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MATHEMATICAL AND NUMERICAL ANALYSIS FOR PREDATOR-PREY SYSTEM IN A POLLUTED ENVIRONMENT

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Abstract. In this paper, we prove existence results for a Predator-prey system in a polluted environment. The existence result is proved by the Schauder fixed-point theorem. Moreover, we construct a combined finite volume - finite element scheme to our model, we establish existence of discrete solutions to this scheme, and show that it converges to a weak solution. The convergence proof is based on deriving series of a priori estimates and using a general L^p compactness criterion. Finally we give some numerical examples.

1. Introduction.

Today the most threatening problem to the society is the change in environment caused by pollution, affecting the long term survival of species, human life style and biodiversity of the habitat.

The rapid economic growth of some countries is also accompanied by the severe deterioration of the environment as evidenced by the polluted air, water, soil erosion, growing number of illness such as cancer caused by company waste, and constantly increased deforestation and desertification.

Therefore the pollution of the environment is a very serious problem in the world today. Organisms are often exposed to a polluted environment and take up toxicant. For that reason, it is important to study the effects of toxicant on populations to determine permanence or extinction.

In the early 1980's Hallam and his colleagues proposed a deterministic modelling approach to the problem of assessing the effects of a pollutant on an ecological system [14, 15, 16]. In particular, Hallam *et al.* [14] studied the effect of a toxicant present in the environment on a single-species population by assuming that its

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growth rate density decreases linearly with the concentration of toxicant but the corresponding carrying capacity does not depend upon the concentration of toxicant present in the environment. Since then, such models have been the subject of many investigations and improvements. Freedman and Shukla [13] studied the effect of toxicant on a single species and on a predator-prey system by taking into account the introduction of toxicant from an external source. Shukla and Dubey [20] studied the simultaneous effect of two toxicants, one being more toxic than the other, on a biological species.

As species do not exist alone in nature, it is more biological significance to study the persistence and extinction of each population in systems of two or more interacting species subjected to toxicant. Dubey [9] proposed a model to study the interaction of two biological species in a polluted environment.

In this paper, the model we considered is based on the following two species model with toxicant effect in a physical domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ over a time span $(0, T)$, $T > 0$, with nonlocal diffusion terms: for $i = 1, 2, 3$

$$\begin{cases} \partial_t u - d_1 \left(\int_\Omega u\,dx\right) \Delta u + \operatorname{div}(uK_1) = F(u, v, C_1), & \text{in} \quad Q_T, \\ \partial_t v - d_2 \left(\int_\Omega v\,dx\right) \Delta v + \operatorname{div}(vK_2) = G(u, v, C_2), & \text{in} \quad Q_T, \\ \partial_t C_1 = H_1(C_3, C_1), \\ \partial_t C_2 = H_2(C_3, C_2), \\ \partial_t C_3 = H_3(C_3). \end{cases} \tag{1}$$

We complete the system (1) with Dirichlet boundary conditions:

$$u = 0 \text{ and } v = 0 \quad \text{on} \quad \Sigma_T := \partial\Omega \times (0, T), \tag{2}$$

where $\partial\Omega$ denotes the boundary of Ω. The nonlinearities F, G, H_1, H_2 and H_3 take the form:

$$\begin{aligned} F(u, v, C_1) &= k(u) - \beta_1 C_1 u - \pi(u)v, \\ G(u, v, C_2) &= -a\,v - \beta_2 C_2 v + e\pi(u)v, \\ H_1(C_1, C_2, C_3) &= k_1 C_3 - g_1 C_1 - m_1 C_1, \\ H_2(C_1, C_2, C_3) &= k_2 C_3 - g_2 C_2 - m_2 C_2, \\ H_3(C_1, C_2, C_3) &= -hC_3. \end{aligned} \tag{3}$$

In our model, $u(x, t)$ and $v(x, t)$ represent the density of the prey population and the predator population at time t, respectively. The functions $C_1(x, t), C_2(x, t)$ and $C_3(x, t)$ represent the concentration of the toxicant in the organism of the prey species, the predator species and the environment at time t, respectively. The constant $-a$ $(a > 0)$ be the natural exponential decay of the predator population and e is the conversion rate from prey to predator. Then, we assume the logistical growth rate of prey reads $k(u) = ru\left(1 - \dfrac{u}{K}\right)$ where $r > 0$ is the natural growth rate of prey and K is the carrying capacity. The predation rate reads $\pi(u) = \dfrac{p\,u}{1 + q\,u}$ with $1/p$ the time spent by a predator to catch a prey and q/p the manipulation time, offering a saturation effect for large densities of preys when $q > 0$. The constants β_1 and β_2 represent the decreasing rate of the intrinsic growth rate associated with the uptake of the toxicant, respectively. The terms $k_i C_3(x, t)$, $-g_i C_i(x, t)$ and $-m_i C_i(x, t)$ $(i = 1, 2)$ stand for the absorving rate of the toxicant from the environment, excretion and depuration rates of the toxicant for the both organisms, respectively. $-hC_3(x,t)$ stands for the loss rate of the toxicant due to volatilization by itself.

In (1), the diffusion rates $d_1 > 0$ and $d_2 > 0$, are taken to depend on the whole of each population in the domain rather than on the local density. This means that the diffusion of individuals is guided by the global state of the population in the medium. For instance, if one wants to model species having the tendency to leave crowded zones, a natural assumption would be to assume that d_i is an increasing function of its argument. Otherwise, if we are dealing with species attracted by the growing population, one will suppose that the nonlocal diffusion d_i decreases. In Yang *et al.* [23], the authors studied a model similar to (1), but without diffusion terms, that is, a system consisting of only ordinary differential equations. In this sense, our model is a slight generalization of the model discussed in [23]. Despite being considered nonlocal diffusion to the dynamics of the two populations, for simplicity we assume no diffusion to the dynamics of pollutants. Then, it might be the case where for an initial condition of the toxicants C_1 and C_2 localized in a compact support (or even more, as sums of dirac deltas localized), the populations of u and v, dissipate in the environment, but the contaminants C_1 and C_2 remain localized in space. A clear example of application of this situation, it can be if one of the two populations (predator or prey), is a plague (e.g. rats, or rabbits) which tends to disperse naturally by the migration of this species, but instead, the associated toxicant is a poison that we want to be ideally localized, this contaminant does not dissipate nor affects the human population.

In this work, we assume that

$$K_1, K_2 \in L^\infty(\Omega, \mathbb{R}^3), \qquad \operatorname{div} K_1, \operatorname{div} K_2 \geq 0, \tag{4}$$

and $d_i : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying: there exist constants $M_i, C > 0$ such that

$$M_i \leq d_i \quad \text{and} \quad |d_i(I_1) - d_i(I_2)| \leq C\,|I_1 - I_2| \tag{5}$$

for all $I_1, I_2 \in \mathbb{R}$, for $i = 1, 2$. The most interesting and real cases for this model are in dimension 3, but in 2-dimension it also has a realistic interpretation. Note that, the parabolic (and elliptic) equations with nonlocal diffusion terms has already been studied from a theoretical point of view by several authors. First, in 1997, M. Chipot and B. Lovat [8] studied the existence and uniqueness of the solutions for a scalar parabolic equation with a nonlocal diffusion term. Existence-uniqueness and long time behavior for other class of nonlocal nonlinear parabolic equations and systems are studied in [1, 18, 7].

Before we state our main results, let us give a relevant definition of a weak solution for our model.

Definition 1.1. A weak solution of (1)-(3) is a nonnegative function $\mathbf{u}$, where $\mathbf{u} = (u, v, C_1, C_2, C_3)$ such that $u, v \in L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$ and $C_1, C_2, C_3 \in$

$C([0,T],L^2(\Omega))$, for $i=1,2,3$

$$\begin{aligned}
-\int_\Omega u_0(x)\varphi_1(x,0)\,dx &- \iint_{Q_T} u\partial_t\varphi_1\,dx\,dt \\
&+\int_0^T d_1\left(\int_\Omega u\,dx\right)\int_\Omega \nabla u\cdot\nabla\varphi_1\,dx\,dt - \iint_{Q_T} uK_1\cdot\nabla\varphi_1\,dx\,dt \\
&= \iint_{Q_T} F(u,v,C_1)\varphi_1\,dx\,dt, \\
-\int_\Omega v_0(x)\varphi_2(x,0)\,dx &- \iint_{Q_T} v\partial_t\varphi_2\,dx\,dt \\
&+\int_0^T d_2\left(\int_\Omega v\,dx\right)\int_\Omega \nabla v\cdot\nabla\varphi_2\,dx\,dt - \iint_{Q_T} vK_2\cdot\nabla\varphi_2\,dx\,dt \\
&= \iint_{Q_T} G(u,v,C_2)\varphi_2\,dx\,dt, \\
-\int_\Omega C_{i,0}(x)\psi_i(x,0)\,dx &- \iint_{Q_T} C_i\partial_t\psi_i\,dx\,dt = \iint_{Q_T} H_i(C_1,C_2,C_3)\psi_i\,dx\,dt,
\end{aligned}$$

for all $\varphi_1,\varphi_2,\psi_1,\psi_2,\psi_3 \in C_c^1(\Omega\times[0,T))$.

Remark 1. Note that all the integrals in Definition 1.1 make sense. In particular from (5) we get

$$\begin{aligned}
&\int_0^T \left[d_1\left(\int_\Omega u\,dx\right)\int_\Omega \nabla u\cdot\nabla\varphi_1\,dx\right]dt \\
&\qquad\le \sup_{t\in[0,T]} d_1\left(\int_\Omega u\,dx\right)\int_0^T\int_\Omega \nabla u\cdot\nabla\varphi_1\,dx\,dt \\
&\qquad\le C\,\|u\|_{L^\infty(0,T;L^2(\Omega))}\,\|\nabla u\|_{L^2(Q_T)}\,\|\nabla\varphi_1\|_{L^2(Q_T)}
\end{aligned}$$

Our first main result is the following existence theorem for weak solutions.

Theorem 1.2. *Assume conditions (4)-(5) hold. If $u_0, v_0, C_{1,0}, C_{2,0}, C_{3,0} \in L^2(\Omega)$, then the system (1)-(3) possesses at least one weak solution.*

We prove existence of solution to the system (1) by applying the Schauder fixed point theorem, deriving a priori estimates, and then passing to the limit in the approximate solutions using compactness arguments.

The next goal is to discretize our model. There are many finite volume schemes to tackle numerically a nonlinear convection-reaction-diffusion system. One of them is the well-known method introduced by Gallouet [11]. In [2, 3] was used this idea by doing a convergence analysis of the method. The inconvenient of this classical finite volume method of Gallouet is the restriction on the admisible meshes which have to be rectangular, Delaunay triangulations or Voronoi meshes. For this reason, an innovative idea was introduced by Eymard, Hilhorst and Vohralik [12] by using a nonconforming finite element method to discretize the diffusion term and the other terms are discretized by means of a cell-centered finite volume scheme on a dual mesh, where the dual volumes are constructed around the sides of the original mesh. This is the method that we will use, besides we do a convergence analysis and show some numerical examples for our problem. Now, we follow [12] in order to do the discretization of the problem (1)-(3). Let Ω be an open bounded polygonal connected subset of $\mathbb{R}^3$ with boundary $\partial\Omega$. We perform a triangulation $\mathcal{T}_h$ of the

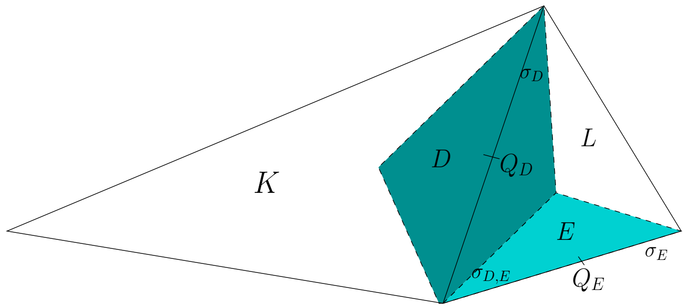


FIGURE 1. Triangles $K, L \in \mathcal{T}_h$ and dual volumes $D, E \in \mathcal{D}_h$

domain Ω, consisting of closed simplices such that $\overline{\Omega} = \cup_{K\in\mathcal{T}_h} K$ and such that if $K, L \in \mathcal{T}_h$ and $K \neq L$, then $K \cap L$ is either an empty set or a common face, edge, or vertex of K and L. We denote by $\mathcal{E}_h$ the set of all sides, by $\mathcal{E}_h^{int}$ the set of all interior sides, by $\mathcal{E}_h^{ext}$ the set of all exterior sides, and by $\mathcal{E}_K$ the set of all sides of an element $K \in \mathcal{T}_h$. We define $h := \max_{K\in\mathcal{T}_h} \text{diam}(K)$ and make the following shape regularity assumption on the family of triangulations $\{\mathcal{T}_h\}_h$:

