NETWORKS AND HETEROGENEOUS MEDIA
©American Institute of Mathematical Sciences
Volume **6**, Number **1**, March **2011**

doi:10.3934/nhm.2011.6.111

pp. **111–126**

HOMOGENIZATION OF CONVECTION-DIFFUSION EQUATION IN INFINITE CYLINDER

Iryna Pankratova

Narvik University College, Postbox 385
8505 Narvik, Norway and
Ecole Polytechnique, Route de Saclay
91128 Palaiseau Cedex, France

Andrey Piatnitski

Narvik University College, Postbox 385
8505 Narvik, Norway and
Lebedev Physical Institute RAS
Leninski ave., 53, 119991 Moscow, Russia

(Communicated by Leonid Berlyand)

Abstract. The paper deals with a periodic homogenization problem for a non-stationary convection-diffusion equation stated in a thin infinite cylindrical domain with homogeneous Neumann boundary condition on the lateral boundary. It is shown that homogenization result holds in moving coordinates, and that the solution admits an asymptotic expansion which consists of the interior expansion being regular in time, and an initial layer.

Introduction. The goal of the paper is to study the asymptotic behaviour of a solution to an initial boundary problem for a convection-diffusion equation defined in a thin infinite cylinder with homogeneous Neumann condition on its lateral boundary. We assume that the coefficients of the equation are periodic in the axial direction of the cylinder and that the period is of the same order as the cylinder diameter. The corresponding parabolic operator takes the form

$$\partial_t u - \mathrm{div}\Big(a\Big(\frac{x}{\varepsilon}\Big)\nabla u\Big) + \frac{1}{\varepsilon}\Big(b\Big(\frac{x}{\varepsilon}\Big), \nabla u\Big); \tag{1}$$

here ε is a small positive parameter, and we assume the standard uniform ellipticity conditions on $a(y)$ and the boundedness of the entries of $a(y)$ and $b(y)$.

Notice that the scaling factor $1/\varepsilon$ is natural for the convection term. Indeed, if one wants to consider the long-term behaviour of a convection-diffusion process described by the equation

$$\partial_s u - \mathrm{div}\big(a(y)\nabla u\big) + \big(b(y), \nabla u\big) = 0$$

in a fixed infinite cylinder, then making the diffusive change of variables

$$x = \varepsilon y,\ t = \varepsilon^2 s$$

leads to a convection-diffusion problem for operator (1) in a thin cylinder.

2000 *Mathematics Subject Classification.* Primary: 35B27, 35B40, 35K20; Secondary: 35B25.
Key words and phrases. Homogenization, convection-diffusion equation, infinite cylinder, asymptotic behaviour.

Closely related problems for a convection-diffusion equation defined in the whole space have been considered in [6] and [1]. It was proved in particular that the homogenization takes place in moving coordinates $(x,t) \longrightarrow \big(x-(\bar{b}/\varepsilon)t, t\big)$ with a constant vector $\bar{b}$.

Homogenization problems for divergence form operators and systems in thin bounded domains have been investigated by many authors, we mention here the works [7], [10] and [11]. General homogenization theory results for parabolic equations can be found in [5] and [12].

In the paper we first prove uniform in ε a priori estimates for the solution. This requires integration in weighted spaces where the solution of the periodic adjoint cell problem is used as a weight. Then we construct the leading terms of the asymptotic expansion in moving coordinates, determine the effective speed, and obtain the estimates for the rate of convergence. Additional difficulty appearing in the problem under consideration is the dimension reduction issue. Indeed, the solutions of the original problem belong to variable Sobolev spaces, which makes the convergence analysis rather involved.

The paper is organized as follows. Section 1 contains the problem setup. In Section 2 we deal with a priori estimates and study auxiliary parabolic cell problems. In Section 3 we construct formal asymptotic expansion which includes the corresponding initial layers, the presence of the initial layer allows us to satisfy the initial condition in higher order approximations. Section 4 is devoted to the convergence analysis.

1. **Problem statement.** Let Q be a bounded domain in $\mathbb{R}^{d-1}$ with the Lipschitz boundary ∂Q. For any $\varepsilon > 0$, we denote by $\mathbb{G}_\varepsilon$ a thin infinite cylinder $\mathbb{R}\times\varepsilon Q$ with the axis directed along x_1. The lateral boundary of the cylinder $\mathbb{G}_\varepsilon$ is denoted by $\Sigma_\varepsilon = \mathbb{R}\times\partial(\varepsilon Q)$. We study the following non-stationary convection-diffusion equation:

$$\begin{cases} \partial_t u^\varepsilon(t,x) + \mathcal{A}_\varepsilon\, u^\varepsilon(t,x) = 0, & (t,x)\in(0,T)\times\mathbb{G}_\varepsilon,\\ \mathcal{B}_\varepsilon\, u^\varepsilon(t,x) = 0, & (t,x)\in(0,T)\times\Sigma_\varepsilon,\\ u^\varepsilon(0,x) = \varphi(x_1), & x\in\mathbb{G}_\varepsilon, \end{cases} \tag{2}$$

where

$$\begin{aligned} \mathcal{A}_\varepsilon u &= -\mathrm{div}\Big(a\big(\frac{x}{\varepsilon}\big)\nabla_x u\Big) + \frac{1}{\varepsilon}\,\big(b\big(\frac{x}{\varepsilon}\big), \nabla_x u\big),\\ \mathcal{B}_\varepsilon u &= \big(a\big(\frac{x}{\varepsilon}\big)\nabla_x u, n\big), \end{aligned} \tag{3}$$

and n stands for the exterior unit normal on Σ_ε. We suppose that the following conditions are fulfilled:

(H1) Q is a Lipschitz bounded domain in $\mathbb{R}^{d-1}$;

(H2) $a_{ij}(y), b_j(y)\in L^\infty(\mathbb{G})$, $i,j=1,...,d$, are 1-periodic functions with respect to y_1;

(H3) The matrix $a(y)$ satisfies the uniform ellipticity condition, that is there exists a positive constant Λ such that, for almost all $x\in\mathbb{R}^d$,

$$\Lambda\,|\xi|^2 \le \sum_{i,j=1}^d a_{ij}(y)\,\xi_i\,\xi_j, \quad \forall\xi\in\mathbb{R}^d. \tag{4}$$

(H4) $\varphi(x_1)\in C_0^\infty(\mathbb{R})$.

Definition 1.1. A function $u^\varepsilon(t,x)$ is said to be a weak solution of problem (2) in $(0,T]\times\mathbb{G}_\varepsilon$ if

$$u^\varepsilon \in L^\infty[\delta,T;L^2_{\rm loc}(\mathbb{G}_\varepsilon)]\cap L^2[0,T;H^1_{\rm loc}(\mathbb{G}_\varepsilon)],\quad \delta\in(0,T)$$

and u^ε satisfies

$$\int_0^T\int_{\mathbb{G}_\varepsilon}\Big\{-u^\varepsilon\,\partial_t\psi+(a^\varepsilon\nabla u^\varepsilon,\nabla\psi)+(b^\varepsilon,\nabla u^\varepsilon)\,\psi\Big\}\,dx\,dt=\int_{\mathbb{G}_\varepsilon}\varphi(x_1)\,\psi(0,x)\,dx$$

for any $\psi\in L^2[0,T;H^1(\mathbb{G}_\varepsilon)]$ such that $\partial_t\psi\in L^2[0,T;L^2(\mathbb{G}_\varepsilon)]$ and $\psi(T,x)=0$.

We are interested in the asymptotic behaviour of $u^\varepsilon(t,x)$, as $\varepsilon\to0$. Notice that, for any $\varepsilon>0$, the existence and the uniqueness of a generalized solution to problem (2) is given by classical theory (see, e.g., [8]).

