough in dimensions $d \geq 2$. It can be readily seen that for the proof, in one dimension, of the results conjectured to be true in $d \geq 2$, the assumption of ferromagnetic coupling is not needed. It is only used later to prove exponential decay of eigenfunctions.

In this work we consider the Glauber dynamics for the d-dimensional nearest neighbour Ising model, with a bounded random potential having absolutely continuous distribution with respect to the Lebesgue measure and prove that conjecture 1 in [1] is true.

That is, there exists a constant C, depending on the distribution of the potential and on the lattice dimension d, such that, at high temperature, the first branch of the spectrum of the generator of the process, at first order in β, coincides, for almost every realization of the potential, with the segment

$$[1 - C\beta, 1 + C\beta]$$

(for a more precise statement see Theorem 2.3). In particular this implies that, at first order in β, the spectral gap is larger than $1 - C\beta$.

We remark that at lower temperatures, but still in the uniqueness region, relaxation is strictly slower than exponential for almost every realization of the potential (see Theorem 3.3 of [2]).

2. **Notations and results.** Consider the lattice $\mathbb{Z}^d$ and the set of bonds of the lattice $\mathbb{B}_d := \left\{\{x,y\} \subset \mathbb{Z}^d : |x-y| = 1\right\}$. We introduce a collection of i.i.d random variables indexed by $\mathbb{B}_d$. On each bond of the lattice we define a random variable

$$\omega_b \in \left[J^-, J^+\right], \ b \in \mathbb{B}_d ,$$

whose probability distribution is absolutely continuous with respect to the Lebesgue measure. The random field ω is a function on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\Omega = \left[J^-, J^+\right]^{\mathbb{B}_d}$, and is ergodic w.r.t. the the group of automorphisms on Ω generated by the lattice shift $\{\theta_z\}_{z\in\mathbb{Z}^d}$

$$\Omega \ni \omega \longmapsto \omega \in \Omega : (\theta_z\omega)_b = \omega_{b-z} \quad z \in \mathbb{Z}^d, b \in \mathbb{B}_d ,$$

where, $\forall \Lambda \subset \mathbb{Z}^d, z \in \mathbb{Z}^d$, $\Lambda - z := \{y \in \mathbb{Z}^d : y = x - z\, ,\, x \in \Lambda\} \subset \mathbb{Z}^d$.

We now consider an Ising spin system in $\mathbb{Z}^d$. Denoting by $\mathcal{S}$ the spin configuration space $\{-1,+1\}^{\mathbb{Z}^d}$ and by σ the spin configuration, let $\{\tau_z\}_{z\in\mathbb{Z}^d}$ be the group of automorphisms of $\mathcal{S}$, generated by the lattice translations

$$\mathcal{S} \ni \sigma \longmapsto \tau_z\sigma \in \mathcal{S} : (\tau_z\sigma)_x = \sigma_{x-z} \quad x, z \in \mathbb{Z}^d$$

and j be the involution of $\mathcal{S}$ given by

$$\mathcal{S} \ni \sigma \longmapsto j(\sigma) = -\sigma \in \mathcal{S} .$$

Let Λ be a finite subset of the lattice. The Hamiltonian of the models studied throughout this paper is

$$H_\Lambda^\omega(\eta|\xi_{\partial\Lambda}) = -\sum_{x,y\in\Lambda : |x-y|=1} \frac{1}{2}\eta_x\omega_{x,y}\eta_y + \sum_{x\in\Lambda,\, y\in\Lambda^c : |x-y|=1} \eta_x\omega_{x,y}\xi_y , \qquad (1)$$

where $\eta \in \mathcal{S}_\Lambda = \{-1,+1\}^\Lambda$ and $\xi_{\partial\Lambda} := (\xi_i)_{i\in\partial\Lambda}$ is a fixed boundary condition.

For any $\beta > 0$ and any realization ω of the potential, let $\mathcal{G}(\beta, \omega)$ be the set of Gibbs states of the system specified by

$$\mu_\Lambda^{\beta,\omega}(d\eta|\sigma_{\partial\Lambda}) := \frac{e^{-\beta H_\Lambda^\omega(\eta|\sigma_{\partial\Lambda})}}{Z_\Lambda^{(d)}(\beta,\omega|\sigma_{\partial\Lambda})}\mu_\Lambda(d\eta) \qquad \Lambda \subset\subset \mathbb{Z}^d$$
$$Z_\Lambda^{(d)}(\beta,\omega|\sigma_{\partial\Lambda}) := \mu_\Lambda\left(e^{-\beta H_\Lambda^\omega(\eta|\sigma_{\partial\Lambda})}\right) .$$

We remark that, for a fixed boundary condition $\xi_{\partial\Lambda}$, the conditional probability measure $\mu_\Lambda^{\beta,\omega}(d\eta|\xi_{\partial\Lambda})$ coincides with the one associated with the formal Hamiltonian

$$H^\omega(\sigma) := -\sum_{x,y\in\mathbb{Z}^d\,:\,|x-y|=1} \frac{1}{2}\sigma_x\omega_{x,y}\sigma_y . \tag{2}$$

The Glauber processes studied in this paper are defined through the generator

$$L(\beta,\omega) f(\sigma) := \sum_{x\in\mathbb{Z}^d} w_x^{\beta,\omega}(\sigma)\left[f(\sigma) - f(\sigma^x)\right] , \tag{3}$$

where the rates $w_x^{\beta,\omega}$ are chosen so that the process is reversible w.r.t. $\mathcal{G}(\beta,\omega)$ and where σ^x represents the configuration in $\mathcal{S}$ such that

$$\sigma_y^x = \begin{cases} \sigma_y & y \neq x \\ -\sigma_y & y = x \end{cases} \qquad y \in \mathbb{Z}^d$$

and f is a cylindrical function in $L^2\left(\mathcal{S}, \mu^{\beta,\omega}\right) := \mathcal{L}(\beta,\omega)$.

We will always consider the generator L a positive operator, so that $S(t) = \exp[-tL]$ will represent the associated semigroup.

In the following, with a little abuse of notation, we will use the same symbol for the operator (3) and for its closure in $\mathcal{L}(\beta,\omega)$ which, by reversibility of the Gibbs measure, is also selfadjoint on $\mathcal{L}(\beta,\omega)$.

Let us define $J := |J^-| \vee |J^+|$ and, by (2), $\forall x \in \mathbb{Z}^d$, $\omega \in \Omega$

$$\Delta_x H^\omega(\sigma) := H^\omega(\sigma) - H^\omega(\sigma^x) = -\sigma_x \sum_{y\colon |x-y|=1} \omega_{x,y}\sigma_y . \tag{4}$$

Then

$$-4dJ \leq |\Delta_x H^\omega(\sigma)| \leq 4dJ . \tag{5}$$

From now on we are only interested in differences such as those in formula (4), which, as long as $x \in \Lambda$, is the same regardless of whether we use (1) or (2). So for simplicity we will be using (2).

In the following we will restrict ourselves to the choice of transition rates from σ to σ^x of the form

$$w_x^{\beta,\omega}(\sigma) = \psi\left(\beta\Delta_x H^\omega(\sigma)\right) , \tag{6}$$

where ψ is a monotone function, so that

$$\psi(-\beta 4dJ) \wedge \psi(\beta 4dJ) \leq w_x^{\beta,\omega}(\sigma) \leq \psi(-\beta 4dJ) \vee \psi(\beta 4dJ) . \tag{7}$$

In particular, we will work out the details for the case of the *heat bath dynamics* as was done in [1]

$$w_{hb,x}^{\beta,\omega}(\sigma) = \psi_{hb}\left(\beta\Delta_x H^\omega(\sigma)\right) = \frac{1}{1 + e^{-\beta\Delta_x H^\omega(\sigma)}} . \tag{8}$$

Our analysis can be applied to any Glauber process with transition rates of the kind given in (6).

The results contained in this paper are:

Theorem 2.1. *There exists a value* $\beta_d^{-1}(J)$ *of the temperature such that, for any* $\beta \in [0, \beta_d(J))$ *and any realization of the potential* ω, *the first non trivial branch of the spectrum of the generator of the heat bath dynamics,* $\sigma_\beta^{(1)}$, *is contained in the interval* $\left[g_d^-(\beta), g_d^+(\beta)\right]$ *where* $g_d^-(\beta), g_d^+(\beta)$ *are analytic functions of* β *such that*

$$g_d^{\pm}(\beta) = 1 \pm 2dJ\beta + o(\beta) \ .$$

For a definition of $\sigma_\beta^{(1)}$ and a discussion of its relevance see Corollary 1 of [1] and Theorem 2.3 of [9].

Theorem 2.2. *There exists a value* $\beta_d^{(1)}$ *of* β *such that, for every* $\beta \in \left[0, \beta_d^{(1)}\right)$ *and almost every realization of the potential* ω, *the first non trivial branch of the spectrum of the generator* $\sigma_\beta^{(1)}$ *satisfies*

$$\left[1 - f_d^-(\beta), 1 + f_d^+(\beta)\right] \subseteq \sigma_\beta^{(1)} ,$$

where $f_d^-(\beta)$, $f_d^+(\beta)$ *are analytic functions of* β *such that*

$$f_d^{\pm}(\beta) = \pm 2dJ\beta + o(\beta) \ .$$

Remark 1. The analyticity of the functions introduced in the above two theorems, does not hold only for the heat bath dynamics, but is guaranteed for any dynamics where ψ is an analytic function. If this is not the case, the statement about analyticity must be dropped from the above theorems.