There exists a positive constant $\kappa_\mathcal{T}$ such that

$$\min_{K\in\mathcal{T}_h} \frac{m(K)}{\text{diam}(K)^d} \geq k_\mathcal{T} \qquad \forall h > 0. \tag{6}$$

The inequality given before is equivalent to the existence of a constant $\theta_\mathcal{T} > 0$ such that

$$\max_{K\in\mathcal{T}_h} \frac{\text{diam(K)}}{\rho_K} \leq \theta_\mathcal{T} \qquad \forall h > 0, \tag{7}$$

where ρ_K is the diameter of the largest ball inscribed in K. Finally, the shape regularity assumption is equivalent to the existence of a constant $\phi_\mathcal{T} > 0$ such that

$$\min_{K\in\mathcal{T}_h} \phi_K \geq \phi_\mathcal{T} \qquad \forall h > 0.$$

Here $\phi_\mathcal{T}$ is the smallest angle of the simplex K

We will next use a dual partition $\mathcal{D}_h$ of Ω such that $\overline{\Omega} = \cup_{D\in\mathcal{D}_h} D$. There is one dual element D associated with each side $\sigma_D \in \mathcal{E}_h$. We construct it by connecting the barycentres of every $K \in \mathcal{T}_h$ that contains σ_D through the vertices of σ_D. For $\sigma_D \in \mathcal{E}_h^{ext}$, the contour of D is completed by the side σ_D itself. We refer to Figure 1, for the two-dimensional case. We denote by Q_D the barycentre of the side σ_D. As for the primal mesh, we set $\mathcal{F}_h, \mathcal{F}_h^{int}, \mathcal{F}_h^{ext}$, and $\mathcal{F}_D$ for the dual mesh sides. We denote by $\mathcal{D}_h^{int}$ the set of all interior and by $\mathcal{D}_h^{ext}$ the set of all boundary dual volumes. We finally denote by $\mathcal{N}(D)$ the set of all adjacents volumes to the volume D,

$$\mathcal{N}(D) := \{E \in \mathcal{D}_h; \exists\sigma \in \mathcal{F}_h^{int} \text{ such that } \sigma = \partial D \cap \partial E\}.$$

Observe that

$$m(K \cap D) = \frac{m(K)}{d+1}$$

for each $K \in \mathcal{T}_h$ and $D \in \mathcal{D}_h$ such that $\sigma_D \in \mathcal{E}_K$. For $E \in \mathcal{N}(D)$, we also set $d_{D,E} := |Q_E - Q_D|, \sigma_{D,E} := \partial D \cap \partial E$, and $K_{D,E}$ the element of $\mathcal{T}_h$ such that $\sigma_{D,E} \subset K_{D,E}$.

We suppose the partition of the time interval $(0,T)$ such that $0 = t_0 < \ldots < t_n < \ldots < t_N = T$ and define $\Delta t_n := t_n - t_{n-1}$ and $\Delta t := \max_{1\le n\le N} \Delta t_n$.

We define the following finite-dimensional spaces:

$$\begin{aligned} X_h &:= \{\varphi_h \in L^2(\Omega); \varphi_{h|K} \text{ is linear } \forall K \in \mathcal{T}_h, \\ &\qquad \varphi_h \text{ is continuous at the points } Q_D, D \in \mathcal{D}_h^{int}\}. \\ X_h^0 &:= \{\varphi_h \in X_h; \varphi_h(Q_D) = 0 \quad \forall D \in \mathcal{D}_h^{ext}\} \end{aligned}$$

The basis of X_h is spanned by the shape functions $\varphi_D, D \in \mathcal{D}_h$, such that $\varphi_D(Q_E) = \delta_{DE}, E \in \mathcal{D}_h, \delta$ being the Kronecker delta. We equip X_h with the seminorm

$$\|u_h\|_{X_h}^2 := \sum_{K\in\mathcal{T}_h} \int_K |\nabla u_h|^2 \, dx.$$

which becomes a norm on X_h^0.

Later we shall need the following Lemma (see for e.g. Lemma 3.1 in [12]):

Lemma 1.3. *Let $u_h \in X_h$ such that $u_h = \sum_{D\in\mathcal{D}_h} u_D\varphi_D$, then*

$$\sum_{\sigma_{D,E}\in\mathcal{F}_h^{int}} diam(K)^{d-2} |u_E - u_D|^2 \le C_{d,k_\tau} \|u_h\|_{X_h}^2,$$

$$\sum_{\sigma_{D,E}\in\mathcal{F}_h^{int}} \frac{m(\sigma_{D,E})}{d_{D,E}} |u_E - u_D|^2 \le C'_{d,k_\tau} \|u_h\|_{X_h}^2,$$

where $C_{d,k_\tau} > 0$ and $C'_{d,k_\tau} > 0$ are constants depending on d and k_τ (recall that k_τ is defined in (6)).

We approximate our model in the following way: Determine vectors $(u_D^n)_{D\in\mathcal{D}_h}$, $(v_D^n)_{D\in\mathcal{D}_h}$, $(C_{i,D}^n)_{D\in\mathcal{D}_h}$, for $n \in \{0,1,\ldots,N\}$ and $i = 1,2,3$, such that for $D \in \mathcal{D}_h^{int}$,

$$\begin{aligned} u_D^0 &= \frac{1}{m(D)} \int_D u_0(x)\, dx, \\ v_D^0 &= \frac{1}{m(D)} \int_D v_0(x)\, dx, \\ C_{i,D}^0 &= \frac{1}{m(D)} \int_D C_{i,0}(x)\, dx, \end{aligned} \tag{8}$$

for $D \in \mathcal{D}_h^{ext}$, $n \in \{0,1,\ldots,N\}$,

$$u_D^n = 0, \qquad v_D^n = 0, \tag{9}$$

and for $D \in \mathcal{D}_h^{int}$, $n \in \{1,2,\ldots,N\}$,

$$\begin{aligned} m(D)\frac{u_D^n - u_D^{n-1}}{\Delta t_n} - d_1 \left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right) \sum_{E\in\mathcal{D}_h^{int}} \mathbb{S}_{D,E}^n u_E^n \\ + \sum_{E\in\mathcal{N}(D)} K_{1,D,E}^n \overline{u_{D,E}^n} = m(D)F_D^n, \end{aligned} \tag{10}$$

$$\begin{aligned} m(D)\frac{v_D^n - v_D^{n-1}}{\Delta t_n} - d_2 \left(\sum_{D_0\in\mathcal{D}_h} m(D_0)v_{D_0}^n\right) \sum_{E\in\mathcal{D}_h^{int}} \mathbb{S}_{D,E}^n v_E^n \\ + \sum_{E\in\mathcal{N}(D)} K_{2,D,E}^n \overline{v_{D,E}^n} = m(D)G_D^n, \end{aligned} \tag{11}$$

$$m(D)\frac{C^n_{i,D}-C^{n-1}_{i,D}}{\Delta t_n}=m(D)H^n_{i,D},\quad i=1,2,3. \tag{12}$$

In the scheme given before we have:

$$K_{i,D,E}:=\frac{1}{\Delta t_n}\int_{t_{n-1}}^{t_n}\int_{\sigma_{D,E}}K_i(x,t)\cdot\vec{\eta}_{D,E}\,d\gamma(x)\,dt \qquad i=1,2$$

for $D\in\mathcal{D}_h^{int}, n\in\{1,2,\dots,N\}$, with $\vec{\eta}_{D,E}$ the unit normal vector of the side $\sigma_{D,E}\in\mathcal{F}_D$, outward to D, and, for $D\in\mathcal{D}_h$ and $n\in\{1,2,\dots,N\}$,

$$F^n_D:=\frac{1}{\Delta t_n m(D)}\int_{t_{n-1}}^{t_n}\int_D F_\varepsilon(x,t)\,dx\,dt,$$
$$G^n_D:=\frac{1}{\Delta t_n m(D)}\int_{t_{n-1}}^{t_n}\int_D G_\varepsilon(x,t)\,dx\,dt,$$
$$H^n_{i,D}:=\frac{1}{\Delta t_n m(D)}\int_{t_{n-1}}^{t_n}\int_D H_i(x,t)\,dx\,dt,\qquad i=1,2,3.$$

We define $\overline{u^n_{D,E}}$ and $\overline{v^n_{D,E}}$, for $D\in\mathcal{D}_h^{int}, E\in\mathcal{N}(D)$, and $n\in\{1,2,\dots,N\}$ as follows:

$$\overline{u^n_{D,E}}:=\begin{cases}u^n_D & \text{if} \quad K^n_{1,D,E}\geq 0\\ u^n_E & \text{if} \quad K^n_{1,D,E}<0\end{cases}$$
$$\overline{v^n_{D,E}}:=\begin{cases}v^n_D & \text{if} \quad K^n_{2,D,E}\geq 0\\ v^n_E & \text{if} \quad K^n_{2,D,E}<0\end{cases}$$

The diffusion matrix $\mathbb{S}^n_{D,E}$ writes in the following form:

$$\mathbb{S}^n_{D,E}=-\sum_{K\in\mathcal{T}_h}(\nabla\varphi_E,\nabla\varphi_D)_{0,K}\quad D,E\in\mathcal{D}_h,\quad n\in\{1,2,\dots,N\}.$$

Remark 2. Note that under condition (4), we deduce easily from Lemma 4.5 in [12] that

$$\begin{aligned}\sum_{D\in\mathcal{D}_h^{int}}u^n_{D,E}\sum_{E\in\mathcal{N}(D)}K^n_{1,D,E}\overline{u^n_{D,E}}\geq 0\\ \sum_{D\in\mathcal{D}_h^{int}}v^n_{D,E}\sum_{E\in\mathcal{N}(D)}K^n_{2,D,E}\overline{v^n_{D,E}}\geq 0.\end{aligned} \tag{13}$$

We will use frequently (13) in the prove of Theorem below.

To simplify the notation, it will always be understood that when h is sent to zero then so is Δt, thereby assuming (without loss of generality) a functional relationship between the spatial and temporal discretization parameters. This is not a real restriction on the time step, but it allows us to write "u_h" and "v_h" instead of "$u_{h,\Delta t}$" and "$v_{h,\Delta t}$" respectively , "$h\to 0$" instead of "$h,\Delta t\to 0$", and so forth.

For the sake of analysis, we introduce the following functions ("piecewise constant" and "piecewise linear and continuous" functions): for $n\in\{1,\dots,N\}$,

$$u_h(x,t)=u^n_D \text{ and } v_h(x,t)=v^n_D$$

for all $(x,t)\in D\times((n-1)\Delta t,n\Delta t]$, with $D\in\mathcal{D}_h$,

$$\tilde{u}_h(x,t)=u^n_h(x) \text{ and } \tilde{v}_h(x,t)=v^n_h(x)$$

for all $(x,t)\in\Omega\times((n-1)\Delta t,n\Delta t]$.

Note that if the below energy estimate (34) is satisfied (see [12]), we get

$$\|w_h - \tilde{w}_h\|_{0,Q_T} \to 0 \text{ as } h \to 0 \text{ for } w = u, v.$$

The definition $\tilde{w}_h$ for $w = u, v$ will be used later to obtain estimates on differences of space and time translates. Moreover, the convergence of w_h is a consequence of the convergence of $\tilde{w}_h$ for $w = u, v$.

In this paper we assume that the following mild time step condition is satisfied:

$$\Delta t < \min(\frac{1}{2r}, \frac{q}{2ep}). \tag{14}$$

This condition will be used to prove the existence of solutions to the scheme. To simplify the notation, let us write $\mathbf{u}_h$ for the vector $(u_h, u_h, C_{1,h}, C_{2,h}, C_{3,h})$.

Our second main result is the following theorem.

Theorem 1.4. *Assume $u_0, v_0, C_{i,0} \in L^2(\Omega)$ for $i = 1, 2, 3$. Then, as $h \to 0$, the finite volume solution $\mathbf{u}_h$, , generated by (8) - (12), converges along a subsequence to a limit $\mathbf{u} = (u, v, C_1, C_2, C_3)$ that is a weak solution of (1).*

The plan of this paper is as follows: In section 2 we prove existence of solutions (proof of Theorem 1.2). In Section 3 we prove that discrete solutions converge, as the discretization parameters tends to zero, to weak solutions (proof of Theorem 1.4). In section 4 we give some numerical examples to our model.