2. Some preliminary results.

2.1. A priori estimates.

In what follows we denote $Y=[0,1)\times Q$,

$$\mathcal{A}v=-{\rm div}_y(a(y)\nabla_y v)+(b(y),\nabla_y v),\quad \mathcal{B}v=(a(y)\nabla_y v,n);$$
$$\mathcal{A}^*p^*=-{\rm div}(a\nabla p^*)-{\rm div}(bp^*),\quad \mathcal{B}^*p^*=(a\nabla p^*,n)+(b,n)p^*,$$

By the Krein-Rutman theorem and the Harnack inequality, the adjoint periodic problem

$$\begin{cases}\mathcal{A}^*\,p^*(y)=0, & y\in Y,\\ \mathcal{B}^*\,p^*(y)=0, & y\in\partial Y,\\ p^*(y)\quad\text{is periodic in } y_1,\end{cases}\tag{5}$$

has a positive solution $p^*(y)\in C(Y)\cap H^1(Y)$ such that

$$0<p^-\le p^*(y)\le p^+<\infty.\tag{6}$$

We fix the choice of p^* by the normalization condition

$$\int_Y p^*(y)\,dy=1.$$

The goal of this section is to obtain a priori estimates for a non-stationary convection-diffusion equation stated in a thin infinite cylinder. Namely, we consider the following non-homogeneous problem:

$$\begin{cases}\partial_t u^\varepsilon(t,x)+\mathcal{A}_\varepsilon\,u^\varepsilon(t,x)=f(t,x), & (t,x)\in(0,T)\times\mathbb{G}_\varepsilon,\\ \mathcal{B}_\varepsilon\,u^\varepsilon(t,x)=\varepsilon\,g(t,x), & (t,x)\in(0,T)\times\Sigma_\varepsilon,\\ u^\varepsilon(0,x)=\varphi(x), & x\in\mathbb{G}_\varepsilon,\end{cases}\tag{7}$$

Multiplying the equation in (7) by $p^*(x/\varepsilon)\,u^\varepsilon(x)$ and integrating the resulting relation by parts over $\mathbb{G}_\varepsilon$, we obtain

$$\begin{aligned}&\frac12\int_{\mathbb{G}_\varepsilon}\frac{\partial(u^\varepsilon)^2}{\partial t}\,p^*\big(\frac{x}{\varepsilon}\big)\,dx+\frac12\int_{\mathbb{G}_\varepsilon}(u^\varepsilon(t,x))^2\,\mathcal{A}^*_\varepsilon p^*\big(\frac{x}{\varepsilon}\big)\,dx+\\ &+\frac12\int_{\Sigma_\varepsilon}(u^\varepsilon(t,x))^2\,\mathcal{B}^*_\varepsilon p^*\big(\frac{x}{\varepsilon}\big)\,d\sigma+\int_{\mathbb{G}_\varepsilon}(a^\varepsilon\nabla u^\varepsilon,\nabla u^\varepsilon)\,p^*\big(\frac{x}{\varepsilon}\big)\,dx=\\ &=\varepsilon\int_{\Sigma_\varepsilon}g(t,x)\,u^\varepsilon(t,x)\,p^*\big(\frac{x}{\varepsilon}\big)\,d\sigma+\int_{\mathbb{G}_\varepsilon}f(t,x)\,u^\varepsilon(t,x)\,p^*\big(\frac{x}{\varepsilon}\big)\,dx.\end{aligned}$$

Here we use the notations

$$\mathcal{A}_\varepsilon^* q(x) = -\mathrm{div}\Big(a\big(\frac{x}{\varepsilon}\big)\nabla q(x)\Big) - \frac{1}{\varepsilon}\,\mathrm{div}\Big(b\big(\frac{x}{\varepsilon}\big)\,q(x)\Big),$$
$$\mathcal{B}_\varepsilon^* q(x) = (a\big(\frac{x}{\varepsilon}\big)\nabla q(x), n) + \frac{1}{\varepsilon}\,(b\big(\frac{x}{\varepsilon}\big), n)\,q(x).$$

Taking into account the definition of $p^*(y)$ we get

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\,p^*\big(\frac{x}{\varepsilon}\big)\,dx + \int\limits_{\mathbb{G}_\varepsilon}(a^\varepsilon\nabla u^\varepsilon, \nabla u^\varepsilon)\,p^*\big(\frac{x}{\varepsilon}\big)\,dx =$$
$$= \varepsilon\int\limits_{\Sigma_\varepsilon} g(t,x)\,u^\varepsilon(t,x)\,p^*\big(\frac{x}{\varepsilon}\big)\,d\sigma + \int\limits_{\mathbb{G}_\varepsilon} f(t,x)\,u^\varepsilon(t,x)\,p^*\big(\frac{x}{\varepsilon}\big)\,dx.$$

Using the positive definiteness of the matrix $a(y)$, bounds (6), and the Cauchy-Bunyakovsky inequality, one can obtain

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\,p^*\big(\frac{x}{\varepsilon}\big)\,dx + \Lambda\,p^-\int\limits_{\mathbb{G}_\varepsilon}|\nabla u^\varepsilon|^2\,dx \le$$
$$+\frac{1}{2\gamma}\Big\{\int\limits_{\mathbb{G}_\varepsilon}(f(t,x))^2\,dx + \varepsilon\int\limits_{\Sigma_\varepsilon}(g(t,x))^2\,d\sigma\Big\}$$
$$+\frac{\gamma}{2}\,p^+\,\varepsilon\int\limits_{\Sigma_\varepsilon}(u^\varepsilon)^2\,d\sigma + \frac{\gamma}{2}\,p^+\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\,dx$$

for any $\gamma > 0$. By the trace theorem

$$\int\limits_{\Sigma_\varepsilon}(u^\varepsilon)^2\,d\sigma \le \frac{C_1}{\varepsilon}\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\,dx + C_2\,\varepsilon\int\limits_{\mathbb{G}_\varepsilon}|\nabla u^\varepsilon|^2\,dx$$

with constants C_1, C_2 independent of ε. Consequently, for a sufficiently small γ,

$$\frac{d}{dt}\Bigg[\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\,p^*\big(\frac{x}{\varepsilon}\big)\,dx + \int\limits_0^t\int\limits_{\mathbb{G}_\varepsilon}|\nabla u^\varepsilon(s,x)|^2\,dxds\Bigg] \le$$
$$+C\left\{\|f\|^2_{L^2(\mathbb{G}_\varepsilon)} + \varepsilon\,\|g\|^2_{L^2(\Sigma_\varepsilon)}\right\} + \Bigg[\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\,p^*\big(\frac{x}{\varepsilon}\big)\,dx + \int\limits_0^t\int\limits_{\mathbb{G}_\varepsilon}|\nabla u^\varepsilon(s,x)|^2\,dxds\Bigg].$$

Integrating with respect to t and applying the Grönwall lemma and the positiveness of the function p^*, one can see that

$$\begin{aligned}&\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\,dx + \int\limits_0^t\int\limits_{\mathbb{G}_\varepsilon}|\nabla u^\varepsilon(s,x)|^2\,dxds \le \\ &\le C\,e^t\left\{\|f\|^2_{L^2[0,T;L^2(\mathbb{G}_\varepsilon)]} + \varepsilon\,\|g\|^2_{L^2[0,T;L^2(\Sigma_\varepsilon)]} + \|\varphi\|^2_{L^2(\mathbb{G}_\varepsilon)}\right\}, \quad t\in(0,T]\end{aligned} \tag{8}$$

where the constant C does not depend on ε, and depends only on Λ, d and Q.