Theorem 2.3. *There exists a value* $\beta_d^*(J) \leq \beta_d^{(1)} \wedge \beta_d(J)$ *of* β *such that, for every* $\beta \in [0, \beta_d^*(J))$ *and almost every realization of the potential* ω, *the first non trivial branch of the spectrum of the generator of the process* $\sigma_\beta^{(1)}$ *is a non random set which coincides with the closed subset of the real line* $\left[1 - h_d^-(\beta), 1 + h_d^+(\beta)\right]$, *where* $h_d^{\pm}(\beta) = \pm 2dJ\beta + o(\beta)$.

The proofs of these theorems rely in part on the approach of [1] and [9] and in part on the lattice gas representation of the system, which we will introduce in the next subsection. More precisely, we will restate the dynamics with rates of the kind (6) in terms of a birth and death process on the set of subsets of the lattice $\mathcal{P}$, which is naturally isomorphic to $\mathcal{S}$, and make use of the setup given in [3] and [4].

2.1. Lattice gas setting.

In [3, 4] we analysed the stochastic dynamics of a system with a ferromagnetic potential constant on $\mathbb{B}_d$, confined in a finite subset Λ of the lattice and subject to free or periodic boundary condition. Making use of a formalism borrowed from quantum mechanics, we were able to represent the restriction of (3) to $\mathcal{S}_\Lambda$, in terms of a selfadjoint operator on $\mathcal{H}_\Lambda := l^2(\mathcal{P}_\Lambda)$ which we showed to be unitarily equivalent to a generator of birth and death process on $\mathcal{P}_\Lambda$. Here we will follow the same approach.

We consider the Hilbert space of complex square summable function on the single site configuration space with respect to the symmetric Bernoulli measure. Namely, $\forall x \in \mathbb{Z}^d$,

$$\mathcal{H}_x := span\left\{|\emptyset\rangle_x, |\mathrm{x}\rangle_x\right\} \cong \mathbb{C}^2$$

$$|\emptyset\rangle_x \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}_x \quad |\mathrm{x}\rangle_x \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}_x$$

$\mathcal{U}_x = M(2,\mathbb{C})$ is the algebra of bounded operators on $\mathcal{H}_x$[1]. Let us define the *spin* operator

$$\mathbf{s}_x \in \mathcal{U}_x : \mathbf{s}_x \begin{cases} |\emptyset\rangle_x = |\mathrm{x}\rangle_x \\ |\mathrm{x}\rangle_x = |\emptyset\rangle_x \end{cases}$$

equivalent to the Pauli matrix $\sigma^{(1)}$

$$\sigma^{(1)} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the *spin flip* operator

$$\mathbf{f}_x \in \mathcal{U}_x : \mathbf{f}_x \begin{cases} |\emptyset\rangle_x = |\emptyset\rangle_x \\ |\mathrm{x}\rangle_x = -|\mathrm{x}\rangle_x \end{cases}$$

equivalent to the Pauli matrix $\sigma^{(3)}$

$$\sigma^{(3)} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Let Λ be any finite subset of the $\mathbb{Z}^d$ lattice. Then we have

$$|\alpha\rangle_\Lambda = \bigotimes_{x\in\alpha} |\mathrm{x}\rangle_x \bigotimes_{x\in\Lambda\setminus\alpha} |\emptyset\rangle_x$$
$$\mathcal{H}_\Lambda = span\left\{|\alpha\rangle_\Lambda : \alpha \subseteq \Lambda\right\}$$

Moreover $\mathcal{U}_\Lambda = M\left(2^{|\Lambda|},\mathbb{C}\right)$ and $\mathcal{C}_\Lambda$ is the algebra of polynomials in $\mathbf{s}_\alpha$ $(\mathbf{f}_\alpha)$ $\forall\alpha \subset \Lambda$. Then

$$\mathbf{s}_\alpha = \bigotimes_{x\in\alpha} \mathbf{s}_x \bigotimes_{x\in\Lambda\setminus\alpha} \mathbf{I}_x ,$$
$$\mathbf{f}_\alpha = \bigotimes_{x\in\alpha} \mathbf{f}_x \bigotimes_{x\in\Lambda\setminus\alpha} \mathbf{I}_x ,$$
$$\mathbf{s}_\emptyset = \mathbf{f}_\emptyset = \mathbf{I}_\Lambda .$$

Now, the generator of any Glauber process on the lattice, which in this representation we denote by $\breve{L}$, can be written in terms of the operators defined above and its generic matrix element becomes

$$\left(\breve{L}\delta_\alpha\right)_\eta = \sum_{x\in\mathbb{Z}^d} \left[w\left(\alpha, \alpha\triangle\{x\}\right)\delta_{\eta,\alpha} - w\left(\alpha\triangle\{x\},\alpha\right)\delta_{\eta,\alpha\triangle\{x\}}\right] , \tag{9}$$

where $\forall\alpha,\gamma \in \Lambda$, $\alpha\triangle\gamma = (\alpha\cup\gamma)\setminus(\alpha\cap\gamma)$ and, with an abuse of notation, we indicate by $w\left(\alpha,\alpha\triangle\{x\}\right)$ the transition rate from the state α to the state $\alpha\triangle\{x\}$.

Since this form of the generator may seem unusual at first glance, here we prove its equivalence to the classical form of generators of birth and death processes on $\mathcal{P}$.

[1]Here, we think of $\mathcal{H}_x$ as spanned by two (orthonormal) vectors labelled by the "empty site" and the "full site" configurations. Consequently any operator acting on the configuration space is lifted to a linear operator acting on $\mathcal{H}_x$ and a probability density on the configuration space becomes a convex combination of the projectors on the subspaces spanned by the basis vectors of $\mathcal{H}_x$.

2.1.1. *Some remarks on birth and death processes for lattice gases.* We denote by $\mathbb{L}(\mathcal{P})$ the linear space of cylinder functions on $\mathcal{P}$ generated by linear combinations of indicator functions of finite subsets of the lattice

$$\mathbb{L}(\mathcal{P}) \ni \varphi = \sum_{\alpha \subset \mathbb{Z}^d \,:\, |\alpha|<\infty} \varphi_\alpha \delta_\alpha$$
$$\forall \alpha \in \mathcal{P}, \quad \mathcal{P} \ni \eta \longmapsto \delta_\alpha(\eta) = \delta_{a,\eta} \in \{0,1\} \ ,$$

where the coefficients φ_α are real numbers.

Usually, see for example [10], the action of the generator of a birth and death process L on $\mathbb{L}(\mathcal{P})$ takes a form which can be expressed in either of the following two representations:

$$L^{(-)}\varphi := \sum_{x\in\alpha} [w(\alpha\backslash\{x\},\alpha)\left(\varphi_{\alpha\backslash\{x\}} - \varphi_\alpha\right)\delta_{\alpha\backslash\{x\}} \tag{10}$$
$$+ w(\alpha,\alpha\backslash\{x\})\left(\varphi_\alpha - \varphi_{\alpha\backslash\{x\}}\right)\delta_\alpha]$$
$$L^{(+)}\varphi := \sum_{x\in\alpha^c} [w(\alpha,\alpha\cup\{x\})\left(\varphi_\alpha - \varphi_{\alpha\cup\{x\}}\right)\delta_\alpha \tag{11}$$
$$+ w(\alpha\cup\{x\},\alpha)\left(\varphi_{\alpha\cup\{x\}} - \varphi_\alpha\right)\delta_{\alpha\cup\{x\}}] \, .$$

Let $\mathcal{P}_0$ be the collection of finite and cofinite subsets of the lattice. These expressions for $(L\varphi)_\alpha$ are mutually equivalent and equivalent to

$$\left(\breve{L}\varphi\right)_\alpha = \sum_{x\in\alpha} w(\alpha,\alpha\backslash\{x\})\left(\varphi_\alpha - \varphi_{\alpha\backslash\{x\}}\right) + \sum_{x\in\alpha^c} w(\alpha,\alpha\cup\{x\})\left(\varphi_\alpha - \varphi_{\alpha\cup\{x\}}\right) \ ,$$

which can be derived from (9) (see (16-19) below). In fact, given the involution of $\mathcal{P}_0$

$$\mathcal{P}_0 \ni \alpha \longmapsto \alpha^c = \mathbb{Z}^d\backslash\alpha \in \mathcal{P}_0 \quad \alpha\subset\mathbb{Z}^d \, , \tag{12}$$

we can define the family of operators $\{\iota_\Lambda\}_{\Lambda\in\mathcal{P}\,:\,|\Lambda|<\infty}$ on $\mathbb{L}(\mathcal{P})$, such that

$$\mathbb{L}(\mathcal{P}) \ni \varphi \longmapsto \phi = \iota_\Lambda\varphi \in \mathbb{L}(\mathcal{P})$$
$$\iota_\Lambda\delta_\alpha = \delta_{\alpha\triangle\Lambda} \quad \alpha\in\mathcal{P} : |\alpha|<\infty$$
$$\iota_\Lambda\varphi = \sum_{\alpha\in\mathcal{P}\,:\,|\alpha|<\infty} \varphi_\alpha\delta_{\alpha\triangle\Lambda} = \sum_{\alpha\in\mathcal{P}\,:\,|\alpha|<\infty} \varphi_{\alpha\triangle\Lambda}\delta_\alpha \, , \quad \varphi_{\alpha\triangle\Lambda} = \varphi_{(\alpha^c\cap\Lambda)\cup(\alpha\cap\Lambda^c)}$$
$$\iota_\Lambda(\iota_\Lambda\varphi) = \varphi \quad \varphi\in\mathbb{L}(\mathcal{P}) \, , \, \Lambda\in\mathcal{P} : |\Lambda|<\infty \, .$$