2. **Existence of weak solutions.** Our proof is based on approximate system to which we can apply the Schauder fixed-point theorem to prove the convergence to weak solutions of the approximations (see for e.g.the works in [6] and [8] where this method is used). For technical reasons, we need to extend the functions F and G so that they become defined for all $u, v, C_i \in \mathbb{R}$ for $i = 1, 2$. We do this by setting:

$$F = \begin{cases} 0, & \text{if } u < 0,\ v \geq 0, \\ k(u) - \beta_1 C_1 u, & \text{if } u \geq 0,\ v < 0, \\ 0, & \text{if } u < 0,\ v < 0. \end{cases}$$

$$G = \begin{cases} -av - \beta_2 C_2 v, & \text{if } u < 0,\ v \geq 0, \\ 0, & \text{if } u \geq 0,\ v < 0, \\ 0, & \text{if } u < 0,\ v < 0. \end{cases}$$

We introduce the following system: for $i = 1, 2, 3$

$$\begin{cases} \partial_t u - d_1 \left(\displaystyle\int_\Omega u\, dx \right) \Delta u + \operatorname{div}(uK_1) = F_\varepsilon(u, v, C_1), & \text{in} \quad Q_T, \\ \partial_t v - d_2 \left(\displaystyle\int_\Omega v\, dx \right) \Delta v + \operatorname{div}(vK_2) = G_\varepsilon(u, v, C_2), & \text{in} \quad Q_T, \\ \partial_t C_i = H_i(C_1, C_2, C_3) & \text{in} \quad Q_T. \end{cases} \tag{15}$$

for each fixed $\varepsilon > 0$. Herein

$$F_\varepsilon(u, v, C_1) = \frac{F(u, v, C_1)}{1 + \varepsilon\, |F(u, v, C_1)|}, \qquad G_\varepsilon(u, v, C_2) = \frac{G(u, v, C_2)}{1 + \varepsilon\, |G(u, v, C_2)|},$$

For $w_i \in L^2(Q_T) \cup L^\infty(0, T; L^2(\Omega)) \quad i = 1, 2$ the mapping

$$t_i \to d_i \left(\int_\Omega w_i(\cdot, t_i)\, dx \right) \quad \text{for} \quad i = 1, 2$$

is clearly measurable and thus belongs to $L^2(0,T) \cup L^\infty(0,T)$. With w_i fixed, let (u, v, C_1, C_2, C_3) be the solution of the system

$$\begin{cases} \partial_t u - d_1 \left(\int_\Omega u\,dx\right) \Delta u + \operatorname{div}(uK_1) = F_\varepsilon(u, v, C_1), & \text{in} \quad Q_T, \\ \partial_t v - d_2 \left(\int_\Omega v\,dx\right) \Delta v + \operatorname{div}(vK_2) = G_\varepsilon(u, v, C_2), & \text{in} \quad Q_T, \\ \partial_t C_1 = H_1(C_3, C_1), \\ \partial_t C_2 = H_2(C_3, C_2), \\ \partial_t C_3 = H_3(C_3). \end{cases} \tag{16}$$

Let $\Theta : (L^\infty(0,T;L^2(\Omega)) \cap L^2(Q_T))^2 \to (L^\infty(0,T;L^2(\Omega)) \cap L^2(Q_T))^2$ such that $\Theta(w_1, w_2) = (u, v)$ solution to

$$\begin{cases} \partial_t u - d_1 \left(\int_\Omega w_1\,dx\right) \Delta u + \operatorname{div}(uK_1) = F_\varepsilon(u, v, C_1), & \text{in} \quad Q_T, \\ \partial_t v - d_2 \left(\int_\Omega w_2\,dx\right) \Delta v + \operatorname{div}(vK_2) = G_\varepsilon(u, v, C_2), & \text{in} \quad Q_T. \end{cases}$$

Observe that for $i = 1, 2, 3$

$$\partial_t C_i = H_i(C_1, C_2, C_3). \tag{17}$$

is an ODE system, then we have the following classical lemma.

Lemma 2.1. *If $C_{1,0}, C_{2,0}, C_{3,0} \in L^2_+(\Omega)$, then the system (17) has a unique solution (C_1, C_2, C_3) with $C_1, C_2, C_3 \in C([0,T], L^2_+(\Omega))$.*

Then the idea is to show that the map Θ has a fixed point. First, let us show that Θ is a continuous mapping. Let $(w_{1,n})_n, (w_{2,n})_n$ be sequences in $L^\infty(0,T;L^2(\Omega)) \cap L^2(Q_T)$ and $w_1, w_2 \in L^\infty(0,T;L^2(\Omega)) \cap L^2(Q_T)$ be such that $(w_{1,n})_n \to w_1$ and $(w_{2,n})_n \to w_2$ in $L^2(Q_T)$ as $n \to \infty$. Define $(u_n, v_n) = \Theta(w_{1,n}, w_{2,n})$, i.e., (u_n, v_n) is the solution of (16) associated with $(w_{1,n}, w_{2,n})$. The objective is to show that (u_n, v_n) converges to $\Theta(w_1, w_2)$ in $(L^2(Q_T))^2$. We begin with the following lemma:

Lemma 2.2. *The solution (u_n, v_n) to problem (16) satisfies*

i) (u_n, v_n) is nonnegative.
ii) The sequence (u_n, v_n) is bounded in $(L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T,L^2(\Omega)))^2$.
iii) The sequence (u_n, v_n) is relatively compact in $(L^2(Q_T))^2$

Proof.
i) The proof is based on the choice of test functions $\varphi_1 = -u_n^-$ and $\varphi_2 = -v_n^-$, where $u_n^- = \max(0, -u_n)$ and $v_n^- = \max(0, -v_n)$. We multiply the first and second equation of (16) by φ_1 and φ_2 respectively. Then integrating over

$(0,t)\times\Omega$, we obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega \left|u_n^-(x,t)\right|^2\,dx =& -\int_0^t d_1\left(\int_\Omega w_1\,dx\right)\int_\Omega \left|\nabla u_n^-\right|^2\,dx\,dt\\ &-\int_0^t\int_\Omega u_n^- K_1\cdot\nabla u_n^-\,dx\,dt+\frac{1}{2}\int_\Omega\left|u_n^-(x,0)\right|^2\,dx\\ &-\int_0^t\int_\Omega F_\varepsilon(u_n,v_n,C_1)u_n^-\,dx\,dt\\ \le& -\int_0^t\left(d_1\left(\int_\Omega w_1\,dx\right)-\frac{M_1}{2}\right)\int_\Omega\left|\nabla u_n^-\right|^2\,dx\,dt\\ &+c\int_0^T\int_\Omega\left|u_n^-\right|^2\,dx\,dt\\ \le& \, c\int_0^T\int_\Omega\left|u_n^-\right|^2\,dx\,dt,\end{aligned}$$

for some constant $c>0$. Herein we have used the nonnegativity of u_0, (5) and Young's inequality. In view of Gronwall's inequality, it follows from this that $u_n^-=0$ a.e. in Ω. Reasoning along the same lines as u_n, we get $v_n^-=0$ a.e. in Ω,

ii) Taking the test function $\varphi_1=u_n$. We multiply the first equation of (16) by φ_1 and integrate over Ω to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega|u_n|^2\,dx &+ d_1\left(\int_\Omega w_1\,dx\right)\int_\Omega|\nabla u_n|^2\,dx\\ &=\int_\Omega u_nK_1\cdot\nabla u_n\,dx+\int_\Omega F_\varepsilon(u_n,v_n,C_1)u_n\,dx\\ &\le\frac{M_1}{2}\int_\Omega|\nabla u_n|^2\,dx+c\int_\Omega|u_n|^2\,dx+r\int_\Omega|u_n|^2\,dx,\end{aligned}$$

for some constant $c>0$. This implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega|u_n|^2\,dx+\left(d_1\left(\int_\Omega w_1\,dx\right)-\frac{M_1}{2}\right)\int_\Omega|\nabla u_n|^2\,dx&\\ \le(c+r)\int_\Omega|u_n|^2\,dx&\end{aligned}\tag{18}$$

Using the nonnegativity of the second term of the left-hand side (18) and the Gronwall's inequality, we obtain

$$\int_\Omega|u_n(x,t)|^2\,dx\le c_1\quad\text{for all}\quad t\in(0,T],$$

for some constant $c_1>0$. Integrating (18) over $(0,T)$ and bearing in mind the previous thing, we get:

$$\int_\Omega|u_n(x,T)|^2\,dx+\frac{M_1}{2}\int_0^T\int_\Omega|\nabla u_n|^2\,dx\,dt\le c_2,$$

for some constant $c_2>0$, which proves

$$\|u_n\|_{L^\infty(0,T;L^2(\Omega))}+\|u_n\|_{L^2(0,T;H_0^1(\Omega))}\le c_3,\tag{19}$$

where $c_3>0$ is a constant independent of n.

Reasoning along the same lines as u_n, we multiply the second equation of (16) by a test function $\varphi_2 = v_n$, thus we get for v_n,

$$\|v_n\|_{L^\infty(0,T;L^2(\Omega))} + \|v_n\|_{L^2(0,T;H_0^1(\Omega))} \le c_4, \tag{20}$$

where $c_4 > 0$ is a constant independent of n.

iii) Finally, taking a test function $\varphi_1 \in L^2(0,T;H_0^1(\Omega))$ and we use the uniform boundedness of (u_n, v_n) in $L^2(0,T;H_0^1(\Omega))$ to have

$$\begin{aligned}
&\left|\int_0^T \int_\Omega \partial_t u_n \varphi_1 \,dx\,dt\right| \\
&\quad \le \left|\int_0^t \int_\Omega F_\varepsilon(u,v,C_1)\varphi_1 \,dx\,dt\right| + \left|\int_0^T d_1\left(\int_\Omega w_1\,dx\right)\int_\Omega \nabla u_n \cdot \nabla\varphi_1\,dx\,dt\right| \\
&\qquad\qquad + \left|\int_0^T \int_\Omega u_n K_1 \cdot \nabla\varphi_1\,dx\,dt\right| \\
&\quad \le \frac{1}{\varepsilon}\left|\int_0^T \int_\Omega \varphi_1\,dx\,dt\right| + \sup_{t\in[0,T]}\left|d_1\left(\int_\Omega w_1\,dx\right)\right|\left|\int_0^T\int_\Omega \nabla u_n\cdot\nabla\varphi_1\,dx\,dt\right| \\
&\qquad\qquad + \|K_1\|_{L^\infty(\Omega,\mathbb{R}^3)}\|u_n\|_{L^2(Q_T)}\|\nabla\varphi_1\|_{L^2(Q_T)} \\
&\quad \le \frac{1}{\varepsilon}T|\Omega|\,\|\varphi_1\|_{L^2(Q_T)} + c_5\|\nabla u_n\|_{L^2(Q_T)}\|\nabla\varphi_1\|_{L^2(Q_T)} \\
&\qquad\qquad + c_6\|u_n\|_{L^2(Q_T)}\|\nabla\varphi_1\|_{L^2(Q_T)} \\
&\quad \le c_7\|\varphi_1\|_{L^2(0,T;H_0^1(\Omega))},
\end{aligned}$$

for some constants $c_5, c_6, c_7 > 0$, where we have used Hölder's inequality. This implies

$$\|\partial_t u_n\|_{L^2(0,T;H^{-1}(\Omega))} \le c_8,$$

where $c_8 > 0$ is a constant independent of n.
Reasoning along the same lines for v_n we get

$$\|\partial_t v_n\|_{L^2(0,T;H^{-1}(\Omega))} \le c_9.$$

where $c_9 > 0$ is a constant independent of n.
Then, $iii)$ is a consequence of $ii)$ and the uniform boundedness of $(\partial_t u_n)_n$ and $(\partial_t v_n)_n$ in $L^2(0,T;H^{-1}(\Omega))$.