For the justification procedure we also need a priori estimates in the case of right-hand side being the divergence of a bounded vector-function. Namely, consider the following problem:

$$\begin{cases} \partial_t u^\varepsilon(t,x) + \mathcal{A}_\varepsilon\, u^\varepsilon(t,x) = \mathrm{div}_x F(t,x), & (t,x)\in(0,T)\times\mathbb{G}_\varepsilon,\\ \mathcal{B}_\varepsilon\, u^\varepsilon(t,x) = -(F,n), & (t,x)\in(0,T)\times\Sigma_\varepsilon,\\ u^\varepsilon(0,x)=0, & x\in\mathbb{G}_\varepsilon, \end{cases} \tag{9}$$

with $F(t,x)$ such that

$$|F(t,x)| \le f_1(t,x)\, e^{-\gamma\,|x_1|}, \quad f_1\in L^\infty((0,T)\times\mathbb{G}_\varepsilon).$$

Multiplying equation in (9) by $p^*(\frac{x}{\varepsilon})u^\varepsilon(x)$ and integrating by parts over $\mathbb{G}_\varepsilon$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\, p^*\big(\frac{x}{\varepsilon}\big)\,dx + \int\limits_{\mathbb{G}_\varepsilon}(a^\varepsilon\nabla u^\varepsilon,\nabla u^\varepsilon)\,p^*\big(\frac{x}{\varepsilon}\big)\,dx = \\ &= -\int\limits_{\mathbb{G}_\varepsilon}(F(t,x),\ \nabla u^\varepsilon(t,x))\,p^*\big(\frac{x}{\varepsilon}\big)\,dx \\ &-\int\limits_{\mathbb{G}_\varepsilon}(F(t,x),\ \nabla p^*\big(\frac{x}{\varepsilon}\big))\,u^\varepsilon(t,x)\,dx \equiv I_1^\varepsilon + I_2^\varepsilon. \end{aligned} \tag{10}$$

Exploiting the Couchy-Bunyakovsky inequality and taking into account (6) one gets

$$\begin{aligned} |I_1^\varepsilon| &\le \frac{p^+}{2\delta}\|F\|^2_{L^2(\mathbb{G}_\varepsilon)} + \frac{p^+\,\delta}{2}\,\|\nabla u^\varepsilon\|_{L^2(\mathbb{G}_\varepsilon)} \\ &\le \frac{p^+}{\delta}\|f_1\|^2_{L^\infty((0,T)\times\mathbb{G}_\varepsilon)}\,\varepsilon^{d-1} + p^+\,\delta\,\|\nabla u^\varepsilon\|_{L^2(\mathbb{G}_\varepsilon)}, \quad \delta = \frac{p^-\Lambda}{p^+}. \end{aligned}$$

The integral I_2^ε is estimated as follows

$$\begin{aligned} |I_2^\varepsilon| &\le \frac{1}{2}\int\limits_{\mathbb{G}_\varepsilon}|\big(F,\nabla p^*\big(\frac{x}{\varepsilon}\big)\big)|^2\,dx + \frac{1}{2}\,\|u^\varepsilon\|^2_{L^2(\mathbb{G}_\varepsilon)} \\ &\le \sum_{n=-\infty}^{+\infty}\|F\|^2_{L^\infty(\varepsilon Y_n)}\int\limits_{\varepsilon Y_n}|\nabla p^*\big(\frac{x}{\varepsilon}\big)|^2\,dx + \frac{1}{2}\,\|u^\varepsilon\|^2_{L^2(\mathbb{G}_\varepsilon)} \\ &\le \sum_{n=-\infty}^{+\infty}\|F\|^2_{L^\infty(\varepsilon Y_n)}\,\varepsilon^{d-2}\int\limits_{Y_n}|\nabla p^*(y)|^2\,dy + \frac{1}{2}\,\|u^\varepsilon\|^2_{L^2(\mathbb{G}_\varepsilon)} \\ &\le C\,\varepsilon^{d-1}\,\frac{1}{\varepsilon}\,\|f_1\|^2_{L^\infty((0,T)\times\mathbb{G}_\varepsilon)} + \frac{1}{2}\,\|u^\varepsilon\|^2_{L^2(\mathbb{G}_\varepsilon)}, \end{aligned}$$

where $Y_n = (n, n+1]\times Q$.

Finally, combining the obtained estimates for I_1^ε and I_2^ε with (10), and using the Grönwall's lemma, for $t\in(0,T]$ one has

$$\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon)^2\,dx + \int\limits_0^t\int\limits_{\mathbb{G}_\varepsilon}|\nabla u^\varepsilon(s,x)|^2\,dxds \le \frac{C\,e^t}{\varepsilon}\,\varepsilon^{d-1}\,\|f_1\|^2_{L^\infty((0,T)\times\mathbb{G}_\varepsilon)}. \tag{11}$$

2.2. **Auxiliary results.** In the sequel we will need the information about the asymptotic behaviour of solutions to parabolic equations, as $t\to\infty$. Consider the initial boundary value problem

$$\begin{cases} \partial_\tau v(\tau,y) + \mathcal{A}\, v(\tau,y) = 0, & (\tau,y)\in(0,\infty)\times Y,\\ \mathcal{B}\, v(\tau,y) = 0, & (\tau,y)\in(0,\infty)\times \partial Y,\\ v(\tau,y) - y_1 - \ \text{periodic}, & \\ v(0,y) = \psi(y), & y\in Y, \end{cases} \tag{12}$$

where $Y = [0,1)\times Q$.

Lemma 2.1. *Suppose conditions* (**H1**)$-$(**H3**) *are fulfilled and* $\psi(y)\in L^2(Y)$. *Then there exists a unique weak solution* v *to problem* (12), *and it stabilizes to a constant* v^∞ *at the exponential rate, as* $\tau\to\infty$, *that is*

$$|v(\tau,y) - v^\infty| \le C_0\, \|\psi\|_{L^2(Y)} e^{-\gamma\,\tau}, \quad y\in Y,\ \tau>0, \tag{13}$$

with positive constants C_0 *and* γ *depending only on* Λ, d *and* Q. *Moreover,* $\nabla_y v$ *stabilizes exponentially to* 0, *as* $\tau\to\infty$:

$$\int\limits_{\tau}^{\tau+1}\int\limits_{Y} |\nabla_y v(s,y)|^2\, dyds \le C\, e^{-2\gamma\,\tau}. \tag{14}$$

The constant v^∞ *is defined by*

$$v^\infty = \int\limits_{Y} \psi(y)\, p^*(y)\, dy,$$

where $p^*(y)$ *solves problem* (5).

Proof. Let us consider two functions

$$m(\tau) = \min_{y\in Y} v(\tau,y), \quad M(\tau) = \max_{y\in Y} v(\tau,y).$$

By the maximum principle, $M(\tau)$ decreases and $m(\tau)$ increases. In view of the linearity of the problem, without loss of generality we assume that $m(\tau_0) = 0$. Since $v\ge 0$, then we can use the Harnack inequality

$$m(\tau_0+1) \ge \alpha\, M(\tau_0+1), \quad \alpha<1$$

to obtain the estimate

$$\text{osc}_{\tau=\tau_0+1} v(\tau,y) \equiv M(\tau_0+1) - m(\tau_0+1) \le (1-\alpha)M(\tau_0) \equiv (1-\alpha)\text{osc}_{\tau=\tau_0} v(\tau,y).$$

Consequently,

$$\text{osc}_{\tau=\tau_0+1} v(\tau,y) \le (1-\alpha)\text{osc}_{\tau=\tau_0} v(\tau,y), \quad \tau_0\ge 0$$

and, obviously, v converges to some constant v^∞, as $\tau\to\infty$

$$|v(\tau,y) - v^\infty| \le C_0\, \|\psi\|_{L^2(Y)} e^{-\gamma\tau},$$

where C and γ depend only on Λ, d and Q.

Let us calculate the constant v^∞. To this end we multiply the equation in (12) by p^* and integrate by parts over the set $(0,\tau)\times Y$. As a result we obtain the following equality:

$$\int\limits_{Y} v(\tau,y)\, p^*(y)\, dy = \int\limits_{Y} \psi(y)\, p^*(y)\, dy.$$

Since v converges uniformly to the constant v^∞, as $\tau \to \infty$, then it follows from the last equality that

$$v^\infty = \int\limits_Y \psi(y)\, p^*(y)\, dy,$$

if p^* is normalized by $\int_Y p^* dy = 1$.