Defining B to be the generator of a pure birth process with rates

$$w(\alpha\backslash\{x\},\alpha)\,\mathbf{1}_\alpha(x) + w(\alpha,\alpha\cup\{x\})\,(1-\mathbf{1}_\alpha(x))$$

and D the generator of a pure death process with rates

$$w(\alpha,\alpha\backslash\{x\})\,\mathbf{1}_\alpha(x) + w(\alpha\cup\{x\},\alpha)\,(1-\mathbf{1}_\alpha(x)) \ ,$$

we may rewrite (10) and (11) in the form

$$\left(L^{(\pm)}\varphi\right)_\alpha = \left(B^{(\pm)}\varphi\right)_\alpha + \left(D^{(\pm)}\varphi\right)_\alpha \ ,$$

where the definition of $B^{(\pm)}$ and $D^{(\pm)}$ is readily understood. Since

$$w(\alpha\backslash\{x\},\alpha) = w(\alpha^c\cup\{x\},\alpha^c) \tag{13}$$
$$w(\alpha,\alpha\backslash\{x\}) = w(\alpha^c,\alpha^c\cup\{x\}) \ ,$$

considering for example (10), for any finite $\Lambda \subset \mathbb{Z}^d$ we have

$$\begin{aligned}\left(\iota_\Lambda B^{(-)}\iota_\Lambda\varphi\right)_\alpha &= \sum_{x\in\alpha\triangle\Lambda} w\left((\alpha\triangle\Lambda)\setminus\{x\},\alpha\triangle\Lambda\right)\left((\iota_\Lambda\varphi)_{((\alpha\triangle\Lambda)\setminus\{x\})} - (\iota_\Lambda\varphi)_{(\alpha\triangle\Lambda)}\right)\\ &= \sum_{x\in\alpha\triangle\Lambda} w\left((\alpha\triangle\Lambda)\setminus\{x\},\alpha\triangle\Lambda\right)\left(\varphi_{((\alpha\triangle\Lambda)\setminus\{x\})\triangle\Lambda} - \varphi_\alpha\right)\end{aligned}$$

and choosing $\Lambda \supset \alpha$, by (13) we get

$$\begin{aligned}\left(\iota_\Lambda B^{(-)}\iota_\Lambda\varphi\right)_\alpha &= \sum_{x\in\alpha^c\cap\Lambda} w\left((\alpha^c\cap\Lambda)\setminus\{x\},\alpha^c\cap\Lambda\right)\left(\varphi_{((\alpha^c\cap\Lambda)\setminus\{x\})^c\cap\Lambda} - \varphi_\alpha\right)\\ &= \sum_{x\in\alpha^c\cap\Lambda} w\left((\alpha\cup\{x\})^c\cap\Lambda,\alpha^c\cap\Lambda\right)\left(\varphi_{\alpha\cup\{x\}} - \varphi_\alpha\right)\\ &= \sum_{x\in\alpha^c\cap\Lambda} w\left(\alpha\cup\{x\}\cup\Lambda^c,\alpha\cup\Lambda^c\right)\left(\varphi_{\alpha\cup\{x\}} - \varphi_\alpha\right)\\ &= \sum_{x\in(\alpha\cup\Lambda^c)^c} w\left((\alpha\cup\Lambda^c)\cup\{x\},\alpha\cup\Lambda^c\right)\left(\varphi_{\alpha\cup\{x\}} - \varphi_\alpha\right)\,.\end{aligned}$$

We now assume the system to be confined in a box Λ with boundary conditions η. Let $\mathcal{P}_\Lambda$ be the set of the subsets of Λ. We can inject $\mathbb{L}(\mathcal{P}_\Lambda)$, the vector space generated by linear combinations of $\delta_\alpha,\ \alpha\subseteq\Lambda$, in $\mathbb{L}(\mathcal{P})$ and consider a naturally defined ι_Λ^η.

$$\mathbb{L}(\mathcal{P}_\Lambda) \ni \varphi \longmapsto \phi^\eta = \iota_\Lambda^\eta\varphi = \iota_\Lambda(\varphi\delta_\eta) \in \mathbb{L}(\mathcal{P}) \tag{14}$$

$$\iota_\Lambda^\eta\delta_\alpha = \delta_{(\alpha\cup\eta)\triangle\Lambda} = \delta_{\Lambda\setminus\alpha\cup\eta} \quad \alpha\subseteq\Lambda\,. \tag{15}$$

Independently of the choice of the boundary conditions η, $\forall\alpha\subseteq\Lambda$

$$\begin{aligned}\left(\iota_\Lambda^\eta B_{\Lambda,\eta}^{(-)}\iota_\Lambda^\eta\varphi\right)_\alpha &= \sum_{x\in\Lambda\setminus\alpha} w\left(\alpha\cup\{x\},\alpha\right)\left(\varphi_{\alpha\cup\{x\}} - \varphi_\alpha\right)\\ &= \left(D_{\Lambda,\eta}^{(+)}\varphi\right)_\alpha \qquad \eta\in\mathcal{P}_{\Lambda^c},\ |\eta|<\infty\,,\end{aligned}$$

where $B_{\Lambda,\eta}^{(\pm)}$ and $D_{\Lambda,\eta}^{(\pm)}$ denote the natural restrictions of $B^{(\pm)}$ and $D^{(\pm)}$, to $\mathbb{L}(\mathcal{P}_\Lambda)$.

To keep notation simple, from now on we will omit to indicate the boundary conditions where there is no risk of ambiguity.

Now, for any realization of $\omega\in\Omega,\ \beta\geq 0$ and $\alpha\in\mathcal{P}$, denoting by $\bar{L}(\beta,\omega)$ the generator of the process given in (3) in this representation, from (9) we get

$$\begin{aligned}\bar{L}(\beta,\omega)\delta_\alpha &= \sum_{\eta\in\mathcal{P}\,:\,|\eta|<\infty} \left(\bar{L}(\beta,\omega)\delta_\alpha\right)_\eta\delta_\eta \\ &= \sum_{x\in\mathbb{Z}^d}\left[w^{\beta,\omega}(\alpha,\alpha\triangle\{x\})\delta_\alpha - w^{\beta,\omega}(\alpha\triangle\{x\},\alpha)\delta_{\alpha\triangle\{x\}}\right]\,.\end{aligned} \tag{16}$$

Then, $\forall\varphi\in\mathbb{L}(\mathcal{P})$, we have

$$\bar{L}(\beta,\omega)\varphi=\sum_{\alpha\in\mathcal{P}\,:\,|\alpha|<\infty}\left(\bar{L}(\beta,\omega)\varphi\right)_{\alpha}\delta_{\alpha} \tag{17}$$

$$\begin{aligned}\bar{L}(\beta,\omega)\varphi&=\sum_{\alpha\in\mathcal{P}\,:\,|\alpha|<\infty}\sum_{x\in\mathbb{Z}^d}\varphi_{\alpha}\left[w^{\beta,\omega}(\alpha,\alpha\triangle\{x\})\delta_{\alpha}-w^{\beta,\omega}(\alpha\triangle\{x\},\alpha)\delta_{\alpha\triangle\{x\}}\right]\\&=\sum_{\alpha\in\mathcal{P}\,:\,|\alpha|<\infty}\sum_{x\in\mathbb{Z}^d}w^{\beta,\omega}(\alpha,\alpha\triangle\{x\})\left(\varphi_{a}-\varphi_{a\triangle\{x\}}\right)\delta_{\alpha}\,.\end{aligned} \tag{18}$$

Notice that, for any $\omega\in\Omega$, (18) takes the form

$$\begin{aligned}\left(\bar{L}(\beta,\omega)\varphi\right)_{\alpha}:=&\sum_{x\in\alpha}w^{\beta,\omega}(\alpha,\alpha\backslash\{x\})\left(\varphi_{\alpha}-\varphi_{\alpha\backslash\{x\}}\right)+\\&+\sum_{x\in\alpha^c}w^{\beta,\omega}(\alpha,\alpha\cup\{x\})\left(\varphi_{\alpha}-\varphi_{\alpha\cup\{x\}}\right)\ ,\quad \alpha\in\mathcal{P}:|\alpha|<\infty\ .\end{aligned} \tag{19}$$

Hence, since $\iota_{\Lambda}D_{\Lambda}^{(+)}\iota_{\Lambda}=B_{\Lambda}^{(-)}$,

$$L_{\Lambda}^{(-)}=B_{\Lambda}^{(-)}+D_{\Lambda}^{(-)}=\iota_{\Lambda}\left(B_{\Lambda}^{(+)}+D_{\Lambda}^{(+)}\right)\iota_{\Lambda}=\iota_{\Lambda}L^{(+)}\iota_{\Lambda}$$

and, $\forall\omega\in\Omega,\ \beta\geq0$, $\alpha\subseteq\Lambda$ and boundary condition η, (19), takes the form

$$\begin{aligned}\left(\bar{L}_{\Lambda}(\beta,\omega,\eta)\varphi\right)_{\alpha}&=\left(D_{\Lambda}^{(-)}(\omega,\beta,\eta)\varphi\right)_{\alpha}+\left(B_{\Lambda}^{(+)}(\omega,\beta,\eta)\varphi\right)_{\alpha}\\&=\left(D_{\Lambda}^{(-)}(\omega,\beta,\eta)\varphi\right)_{\alpha}+\left(\iota_{\Lambda}D_{\Lambda}^{(-)}(\omega,\beta,\eta)\iota_{\Lambda}\varphi\right)_{\alpha}\,.\end{aligned} \tag{20}$$

It is worth to notice that, for any realization of the potential, $\bar{L}_{\Lambda}(\beta,\omega,\eta)$ commutes with ι_{Λ}.