□

From Lemma 2.1 and 2.2, there exist functions $u, v \in L^2(0,T;H_0^1(\Omega))$ such that, up to extracting subsequences if necessary,

$$u_n \to u \quad \text{strongly in} \quad L^2(Q_T), \qquad v_n \to v \quad \text{strongly in} \quad L^2(Q_T)$$

and from this the continuity of Θ on $(L^\infty(0,T;L^2(\Omega)) \cap L^2(Q_T))^2$ follows.
We observe that, from Lemma 2.2, Θ is bounded in the set $\mathcal{W}_u \times \mathcal{W}_v$

$$\mathcal{W}_u = \{u \in L^2(0,T;H_0^1(\Omega)) : \partial_t u \in L^2(0,T;H^{-1}(\Omega))\},$$

$$\mathcal{W}_v = \{v \in L^2(0,T;H_0^1(\Omega)) : \partial_t v \in L^2(0,T;H^{-1}(\Omega))\}.$$

By the results of [19], $\mathcal{W}_\kappa \hookrightarrow L^2(Q_T)$ is compact with $\kappa = u, v$; therefore, Θ is compact. Moreover it is easy to obtain the uniqueness of the solutions to (16). Then, by the Schauder fixed point theorem, the operator Θ has a fixed point $(u_\varepsilon, v_\varepsilon)$ such that $\Theta(u_\varepsilon, v_\varepsilon) = (u_\varepsilon, v_\varepsilon)$. This implies that there exists a solution $(u_\varepsilon, v_\varepsilon, C_1, C_2, C_3)$ of

$$\begin{aligned}
&\int_0^T \langle \partial_t u_\varepsilon \varphi_1 \rangle \, dt + \int_0^T d_1 \left(\int_\Omega u_\varepsilon \, dx \right) \int_\Omega \nabla u_\varepsilon \cdot \nabla \varphi_1 \, dx \, dt \\
&\qquad - \int_0^T \int_\Omega u_\varepsilon K_1 \cdot \nabla \varphi_1 \, dx \, dt = \int_0^T \int_\Omega F_\varepsilon(u_\varepsilon, v_\varepsilon, C_1) \varphi_1 \, dx \, dt, \\
&\int_0^T \langle \partial_t v_\varepsilon \varphi_2 \rangle \, dt + \int_0^T d_2 \left(\int_\Omega v_\varepsilon \, dx \right) \int_\Omega \nabla v_\varepsilon \cdot \nabla \varphi_2 \, dx \, dt \\
&\qquad - \int_0^T \int_\Omega v_\varepsilon K_2 \cdot \nabla \varphi_2 \, dx \, dt = \int_0^T \int_\Omega G_\varepsilon(u_\varepsilon, v_\varepsilon, C_2) \varphi_2 \, dx \, dt, \\
&- \int_\Omega C_{i,0}(x) \psi_i(x, 0) \, dx - \int_0^T \int_\Omega C_i \partial_t \psi_i \, dx \, dt = \int_0^T \int_\Omega H_i(C_1, C_2, C_3) \psi_i \, dx \, dt,
\end{aligned}$$

for $i = 1, 2, 3$, for all $\varphi_1, \varphi_2 \in L^2(0, T; H_0^1(\Omega))$ and $\psi_1, \psi_2, \psi_3 \in C_c^1(\Omega \times [0, T))$.

We have shown that the problem (15) admits a solution $(u_\varepsilon, v_\varepsilon, C_1, C_2, C_3)$. The goal now is to send the regularization parameter ε to zero in sequences of such solutions to obtain weak solutions of the original system (1)-(3). Note that, for each fixed $\varepsilon > 0$, we have shown the existence of a solution $(u_\varepsilon, v_\varepsilon, C_1, C_2, C_3)$ to (15) such that

$$u_\varepsilon \geq 0 \quad \text{and} \quad v_\varepsilon \geq 0 \qquad \text{for a.e.}(x, t) \in Q_T.$$

Lemma 2.3. *There exist constants $c_{10}, c_{11}, c_{12} > 0$ not depending on ε such that the solution $(u_\varepsilon, v_\varepsilon)$ satisfies*

$$\|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|u_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \leq c_{10}, \tag{21}$$

$$\|v_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|v_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \leq c_{11}. \tag{22}$$

$$\|F_\varepsilon(u_\varepsilon, v_\varepsilon, C_1)\|_{L^1(Q_T)} + \|G_\varepsilon(u_\varepsilon, v_\varepsilon, C_2)\|_{L^1(Q_T)} \leq c_{12}. \tag{23}$$

Proof. By the (weak) lower semicontinuity properties of norms, the estimates (19), and (20) hold with u_n and v_n replaced by u_ε and v_ε respectively. Moreover, the constants c_3, c_4 are independent of ε (consult the proof of Lemma 2.2). Besides, from (21) and (22) we get

$$\begin{aligned}
&\iint_{Q_T} |F_\varepsilon(u_\varepsilon, v_\varepsilon, C_1)| \, dx \, dt + \iint_{Q_T} |F_\varepsilon(u_\varepsilon, v_\varepsilon, C_1)| \, dx \, dt \\
&\qquad \leq c_{13} \iint_{Q_T} \left(|u_\varepsilon|^2 + |v_\varepsilon|^2 + |C_1|^2 + |C_2|^2 \right) dx \, dt \leq c_{14},
\end{aligned}$$

where $c_{13} > 0$ and c_{14} are constants independent of ε. □

From Lemma 2.3 we have that $u_\varepsilon, v_\varepsilon$ are bounded in $L^2(0, T; H_0^1(\Omega))$ and $\partial_t u_\varepsilon, \partial_t v_\varepsilon$ are bounded in $L^2(0, T; H^{-1}(\Omega)) + L^1(Q_T)$. Note that $L^1(\Omega) \subset H^{-s}(\Omega)$ for $s > 0$ large. Then, $\partial_t u_\varepsilon, \partial_t v_\varepsilon$ are bounded in $L^1(0, T; H^{-s}(\Omega))$ for $s > 0$ large. As $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-s}(\Omega)$, therefore, possible at the cost of extracting subsequences denoted $u_\varepsilon, v_\varepsilon$, see e.g. [19], we can assume that there exist u, v in $L^2(0, T; H_0^1(\Omega))$ such that as ε goes to 0

$$\begin{cases} u_\varepsilon \to u, v_\varepsilon \to v \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \\ u_\varepsilon \rightharpoonup u, v_\varepsilon \rightharpoonup v \text{ weakly in } L^2(0, T; H_0^1(\Omega)). \end{cases} \tag{24}$$

The consequence of (24) and Vitali's theorem:

$F_\varepsilon(u_\varepsilon, v_\varepsilon, C_1) \to F(u, v, C_1)$ and $G_\varepsilon(u_\varepsilon, v_\varepsilon, C_2) \to G(u, v, C_2)$ strongly in $L^1(Q_T)$.

Finally, using the following weak formulation

$$
\begin{aligned}
-\int_\Omega u_{0,\varepsilon}(x)\varphi_1(x,0)\,dx - \iint_{Q_T} & u_\varepsilon \partial_t \varphi_1 \,dx\,dt \\
& + \int_0^T d_1\left(\int_\Omega u_\varepsilon\,dx\right)\int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi_1\,dx\,dt \\
& - \iint_{Q_T} u_\varepsilon K_1 \cdot \nabla\varphi_1\,dx\,dt = \iint_{Q_T} F_\varepsilon(u_\varepsilon, v_\varepsilon, C_1)\varphi_1\,dx\,dt, \\
-\int_\Omega v_{0,\varepsilon}(x)\varphi_2(x,0)\,dx - \iint_{Q_T} & v_\varepsilon \partial_t \varphi_2 \,dx\,dt \\
& + \int_0^T d_2\left(\int_\Omega v_\varepsilon\,dx\right)\int_\Omega \nabla v_\varepsilon \cdot \nabla\varphi_2\,dx\,dt \\
& - \iint_{Q_T} v_\varepsilon K_2 \cdot \nabla\varphi_2\,dx\,dt = \iint_{Q_T} G_\varepsilon(u_\varepsilon, v_\varepsilon, C_2)\varphi_2\,dx\,dt,
\end{aligned}
$$

for all $\varphi_1, \varphi_2, \phi \in C_c^1(\Omega \times [0,T))$, we can let $\varepsilon \to 0$ and obtain a weak solution.

3. Finite volume scheme.

3.1. Existence of solutions to the combined finite volume - finite element scheme.

The existence of a solution to the scheme (8)-(12) is given in the following proposition.

Proposition 1. *Let $\mathcal{D}_h$ be a discretization of Q_T. Then the problem (8)-(12) admits at least one solution $(u_D^n, v_D^n, C_{1,D}^n, C_{2,D}^n, C_{3,D}^n)_{(D,n)\in\mathcal{D}_h\times\{1,\dots,N\}}$.*

Proof. First, we introduce the Hilbert space

$$E_h := X_h^0(\Omega) \times X_h^0(\Omega),$$

under the norm

$$\|U_h\|_{X_h^0}^2 := \sum_{K\in\mathcal{T}_h} \int_K |\nabla U_h|^2\,dx,$$

where $U_h = (u_h, v_h)$. Let $\Phi_h = (\varphi_u, \varphi_v) \in E_h$ and define the discrete bilinear forms

$$T_h(U_h^n, \Phi_h) = \sum_{D\in\mathcal{D}_h} m(D)\big(u_D^n\varphi_u + v_D^n\varphi_v\big),$$

$$c_h(U_h^n, \Phi_h) = \sum_{D\in\mathcal{D}_h} m(D)\big(F_D^n\varphi_u + G_D^n\varphi_v\big),$$

$$
\begin{aligned}
a_h(U_h^n, \Phi_h) = \sum_{D\in\mathcal{D}_h} \Bigg(& d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right) \sum_{E\in\mathcal{D}_h^{int}} \mathbb{S}_{D,E}^n u_E^n \varphi_u \\
& + d_2\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)v_{D_0}^n\right) \sum_{E\in\mathcal{D}_h^{int}} \mathbb{S}_{D,E}^n v_E^n \varphi_v \Bigg),
\end{aligned}
$$

and

$$b_h(U_h^n,\Phi_h)=\sum_{D\in\mathcal{D}_h}\left(\sum_{E\in\mathcal{N}(D)}K_{1,D,E}^n\overline{u_{D,E}^n}\varphi_u+\sum_{E\in\mathcal{N}(D)}K_{2,D,E}^n\overline{v_{D,E}^n}\varphi_v\right)$$

Multiplying (10) and (11) by φ_u and φ_v, respectively, we get the equation

$$\frac{1}{\Delta t_n}\Big(T_h(U_h^n,\Phi_h)-T_h(U_h^{n-1},\Phi_h)\Big)-a_h(U_h^n,\Phi_h)+b_h(U_h^n,\Phi_h) - c_h(U_h^n,\Phi_h)=0.$$

We define the mapping $\mathcal{P}$ from E_h into itself

$$[\mathcal{P}(U_h^n),\Phi_h]=\frac{1}{\Delta t_n}(T_h(U_h^n,\Phi_h)-T_h(U_h^{n-1},\Phi_h))-a_h(U_h^n,\Phi_h) + b_h(U_h^n,\Phi_h)-c_h(U_h^n,\Phi_h),$$

for all $\Phi_h\in E_h$.

Note that the continuity of the mapping $\mathcal{P}$ follows from the continuity of the discrete forms $a_h(\cdot,\cdot)$, $b_h(\cdot,\cdot)$, $c_h(\cdot,\cdot)$ and $T_h(\cdot,\cdot)$. Our goal now is to show that

$$[\mathcal{P}(U_h^n),U_h^n]>0\quad\text{for }\|U_h^n\|_{E_h}=r>0, \tag{25}$$

for a sufficiently large r. This implies that (see e.g. [17] and [21])

$$\mathcal{P}(U_h^n)=0.$$

We observe that

$$\begin{aligned}
[\mathcal{P}(U_h^n),U_h^n]\geq{}&\frac{1}{\Delta t_n}\sum_{D\in\mathcal{D}_h}m(D)\left|u_D^n\right|^2+\frac{1}{\Delta t_n}\sum_{D\in\mathcal{D}_h}m(D)\left|v_D^n\right|^2\\
&+M_1\left\|u_h^n\right\|_{X_h^0}^2+M_2\left\|v_h^n\right\|_{X_h^0}^2\\
&+\frac{1}{2}\sum_{D\in\mathcal{D}_h^{int}}\left|u_D^n\right|^2\sum_{E\in\mathcal{N}(D)}K_{1,D,E}^n\\
&+\frac{1}{2}\sum_{D\in\mathcal{D}_h^{int}}\left|v_D^n\right|^2\sum_{E\in\mathcal{N}(D)}K_{2,D,E}^n\\
&-\sum_{D\in\mathcal{D}_h}m(D)F_D^nu_D^n-\sum_{D\in\mathcal{D}_h}m(D)G_D^nv_D^n\\
&-\frac{1}{2\Delta t_n}\sum_{D\in\mathcal{D}_h}m(D)\left|u_D^n\right|^2-C(\Delta t_n)\sum_{D\in\mathcal{D}_h}m(D)\left|u_D^{n-1}\right|^2\\
&-\frac{1}{2\Delta t_n}\sum_{D\in\mathcal{D}_h}m(D)\left|v_D^n\right|^2-C'(\Delta t_n)\sum_{D\in\mathcal{D}_h}m(D)\left|v_D^{n-1}\right|^2\\
\geq{}&\Big(\frac{1}{2\Delta t_n}-r\Big)\sum_{D\in\mathcal{D}_h}m(D)\left|u_D^n\right|^2\\
&+\Big(\frac{1}{2\Delta t_n}-ep/q\Big)\sum_{D\in\mathcal{D}_h}m(D)\left|v_D^n\right|^2\\
&+M_1\left\|u_h^n\right\|_{X_h^0}^2+M_2\left\|v_h^n\right\|_{X_h^0}^2-C(\Delta t_n)\sum_{D\in\mathcal{D}_h}m(D)\left|u_D^{n-1}\right|^2\\
&-C'(\Delta t_n)\sum_{D\in\mathcal{D}_h}m(D)\left|v_D^{n-1}\right|^2
\end{aligned}$$

Then, we have

$$\begin{aligned}[\mathcal{P}(U_h^n), U_h^n] \geq M_1 \|u_h^n\|_{X_h^0}^2 + M_2 \|v_h^n\|_{X_h^0}^2 - C(\Delta t_n) \sum_{D\in\mathcal{D}_h} m(D) \left|u_D^{n-1}\right|^2 \\ - C'(\Delta t_n) \sum_{D\in\mathcal{D}_h} m(D) \left|v_D^{n-1}\right|^2 .\end{aligned} \tag{26}$$

Herein, we have used (5), (4), (13), (14) and Young's inequality. Finally, for a given u_h^{n-1}, v_h^{n-1} we deduce from (26) that (25) holds for r large enough (recall that $\|U_h^n\|_{E_h} = r$). Then, we obtain the existence of at least one solution to the scheme (8)-(12). ☐

Remark 3. Note that we only proved the existence for u and v because the equations for $C_i, i = 1,2,3$ are an ODE system then the existence and convergence of the solution generated by the scheme are trivial.