Now we prove estimate (14). Note that the function $w = v - v^\infty$ solves the same equation as v, and satisfies the initial condition $w(0,y) = \psi - v^\infty$. Multiplying the equation by w, integrating by parts and applying the Cauchy-Bunyakovski inequality gives

$$\int\limits_Y |w(\tau+1,y)|^2\, dy + \Lambda \int\limits_\tau^{\tau+1} \int\limits_Y |\nabla w|^2\, dyds \le$$
$$\Lambda^{-1} \frac{1}{2\eta} \int\limits_\tau^{\tau+1} \int\limits_Y |w(s,y)|^2\, dyds + \Lambda^{-1} \frac{\eta}{2} \int\limits_\tau^{\tau+1} \int\limits_Y |\nabla_y w(s,y)|^2\, dyds + \int\limits_Y |w(\tau,y)|^2\, dy,$$

and, consequently, choosing $\eta < 2\Lambda^2$ and using (13), we obtain

$$\int\limits_\tau^{\tau+1} \int\limits_Y |\nabla w|^2\, dyds \le C\, e^{-2\gamma\tau}.$$

□

The next lemma generalizes the result of Lemma 2.1 to the non-homogeneous case. Consider the boundary value problem

$$\begin{cases} \partial_\tau v(\tau,y) + \mathcal{A}\, v(\tau,y) = f(\tau,y) + \mathrm{div}_y F(\tau,y), & (\tau,y) \in (0,\infty) \times Y, \\ \mathcal{B}\, v(\tau,y) = g(\tau,y) - (F(\tau,y), n), & (\tau,y) \in (0,\infty) \times \partial Y, \\ v(0,y) = 0, & y \in Y, \end{cases} \tag{15}$$

where $f \in L^2[0,\infty; L^2(Y)]$, $F \in L^2[0,\infty; L^2(Y)^d]$ and $g \in L^2[0,\infty; L^2(\partial Y)]$ decay exponentially, as $\tau \to \infty$, that is

$$\int\limits_\tau^{\tau+1} \|f(s,\cdot)\|^2_{L^2(Y)}\, ds \le C\, e^{-\gamma_1 \tau}; \quad \int\limits_\tau^{\tau+1} \|F(s,\cdot)\|^2_{L^2(Y)^d}\, ds \le C\, e^{-\gamma_1 \tau};$$
$$\int\limits_\tau^{\tau+1} \|g(s,\cdot)\|^2_{L^2(\partial Y)}\, ds \le C\, e^{-\gamma_1 \tau}, \quad \gamma_1 > 0.$$

Lemma 2.2. *Under the assumptions being made, a solution of problem* (15) *satisfies the estimates*

$$\int\limits_\tau^{\tau+1} \|v(s,\cdot) - v^\infty\|^2_{L^2(Y)}\, ds \le C\, e^{-\tilde\gamma\tau}, \tag{16}$$

$$\int\limits_\tau^{\tau+1} \|\nabla v(s,\cdot)\|^2_{L^2(Y)}\, ds \le C\, e^{-\tilde\gamma\tau}, \quad \tilde\gamma > 0. \tag{17}$$

Here C depends on Λ, d and Q. The constant v^∞ is determined by

$$v^\infty = \int_0^\infty \int_Y f(\tau,y)\,p^*(y)\,dy\,d\tau - \int_0^\infty \int_Y (F(\tau,y), \nabla p^*(y))\,dy\,d\tau + \int_0^\infty \int_{\partial Y} g(\tau,y)\,p^*(y)\,d\sigma\,d\tau,$$

p^* being a solution of (5).

Proof. First of all we represent the functions on the right-hand side of (15) as the sums of functions with finite supports, that is

$$f(\tau,y) = \sum_{m=-\infty}^{+\infty} f_m(\tau,y), \quad F(\tau,y) = \sum_{m=-\infty}^{+\infty} F_m(\tau,y), \quad g(\tau,y) = \sum_{m=-\infty}^{+\infty} g_m(\tau,y),$$

where $f_m(\tau,y) = f(\tau,y)\,\chi_{[m,m+1)}$, $F_m(\tau,y) = F(\tau,y)\,\chi_{[m,m+1)}$, $g_m(\tau,y) = g(\tau,y)\,\chi_{[m,m+1)}$, and $\chi_{[m,m+1)} = \chi_{[m,m+1)}(\tau)$ is the characteristic function of the interval $[m, m+1)$.

Due to the linearity of the problem, the solution v of (15) can be represented in the form

$$v(\tau,y) = \sum_{m=-\infty}^{+\infty} v_m(\tau,y),$$

where v_m solves the problem

$$\begin{cases} \partial_\tau v_m + \mathcal{A}\,v_m = f_m(\tau,y) + \mathrm{div}_y F_m(\tau,y), & (\tau,y) \in (0,\infty)\times Y, \\ \mathcal{B}\,v_m = g_m(\tau,y) - (F_m(\tau,y), n), & (\tau,y) \in (0,\infty)\times \partial Y, \\ v_m(0,y) = 0, & y \in Y. \end{cases} \tag{18}$$

Notice that, in view of the uniqueness of the solution, $v_m(\tau,y) = 0$ for $\tau \in [0,m)$. Then, multiplying the equation in (18) by v_m and integrating over $(m-1,m+2)\times Y$, we obtain

$$\int_Y (v_m(m+2,y))^2\,dy \le C\Big(\int_m^{m+1} \|f(s,\cdot)\|^2_{L^2(Y)}\,ds + \int_m^{m+1} \|g(s,\cdot)\|^2_{L^2(\partial Y)}\,ds + \int_m^{m+1} \|F(s,\cdot)\|^2_{L^2(Y)}\,ds\Big) \le C\,e^{-\gamma_1 m}.$$

By Lemma 2.1, v_m satisfies estimate (13) in $(m+2,\infty)\times Y$, namely,

$$|v_m(\tau,y) - v_m^\infty| \le C\,e^{-\gamma_1 m}\,e^{-\gamma\,(\tau-m-2)} \le C\,e^{-\tilde\gamma\tau}, \quad \tau > m+2,$$

for some constant v_m^∞, where $\tilde\gamma = \min\{\gamma,\gamma_1\}$. In view of the maximum principle,

$$v_m^\infty \le Ce^{-\gamma_1 m}.$$

Let us show that $v = \sum v_m$ stabilizes to $v^\infty = \sum v_m^\infty$, as $\tau \to \infty$. To this end we estimate the L^2-norm of the difference $v - v^\infty$.

$$\int_N^{N+1} \|v(s,\cdot) - v^\infty\|^2_{L^2(Y)}\, ds = \int_N^{N+1} \|\sum_{m=0}^{+\infty} (v_m(s,\cdot) - v_m^\infty)\|^2_{L^2(Y)}\, ds =$$
$$= \int_N^{N+1} \|\Big\{\sum_{m\le N-2} + \sum_{m\ge N-1}\Big\}(v_m(s,\cdot) - v_m^\infty)\|^2_{L^2(Y)}\, ds \le$$
$$\le C_1\, N^2\, e^{-2\gamma N} + C_2\, e^{-2\gamma_1 N} \le C\, e^{-\tilde\gamma N}, \quad \tilde\gamma > 0.$$

The exponential decay of ∇v can be proved in much the same way as in the homogeneous case. □

3. Asymptotic expansion.

3.1. Formal inner expansion.

Following the ideas in [6] and [1], we are looking for an approximate solution in the form

$$u^\varepsilon \sim u_0\big(t\,,\, x_1 - \varepsilon^{-1}\,\bar b_1 t\big) + \sum_{k=1}^{\infty} \varepsilon^k\, v_k\big(t\,,\, x_1 - \varepsilon^{-1}\,\bar b_1 t\,,\, y\big), \quad y = \frac{x}{\varepsilon}, \tag{19}$$

where v_k, $k \ge 1$, are unknown functions which are 1-periodic in y_1, the constant $\bar b_1$ is to be determined.