We remark that the equivalence between (18) and the generator of process defined in (3) can be deduced comparing the associated Dirichlet forms.

3. **Proof of the Theorems.**

Replacing φ by δ_{η} for a fixed $\eta\subseteq\Lambda$ in (19), we get the generic matrix element of (17) and then of (18) as operators acting on $\mathcal{H}_{\Lambda}$. We can then transform (18) into a selfadjoint operator $\tilde{L}_{\Lambda}^{s}(\beta,\omega)$ on $\mathcal{H}_{\Lambda}$ through the unitary mapping from $\mathcal{H}_{\Lambda}(\beta,\omega):=l^2\left(\mathcal{P}_{\Lambda},\mu_{\Lambda}^{\beta,\omega}\right)$ (which is isomorphic to the restriction of $\mathcal{L}(\beta,\omega)$ to Λ) to $\mathcal{H}_{\Lambda}$ given by the multiplication of the elements of $\mathcal{H}_{\Lambda}(\beta,\omega)$ by $\sqrt{\frac{\mu_{\Lambda}^{\beta,\omega}}{\mu_{\Lambda}}}$.

We will give a relative bound of the Dirichlet form of $\tilde{L}_{\Lambda}^{s}(\beta,\omega)$ in terms of the Dirichlet form of the generator of the independent process $\bar{L}_{\Lambda}$ and make use of standard perturbation theory to give a lower bound for the spectral gap of $\tilde{L}_{\Lambda}^{s}(\beta,\omega)$, $g_d^{-,\Lambda}(\beta)$, for small values of $\beta>0$ and for any $\omega\in\Omega$. These bounds will turn out to be independent of Λ, which implies in particular $g_d^{-,\Lambda}(\beta)=g_d^{-}(\beta)$, and therefore extend to the infinite volume setting. We get $g_d^{+}(\beta)$ by applying the same argument to the operator $\hat{L}_{\Lambda}^{s}(\beta,\omega)\geq\tilde{L}_{\Lambda}^{s}(\beta,\omega)$ on $\mathcal{H}_{\Lambda}$, which is also unitary equivalent to a generator of a Glauber process for the Ising model reversible with respect to $\mu_{\Lambda}^{\beta,\omega}$. The proof of Theorem 2 relies on two results. First a theorem of Minlos (Theorem 2.2 of [9]) which gives detailed information on the first branch of the spectrum for constant realizations. Second on the part 2) of Theorem 3 of

[1], which proves that the first branch of the spectrum for a constant realization is contained in the first branch of the spectrum with random coupling.

Finally, since the family of operators and spaces $(L(\beta,\omega), \mathcal{L}(\beta,\omega))$ is a metrically transitive family with respect to lattice translations, from general results of spectral theory for random operators (see [11] and Remark 4 of [1]), it will follow that the spectrum of $L(\beta,\omega)$ is non-random for $\mathbb{P}$-a.e. ω. This remark, together with the two previous results, will then prove Theorem 3.

Let us consider the heat bath case. Given a finite portion of the lattice Λ and a realization of the potential ω, assuming for example periodic boundary condition, the restriction of the generator of the process given in (18) to $\mathcal{P}_\Lambda$, takes the form (17), (19), where

$$w^{\beta,\omega}(\alpha, \alpha\triangle\{x\}) = \psi_{hb}(\beta\Delta_x H_\alpha^\omega) = \frac{1}{1+e^{-\beta\Delta_x H_\alpha^\omega}}$$

and

$$\begin{aligned}\Delta_x H_\alpha^\omega &= H_\alpha(\omega) - H_{\alpha\triangle\{x\}}(\omega) \\ &= \mathbf{1}_\alpha(x)\left[H_\alpha(\omega) - H_{\alpha\setminus\{x\}}(\omega)\right] + \mathbf{1}_{\alpha^c}(x)\left[H_\alpha(\omega) - H_{\alpha\cup\{x\}}(\omega)\right]\end{aligned} \tag{21}$$

representing respectively (8) and (4) in the lattice gas framework. Here

$$H_\alpha(\omega) = \sum_{b\in\mathbb{B}_d}\omega_b - 2\sum_{b\in\partial\alpha}\omega_b.$$

Although infinite, $\sum_{b\in\mathbb{B}_d}\omega_b$ is an harmless constant since transition rates are functions only of $\Delta_x H_\alpha^\omega$.

Following [3], since $\mathcal{H}_\Lambda \cong \bigoplus_{n=0}^{|\Lambda|}\mathcal{H}_\Lambda^{(n)}$, with $\mathcal{H}_\Lambda^{(0)} \equiv \mathbb{R}$ and

$$\mathcal{H}_\Lambda^{(n)} := \{|\alpha\rangle \in \mathcal{H}_\Lambda : |\alpha| = n\},$$

we denote by U_Λ the unitary operator

$$\begin{aligned} U_\Lambda &: \mathcal{H}_\Lambda \longrightarrow \mathcal{H}_\Lambda \\ U_\Lambda|\alpha\rangle &= \frac{1}{2^{\frac{|\Lambda|}{2}}}\sum_{\gamma\subseteq\Lambda}(-1)^{|\alpha\cap\gamma|}|\gamma\rangle \quad \alpha\subseteq\Lambda,\end{aligned} \tag{22}$$

and by E_Λ, the representation of the involution ι_Λ introduced in (14) as an operator on $\mathcal{H}_\Lambda$, that is

$$\begin{aligned} E_\Lambda &: \mathcal{H}_\Lambda \longrightarrow \mathcal{H}_\Lambda \\ E_\Lambda|\alpha\rangle &= |\Lambda\setminus\alpha\rangle \quad \alpha\subseteq\Lambda.\end{aligned} \tag{23}$$

Now, for any $\alpha\subseteq\Lambda$, $E_\Lambda\frac{|\alpha\rangle\pm|\alpha^c\rangle}{\sqrt{2}} = \pm\frac{|\alpha\rangle\pm|\alpha^c\rangle}{\sqrt{2}}$, hence $\mathcal{H}_\Lambda = \mathcal{H}_\Lambda^+ \oplus \mathcal{H}_\Lambda^-$, where

$$\mathcal{H}_\Lambda^\pm := span\{\frac{|\alpha\rangle\pm|\alpha^c\rangle}{\sqrt{2}} : \alpha\subseteq\Lambda\}$$

Moreover, setting

$$\bar{E}_\Lambda := U_\Lambda E_\Lambda U_\Lambda : \mathcal{H}_\Lambda \longrightarrow \mathcal{H}_\Lambda, \tag{24}$$

since

$$\begin{aligned}\delta_{\alpha,\gamma} &= \langle\alpha|\gamma\rangle = \langle\alpha|U_\Lambda U_\Lambda|\gamma\rangle = 2^{-|\Lambda|}\sum_{\eta\subseteq\Lambda}(-1)^{|\alpha\cap\eta|+|\gamma\cap\eta|} \\ &= 2^{-|\Lambda|}\sum_{\eta\subseteq\Lambda}(-1)^{|(\alpha\triangle\gamma)\cap\eta|},\end{aligned} \tag{25}$$

then, for any $|\alpha\rangle \in \mathcal{H}_\Lambda$,

$$\bar{E}_\Lambda |\alpha\rangle = \sum_{\gamma,\eta\subseteq\Lambda} \frac{(-1)^{|\alpha\cap\gamma|+|(\Lambda\setminus\gamma)\cap\eta|}}{2^{|\Lambda|}} |\eta\rangle = \sum_{\gamma,\eta\subseteq\Lambda} \frac{(-1)^{|\alpha\cap\gamma|-|\gamma\cap\eta|+|\eta|}}{2^{|\Lambda|}} |\eta\rangle = (-1)^{|\alpha|} |\alpha\rangle \ ,$$

so that $\mathcal{H}_\Lambda$ can also be decomposed as direct sum of $\overline{\mathcal{H}}_\Lambda^+ := \bigoplus_{n\geq 0\,:\,2n\in\{0,..,|\Lambda|\}} \mathcal{H}_\Lambda^{(2n)}$ and $\overline{\mathcal{H}}_\Lambda^- := \bigoplus_{n\geq 0\,:\,2n+1\in\{0,..,|\Lambda|\}} \mathcal{H}_\Lambda^{(2n+1)}$. Clearly $U_\Lambda \mathcal{H}_\Lambda^\pm = \overline{\mathcal{H}}_\Lambda^\pm$. Furthermore, by (25), $\mathcal{H}_\Lambda = \mathcal{H}_{\Lambda,+} \oplus \mathcal{H}_{\Lambda,-}$, where $U_\Lambda \mathcal{H}_{\Lambda,\pm} = \pm\mathcal{H}_{\Lambda,\pm}$.