3.2. **Nonnegativity.** We have the following lemma.

Lemma 3.1. *Let $(u_D^n, v_D^n, C_{1,D}^n, C_{2,D}^n, C_{3,D}^n)$ for $D \in \mathcal{D}_h$, and $n \in \{0,\ldots,N\}$ be a solution of the combined finite volume - finite element scheme (8) - (12). Then, $(u_D^n, v_D^n, C_{1,D}^n, C_{2,D}^n, C_{3,D}^n)$ for $D \in \mathcal{D}_h$ and $n \in \{0,\ldots,N\}$ is nonnegative.*

Proof. Note that since (12) is a discrete of an ODE equation, it is easy to obtain the nonnegativity of $C_{i,D}^n$ for $D \in \mathcal{D}_h$ and $n \in \{0,\ldots,N\}$.
Multiplying (10) by $-\Delta t_n u_D^{n\,-}$, we find that

$$\begin{aligned}-m(D)u_D^{n\,-}(u_D^n - u_D^{n-1}) + \Delta t_n d_1 \left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right) \sum_{E\in\mathcal{D}_h^{int}} \mathbb{S}_{D,E}^n u_E^n u_D^{n\,-} \\ - \Delta t_n \sum_{E\in\mathcal{N}(D)} K_{1,D,E}^n \overline{u_{D,E}^n} u_D^{n\,-} \\ + m(D)\Delta t_n F_D^n u_D^{n\,-} = 0.\end{aligned} \tag{27}$$

Observe that for $1 < N_0 \leq N$

$$\begin{aligned}\sum_{n=1}^{N_0} \Delta t_n d_1 \left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right) \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{D}_h^{int}} \mathbb{S}_{D,E}^n u_E^n u_D^{n\,-} \\ = \sum_{n=1}^{N_0} \Delta t_n d_1 \left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right) \sum_{D\in\mathcal{D}_h} u_D^{n\,-} \sum_{E\in\mathcal{D}_h^{int}} \mathbb{S}_{D,E}^n u_E^n \\ = -\sum_{n=1}^{N_0} \Delta t_n d_1 \left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right) \sum_{K\in\mathcal{T}_h} (\nabla u_h^{n\,-}, \nabla u_h^n)_{0,K} \\ = \sum_{n=1}^{N_0} \Delta t_n d_1 \left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right) \sum_{K\in\mathcal{T}_h} \int_K \left|\nabla u_h^{n\,-}\right|^2 dx \\ \geq M_1 \sum_{n=1}^{N_0} \Delta t_n \|u_h^n\|_{X_h^0}^2 \geq 0,\end{aligned} \tag{28}$$

and

$$\begin{aligned}\sum_{n=1}^{N_0}\Delta t_n \sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)} K_{1,D,E}^n\overline{u_{D,E}^n}u_D^{n\,-}\\ &= \sum_{n=1}^{N_0}\Delta t_n \sum_{D\in\mathcal{D}_h} u_D^{n\,-}\sum_{E\in\mathcal{N}(D)} K_{1,D,E}^n\overline{u_{D,E}^n}\\ &= \sum_{n=1}^{N_0}\Delta t_n \sum_{\substack{\sigma_{D,E}\in\mathcal{T}_h\\ K_{1,D,E}^n\ge 0}} K_{1,D,E}^n\left(u_D^n(u_D^{n\,-}-u_E^{n\,-})\right).\end{aligned} \tag{29}$$

Let us introduce a function H,

$$H(s)=h(s)s-\int_0^s h(\tau)d\tau, \qquad s\in\mathbb{R}.$$

Taking $h(s)=-s^-$ a nondecreasing function, we get

$$\left(H(u_D^n)-H(u_E^n)\right)\le\left(h(u_D^n)-h(u_E^n)\right)u_D^n$$

which implies

$$u_D^n\left(u_D^{n\,-}-u_E^{n\,-}\right)\le-\left(\frac{\left|u_D^{n\,-}\right|^2}{2}-\frac{\left|u_E^{n\,-}\right|^2}{2}\right)$$

Using the previous inequality in (29), we obtain

$$\begin{aligned}\sum_{n=1}^{N_0}\Delta t_n \sum_{D\in\mathcal{D}_h}\sum_{E\in\mathcal{N}(D)} K_{1,D,E}^n\overline{u_{D,E}^n}u_D^{n\,-}\\ &\le-\sum_{n=1}^{N_0}\Delta t_n \sum_{\substack{\sigma_{D,E}\in\mathcal{F}_h^{int}\\ K_{1,D,E}^n\ge 0}} K_{1,D,E}^n\left(\frac{\left|u_D^{n\,-}\right|^2}{2}-\frac{\left|u_E^{n\,-}\right|^2}{2}\right)\\ &\le-\sum_{k=1}^{N_0}\Delta t_n\sum_{D\in\mathcal{D}_h^{int}}\frac{\left|u_D^{n\,-}\right|^2}{2}\sum_{E\in\mathcal{N}(D)}K_{1,D,E}^n\le 0.\end{aligned}$$

Besides, we have

$$\sum_{n=1}^{N_0}\Delta t_n\sum_{D\in\mathcal{D}_h} m(D)F_D^n u_D^{n\,-}=0.$$

Let $f\in C^2$ function. By using a Taylor expansion we find

$$f(b)=f(a)+f'(a)(b-a)+\frac{1}{2}f''(\xi)(b-a)^2, \tag{30}$$

for some ξ between a and b. Using the Taylor expansion (30) on the sequence $f(u_D^n)$ with $f(\rho)=\int_0^{\rho^-}s\,ds$, $a=u_D^n$ and $b=u_D^{n-1}$. We find

$$u_D^{n\,-}(u_D^n-u_D^{n-1})=\frac{\left|u_D^{n-1\,-}\right|^2}{2}-\frac{\left|u_D^{n\,-}\right|^2}{2}-\frac{1}{2}f''(\xi)\left(u_D^n-u_D^{n-1}\right)^2.$$

We observe from the definition of f that $f''(\rho) = 1 > 0$, which implies

$$u_D^{n\,-}(u_D^n - u_D^{n-1}) \leq \frac{\left|u_D^{n-1\,-}\right|^2}{2} - \frac{\left|u_D^{n\,-}\right|^2}{2}. \tag{31}$$

Now, using (28)-(31) to deduce from (27)

$$\sum_{n=1}^{N_0}\Big(\frac{\left|u_D^{n\,-}\right|^2}{2} - \frac{\left|u_D^{n-1\,-}\right|^2}{2}\Big) + M_1 \left\|u_h^n\right\|^2_{X_h^0} + \sum_{n=1}^{N_0} \Delta t_n \sum_{D\in\mathcal{D}_h^{int}} \frac{\left|u_D^{n\,-}\right|^2}{2} \sum_{E\in\mathcal{N}(D)} K_{1,D,E}^n \leq 0.$$

This implies that

$$\frac{1}{2}\Big(\left|u_D^{N_0\,-}\right|^2 - \left|u_D^{0\,-}\right|^2\Big) \leq 0.$$

Since u_D^0 is nonnegative, the result is $u_D^{n\,-} = 0$ for all $1 \leq n \leq N$ and all $D \in \mathcal{D}_h$. Along the same lines as u_D^n, we obtain the nonnegativity of the discrete solution v_D^n for all $1 \leq n \leq N$ and all $D \in \mathcal{D}_h$. □

3.3. **A priori estimates.** The goal now is to establish several a priori (discrete energy) estimates for the combined finite volume - finite element scheme, which eventually will imply the desired convergence results.

Proposition 2. *Let $(u_D^n, v_D^n, C_{1,D}^n, C_{2,D}^n, C_{3,D}^n)_{D\in\mathcal{D}_h, n\in\{0,\ldots,N\}}$ be a solution of the scheme (8)-(12). Then there exist constants $C_1, C_2, C_3 > 0$, depending on Ω, T, u_0, v_0, such that*

$$\max_{n\in\{1,\ldots,N\}} \sum_{D\in\mathcal{D}_h} m(D)\left(\left|C_{1,D}^n\right|^2 + \left|C_{2,D}^n\right|^2 + \left|C_{3,D}^n\right|^2\right) \leq C_1, \tag{32}$$

$$\max_{n\in\{1,\ldots,N\}} \sum_{D\in\mathcal{D}_h} m(D)\left(\left|u_D^n\right|^2 + \left|v_D^n\right|^2\right) \leq C_2, \tag{33}$$

$$\sum_{n=1}^{N} \Delta t_n \left(\|u_h^n\|^2_{X_h^0} + \|v_h^n\|^2_{X_h^0}\right) \leq C_3. \tag{34}$$

Proof. First since (12) is a discrete of an ODE equation, it is easy to get (32). Second, we multiply (10) and (11) by $\Delta t_n u_D^n$, $\Delta t_n v_D^n$, respectively, and add together the outcomes. Summing the resulting equation over D and n yields

$$E_1 + E_2 + E_3 = E_4,$$

where

$$E_1 = \sum_{n=1}^{N_0} \sum_{D \in \mathcal{D}_h^{int}} m(D)(u_D^n - u_D^{n-1})u_D^n + \sum_{n=1}^{N_0} \sum_{D \in \mathcal{D}_h^{int}} m(D)(v_D^n - v_D^{n-1})v_D^n$$

$$E_2 = -\sum_{n=1}^{N_0} \Delta t_n d_1 \left(\sum_{D_0 \in \mathcal{D}_h} m(D_0) u_{D_0}^n \right) \sum_{D \in \mathcal{D}_h^{int}} u_D^n \sum_{E \in \mathcal{D}_h^{int}} \mathbb{S}_{D,E}^n u_E^n$$
$$- \sum_{n=1}^{N_0} \Delta t_n d_2 \left(\sum_{D_0 \in \mathcal{D}_h} m(D_0) v_{D_0}^n \right) \sum_{D \in \mathcal{D}_h^{int}} v_D^n \sum_{E \in \mathcal{D}_h^{int}} \mathbb{S}_{D,E}^n v_E^n$$

$$E_3 = \sum_{n=1}^{N_0} \Delta t_n \sum_{D \in \mathcal{D}_h^{int}} u_D^n \sum_{E \in \mathcal{N}(D)} K_{1,D,E}^n \overline{u_{D,E}^n}$$
$$+ \sum_{n=1}^{N_0} \Delta t_n \sum_{D \in \mathcal{D}_h^{int}} v_D^n \sum_{E \in \mathcal{N}(D)} K_{2,D,E}^n \overline{v_{D,E}^n}$$

$$E_4 = \sum_{n=1}^{N_0} \Delta t_n \sum_{D \in \mathcal{D}_h^{int}} m(D) F_D^n u_D^n + \sum_{n=1}^{N_0} \Delta t_n \sum_{D \in \mathcal{D}_h^{int}} m(D) G_D^n v_D^n,$$

where $1 < N_0 \leq N$. From the inequality "$a(a-b) \geq \frac{1}{2}(a^2 - b^2)$", we obtain

$$E_1 \geq \frac{1}{2} \sum_{n=1}^{N_0} \sum_{D \in \mathcal{D}_h^{int}} m(D) \left(|u_D^n|^2 - \left|u_D^{n-1}\right|^2 \right)$$
$$+ \frac{1}{2} \sum_{n=1}^{N_0} \sum_{D \in \mathcal{D}_h^{int}} m(D) \left(|v_D^n|^2 - \left|v_D^{n-1}\right|^2 \right)$$
$$= \frac{1}{2} \sum_{D \in \mathcal{D}_h^{int}} m(D) \left(\left|u_D^{N_0}\right|^2 - \left|u_D^0\right|^2 \right)$$
$$+ \frac{1}{2} \sum_{D \in \mathcal{D}_h^{int}} m(D) \left(\left|v_D^{N_0}\right|^2 - \left|v_D^0\right|^2 \right).$$