Substituting (19) into (2) and collecting power-like terms in front of ε^{-1} in the equation and of ε^0 in the boundary condition, we obtain the following periodic problem for the unknown function v_1:

$$\begin{cases} \mathcal{A}_y\, v_1(t\,,\, x_1 - \varepsilon^{-1}\bar b_1 t\,,\, y) \\ = \big(\partial_{y_i} a_{i1}(y) - b_1(y) + \bar b_1\big)\ \partial_{x_1} u_0(t\,,\, x_1 - \varepsilon^{-1}\bar b_1 t), & y \in Y, \\ \mathcal{B}_y\, v_1(t\,,\, x_1 - \varepsilon^{-1}\bar b_1 t\,,\, y) = -a_{i1}(y)\, n_i\, \partial_{x_1} u_0(t\,,\, x_1 - \varepsilon^{-1}\bar b_1 t), & y \in \partial Y; \end{cases} \tag{20}$$

Setting

$$\bar b_1 = \int_Y (a_{i1}(y)\, \partial_{y_i} p^*(y) + b_1(y)\, p^*(y))\, dy, \tag{21}$$

we guarantee that a solution to problem (20) exists. The specific form of the right-hand side of (20) suggests the following representation of v_1:

$$v_1(t\,,\, x_1 - \varepsilon^{-1}\bar b_1 t\,,\, y) = N_1(y)\, \partial_{x_1} u_0(t\,,\, x_1 - \varepsilon^{-1}\bar b_1 t) + u_1(t\,,\, x_1 - \varepsilon^{-1}\bar b_1 t),$$

where a Y-periodic function N_1 solves the problem

$$\begin{cases} \mathcal{A}_y\, N_1(y) = \partial_{y_i} a_{i1}(y) - b_1(y) + \bar b_1, & y \in Y, \\ \mathcal{B}_y\, N_1(y) = -a_{i1}(y)\, n_i, & y \in \partial Y; \end{cases} \tag{22}$$

Similarly, we get the problem for v_2

$$\left\{\begin{array}{l} \mathcal{A}_y\, v_2(t\,,\,x_1-\varepsilon^{-1}\bar b_1 t\,,\,y)=-\partial_t u_0\big(t,x_1-\dfrac{\bar b_1}{\varepsilon}\,r\big)\Big|_{r=t} \\ +\big\{a_{11}(y)+\partial_{y_i}(a_{i1}(y)N_1(y))+a_{1j}(y)\partial_{y_j}N_1(y) \\ -b_1(y)\,N_1(y)+\bar b_1\,N_1(y)\big\}\ \partial^2_{x_1}u_0\big(t,x_1-\dfrac{\bar b_1}{\varepsilon}\,t\big) \\ +\big\{\partial_{y_i}a_{i1}(y)-b_1(y)+\bar b_1\big\}\ \partial_{x_1}u_1\big(t,x_1-\dfrac{\bar b_1}{\varepsilon}\,t\big), \qquad y\in Y, \\ \mathcal{B}_y\, v_2(y)=-a_{i1}(y)\,N_1(y)\,n_i\ \partial^2_{x_1}u_0\big(t,x_1-\dfrac{\bar b_1}{\varepsilon}\,t\big), \quad y\in\partial Y. \end{array}\right. \tag{23}$$

The compatibility condition for (23) gives rise to the Cauchy problem for u_0

$$\left\{\begin{array}{ll} \partial_t u_0(t,x_1)=a_{11}^{\rm hom}\,\partial^2_{x_1}u_0(t,x_1), & (t,x_1)\in(0,T)\times\mathbb{R}, \\ u_0(0,x_1)=\varphi(x_1), & x_1\in\mathbb{R}, \end{array}\right. \tag{24}$$

where the constant $a_{11}^{\rm hom}$ is defined by

$$\begin{aligned} a_{11}^{\rm hom}=&\int\limits_Y\big[a_{11}(y)+a_{1j}(y)\partial_{y_j}N_1(y)-b_1(y)\,N_1(y)\big]\,p^*(y)\,dy \\ &+\int\limits_Y\big[\bar b_1\,N_1(y)p^*(y)-a_{i1}(y)\,N_1(y)\,\partial_{y_i}p^*(y)\big]\,dy. \end{aligned}$$

The positiveness of $a_{11}^{\rm hom}$ has been proved in [9].

Lemma 3.1. *The constant $a_{11}^{\rm hom}$ is strictly positive.*

The form of the right-hand side of the equation in (23) suggests the following representation for the solution v_2:

$$\begin{aligned} v_2(t\,,\,x_1-\varepsilon^{-1}\bar b_1 t\,,\,y)&=N_2(y)\,\partial^2_{x_1}u_0(t\,,\,x_1-\varepsilon^{-1}\bar b_1 t) \\ +N_1(y)\,\partial_{x_1}u_1(t\,,\,x_1&-\varepsilon^{-1}\bar b_1 t)+u_2(t\,,\,x_1-\varepsilon^{-1}\bar b_1 t) \end{aligned}$$

with y_1-periodic function N_2 being a solution of the problem

$$\left\{\begin{array}{ll} \mathcal{A}\,N_2(y)=a_{11}(y)+\partial_{y_i}(a_{i1}(y)N_1(y))+a_{1j}(y)\partial_{y_j}N_1(y) & \\ \qquad\qquad -b_1(y)\,N_1(y)+\bar b_1\,N_1(y)-a_{11}^{\rm hom}, & y\in Y, \\ \mathcal{B}\,N_2(y)=-a_{i1}(y)\,n_i\,N_1(y), & y\in\partial Y; \end{array}\right. \tag{25}$$

Similarly, we obtain a boundary value problem for v_3

$$\begin{cases} \mathcal{A}_y\, v_3(t\,,\, x_1-\varepsilon^{-1}\bar b_1 t\,,\, y) = -N_1(y)\partial_t\partial_{x_1}u_0\big(t,x_1-\dfrac{\bar b_1}{\varepsilon}\,r\big)\Big|_{r=t} \\ -\partial_t u_1\big(t,x_1-\dfrac{\bar b_1}{\varepsilon}\,r\big)\Big|_{r=t} + \Big[a_{11}(y)N_1(y)+\partial_{y_i}(a_{i1}(y)N_2(y)) \\ +a_{1j}(y)\partial_{y_j}N_2(y) - b_1(y)\,N_2(y)+\bar b_1\,N_2(y)\Big]\,\partial^3_{x_1}u_0\big(t,x_1-\dfrac{\bar b_1}{\varepsilon}\,t\big) \\ +\Big[a_{11}(y)+\partial_{y_i}(a_{i1}(y)N_1(y))+a_{1j}(y)\partial_{y_j}N_1(y) \\ +b_1(y)\,N_1(y)+\bar b_1\,N_1(y)\Big]\,\partial^2_{x_1}u_1\big(t,x_1-\dfrac{\bar b_1}{\varepsilon}\,t\big) \\ +\Big[\partial_{y_i}a_{i1}(y)-b_1(y)+\bar b_1\Big]\,\partial_{x_1}u_2\big(t,x_1-\dfrac{\bar b_1}{\varepsilon}\,t\big), \qquad y\in Y, \\ \mathcal{B}_y\,v_3(y) = -a_{i1}(y)\,N_2(y)\,n_i\;\partial^3_{x_1}u_0\big(t,x_1-\dfrac{\bar b_1}{\varepsilon}\,t\big) \\ -a_{i1}(y)\,N_1(y)\,n_i\;\partial^2_{x_1}u_1\big(t,x_1-\dfrac{\bar b_1}{\varepsilon}\,t\big) \\ -a_{i1}(y)\,n_i\;\partial_{x_1}u_2\big(t,x_1-\dfrac{\bar b_1}{\varepsilon}\,t\big), \qquad y\in\partial Y. \end{cases} \tag{26}$$

From the compatibility condition for (26) we derive the equation for u_1:

$$\partial_t u_1(t,x_1) = a^{\rm hom}_{11}\partial^2_{x_1}u_1(t,x_1) + h_3\,\partial^3_{x_1}u_0(t,x_1), \quad (t,x_1)\in(0,T)\times\mathbb{R},$$

where

$$h_3 = \int\limits_Y \big(-a^{\rm hom}_{11}N_1p^* + a_{11}N_1p^* - a_{i1}N_2\partial_{y_i}p^* + b_1N_2p^* \\ + a_{1j}\partial_{y_j}N_2p^* + \bar b_1\,N_2p^*\big)\,dy. \tag{27}$$