If, for any $x \in \Lambda$, ℓ_x^Λ, $\ell_x^{\Lambda,\perp}$ denote the mutually orthogonal projectors on $\mathcal{H}_\Lambda$ such that

$$\ell_x^{\Lambda,\perp} := I_\Lambda - \ell_x^\Lambda \,;\ \ell_x^\Lambda |\alpha\rangle = \mathbf{1}_\alpha(x) |\alpha\rangle \quad \alpha \subseteq \Lambda\,. \tag{26}$$

We have

$$\begin{aligned} \bar{\ell}_x^\Lambda &= U_\Lambda \ell_x^\Lambda U_\Lambda; \quad \bar{\ell}_x^{\Lambda,\perp} = U_\Lambda \ell_x^{\Lambda,\perp} U_\Lambda, \\ \ell_x^\Lambda &= E_\Lambda \ell_x^{\Lambda,\perp} E_\Lambda; \quad \ell_x^{\Lambda,\perp} = E_\Lambda \ell_x^\Lambda E_\Lambda, \\ &[E_\Lambda, \bar{\ell}_x^\Lambda] = [\bar{E}_\Lambda, \ell_x^\Lambda] = 0. \end{aligned} \tag{27}$$

We also denote by

$$\frac{e^{-\frac{\beta}{2}H_\Lambda^\omega}}{\left(Z_\Lambda^{(d)}(\beta,\omega)\right)^{\frac{1}{2}}} : \mathcal{H}_\Lambda(\beta,\omega) \longrightarrow \mathcal{H}_\Lambda$$

the matrix representation of the multiplication operator by $\sqrt{\frac{\mu_\Lambda^{\beta,\omega}}{\mu_\Lambda}}$.

In [3, 4], comparing Dirichlet forms, we also showed that $\bar{L}_\Lambda^s(\beta,\omega)$ admits the representation

$$\tilde{L}_\Lambda^s(\beta,\omega) = \sum_{x\in\Lambda} \tilde{L}_{x,\Lambda}^s(\beta,\omega) \ ,$$

whose matrix elements, by the definition of $\Delta_x H_\alpha^\omega$, are, for any two vectors $|\alpha\rangle, |\gamma\rangle$ of the basis of $\mathcal{H}_\Lambda$

$$\begin{aligned} &\langle\gamma| \tilde{L}_\Lambda^s(\beta,\omega) |\alpha\rangle = \sum_{x\in\Lambda} \langle\gamma| \tilde{L}_{x,\Lambda}^s(\beta,\omega) |\alpha\rangle \\ &\langle\gamma| \tilde{L}_{x,\Lambda}^s(\beta,\omega) |\alpha\rangle \\ &= \langle\gamma| \left\{ \mathbf{1}_\alpha(x) \frac{1}{\cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega} \left[\bar{\ell}_x^\Lambda + I_\Lambda \frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega} - 1}{2} \right] + \right. \\ &\qquad \left. + \mathbf{1}_{\alpha^c}(x) \frac{1}{\cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega} \left[\bar{\ell}_x^\Lambda + I_\Lambda \frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega} - 1}{2} \right] \right\} |\alpha\rangle \\ &= \langle\gamma| \left\{ \mathbf{1}_\alpha(x) \frac{1}{\cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega} \left[\frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega} + 1}{2} \bar{\ell}_x^\Lambda + \frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega} - 1}{2} \bar{\ell}_x^{\Lambda,\perp} \right] + \right. \\ &\qquad \left. + \mathbf{1}_{\alpha^c}(x) \frac{1}{\cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega} \left[\frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega} + 1}{2} \bar{\ell}_x^\Lambda + \frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega} - 1}{2} \bar{\ell}_x^{\Lambda,\perp} \right] \right\} |\alpha\rangle \ . \end{aligned} \tag{28}$$

3.1. **Proof of Theorem 2.1.** Let us set

$$L_{\Lambda} := \sum_{x\in\Lambda} \ell_x^{\Lambda}\,; \quad \bar{L}_{\Lambda} = U_{\Lambda} L_{\Lambda} U_{\Lambda} = \sum_{x\in\Lambda} \bar{\ell}_x^{\Lambda}\,,$$
$$L_{\Lambda} \left|\alpha\right\rangle = |\alpha| \left|\alpha\right\rangle \quad \alpha \subseteq \Lambda\,.$$

Lemma 3.1. *For any* $\left|u\right\rangle \in \mathcal{H}_{\Lambda}^{\pm}$,

$$\left\langle u\right| \bar{L}_{\Lambda} \left|u\right\rangle \leq 2 \left\langle u\right| L_{\Lambda} \left|u\right\rangle \ . \tag{29}$$

Proof. We first notice that, for any $x \in \Lambda$, $a : \mathcal{P}_{\Lambda} \times \mathcal{P}_{\Lambda} \longrightarrow \mathbb{R}$,

$$\sum_{\alpha\subseteq\Lambda} a_{\alpha,\alpha\cup\{x\}} \mathbf{1}_{\alpha^c}(x) = \sum_{\alpha\subseteq\Lambda} \mathbf{1}_{\alpha}(x)\, a_{\alpha\backslash\{x\},\alpha}\,.$$

Then, for any $\left|u\right\rangle \in \mathcal{H}_{\Lambda}^{\pm}$, we get

$$\begin{aligned}
\left\langle u\right| \bar{L}_{\Lambda} \left|u\right\rangle &= \sum_{x\in\Lambda}\sum_{\alpha\subseteq\Lambda} \frac{u_{\alpha}^2 - u_{\alpha}u_{\alpha\triangle\{x\}}}{2} = \sum_{x\in\Lambda}\sum_{\alpha\subseteq\Lambda} \left(\frac{u_{\alpha} - u_{\alpha\triangle\{x\}}}{2}\right)^2 \\
&\leq \sum_{x\in\Lambda}\sum_{\alpha\subseteq\Lambda} \frac{u_{\alpha}^2 + u_{\alpha\triangle\{x\}}^2}{2} = \sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha} \frac{u_{\alpha}^2 + u_{\alpha\backslash\{x\}}^2}{2} + \sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha^c} \frac{u_{\alpha}^2 + u_{\alpha\cup\{x\}}^2}{2} \\
&= \sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha} u_{\alpha}^2 + \sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha^c} u_{\alpha}^2 = \sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha} u_{\alpha}^2 + \sum_{\alpha^c\subseteq\Lambda}\sum_{x\in\alpha^c} u_{\alpha^c}^2 = 2\left\langle u\right| L_{\Lambda} \left|u\right\rangle \ .
\end{aligned}$$

□

Remark 2. From (1) it follows that, for any $\omega \in \Omega$, H_{α}^{ω} depends on α only through the subset of $\mathbb{B}_{\Lambda}$, $\partial\alpha := \{b \in \mathbb{B}_{\Lambda} : |b \cap \alpha| = 1\}$ then, because $\partial\alpha = \partial\alpha^c$, by (20), (23) and (27), for any realization of the potential $H_{\alpha}(\omega) = H_{\alpha^c}(\omega)$. Hence, for any $\beta \geq 0$ and $\omega \in \Omega$, $\tilde{L}_{\Lambda}^{s}(\beta,\omega)$ commutes with E_{Λ}. The ground state of $\tilde{L}_{\Lambda}^{s}(\beta,\omega)$, that is to say

$$\left|g_{\Lambda}(\beta,\omega)\right\rangle := \sum_{\alpha\subseteq\Lambda} g_{\alpha}^{\Lambda}(\beta,\omega) \left|\alpha\right\rangle \ ,$$

where $g_{\alpha}^{\Lambda}(\beta,\omega) := \frac{e^{-\frac{\beta}{2}H_{\alpha}(\omega)}}{\left(Z_{\Lambda}^{(d)}(\beta,\omega)\right)^{\frac{1}{2}}}$, belongs to $\mathcal{H}_{\Lambda}^{+}$.

Now let β and ω be fixed. For any vector $|u\rangle \in \mathcal{H}_\Lambda$, by (28) the Dirichlet form associated to $\tilde{L}^s_\Lambda(\beta,\omega)$ can be written in the following way

$$\begin{aligned}\langle u|\,\tilde{L}^s_\Lambda(\beta,\omega)\,|u\rangle &= \sum_{\alpha\subseteq\Lambda}\sum_{x\in\Lambda}\frac{1}{\cosh\frac{\beta}{2}\Delta_x H^\omega_\alpha}\left[\left(\frac{u_\alpha - u_{\alpha\triangle\{x\}}}{2}\right)^2 + u^2_\alpha\frac{e^{\frac{\beta}{2}\Delta_x H^\omega_\alpha}-1}{2}\right] \qquad (30)\\ &= \sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha}\frac{1}{\cosh\frac{\beta}{2}\Delta_x H^\omega_\alpha}\left[\left(\frac{u_\alpha - u_{\alpha\setminus\{x\}}}{2}\right)^2 + u^2_\alpha\frac{e^{\frac{\beta}{2}\Delta_x H^\omega_\alpha}-1}{2}\right]+\\ &\quad+\sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha^c}\frac{1}{\cosh\frac{\beta}{2}\Delta_x H^\omega_\alpha}\left[\left(\frac{u_\alpha - u_{\alpha\cup\{x\}}}{2}\right)^2 + u^2_\alpha\frac{e^{\frac{\beta}{2}\Delta_x H^\omega_\alpha}-1}{2}\right]\\ &= \sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha}\frac{1}{\cosh\frac{\beta}{2}\Delta_x H^\omega_\alpha}\left[\frac{\left(u_\alpha - u_{\alpha\setminus\{x\}}\right)^2}{2} + u^2_\alpha\frac{e^{\frac{\beta}{2}\Delta_x H^\omega_\alpha}-1}{2}\right.\\ &\quad\left.+u^2_{\alpha\setminus\{x\}}\frac{e^{-\frac{\beta}{2}\Delta_x H^\omega_\alpha}-1}{2}\right].\end{aligned}$$

Clearly, $\forall\omega\in\Omega,\ \tilde{L}^s_\Lambda(\beta=0,\omega)=\bar{L}_\Lambda$.