On the other hand, we have the following

$$E_2 \geq M_1 \sum_{n=1}^{N_0} \Delta t_n \|u_h^n\|_{X_h^0}^2 + M_2 \sum_{n=1}^{N_0} \Delta t_n \|v_h^n\|_{X_h^0}^2,$$

from (13)

$$E_3 \geq 0.$$

and from the definition of F and G

$$E_4 \leq r \sum_{n=1}^{N_0} \Delta t_n \sum_{D \in \mathcal{D}_h^{int}} m(D) |u_D^n|^2 + ep/q \sum_{n=1}^{N_0} \Delta t_n \sum_{D \in \mathcal{D}_h^{int}} m(D) |v_D^n|^2.$$

Collecting the previous inequalities we obtain

$$\begin{aligned}
&\frac{1}{2}\sum_{D\in\mathcal{D}_h^{int}} m(D)\left(\left|u_D^{N_0}\right|^2-\left|u_D^0\right|^2\right)+\frac{1}{2}\sum_{D\in\mathcal{D}_h^{int}} m(D)\left(\left|v_D^{N_0}\right|^2-\left|v_D^0\right|^2\right)\\
&\quad+M_1\sum_{n=1}^{N_0}\Delta t_n\left\|u_h^n\right\|_{X_h^0}^2+M_2\sum_{n=1}^{N_0}\Delta t_n\left\|v_h^n\right\|_{X_h^0}^2 \qquad (35)\\
&\le r\sum_{n=1}^{N_0}\Delta t_n\sum_{D\in\mathcal{D}_h^{int}} m(D)\left|u_D^n\right|^2+ep/q\sum_{n=1}^{N_0}\Delta t_n\sum_{D\in\mathcal{D}_h^{int}} m(D)\left|v_D^n\right|^2.
\end{aligned}$$

This implies

$$\begin{aligned}
&\sum_{D\in\mathcal{D}_h^{int}} m(D)\left|u_D^{N_0}\right|^2+\sum_{D\in\mathcal{D}_h^{int}} m(D)\left|v_D^{N_0}\right|^2\\
&\qquad\le\|u_0\|_{L^2(\Omega)}^2+\|v_0\|_{L^2(\Omega)}^2+2r\sum_{n=1}^{N_0}\Delta t_n\sum_{D\in\mathcal{D}_h^{int}} m(D)\left|u_D^n\right|^2 \qquad (36)\\
&\qquad\quad+2ep/q\sum_{n=1}^{N_0}\Delta t_n\sum_{D\in\mathcal{D}_h^{int}} m(D)\left|v_D^n\right|^2.
\end{aligned}$$

In view of (36), this implies that there exist constants $C_4, C_5, C_6 > 0$ such that

$$\begin{aligned}
&\sum_{D\in\mathcal{D}_h^{int}} m(D)\left|u_D^{N_0}\right|^2+\sum_{D\in\mathcal{D}_h^{int}} m(D)\left|v_D^{N_0}\right|^2\\
&\qquad\le C_4+C_5\sum_{n=1}^{N_0}\Delta t_n\sum_{D\in\mathcal{D}_h^{int}} m(D)\left|u_D^n\right|^2 \qquad (37)\\
&\qquad\quad+C_6\sum_{n=1}^{N_0}\Delta t_n\sum_{D\in\mathcal{D}_h^{int}} m(D)\left|v_D^n\right|^2.
\end{aligned}$$

An application of Gronwall's inequality, we deduce from (37)

$$\sum_{D\in\mathcal{D}_h^{int}} m(D)\left|u_D^{N_0}\right|^2+\sum_{D\in\mathcal{D}_h^{int}} m(D)\left|v_D^{N_0}\right|^2\le C_7, \qquad (38)$$

for any $N_0\in\{1,\dots,N\}$ and some constant $C_7>0$. Then

$$\max_{n\in\{1,\dots,N\}}\sum_{D\in\mathcal{D}_h^{int}} m(D)\left|u_D^n\right|^2+\max_{n\in\{1,\dots,N\}}\sum_{D\in\mathcal{D}_h^{int}} m(D)\left|v_D^n\right|^2\le C_7.$$

Moreover, we obtain from (35) and (38) the existence of a constant $C_8>0$ such that

$$\sum_{n=1}^{N_0}\Delta t_n\left\|u_h^n\right\|_{X_h^0}^2+\sum_{n=1}^{N_0}\Delta t_n\left\|v_h^n\right\|_{X_h^0}^2\le C_8.$$

□

3.4. **Convergence of the combined finite volume - finite element scheme.** In this section we derive estimates on differences of space and time translates of the functions $\tilde{u}_h, \tilde{v}_h$ which imply that the sequences $\tilde{u}_h, \tilde{v}_h$ are relatively compact in $L^2(Q_T)$.

Lemma 3.2. *There exists a positive constant $C > 0$ depending on Ω, T, u_0, v_0 such that*

$$\iint_{\Omega' \times (0,T)} |\tilde{w}_h(x+y,t) - \tilde{w}_h(x,t)|^2 \, dx\, dt \le C\,|y|\,(|y| + 2h), \quad \tilde{w}_h = \tilde{u}_h, \tilde{v}_h \tag{39}$$

for all $y \in \mathbb{R}^3$ with $\Omega' = \{x \in \Omega,\ [x, x+y] \subset \Omega\}$, and

$$\iint_{\Omega \times (0,T-\tau)} |\tilde{w}_h(x,t+\tau) - \tilde{w}_h(x,t)|^2 \, dx\, dt \le C(\tau + \Delta t), \quad \tilde{w}_h = \tilde{u}_h, \tilde{v}_h \tag{40}$$

for all $\tau \in (0,T)$.

Proof. To prove (39), we define a function $\chi_\sigma(x)$ for each $\sigma \in \mathcal{F}_h^{int}$ by

$$\chi_\sigma(x) = \begin{cases} 1, & \text{if } \sigma \cap [x, x+y] \neq \emptyset, \\ 0, & \text{if } \sigma \cap [x, x+y] = \emptyset. \end{cases}$$

A simple geometrical consideration leads to

$$|\tilde{w}_h(x+y,t) - \tilde{w}_h(x,t)| \le \sum_{\sigma_{D,E} \in \mathcal{F}_h^{int}} |w_E^n - w_D^n|\, \chi_{\sigma_{D,E}}(x)$$

for a.e. $x \in \Omega$ and for $t \in (t_{n-1}, t_n]$, considering that w_h is piecewise constant on $\mathcal{D}_h$, the boundary condition (9). The last inequality is not valid for $x \in \Omega$ such that the segment $[x, x+y]$ intersects some vertex of the dual mesh. The Cauchy-Schwarz inequality yields

$$\begin{aligned} &|\tilde{w}_h(x+y,t) - \tilde{w}_h(x,t)|^2 \\ &\qquad \le \sum_{\sigma_{D,E} \in \mathcal{F}_h^{int}} \chi_{\sigma_{D,E}}(x) \mathrm{diam}(K_{D,E}) \sum_{\sigma_{D,E} \in \mathcal{F}_h^{int}} \frac{|w_E^n - w_D^n|^2}{\mathrm{diam}(K_{D,E})} \chi_{\sigma_{D,E}}(x) \end{aligned} \tag{41}$$

for a.e. $x \in \Omega$ and for $t \in (t_{n-1}, t_n]$. We know that (see [22])

$$\sum_{\sigma_{D,E} \in \mathcal{F}_h^{int}} \chi_{\sigma_{D,E}}(x) \mathrm{diam}(K_{D,E}) \le C_{d,\tau}(|y| + h), \tag{42}$$

where $C_{d,\tau} > 0$ is a constant depending on d and $\phi_{\mathcal{T}} > 0$ (the smallest angle of the simplex K).

Now we integrate (41) over Q_T, this gives

$$\begin{aligned} &\int_0^T \int_\Omega (w_h(x+y,t) - w_h(x,t))^2 \, dx\, dt \\ &\qquad \le \frac{C_{d,\tau}}{d}(|y| + h) \sum_{n=1}^N \Delta t_n \sum_{\sigma_{D,E} \in \mathcal{F}_h^{int}} \frac{(w_E^n - w_D^n)^2}{d_{D,E}} \int_\Omega \chi_{\sigma_{D,E}}(x)\, dx \\ &\qquad \le \frac{C_{d,\tau}}{d} \sum_{n=1}^N \Delta t_n \sum_{\sigma_{D,E} \in \mathcal{F}_h^{int}} m(\sigma_{D,E}) \frac{(w_E^n - w_D^n)^2}{d_{D,E}} \,|y|\,(|y| + h), \end{aligned} \tag{43}$$

where we have used (42) and the following

$$d_{D,E} \leq \frac{\operatorname{diam}(K_{D,E})}{d}, \quad \int_\Omega \chi_{\sigma_{D,E}}(x)\,dx \leq m(\sigma_{D,E})\,|y|\,.$$

Finally using the apriori estimate (34) in (43) to deduce (39) □

From Proposition 2, Lemma 3.2 and Kolmogorov's compactness criterion (see, e.g., [24]), there exists a subsequence of $(u_h, v_h, C_{1,h}, C_{2,h}, C_{3,h})$, which converges strongly in $L^2(Q_T)$ and a.e. in Q_T. Moreover we have

$$F(u_h, v_h) \to F(u, v) \text{ and } G(u_h, v_h) \to G(u, v)$$

strongly in $L^1(Q_T)$ (from Vitali theorem).

Our final goal is to prove that the limit functions u, v, C_1, C_2, C_3 constitute a weak solution of the system (1). For this purpose, we introduce

$$\Psi := \Big\{\varphi \in C^{2,1}(\overline{\Omega} \times [0,T]), \varphi = 0 \text{ on } \partial\Omega \times [0,T], \varphi(.,T) = 0\Big\}$$

Taking an arbitrary $\varphi_1 \in \Psi$, we multiply the discrete equation (10) by $\Delta t_n \varphi(Q_D, t^{n-1})$ for all $D \in \mathcal{D}_h$ and $n \in \{1, \dots, N\}$. Summing the result over D and n yields

$$T_T + T_D + T_C = T_R,$$

where

$$T_T := \sum_{n=1}^{N} \sum_{D\in\mathcal{D}_h} (u_D^n - u_D^{n-1})\varphi_1(Q_D, t_{n-1})m(D),$$

$$T_D :=$$

$$\sum_{n=1}^{N} \Delta t_n d_1 \left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right) \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{D}_h} u_E^n \sum_{K\in\mathcal{T}_h} \Big(\nabla\varphi_E, \nabla\varphi_D\Big)_{0,K} \varphi_1(Q_D, t_{n-1}),$$

$$T_C := \sum_{n=1}^{N} \Delta t_n \sum_{D\in\mathcal{D}_h} \sum_{E\in\mathcal{N}(D)} K_{1,D,E}^n \overline{u_{D,E}^n}\, \varphi_1(Q_D, t_{n-1}),$$

$$T_R := \sum_{n=1}^{N} \Delta t_n \sum_{D\in\mathcal{D}_h} F_D^n \varphi_1(Q_D, t_{n-1})m(D).$$

We have to show that each of the terms defined above converges to its continuous version as h and Δ_t tend to zero. We will not do the proof for the convergence of T_T because is similar to that in [12] , and the convergence for the term T_R is easy by using Vitali. The proof for the convergence of T_C will be omitted since it is similar to that in [12]. Therefore, we will concentrate our attention in the proof of the convergence of T_D. Observe that (recall that $u_h^n \in X_h$)

$$T_D :=$$

$$\sum_{n=1}^{N} \Delta t_n\, d_1 \left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right) \sum_{K\in\mathcal{T}_h} \int_K \nabla u_h^n(x) \cdot \nabla\Big(\sum_{D\in\mathcal{D}_h} \varphi_1(Q_D, t_{n-1})\varphi_D(x)\Big)\,dx,$$

Next, we will prove that

$$\sum_{n=1}^{N}\Delta t_n d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right)\sum_{K\in\mathcal{T}_h}\int_K \nabla u_h^n(x)\cdot\nabla\Big(\sum_{D\in\mathcal{D}_h}\varphi_1(Q_D,t_{n-1})\varphi_D\Big)\,dx$$
$$-\sum_{n=1}^{N}\Delta t_n d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right)\sum_{K\in\mathcal{T}_h}\int_K \nabla u_h^n(x)\cdot\nabla\varphi_1(x,t_{n-1})\,dx\longrightarrow 0, \tag{44}$$

as $h\to 0$.