Naturally, v_3 can be represented as the sum

$$v_3\big(t,x_1-\frac{\bar b_1}{\varepsilon}\,t\,,\,y\big) = N_3(y)\,\partial^3_{x_1}u_0\big(t,x_1-\frac{\bar b_1}{\varepsilon}\,t\big) + N_2(y)\,\partial^2_{x_1}u_1\big(t,x_1-\frac{\bar b_1}{\varepsilon}\,t\big) \\ + N_1(y)\,\partial_{x_1}u_2\big(t,x_1-\frac{\bar b_1}{\varepsilon}\,t\big) + u_3\big(t,x_1-\frac{\bar b_1}{\varepsilon}\,t\big),$$

with N_3 being a y_1-periodic solution of the cell problem

$$\begin{cases} \mathcal{A}\,N_3(y) = a_{11}(y)N_1(y)+\partial_{y_i}(a_{i1}(y)N_2(y))+a_{1j}(y)\partial_{y_j}N_2(y) \\ \qquad -b_1(y)\,N_2(y)+\bar b_1\,N_2(y)-a^{\rm hom}_{11}N_1(y)-h_3, \quad y\in Y, \\ \mathcal{B}\,N_3(y) = -a_{i1}(y)\,n_i\,N_2(y), \qquad y\in\partial Y. \end{cases} \tag{28}$$

Arguing as above, one can derive the equation for u_2

$$\partial_t u_2(t,x_1) = a^{\rm hom}_{11}\partial^2_{x_1}u_2(t,x_1) \\ + h_4\,\partial^4_{x_1}u_0(t,x_1) + h_3\,\partial^3_{x_1}u_1(t,x_1),$$

where the constant h_4 is defined by

$$h_4 = \int\limits_Y \big(-a^{\rm hom}_{11}N_2p^* + a_{11}N_2p^* - a_{i1}N_3\partial_{y_i}p^* + b_1N_3p^* \\ + a_{1j}\partial_{y_j}N_3p^* + \bar b_1\,N_3p^* - h_3N_1\big)\,dy. \tag{29}$$

Notice that determining initial conditions for u_1 and u_2 requires constructing initial layer correctors, which is done in Section 3.2.

Finally, as an inner approximate solution we take first three terms of (19)

$$\begin{aligned} u_\infty^\varepsilon(t,x) = u_0\big(t\,,\,x_1 - \frac{\bar b_1}{\varepsilon}\,t\big) + \varepsilon\, N_1\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1} u_0(t\,,\,x_1 - \frac{\bar b_1}{\varepsilon}\,t) \\ +\varepsilon\, u_1\big(t\,,\,x_1 - \frac{\bar b_1}{\varepsilon}\,t\big) + \varepsilon^2\, N_2\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1}^2 u_0(t\,,\,x_1 - \frac{\bar b_1}{\varepsilon}\,t) \\ +\varepsilon^2\, N_1\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1} u_1(t\,,\,x_1 - \frac{\bar b_1}{\varepsilon}\,t) + \varepsilon^2 u_2(t\,,\,x_1 - \frac{\bar b_1}{\varepsilon}\,t). \end{aligned}$$

3.2. **Initial layers.** The leading term $u_0(t,x_1)$ satisfies the initial condition $u_0(0,x_1) = \varphi(x_1)$. We introduce the initial layer functions, which will allow us to satisfy the initial condition up to the second power of ε. Consider the function $\phi_1(\tau,y)$ which is a solution to the problem

$$\begin{cases} \partial_\tau \phi_1 + \mathcal{A}_y \phi_1 = 0, & (\tau,y)\in(0,\infty)\times Y, \\ \mathcal{B}_y\phi_1 = 0, & (\tau,y)\in(0,\infty)\times \partial Y, \\ \phi_1(0,y) = -N_1(y). \end{cases} \tag{30}$$

By Lemma 2.1, ϕ_1 stabilizes to a constant $\overline{\phi_1}$, as $\tau\to\infty$, at the exponential rate. The constant $\overline{\phi_1}$ can be calculated as follows

$$\overline{\phi_1} = -\int\limits_Y N_1(y)\,p^*(y)\,dy. \tag{31}$$

We use this constant to set the initial value for u_1: $u_1(0,x_1) = \overline{\phi_1}\,\varphi'(x_1)$. In this way

$$\begin{aligned} \Big[u_0\big(t\,,\,x_1 - \frac{\bar b_1}{\varepsilon}\,t\big) + \varepsilon\, N_1\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1} u_0(t\,,\,x_1 - \frac{\bar b_1}{\varepsilon}\,t) + \varepsilon\, u_1\big(t\,,\,x_1 - \frac{\bar b_1}{\varepsilon}\,t\big) \\ + \varepsilon\big(\phi_1\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big) - \overline{\phi_1}\big)\,\varphi'(x_1)\Big]\Big|_{t=0} = \varphi(x_1). \end{aligned}$$

Similarly, we introduce $\phi_2(\tau,y)$ such that:

$$\begin{cases} \partial_\tau \phi_2 + \mathcal{A}_y \phi_2 = 0, & (\tau,y)\in(0,\infty)\times Y, \\ \mathcal{B}_y\phi_2 = 0, & (\tau,y)\in(0,\infty)\times \partial Y, \\ \phi_2(0,y) = -N_2(y); \end{cases} \tag{32}$$

The constant to which ϕ_2 stabilizes, as $\tau\to\infty$, we denote by $\overline{\phi_2}$

$$\overline{\phi_2} = -\int\limits_Y N_2(y)\,p^*(y)\,dy, \tag{33}$$

and set

$$u_2(0,x_1) = \overline{\phi_2}\,\varphi''(x_1) + \overline{\phi_1}\,\varphi''(x_1).$$

In this way the boundary value problems for u_1 and u_2 take the form

$$\begin{cases} \partial_t u_1(t,x_1) = a_{11}^{\rm hom}\partial_{x_1}^2 u_1(t,x_1) + h_3\,\partial_{x_1}^3 u_0(t,x_1), & (t,x_1)\in(0,T)\times\mathbb{R}, \\ u_1(0,x_1) = \overline{\phi_1}\,\varphi'(x_1), \quad x_1\in\mathbb{R}; \end{cases} \tag{34}$$

$$\left\{\begin{array}{l}\partial_t u_2(t,x_1) = a_{11}^{\hom}\partial^2_{x_1} u_2(t,x_1)\\ +h_4\,\partial^4_{x_1} u_0(t,x_1) + h_3\,\partial^3_{x_1} u_1(t,x_1),, \quad (t,x_1)\in(0,T)\times\mathbb{R},\\ u_2(0,x_1) = \overline{\phi_2}\,\varphi''(x_1) + \overline{\phi_1}\,\varphi''(x_1)\end{array}\right. \tag{35}$$

with the constants h_3, h_4 defined in (27), (29). Then

$$\Big[\varepsilon^2 N_2\big(\frac{x}{\varepsilon}\big)\,\partial^2_{x_1} u_0(t\,,\,x_1-\frac{\bar b_1}{\varepsilon}\,t) + \varepsilon^2 N_1\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1} u_1(t\,,\,x_1-\frac{\bar b_1}{\varepsilon}\,t) +$$
$$+ +\varepsilon^2 q_2\big(\frac{x}{\varepsilon}\big)\,g(x_1) + \varepsilon^2 u_2(t\,,\,x_1-\frac{\bar b_1}{\varepsilon}\,t) + \varepsilon^2\big(\phi_2\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big) - \overline{\phi_2}\big)\,\varphi''(x_1) +$$
$$+\varepsilon^2\big(\phi_1\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big) - \overline{\phi_1}\big)\,\varphi''(x_1)\Big]\Big|_{t=0} = 0.$$

Denote

$$u^\varepsilon_{il}(t,x) = \varepsilon\big(\phi_1\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big) - \overline{\phi_1}\big)\,\varphi'(x_1) + \varepsilon^2\big(\phi_2\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big) - \overline{\phi_2}\big)\,\varphi''(x_1) +$$
$$+\varepsilon^2\big(\phi_1\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big) - \overline{\phi_1}\big)\,\varphi''(x_1). \tag{36}$$

We summarize this section by writing down the formal asymptotic expansion for a solution u^ε of problem (2) which has been constructed above. It reads

$$U^\varepsilon(t,x) = u_0\big(t\,,\,x_1-\frac{\bar b_1}{\varepsilon}\,t\big) + \varepsilon\,N_1\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1} u_0(t\,,\,x_1-\frac{\bar b_1}{\varepsilon}\,t)$$
$$+\varepsilon\,u_1\big(t\,,\,x_1-\frac{\bar b_1}{\varepsilon}\,t\big) + \varepsilon^2 N_2\big(\frac{x}{\varepsilon}\big)\,\partial^2_{x_1} u_0(t\,,\,x_1-\frac{\bar b_1}{\varepsilon}\,t) \tag{37}$$
$$+\varepsilon^2 N_1\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1} u_1(t\,,\,x_1-\frac{\bar b_1}{\varepsilon}\,t) + \varepsilon^2 u_2(t\,,\,x_1-\frac{\bar b_1}{\varepsilon}\,t) + u^\varepsilon_{il}(t,x).$$

Here u_0 is a solution of the homogenized problem (24); N_1, N_2 solve auxiliary cell problems (22), (25); u_1 and u_2 are solutions of nonhomogeneous Cauchy problems (34), (35); the initial layer u^ε_{il} is given by (30)-(33) and (36). Notice that the approximate solution satisfies the initial condition: $U^\varepsilon(0,x) = \varphi(x_1)$.