Proposition 1. *Let $\beta\geq 0$, $\omega\in\Omega$ be fixed and Λ be such that $|\Lambda|=2N$, $N\in\mathbb{N}$. For any $|v\rangle\in\mathcal{H}^\pm_\Lambda$,*

$$\langle v|\,U_\Lambda\tilde{L}^s_\Lambda(\beta,\omega)\,U_\Lambda\,|v\rangle \leq (1+2b_{dJ}(\beta))\,\langle v|\,L_\Lambda\,|v\rangle\ , \qquad (31)$$

where $b_{dJ}(\beta)$ is an analytic function of β such that $b_{dJ}(\beta)=c_{dJ}\beta+o(\beta)$.

Proof. Since by the previous remark it follows that $U_\Lambda\tilde{L}^s_\Lambda(\beta,\omega)\,U_\Lambda$ commutes with $\bar{E}_\Lambda$, we can restrict ourselves to vectors in $\overline{\mathcal{H}}^\pm_\Lambda$. Let us take $|v\rangle\in\overline{\mathcal{H}}^\pm_\Lambda$, then $U_\Lambda|v\rangle=|u\rangle\in\mathcal{H}^\pm_\Lambda$. From (30) it follows that

$$\begin{aligned}\frac{1}{2}\sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha^c}\frac{e^{\frac{\beta}{2}\Delta_x H^\omega_\alpha}-1}{\cosh\frac{\beta}{2}\Delta_x H^\omega_\alpha}u^2_\alpha &= \frac{1}{2}\sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha^c}\frac{e^{\frac{\beta}{2}\Delta_x H^\omega_{\alpha^c}}-1}{\cosh\frac{\beta}{2}\Delta_x H^\omega_{\alpha^c}}u^2_{\alpha^c}\\ &= \frac{1}{2}\sum_{\alpha^c\subseteq\Lambda}\sum_{x\in\alpha^c}\frac{e^{\frac{\beta}{2}\Delta_x H^\omega_{\alpha^c}}-1}{\cosh\frac{\beta}{2}\Delta_x H^\omega_{\alpha^c}}u^2_{\alpha^c}\\ &= \frac{1}{2}\sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha}\frac{e^{\frac{\beta}{2}\Delta_x H^\omega_\alpha}-1}{\cosh\frac{\beta}{2}\Delta_x H^\omega_\alpha}u^2_\alpha\,.\end{aligned}$$

Thus,

$$\langle u|\,\tilde{L}^s_\Lambda(\beta,\omega)\,|u\rangle = \sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha}\frac{1}{\cosh\frac{\beta}{2}\Delta_x H^\omega_\alpha}\left[\frac{\left(u_\alpha-u_{\alpha\setminus\{x\}}\right)^2}{2}+u^2_\alpha\left(e^{\frac{\beta}{2}\Delta_x H^\omega_\alpha}-1\right)\right].$$

Therefore,

$$\begin{aligned}\langle v|\,U_\Lambda\tilde{L}^s_\Lambda(\beta,\omega)\,U_\Lambda\,|v\rangle &= \langle u|\,\tilde{L}^s_\Lambda(\beta,\omega)\,|u\rangle\\ &\leq \langle v|\,L_\Lambda\,|v\rangle + b_J(\beta)\,\langle u|\,L_\Lambda\,|u\rangle\\ &= \langle v|\,L_\Lambda\,|v\rangle + b_J(\beta)\,\langle v|\,\bar{L}_\Lambda\,|v\rangle\ ,\end{aligned}$$

Moreover, if $v \in \mathcal{H}_\Lambda^\pm \cap \overline{\mathcal{H}}_\Lambda^\pm$, by (29),

$$\langle v| L_\Lambda |v\rangle + b_J(\beta) \langle v| \bar{L}_\Lambda |v\rangle \le (1 + 2b_J(\beta)) \langle v| L_\Lambda |v\rangle \ ,$$

where $b_{dJ}(\beta) := \max_{z\in[0,4dJ]} \left[\frac{e^{\frac{\beta}{2}z}-1}{\cosh\frac{\beta}{2}z}\right] = \frac{e^{2\beta dJ}-1}{\cosh 2\beta dJ}$. □

In [3, 4] we introduced a new form for the generator of stochastic Ising model with transition rates

$$w^{\beta,\omega}(\alpha, \alpha\triangle\{x\}) = \frac{1+e^{\beta\Delta_x H_\alpha^\omega}}{4}$$

whose generic matrix element as an operator acting on $\mathcal{H}_\Lambda$ is

$$\langle\gamma| \hat{L}_\Lambda^s(\beta,\omega) |\alpha\rangle = \langle\gamma| \sum_{x\in\Lambda} \hat{L}_x^s(\beta,\omega) |\alpha\rangle \tag{32}$$

$$\begin{aligned}
&\langle\gamma| \hat{L}_x^s(\beta,\omega) |\alpha\rangle \\
&= \langle\gamma| \left\{ \mathbf{1}_\alpha(x) \cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega \left[\bar{\ell}_x^\Lambda + I_\Lambda \frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega}-1}{2}\right] + \right. \\
&\qquad \left. +\mathbf{1}_{\alpha^c}(x) \cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega \left[\bar{\ell}_x^\Lambda + I_\Lambda \frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega}-1}{2}\right]\right\} |\alpha\rangle \\
&= \langle\gamma| \left\{ \mathbf{1}_\alpha(x) \cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega \left[\frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega}+1}{2}\bar{\ell}_x^\Lambda + \frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega}-1}{2}\bar{\ell}_x^{\Lambda,\perp}\right] + \right. \\
&\qquad \left. +\mathbf{1}_{\alpha^c}(x) \cosh\frac{\beta}{2}\Delta_x H_\alpha^\omega \left[\frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega}+1}{2}\bar{\ell}_x^\Lambda + \frac{e^{\frac{\beta}{2}\Delta_x H_\alpha^\omega}-1}{2}\bar{\ell}_x^{\Lambda,\perp}\right]\right\} |\alpha\rangle \ .
\end{aligned}$$

We also showed that $\hat{L}_\Lambda^s(\beta,\omega)$ admits the representation

$$\hat{L}_\Lambda^s(\beta,\omega) = \sum_{x\in\Lambda} U_\Lambda e^{\frac{\beta}{2}\mathbf{H}_\Lambda(\omega)} \ell_x^\Lambda e^{-\beta\mathbf{H}_\Lambda(\omega)} \ell_x^\Lambda e^{\frac{\beta}{2}\mathbf{H}_\Lambda(\omega)} U_\Lambda \,,$$

where

$$\begin{aligned}
\mathbf{H}_\Lambda(\omega) &:= \sum_{\alpha\subseteq\Lambda} H_\alpha(\omega) |\alpha\rangle\langle\alpha| \simeq \mathbf{H}_\Lambda(\omega) = \sum_{b\in\mathbb{B}_\Lambda} \omega_b \mathbf{s}_b \,, \\
\mathbf{s}_b &:= \mathbf{1}_b(x) \mathbf{1}_b(y) (1-\delta_{x,y}) \mathbf{s}_x \mathbf{s}_y \,,
\end{aligned} \tag{33}$$

so that, $\forall x \in \Lambda$, $\alpha \subseteq \Lambda$, $\mathbf{s}_x |\alpha\rangle = |\alpha\triangle\{x\}\rangle$ (we prefer to work in the representation where the spin flip operator is diagonal). By (28) and (32), for any two basis vectors of $\mathcal{H}_\Lambda$, $|\alpha\rangle, |\gamma\rangle$ we have

$$\left|\langle\gamma| \tilde{L}_x^s(\beta,\omega) |\alpha\rangle\right| \le \left|\langle\gamma| \hat{L}_x^s(\beta,\omega) |\alpha\rangle\right| \quad x \in \Lambda \,,$$

moreover, the first order terms in the expansion for small β of $\langle\gamma| \tilde{L}_\Lambda^s(\beta,\omega) |\alpha\rangle$ and $\langle\gamma| \hat{L}_\Lambda^s(\beta,\omega) |\alpha\rangle$ are equal for every Λ. Clearly, $\hat{L}_\Lambda^s(\beta,\omega)$ also commutes with E_Λ and for any $|u\rangle \in \mathcal{H}_\Lambda$, since $|u\rangle = |u^+\rangle + |u^-\rangle$, $|u^\pm\rangle \in \mathcal{H}_\Lambda^\pm$, we have

$$\begin{aligned}
&\left\langle u^{\pm}\right| \hat{L}_{\Lambda}^{s}\left(\beta,\omega\right)\left|u^{\pm}\right\rangle \\
&=\sum_{\alpha\subseteq\Lambda}\sum_{x\in\Lambda}\cosh\frac{\beta}{2}\Delta_{x}H_{\alpha}^{\omega}\left[\left(\frac{u_{\alpha}^{\pm}-u_{\alpha\triangle\{x\}}^{\pm}}{2}\right)^{2}+\left(u_{\alpha}^{\pm}\right)^{2}\frac{e^{\frac{\beta}{2}\Delta_{x}H_{\alpha}^{\omega}}-1}{2}\right] \qquad (34)\\
&=\sum_{\alpha\subseteq\Lambda}\sum_{x\in\alpha}\cosh\frac{\beta}{2}\Delta_{x}H_{\alpha}^{\omega}\left[\left(\frac{u_{\alpha}^{\pm}-u_{\alpha\backslash\{x\}}^{\pm}}{2}\right)^{2}+\left(u_{\alpha}^{\pm}\right)^{2}\frac{e^{\frac{\beta}{2}\Delta_{x}H_{\alpha}^{\omega}}-1}{2}\right].
\end{aligned}$$