Set

$$I_{\varphi_1}(\cdot,t_{n-1}) := \sum_{D\in\mathcal{D}_h}\varphi_1(Q_D,t_{n-1})\varphi_D$$

and

$$T_{D_1} := \sum_{n=1}^{N}\Delta t_n\, d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right)\sum_{K\in\mathcal{T}_h}\int_K \nabla u_h^n(x)\cdot\nabla\Big(I_{\varphi_1}(x,t_{n-1})-\varphi_1(x,t_{n-1})\Big)\,dx.$$

Then using the Cauchy-Schwarz inequality, we estimate

$$\begin{aligned}
|T_{D_1}| &\le C\sum_{n=1}^{N}\Delta t_n\, d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right)\|u_h^n\|_{X_h}\,\|I_{\varphi_1}(\cdot,t_{n-1})-\varphi_1(\cdot,t_{n-1})\|_{X_h}\\
&\le C(C_I,\theta_{\mathcal{T}})h\sum_{n=1}^{N}\Delta t_n\, d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right)\|u_h^n\|_{X_h}\Big(\sum_{K\in\mathcal{T}_h}|\varphi_1(\cdot,t_{n-1})|_{2,K}^2\Big)\\
&\le C(C_I,C_{\varphi_1},\theta_{\mathcal{T}})h\sum_{n=1}^{N}\Delta t_n\, d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right)\|u_h^n\|_{X_h}\\
&\le C(C_I,C_{\varphi_1},\theta_{\mathcal{T}})h\Big(\sum_{n=1}^{N}\Delta t_n\,\|u_h^n\|_{X_h}^2\Big)^{1/2}\Big(\sum_{n=1}^{N}\Delta t_n\Big)^{1/2},
\end{aligned}$$

where $\theta_{\mathcal{T}}$ is given by (7), C_I comes from the interpolation estimate. Thus (44) is obtained by using (34).

Our next goal is to show that

$$\sum_{n=1}^{N}\Delta t_n\, d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right)\sum_{K\in\mathcal{T}_h}\int_K \nabla u_h^n(x)\cdot\nabla\varphi_1(x,t_{n-1})\,dx$$
$$\longrightarrow\int_0^T d_1\left(\int_\Omega u\,dx\right)\int_\Omega \nabla u(x,t)\cdot\nabla\varphi_1(x,t)\,dx\,dt, \tag{45}$$

as $h,\Delta t\to 0$.

We introduce the following integrals

$$T_{D_2} := \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} d_1 \left(\sum_{D_0 \in \mathcal{D}_h} m(D_0) u_{D_0}^n \right) \sum_{K \in \mathcal{T}_h} \int_K \nabla u_h^n(x) \cdot \Big(\nabla \varphi_1(x, t_{n-1}) - \nabla \varphi_1(x,t) \Big) \, dx \, dt,$$

$$T_{D_3} := \int_0^T d_1 \left(\sum_{D_0 \in \mathcal{D}_h} m(D_0) u_{D_0}^n \right) \sum_{K \in \mathcal{T}_h} \int_K \nabla u_h^n(x,t) \cdot \nabla \varphi_1(x,t) \, dx \, dt - \int_0^T d_1 \left(\int_\Omega u \, dx \right) \int_\Omega \nabla u(x,t) \cdot \nabla \varphi_1(x,t) \, dx \, dt,$$

The convergence (45) is a consequence of the convergence of T_{D_2} and T_{D_3} to zero when $h, \Delta t \to 0$. Observe that for $t \in (t_{n-1}, t_n]$, we have

$$|\nabla \varphi_1(x, t_{n-1}) - \nabla \varphi_1(x,t)| \le g(\Delta t),$$

where g satisfies $g(\Delta t) > 0$ and $g(\Delta t) \to 0$ as $\Delta t \to 0$. Therefore,

$$\begin{aligned} |T_{D_2}| &\le C g(\Delta t) \sum_{n=1}^{N} \Delta t_n \, d_1 \left(\sum_{D_0 \in \mathcal{D}_h} m(D_0) u_{D_0}^n \right) \sum_{K \in \mathcal{T}_h} \left| \nabla u_{h|K}^n \right| m(K) \\ &\le C g(\Delta t) C_3^{1/2} T^{1/2} m(\Omega)^{1/2}. \end{aligned}$$

Herein, we have used the Cauchy-Schwarz inequality and the apriori estimate (34).

Next, we show that

$$T'_{D_3} := \int_0^T d_1 \left(\sum_{D_0 \in \mathcal{D}_h} m(D_0) u_{D_0}^n \right) \sum_{K \in \mathcal{T}_h} \int_K \big(\nabla u_h^n(x,t) - \nabla u(x,t) \big) \cdot w(x,t) \, dx \, dt \longrightarrow 0 \tag{46}$$

as $h, \Delta t \to 0$ for all $\mathbf{w} \in [C^1(\overline{Q_T})]^3$. Using the Green theorem for u and $\mathbf{w}$, we obtain (recall that $u_h^n \notin H_0^1(\Omega)$)

$$\begin{aligned} T'_{D_3} :=& \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} d_1 \left(\sum_{D_0 \in \mathcal{D}_h} m(D_0) u_{D_0}^n \right) \sum_{K \in \mathcal{T}_h} \int_K \nabla u_h^n(x) \cdot \mathbf{w}(x,t) \, dx \, dt \\ &+ \int_0^T d_1 \left(\sum_{D_0 \in \mathcal{D}_h} m(D_0) u_{D_0}^n \right) \int_\Omega u(x,t) \nabla \cdot \mathbf{w}(x,t) \, dx \, dt \\ =& \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} d_1 \left(\sum_{D_0 \in \mathcal{D}_h} m(D_0) u_{D_0}^n \right) \sum_{K \in \mathcal{T}_h} \int_K -u_h^n(x) \nabla \cdot \mathbf{w}(x,t) \, dx \, dt \\ &+ \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} d_1 \left(\sum_{D_0 \in \mathcal{D}_h} m(D_0) u_{D_0}^n \right) \sum_{K \in \mathcal{T}_h} \int_{\partial K} u_h^n(x) \, \mathbf{w}(x,t) \cdot \mathbf{n} \, d\gamma(x) \, dt \\ &+ \int_0^T d_1 \left(\sum_{D_0 \in \mathcal{D}_h} m(D_0) u_{D_0}^n \right) \int_\Omega u(x,t) \nabla \cdot \mathbf{w}(x,t) \, dx \, dt. \end{aligned}$$

Reordering the summation by sides in the second term of T'_{D_3} as follows

$$\begin{aligned}
&\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right)\sum_{K\in\mathcal{T}_h}\int_{\partial K} u_h^n(x)\mathbf{w}(x,t)\cdot\mathbf{n}d\gamma(x)\,dt\\
=&\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right)\left(\sum_{\sigma_{K,L}\in\mathcal{E}_h^{int}}\int_{\sigma_{K,L}}(u_{h|K}^n-u_{h|L}^n)\mathbf{w}(x,t)\right.\\
&\qquad\left.\cdot\,\mathbf{n}_{K,L}d\gamma(x)+\sum_{\sigma_{K,L}\in\mathcal{E}_h^{ext}}\int_{\sigma_K} u_{h|K}^n\mathbf{w}(x,t)\cdot\mathbf{n}_K d\gamma(x)\right)dt := T''_{D_3},
\end{aligned}$$

we can estimate T''_{D_3} in the same form as in [12] for (6.13) and obtain the following

$$\begin{aligned}
\left|T''_{D_3}\right| &\le C_{\mathbf{w}}2h\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right)\sum_{K\in\mathcal{T}_h}\left|\nabla u_{h|K}^n\right|\operatorname{diam}(K)^3\,dt\\
&\le \frac{C_{\mathbf{w}}}{\kappa_{\mathcal{T}}}2h\sum_{n=1}^{N}\Delta t_n d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right)\sum_{K\in\mathcal{T}_h}\left|\nabla u_{h|K}^n\right| m(K)\,dt\\
&\le \frac{C_{\mathbf{w}}}{\kappa_{\mathcal{T}}}2hC_3^{1/2}T^{1/2}m(\Omega)^{1/2}
\end{aligned}$$

using the fact that each $\nabla u_{h|K}^n$ is in the summation over all sides just 4-times, $m(\sigma_D)\le \operatorname{diam}(K)^2/2$ and $\operatorname{diam}(\sigma_D)\le\operatorname{diam}(K)\le h$ for all $\sigma_D\in\mathcal{E}_K$, (6), the Cauchy-Schwarz inequality, and the apriori estimate (34). Thus $T''_{D_3}\to 0$ as $h\to 0$. To conclude that $T'_{D_3}\to 0$ as $h\to 0$, it remains to show that

$$\begin{aligned}
&-\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right)\sum_{K\in\mathcal{T}_h}\int_K u_h^n(x)\nabla\cdot\,\mathbf{w}(x,t)\,dx\,dt\\
&\qquad+\int_0^T d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right)\int_\Omega u(x,t)\nabla\cdot\mathbf{w}(x,t)\,dx\,dt\to 0.
\end{aligned}$$

This is immediate, since we can rewrite it as

$$\int_0^T d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right)\int_\Omega (u(x,t)-u_h(x,t))\nabla\cdot\mathbf{w}(x,t)\,dx\,dt\to 0.$$

which is a consequence of the strong $L^2(Q_T)$ convergence of u_h to u.

We next show that the density of the set $[C^1(\overline{Q_T})]^3$ in $[L^2(Q_T)]^3$ and (46) imply a weak convergence of ∇u_h to ∇u. In fact, let $\mathbf{w}\in[L^2(Q_T)]^3$ be given and let $\mathbf{w}_n$ be a sequence of $[C^1(\overline{Q_T})]^3$ functions converging in $[L^2(Q_T)]^3$ to $\mathbf{w}$. Then

$$\begin{aligned}
&\int_0^T d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right)\int_\Omega(\nabla u_h-\nabla u)\cdot\mathbf{w}\,dx\,dt\\
&\qquad=\int_0^T d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right)\int_\Omega(\nabla u_h-\nabla u)\cdot\mathbf{w}_n\,dx\,dt\\
&\qquad\quad+\int_0^T d_1\left(\sum_{D_0\in\mathcal{D}_h} m(D_0)u_{D_0}^n\right)\int_\Omega(\nabla u_h-\nabla u)\cdot(\mathbf{w}-\mathbf{w}_n)\,dx\,dt
\end{aligned}$$

The second term of the above expression tends to zero as $n \to \infty$ by the Cauchy-Schwarz inequality. Therefore, the whole expression tends to zero as $h \to 0$ for each $\mathbf{w} \in [L^2(Q_T)]^3$, using (46) for the first term.

Finally, we conclude that $T_{D_3} \to 0$ as $h \to 0$. We can write

$$T_{D_3} := \int_0^T \left(d_1\left(\sum_{D_0 \in \mathcal{D}_h} m(D_0) u_{D_0}^n \right) - d_1\left(\int_\Omega u\,dx \right) \right) \int_\Omega \nabla u_h^n \cdot \nabla \varphi_1 \,dx\,dt$$
$$- \int_0^T d_1\left(\int_\Omega u\,dx \right) \int_\Omega \left(\nabla u - \nabla u_h^n \right) \cdot \nabla \varphi_1 \,dx\,dt,$$

The first term of the above expression tends to zero as $h \to 0$, using the boundedness of $|\nabla \varphi_1|$, the apriori estimate (34), and the Cauchy-Schwarz inequality. The second term converges to zero by the weak convergence of ∇u_h^n to ∇u shown before. Altogether, combining (44) and (45) gives

$$T_D \to \int_0^T d_1\left(\int_\Omega u\,dx \right) \int_\Omega \nabla u \cdot \nabla \varphi_1 \,dx\,dt \qquad \text{as} \quad h, \Delta t \to 0$$

4. **Numerical results.** The numerical study that is done in this work, does not try to be an exhaustive study of the influence of the non-locality of the diffusion on the behavior of the solution, but we show by means of examples and with a particular model of non-local diffusion that this non-locality alters substantially the behavior of the populations.