4. **Justification procedure.** In thin domain $\mathbb{G}_\varepsilon$ it is natural to introduce the following notion of convergence (see, for example, [4], [13]).

Definition 4.1. We say that $f_\varepsilon(t,x)$ converges strongly to zero in $L^2[0,T;H^1(\mathbb{G}_\varepsilon)]$ if

$$\varepsilon^{-\frac{(d-1)}{2}}\,\|f_\varepsilon\|_{L^2[0,T;H^1(\mathbb{G}_\varepsilon)]} \longrightarrow 0, \quad \varepsilon\to 0.$$

The normalization factor $\varepsilon^{-\frac{(d-1)}{2}}$ appears due to the fact that the norm of a fixed nontrivial $C_0^\infty(\mathbb{R})$ function $\varphi(x_1)$ in the space $L^2[0,T;H^1(\mathbb{G}_\varepsilon)]$ is of order $\varepsilon^{\frac{(d-1)}{2}}$.

The following theorem is the main result of the paper.

Theorem 4.2. *Let conditions* (**H1**) − (**H4**) *be fulfilled. Then the difference between the exact solution u^ε of problem* (13) *and the approximate solution U^ε given by* (37), *converges in $L^2[0,T;H^1_{loc}(\mathbb{G}_\varepsilon)]$ to zero, as $\varepsilon\to 0$. Moreover, the following estimate holds:*

$$\int\limits_{\mathbb{G}_\varepsilon}(u^\varepsilon - U^\varepsilon)^2\,dx + \int\limits_0^t\int\limits_{\mathbb{G}_\varepsilon}|\nabla(u^\varepsilon(s,x) - U^\varepsilon(s,x))|^2\,dxds \le C\,\varepsilon^2\,\varepsilon^{d-1}. \tag{38}$$

Proof. In order to estimate the norm (in the appropriate space) of the difference $u^\varepsilon - U^\varepsilon$ between the exact and the approximate solutions, we calculate first $\mathcal{A}_\varepsilon(u^\varepsilon - U^\varepsilon)$ and $\mathcal{B}_\varepsilon(u^\varepsilon - U^\varepsilon)$, and then make use of a priori estimates (8), (11). Straightforward computations yield

$$\mathcal{A}_\varepsilon(u^\varepsilon(t,x) - U^\varepsilon(t,x)) = \varepsilon\,(R_1^\varepsilon(t,x) + R_2^\varepsilon(t,x)) + o(\varepsilon), \quad \varepsilon \to 0,$$

where

$$\begin{aligned}
&R_1^\varepsilon(t,x) = -\sum_{k=0}^{1} N_k\big(\frac{x}{\varepsilon}\big)\partial_t\partial_{x_1}^k u_{1-k}(t\,,\,x_1 - \frac{\bar b_1}{\varepsilon}\,r)\Big|_{r=t}\\
&+\bar b_1 \sum_{k=0}^{2} N_k\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1}^{k+1}u_{2-k}(t\,,\,x_1 - \frac{\bar b_1}{\varepsilon}\,t)\\
&\sum_{k=0}^{1} a_{11}\big(\frac{x}{\varepsilon}\big)\,N_k\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1}^{k+2}u_{1-k}(t\,,\,x_1 - \frac{\bar b_1}{\varepsilon}\,t)\\
&+\sum_{k=1}^{2} a_{1j}\big(\frac{x}{\varepsilon}\big)\,\partial_{y_j}N_k(y)\Big|_{y=x/\varepsilon}\,\partial_{x_1}^{k+1}u_{2-k}(t\,,\,x_1 - \frac{\bar b_1}{\varepsilon}\,t)\\
&+\sum_{k=0}^{2} b_1\big(\frac{x}{\varepsilon}\big)\,N_k\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1}^{k+1}u_{2-k}(t\,,\,x_1 - \frac{\bar b_1}{\varepsilon}\,t)\\
&+\bar b_1 \sum_{k=1}^{2}\big(\phi_k(\tau,y) - \overline{\phi_k}\big)\,\varphi'''(x_1) + a_{11}(y)\,\big(\phi_1\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big) - \overline{\phi_1}\big)\,\varphi'''(x_1)\\
&+\sum_{k=1}^{2} a_{1j}(y)\,\nabla_y\phi_k(\tau,y)\Big|_{y=x/\varepsilon,\tau=t/\varepsilon^2}\,\varphi'''(x_1)\\
&+b_1(y)\sum_{k=1}^{2}\big(\phi_k\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big) - \overline{\phi_k}\big)\,\varphi'''(x_1).
\end{aligned}$$

$$\begin{aligned}
&R_2^\varepsilon(t,x) = \sum_{k=0}^{2}\partial_{y_i}(a_{i1}(y)\,N_k(y))\Big|_{y=x/\varepsilon}\,\partial_{x_1}^{k+1}u_{2-k}(t\,,\,x_1 - \frac{\bar b_1}{\varepsilon}\,t)\\
&+\sum_{k=1}^{2}\partial_{y_i}(a_{i1}(y)(\phi_k(\tau,y) - \overline{\phi_k}))\Big|_{y=x/\varepsilon,\tau=t/\varepsilon^2}\,\varphi'''(x_1).
\end{aligned}$$

Similarly,

$$\mathcal{B}_\varepsilon(u^\varepsilon(t,x) - U^\varepsilon(t,x)) = \varepsilon^2\,R_3^\varepsilon(t,x)$$

with

$$\begin{aligned}
&R_3^\varepsilon(t,x) = -\sum_{k=0}^{2} a_{i1}\big(\frac{x}{\varepsilon}\big)\,n_i\,N_k\big(\frac{x}{\varepsilon}\big)\,\partial_{x_1}^{k+1}u_{2-k}(t\,,\,x_1 - \frac{\bar b_1}{\varepsilon}\,t)\\
&a_{i1}(y)n_i\sum_{k=1}^{2}\big(\phi_k\big(\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}\big) - \overline{\phi_k}\big)\,\varphi'''(x_1).
\end{aligned}$$

By a priori estimates (8) and (11),

$$\begin{aligned}
&\int_{\mathbb{G}_\varepsilon}(u^\varepsilon - U^\varepsilon)^2\,dx + \int_0^t\int_{\mathbb{G}_\varepsilon}|\nabla(u^\varepsilon(s,x) - U^\varepsilon(s,x))|^2\,dxds\\
&\le C\,e^T\,\Big\{\|\varepsilon\,R_1^\varepsilon\|^2_{L^2[0,T;L^2(\mathbb{G}_\varepsilon)]} + \varepsilon\,\|\varepsilon^2\,R_3^\varepsilon\|^2_{L^2[0,T;L^2(\Sigma_\varepsilon)]} + \frac{1}{\varepsilon}\,\|\varepsilon R_2^\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{G}_\varepsilon)}\Big\}.
\end{aligned}$$

In order to estimate $R_1^\varepsilon, R_2^\varepsilon$ and R_3^ε, we analyze properties of the solutions u_0, u_1 and u_2 of problems (24), (34) and (35). For u_0 the well-known integral Poisson formula takes place:

$$u_0(t,x_1) = \frac{\theta(t)}{\sqrt{4\pi a_{11}^{\rm hom} t}} \int\limits_{\mathbb{R}} \varphi(\xi)\, e^{-\frac{|x_1-\xi|^2}{4a_{11}^{\rm hom}t}}\, d\xi.$$