Proceding as in Proposition 1, we get

$$\left\langle u\right| \hat{L}_{\Lambda}^{s}\left(\beta,\omega\right)\left|u\right\rangle \leq\left(1+2b_{dJ}^{\prime}\left(\beta\right)\right)\left\langle u\right|\bar{L}_{\Lambda}\left|u\right\rangle \qquad u\in\mathcal{H}_{\Lambda}^{\pm}\cap\overline{\mathcal{H}}_{\Lambda}^{\pm}\,, \qquad (35)$$

where $b_{dJ}^{\prime}\left(\beta\right):=\max_{z\in[0,4dJ]}\left[\left(e^{\frac{\beta}{2}z}-1\right)\cosh\frac{\beta}{2}z+\cosh\frac{\beta}{2}z-1\right]=\frac{e^{2\beta dJ}-1}{2}$. Comparing the Dirichlet forms of $\hat{L}_{\Lambda}^{s}\left(\beta,\omega\right)$ and $\tilde{L}_{\Lambda}^{s}\left(\beta,\omega\right)$, we proved in [4] that this process converges to the equilibrium state at high temperature faster than the heat-bath process.

Remark 3. The relative bounds (31) and (35) are independent of Λ and extend straightforwardly to the quadratic forms associated to the operators $\tilde{L}^{s}\left(\beta,\omega\right)$ and $\hat{L}^{s}\left(\beta,\omega\right)$ acting on $\mathcal{H}$. Therefore, by standard arguments of perturbation theory (see [7] Theorem VI.3.4) (31) implies the analyticity of the projectors

$$P_{n}\left(\beta,\omega\right):=\oint_{\{z\in\mathbb{C}:\,|z-n|\leq r(\beta)\}}\frac{dz}{2\pi i}\frac{1}{Iz-A\left(\beta,\omega\right)}\qquad n\in\mathbb{N}\,,$$

where $A\left(\beta,\omega\right)$ is either $\tilde{L}^{s}\left(\beta,\omega\right)$ or $\hat{L}^{s}\left(\beta,\omega\right)$, for sufficiently small values of β.

3.1.1. *Lower bound* $g_{d}^{-}\left(\beta\right)$. By Remark 3, we can make use of perturbation theory and, for sufficently small values of β and any realization of the potential, we can write

$$\hat{L}_{\Lambda}^{s}\left(\beta,\omega\right)=U_{\Lambda}L_{\Lambda}U_{\Lambda}+\beta U_{\Lambda}T_{\Lambda}^{(1)}\left(\omega\right)U_{\Lambda}+\bar{T}_{\Lambda}\left(\beta,\omega\right)\,,$$

where

$$T_{\Lambda}^{(1)}\left(\omega\right):=\frac{1}{2}\sum_{x\in\Lambda}[[\mathbf{H}_{\Lambda}\left(\omega\right),\ell_{x}^{\Lambda}],\ell_{x}^{\Lambda}]$$

is the first term in the expansion of $\hat{L}_{\Lambda}^{s}\left(\beta,\omega\right)$ and $\bar{T}_{\Lambda}\left(\beta,\omega\right)$ is such that

$$\left\langle u\right|\bar{T}_{\Lambda}\left(\beta,\omega\right)\left|u\right\rangle \leq\beta^{2}C\left(d,J\right)\,,$$

with $C\left(d,J\right)$ a positive constant.

Since, by definition of U_{Λ}, $U_{\Lambda}\hat{L}_{\Lambda}^{s}\left(\beta,\omega\right)U_{\Lambda}$ and $\hat{L}_{\Lambda}^{s}\left(\beta,\omega\right)$ have the same spectrum, the eigenspace corresponding to the first non-trivial eigenvalue of the unperturbed generator, $\xi_{1}\left(L_{\Lambda}\right)=1$, is $span\{|y\rangle:y\in\Lambda\}$ and

$$\begin{aligned}
\left\langle z\right|T_{\Lambda}^{(1)}\left(\omega\right)\left|y\right\rangle &=\frac{1}{2}\left\langle z\right|\sum_{x\in\Lambda}[[\mathbf{H}_{\Lambda}\left(\omega\right),\ell_{x}^{\Lambda}],\ell_{x}^{\Lambda}]\left|y\right\rangle \qquad (36)\\
&=\frac{1}{2}\sum_{x\in\Lambda}\left\langle z\right|\mathbf{H}_{\Lambda}\left(\omega\right)\left|y\right\rangle\left(\delta_{x,y}+\delta_{x,z}-2\delta_{z,x}\delta_{x,y}\right)\\
&=\left\langle z\right|\mathbf{H}_{\Lambda}\left(\omega\right)\left|y\right\rangle-\delta_{z,y}\left\langle y\right|\mathbf{H}_{\Lambda}\left(\omega\right)\left|y\right\rangle\,,
\end{aligned}$$

where by (33)

$$\langle z|\,\mathbf{H}_{\Lambda}(\omega)\,|y\rangle = \sum_{b\in\mathbb{B}_{\Lambda}} \omega_b \langle z|\,\mathbf{s}_b\,|y\rangle = \sum_{b\in\mathbb{B}_{\Lambda}} \omega_b \langle z|\{y\}\triangle b\rangle = \sum_{b\in\mathbb{B}_{\Lambda}} \omega_b \mathbf{1}_{\{z,y\}}(b) = \omega_{z,y}\,. \tag{37}$$

Moreover, looking at the expansion in β of the Dirichlet forms of $\tilde{L}^s_{\Lambda}(\beta,\omega)$ and $\hat{L}^s_{\Lambda}(\beta,\omega)$, we realize that these operators coincides up to first order. Hence, we get

$$\xi_1\left(\tilde{L}^s_{\Lambda}(\beta,\omega)\right) \geq g_d^-(\beta)\ ,$$

with $g_d^-(\beta)$ analytic function of β such that

$$g_d^-(\beta) := 1 - \beta \sup_{z\in\Lambda} \sum_{y\in\Lambda} |\omega_{x,y}| + o(\beta) = 1 - 2\beta d J + O\left(\beta^2\right)\ . \tag{38}$$

Notice that all the above estimates, which are independent of Λ, hold in infinite volume as well.

Remark 4. Since the ω's are bounded, the last result implies the existence of a value of $\beta_d(J)$ smaller than the critical one $\beta_c(d,\omega)$, such that for $\mathbb{P}$ a.e. ω, if $\beta \in [0,\beta_d(J))$, the process is ergodic. Hence, by the reversibility with respect to the Gibbs measure, we get the uniqueness of the Gibbs state. Furthermore, the unique element $\mu^{\beta,\omega}$ of $\mathcal{G}(\beta,\omega)$ has the property

$$\mu^{\beta,\omega}(A) = \mu^{\beta,\theta_z\omega}(\tau_z A) \quad A\subset\mathcal{S},\ z\in\mathbb{Z}^d\,, \tag{39}$$

where

$$\tau_z A := \left\{\sigma\in\mathcal{S} : \forall x\in\mathbb{Z}^d \quad \sigma_x = \eta_{x-z} = (\tau_z\eta)_x\,,\ \eta\in A\right\}\ .$$

Let $\{\Theta_z\}_{z\in\mathbb{Z}^d}$ be the unitary group of operators on $\mathcal{L}(\beta,\omega)$ generated by the group $\{\tau_z\}_{z\in\mathbb{Z}^d}$ that is,

$$(\Theta_z\varphi)(\sigma) = \varphi\left(\tau_z^{-1}\sigma\right) \quad \varphi\in\mathcal{L}(\beta,\omega)\ .$$

Then, by the previous remark, we get that $\forall z\in\mathbb{Z}^d$ the Hilbert spaces $\mathcal{L}(\beta,\omega)$ and $\mathcal{L}\left(\beta,\theta_z^{-1}\omega\right)$ are unitary equivalent (isomorphic) via the unitary mapping Θ_z

$$\Theta_z : \mathcal{L}(\beta,\omega) \longmapsto \mathcal{L}\left(\beta,\theta_z^{-1}\omega\right)$$

and from the representation (3) of $L(\beta,\omega)$, we have

$$\Theta_z L(\beta,\omega)\,\Theta_z^{-1} = L\left(\beta,\theta_z^{-1}\omega\right)\ ,$$

which implies that, at least for $\beta\in[0,\beta_d(J))$, the family of operators and spaces $(L(\beta,\omega),\mathcal{L}(\beta,\omega))$ is a metrically transitive family with respect to the unitary group of lattice translation $\{\Theta_z\}_{z\in\mathbb{Z}^d}$. Hence, (see [11] and Remark 4 of [1]) the spectrum of $L(\beta,\omega)$ is non-random for $\mathbb{P}$-a.e. ω.

Remark 5. To get an upper bound for the spectral gap of the generator of the process we can compute the Dirichlet form of $L(\beta,\omega)$ with respect to the function of the empirical magnetization

$$\phi_\Lambda := \sum_{x\in\Lambda}\frac{\sigma_x}{|\Lambda|} - \mu^{\beta,\omega}\left(\sum_{x\in\Lambda}\frac{\sigma_x}{|\Lambda|}\right)\ .$$

We have

$$\langle \phi_\Lambda, L(\beta,\omega)\phi_\Lambda\rangle_{\beta,\omega} = \frac{1}{2}\int \mu^{\beta,\omega}(d\sigma)\sum_{x\in\mathbb{Z}^d} w_x^{\beta,\omega}(\sigma)\left[\sum_{y\in\Lambda}\frac{\sigma_y}{|\Lambda|}(1-2\delta_{x,y}) - \sum_{y\in\Lambda}\frac{\sigma_y}{|\Lambda|}\right]^2$$
$$= 2\int \mu^{\beta,\omega}(d\sigma)\sum_{x\in\Lambda}\frac{w_x^{\beta,\omega}(\sigma)}{|\Lambda|^2}.$$

Dividing by the $\mathcal{L}(\beta,\omega)$ norm of ϕ_Λ

$$\frac{1}{|\Lambda|^2}\sum_{x,y\in\Lambda}\left[\mu^{\beta,\omega}(\sigma_x\sigma_y) - \mu^{\beta,\omega}(\sigma_x)\mu^{\beta,\omega}(\sigma_y)\right] ,$$

we have that the spectral gap is smaller than

$$\frac{2\int \mu^{\beta,\omega}(d\sigma)\sum_{x\in\Lambda} w_x^{\beta,\omega}(\sigma)}{\sum_{x,y\in\Lambda}\left[\mu^{\beta,\omega}(\sigma_x\sigma_y) - \mu^{\beta,\omega}(\sigma_x)\mu^{\beta,\omega}(\sigma_y)\right]}.$$

By the ergodicity of the random field ω with respect to the lattice translations, the last expression becomes

$$\frac{2\int \mathbb{P}(d\omega)\int \mu^{\beta,\omega}(d\sigma)\, w_0^{\beta,\omega}(\sigma)}{\int \mathbb{P}(d\omega)\sum_{y\in\mathbb{Z}^d}\left[\mu^{\beta,\omega}(\sigma_0\sigma_y) - \mu^{\beta,\omega}(\sigma_0)\mu^{\beta,\omega}(\sigma_y)\right]}, \tag{40}$$

where

$$\chi^{d,\omega}(\beta) := \sum_{x\in\mathbb{Z}^d}\left[\mu^{\beta,\omega}(\sigma_x\sigma_0) - \mu^{\beta,\omega}(\sigma_x)\mu^{\beta,\omega}(\sigma_0)\right] , \quad \omega\in\Omega ,$$

is the *susceptibility* relative to a realization of the potential. We could now get estimates for (40) at small values of β through a cluster expansion. We will not pursue this here but rather get a bound by different means in 3.1.2 below. In the ferromagnetic case ($\omega_b \geq J^- > 0, \forall b\in\mathbb{B}_d$), by the Griffiths inequalities (see for example [8] page 186), we have that $\chi^{d,\omega}(\beta)$ is larger than or equal to the susceptibility relative to the configuration of the potential constantly equal to J^-, $\chi^{d,J^-}(\beta)$, which is known to be a function of β diverging when β approachs its critical value $\beta_c(d,\omega)$ from below. In particular, in the two-dimensional case, $\chi^{2,J^-}(\beta)$ is proportional to $|\beta-\beta_c(2,J^-)|^{-\frac{7}{4}}$ ([6] Theorem 2.11). Then by (7), (40) is smaller than

$$\frac{2\psi(-\beta 4dJ)\vee\psi(\beta 4dJ)}{\chi^{d,J^-}(\beta)}.$$

3.1.2. *Upper bound* $g_d^+(\beta)$. Since, for any $\beta\geq 0$ and $\omega\in\Omega$,

$$\langle u|\,\tilde{L}_\Lambda^s(\beta,\omega)\,|u\rangle \leq \langle u|\,\hat{L}_\Lambda^s(\beta,\omega)\,|u\rangle , \quad |u\rangle\in\mathcal{H}_\Lambda ,$$

from (36) and (37) we get $\xi_1\left(\tilde{L}_\Lambda^s(\beta,\omega)\right)\leq g_d^+(\beta)$, with $g_d^+(\beta)$ analytic function of β such that

$$g_d^+(\beta) := 1+\beta\sup_{z\in\Lambda}\sum_{y\in\Lambda}|\omega_{x,y}| + o(\beta) = 1+2\beta dJ + O\left(\beta^2\right) . \tag{41}$$

3.1.3. $\sigma_\beta^{(1)}\subseteq\left[g_d^-(\beta J), g_d^+(\beta J)\right]$. We just notice that, for β smaller than $\beta_d(J)$, we have $g_d^-(\beta) = 1-2\beta Jd + O\left(\beta^2\right)$ and $g_d^+(\beta) = 1+2\beta Jd + O\left(\beta^2\right)$, which implies that, for such values of β,

$$\sigma_\beta^{(1)}\subseteq[1-2\beta Jd, 1+2\beta Jd] . \tag{42}$$

3.2. **Proof of Theorem 2.2.** Here we mimic the second part of the proof of Theorem 3 in [1] and consider Ω as a topological space endowed with the Schwartz topology, which we will denote by $\mathcal{D}_{\mathbb{B}_d}$. We will denote by $supp\mathbb{P}$ the support of $\mathbb{P}$ as a function on $\mathcal{D}_{\mathbb{B}_d}$. Let ζ be any realization of the potential constant on $\mathbb{B}_d$ which belongs to the support of $\mathbb{P}$, namely

$$\zeta = \{\omega_b = \zeta \in \mathbb{R},\ \forall b \in \mathbb{B}\} \in supp\mathbb{P}$$

and denote by $\mathcal{C}_{\mathbb{B}_d} \subset \mathcal{D}_{\mathbb{B}_d}$ the collection of all such realizations of the potential.

Theorem 3 of [1] uses the explicit representation of the matrix elements of the generator for the one dimension model to prove the weak continuity of the spectral measure. All is really needed is that the matrix elements of the generator and thus the semigroup are smooth functions of the potential. In higher dimension we rely on (28), which in particular ensures the necessary regularity.

It is proved in Theorem 2.2 of [9] that, for any constant realizations ζ of the potential in $supp\mathbb{P}$, there exists a value $\beta_d^{(1)}(\zeta) > 0$ such that, for any $|\beta| < \beta_d^{(1)}(\zeta)$, we obtain

$$[1 - a_d(\beta\zeta), 1 + a_d(\beta\zeta)] \subseteq \sigma_\beta^{(1)},$$

with

$$a_d(r) := \max_{\lambda \in \mathbb{T}^d} |a_d(\lambda, r)| \ . \tag{43}$$

For the definition of $a_d(\lambda, r)$ see Theorem 2.3 of [9]. Consequently, if $\beta_d^{(1)} := \inf_\zeta \beta_d^{(1)}(\zeta) > 0$ and $\beta_d^*(J) := \beta_d(J) \wedge \beta_d^{(1)}$, then, $\forall \beta \in [0, \beta_d^*(J))$,

$$[1 - \bar{a}_d(\beta), 1 + \bar{a}_d(\beta)] \subseteq \bigcup_{\zeta \in \mathcal{C}_{\mathbb{B}_d}} \sigma_\beta^{(1)}\left(L^{(1)}(\beta,\zeta)\right) \subseteq \sigma_\beta^{(1)}, \tag{44}$$

with

$$\bar{a}_d(\beta) := \max_{\zeta \in \mathcal{C}_{\mathbb{B}_d}} a_d(\beta\zeta) \ ,$$

which is an analytic function of β. Thus $f_d^{\pm}(\beta) := \pm\bar{a}_d(\beta)$. Since $\bar{a}_d(\beta) = a_d(\beta\zeta)$ for ζ such that $|\zeta| = J$, then for small values of β, $\bar{a}_d(\beta) = 2dJ\beta + o(\beta)$, where the linear term in β is the same in the expansion of (38), as well as in (41).

3.3. **Proof of Theorem 2.3.** Because the family of operators and spaces $(L(\beta,\omega), \mathcal{L}(\beta,\omega))$ is metrically transitive with respect to lattice translations, $\sigma_\beta^{(1)}$ is a non random set (see Remark 2). Thus, for every $\beta \in [0, \beta_d^*(J))$, at first order in β, by (42) and (44) we obtain

$$[1 - 2dJ\beta, 1 + 2dJ\beta] \subseteq \sigma_\beta^{(1)} \subseteq [1 - 2\beta dJ, 1 + 2\beta dJ] \ .$$

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E-mail address: gianfelice@mat.unical.it
E-mail address: isopi@mat.uniroma1.it