Here θ is the unit step function, that is $\theta(t)=1$ for $t\ge 0$, and $\theta(t)=0$ when $t<0$. Moreover, similar formula is valid for any derivative of u_0 with respect to x_1:

$$\partial_{x_1}^{(k)} u_0(t,x_1) = \frac{\theta(t)}{\sqrt{4\pi a_{11}^{\rm hom} t}} \int\limits_{\mathbb{R}} \partial_\xi^{(k)}\varphi(\xi)\, e^{-\frac{|x_1-\xi|^2}{4a_{11}^{\rm hom}t}}\, d\xi.$$

Bearing in mind that φ has finite support, one can see that

$$|\partial_{x_1}^{(k)} u_0(t,x_1)| \le C\, e^{-\alpha|x_1|^2}, \quad C,\alpha>0, \tag{39}$$

where α depends on T. Similarly, the following integral representation of $\partial_{x_1}^{(k)} u_1$ is valid:

$$\partial_{x_1}^{(k)} u_1(t,x_1) = \frac{\theta(t)\,\bar\phi_1}{\sqrt{4\pi a_{11}^{\rm hom} t}} \int\limits_{\mathbb{R}} \partial_\xi^{(k+1)}\varphi(\xi)\, e^{-\frac{|x_1-\xi|^2}{4a_{11}^{\rm hom}t}}\, d\xi$$
$$+\int\limits_0^t \frac{h_3}{\sqrt{4\pi a_{11}^{hom}(t-\tau)}}\, d\tau \int\limits_{\mathbb{R}} \partial_{x_1}^{3+k} u_0(\tau,\xi)\, e^{-\frac{|x_1-\xi|^2}{4a_{11}^{hom}(t-\tau)}}\, d\xi = I_1 + I_2.$$

Arguing as above we obtain

$$|I_1| \le C\, e^{-\alpha|x_1|^2}, \quad \alpha>0.$$

Let us estimate I_2.

$$|I_2| \le \Big\{ \int\limits_0^t d\tau \int\limits_{|x_1-\xi|\le 2|x_1|} d\xi + \int\limits_0^t d\tau \int\limits_{|x_1-\xi|> 2|x_1|} d\xi \Big\} \frac{h_3\, \partial_{x_1}^{3+k} u_0(\tau,\xi)}{\sqrt{4\pi a_{11}^{hom}(t-\tau)}}\, e^{-\frac{|x_1-\xi|^2}{4a_{11}^{hom}(t-\tau)}}.$$

In view of (39), for ξ satisfying $|x_1-\xi|\le 2|x_1|$,

$$\big|\partial_{x_1}^{3+k} u_0(\tau,\xi)\, e^{-\frac{|x_1-\xi|^2}{4a_{11}^{hom}(t-\tau)}}\big| \le C\, e^{-\alpha|\xi|^2}\, e^{-\frac{|x_1-\xi|^2}{4a_{11}^{hom}T}} \le C\, e^{-\alpha_1|x_1|^2},\ \alpha_1>0,$$

thus,

$$\int\limits_0^t d\tau \int\limits_{|x_1-\xi|\le 2|x_1|} \frac{h_3\, \partial_{x_1}^{3+k} u_0(\tau,\xi)}{\sqrt{4\pi a_{11}^{hom}(t-\tau)}}\, e^{-\frac{|x_1-\xi|^2}{4a_{11}^{hom}(t-\tau)}}\, d\xi$$
$$\le C\,|x_1|\, e^{-\alpha_1|x_1|^2} \int\limits_0^t \frac{1}{\sqrt{t-\tau}} d\tau \le C\, e^{-\alpha_1|x_1|^2}.$$

Noticing that

$$\frac{1}{\sqrt{4\pi a_{11}^{\rm hom} t}} \int\limits_{\mathbb{R}} e^{-\frac{|x_1-\xi|^2}{4a_{11}^{\rm hom}t}}\, d\xi = 1,$$

one has

$$\int_0^t d\tau \int_{|x_1-\xi|>2|x_1|} \frac{h_3\,\partial_{x_1}^{3+k} u_0(\tau,\xi)}{\sqrt{4\pi a_{11}^{hom}(t-\tau)}}\, e^{-\frac{|x_1-\xi|^2}{4a_{11}^{hom}(t-\tau)}}\, d\xi \le C\, e^{-\alpha_1|x_1|^2},\ \alpha_1>0.$$

In this way we see that u_1 satisfies the estimate

$$|\partial_{x_1}^k u_1(t,x_1)| \le C\, e^{-\alpha_1|x_1|^2},\ \alpha_1>0,\ t\ge 0,\ x\in\mathbb{R}. \tag{40}$$

Arguing as above, one can see that analogous estimate holds for u_2 solving problem (35).

$$|\partial_{x_1}^k u_2(t,x_1)| \le C\, e^{-\alpha_1|x_1|^2},\ \alpha_1>0,\ t\ge 0,\ x\in\mathbb{R}. \tag{41}$$

Bearing in mind the boundedness of the coefficients a_{ij}, b_j, properties of N_1 and N_2 as the solutions of (22), (25), and bounds (39)-(41), one can check the validity of (38). Note that the exponential decay of the initial layer functions is used while estimating the corresponding terms. □

REFERENCES

[1] G. Allaire and A. Raphael, *Homogenization of a convection-diffusion model with reaction in a porous medium*, (English, French summary) C. R. Math. Acad. Sci. Paris, **344** (2007), 523–528.

[2] D. G. Aronson and J. Serrin, *Local behavior of solutions of quasilinear parabolic equations,* Arch. Rational Mech. Anal., **25** (1967), 81–122.

[3] D. G. Aronson, *Non-negative solutions of linear parabolic equations,* Ann. Scuola Norm. Sup. Pisa (3), **22** (1968), 607–694.

[4] N. S. Bakhvalov and G. P. Panasenko, "Homogenization: Averaging Processes in Periodic Media," Kluwer, Dordrecht/Boston/London, 1989.

[5] A. Bensoussan, J.-L. Lions and G. Papanicolaou, "Asymptotic Analysis for Periodic Structure," Studies in Mathematics and its Applications, 5. North-Holland Publishing Co., Amsterdam-New York, 1978.

[6] P. Donato and A. Piatnitski, *Averaging of nonstationary parabolic operators with large lower order terms*, Multi Scale Problems and Asymptotic Analysis, GAKUTO Internat. Ser. Math. Sci. Appl., **24** (2005), 153–165.

[7] M. V. Kozlova and G. P. Panasenko, *Averaging a three-dimensional problem of elasticity theory in a nonhomogeneous rod,* Comput. Math. Math. Phys., **31** (1992), 128–131.

[8] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva, "Linear and Quasilinear Equations of Parabolic Type," Translations of Mathematical Monographs, Vol. **23**, American Mathematical Society, Providence, R.I. 1967.

[9] I. Pankratova and A. Piatnitski, *On the behaviour at infinity of solutions to stationary convection-diffusion equation in a cylinder*, DCDS-B, **11** (2009), 935–970.

[10] L. Trabucho and J. M. Viaño, *Derivation of generalized models for linear elastic beams by asymptotic expansion methods,* Applications of multiple scaling in mechanis (Paris, 1986), Rech. Math. Appl., **4**, Masson, Paris (1987), 302–315.

[11] Z. Tutek, *A homogenized model of rod in linear elasticity,* Applications of multiple scaling in mechanis (Paris, 1986), Rech. Math. Appl., **4**, Masson, Paris (1987), 302–315.

[12] V. V. Zhikov, S. M. Kozlov and O. A. Oleinik, "Homogenization of Differential Operators and Integral Functionals," Springer-Verlag, Berlin, 1994.

[13] V. V. Zhikov, *On an extension and an application of the two-scale convergence method*, Sb. Math., **191** (2000), 973–1014.

Received February 2010; revised May 2010.

E-mail address: iripan@hin.no
E-mail address: andrey@sci.lebedev.ru