We now show some numerical experiments in two dimension. Let us consider a two-dimensional closed rectangular habitat $\Omega =]0,1[\times]0,1[$. The calculations were based on the above combined finite volume - finite element scheme. We performed a triangulation of the domain Ω using the free software *triangle*, a two-dimensional quality mesh generator and delaunay triangulator. In our triangulation, the maximum triangle area is 0.000062. For our simulations, we use a time step $\Delta t = 0.01$, and we take the following ecological parameters:

$$r = 0.3, \quad a = 0.3, \quad \beta_1 = 5, \quad \beta_2 = 2, \quad e = 0.9, \quad p = 0.9, \quad q = 0.2$$
$$k_1 = 0.5, \quad g_1 = 0.3, \quad m_1 = 0.35, \quad k_2 = 0.2, \quad g_2 = 0.2, \quad m_2 = 0.2, \quad h = 0.4$$

for the carrying capacity parameter we take $K = 2$. The values of the parameters were taken from [4] and [23]

To do the numerical results, we divided the domain $[0,1] \times [0,1]$ in four regions:

$$R_1 = [0, 0.5] \times [0, 0.5]; \quad R_2 = [0.5, 1] \times [0, 0.5];$$
$$R_3 = [0, 0.5] \times [0.5, 1]; \quad R_4 = [0.5, 1] \times [0.5, 1];$$

In Figure 2 we can observe the evolution in time of the toxicants. The evolution of the toxicants depends neither on the populations nor on the parameters (diffusion, convection). On the contrary, the behavior of preys and predators depends on the distribution of the contaminants. We have the following initial conditions for the contaminants, for $t = 0$ we have $C_1 = 0$, $C_2 = 0$ and $C_3 = 0.3$. We observed that the behavior of the contaminants is exponentially decaying, for this reason for large times, we do not have contaminant in the environment, neither in the organism of the preys and predators. However, even having exponential decay, the initial presence of the contaminants afects the solution in both cases. When we

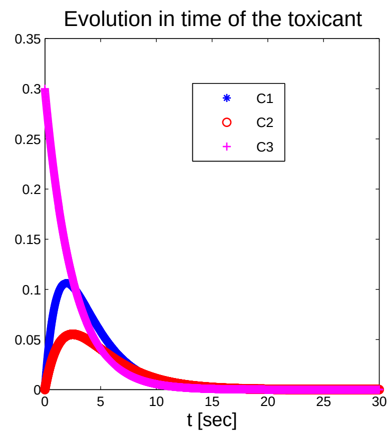


FIGURE 2. The behavior of the contaminants C_1, C_2, and C_3, for $t = 0$ we have $C_1 = 0$, $C_2 = 0$ and $C_3 = 0.3$.

have constant diffusion (see cases(2) and (3) in Figure 4 and 5) as much as we have non-local diffusion (see cases(2) and (3) in Figure 6 and 7)

Let us first precise the initial conditions. The initial distribution of predators and preys is given in the following form: a horizontal band of preys crossing a vertical band of predators, see Figure 3

In Figure 4, we use constant difusion $d_u = 0.001, d_v = 0.001$ and we can observed the predator prey interaction for 4500 time step. In case(1),$C_1 = 0, C_2 = 0$, and $C_3 = 0$ for all the domain Ω, we can observe clearly the effect of the diffusion of the two populations. At first the predators are attracted to the coincidence domain, which grows due to the diffusion, while we can see an evasion of the prey population toward the lateral edges of the horizontal band. Simultaneously this horizontal band grows with diffusion in the ortogonal direction. The two population interact and grow or decay. In case(2),$C_1 \neq 0, C_2 \neq 0$, and $C_3 \neq 0$, as in the previous case, we can observe the effect of the diffusion of the two populations and in this case we have a constant contaminant in all the domain. And due to the contaminant the diffusion is a little slower. At first the predators are attracted to the coincidence domain, which grows due to the diffusion, while we can see an evasion of the prey population toward the lateral edges of the horizontal band. Simultaneously this horizontal band grows with diffusion in the ortogonal direction. In case(3), we can observe the effect of the diffusion of the two populations and the contaminant. As we said the contaminant is not in the whole domain, it is only in some regions. The contaminant in these regions is distributed in the following form:

R_1: $C_1 = 0, C_2 \neq 0, C_3 \neq 0$, R_2: $C_1 \neq 0, C_2 \neq 0, C_3 \neq 0$, R_3: $C_1 = 0, C_2 = 0, C_3 = 0$, and R_4: $C_1 \neq 0, C_2 = 0, C_3 \neq 0$.

In R_1 we have contaminant in the environment and concentration of the toxicant in the organism of the predators. In R_2, contaminant in the environment and concentration of the toxicant in the organism of the predators and preys. In R_3 we do not have any contaminant and finally in R_4 we have contaminant in the environment and concentration of the toxicant in the organism of the preys.

At first the predators are attracted to the coincidence domain, which grows due to the diffusion, while we can see an evasion of the most prey population toward

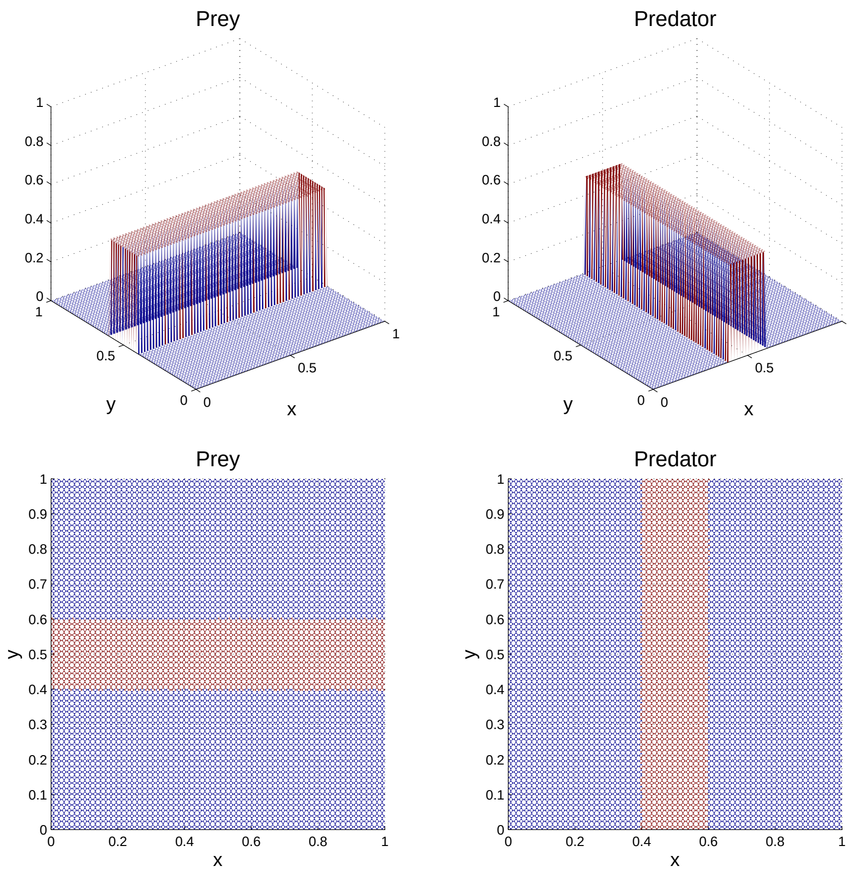
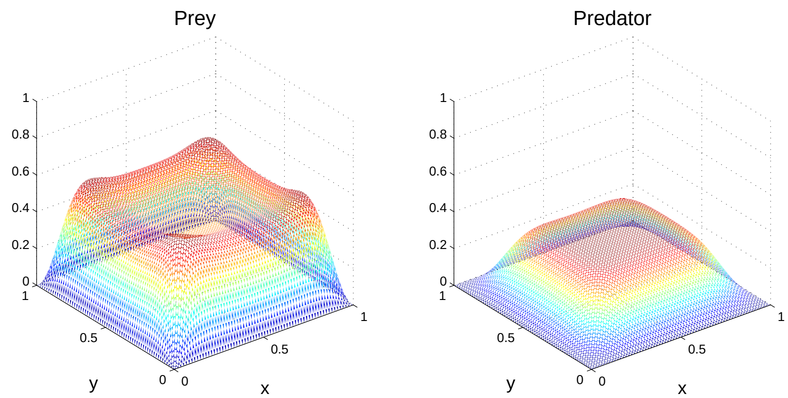


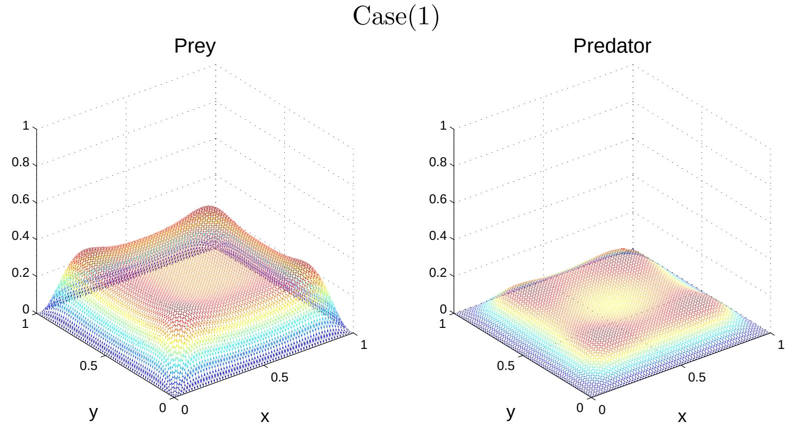
FIGURE 3. Initial data

the left lateral edge of the horizontal band where there is not contaminant. Then we can observed that both populations go out to the region with contaminant. As the time passes the toxicant in the environment disappears and for this reason the preys can escape of the predators and go to the right side of the domain.

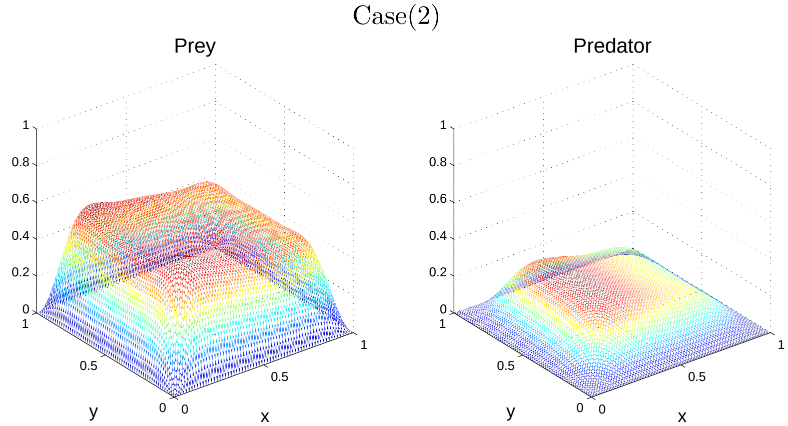
In Figure 5, we use constant diffusion $d_u = 0.001$, $d_v = 0.001$ and convection $K_1 = (0, 0.02)$, $K_2 = (0.02, 0)$, case(1), $C_1 = 0, C_2 = 0$, and $C_3 = 0$ for all the domain Ω, we can observe clearly the effect of the diffusion and the velocity of convection of the two populations. At first the predators are attracted to the coincidence domain, which grows due to the diffusion, while we can see an evasion of the prey population toward the lateral edges of the horizontal band. Then we can observe that the displacement of the populations is given by the direction of its respective speed of convection. The preys try to evade the predators. In case(2), $C_1 \neq 0, C_2 \neq 0$, and $C_3 \neq 0$, as in the previous case we can observe clearly the effect of the diffusion and the velocity of convection of the two populations, and some little changes because of the contaminant in all the domain. At first the predators are attracted to the coincidence domain, which grows due to the diffusion, while we can see an evasion of the prey population toward the lateral edges of the horizontal band. Then we can observe that the displacement of the populations is given by the direction of its respective speed of convection. The preys try to evade the predators.



Case(1)



Case(2)



Case(3)

FIGURE 4. Prey-predator interaction with diffusion $d_u = 0.001$ and $d_v = 0.001$, after 4500 time step. Case(1) without contaminant; Case(2) constant contaminant in all the domain; Case(3) contaminant in some regions of Ω.

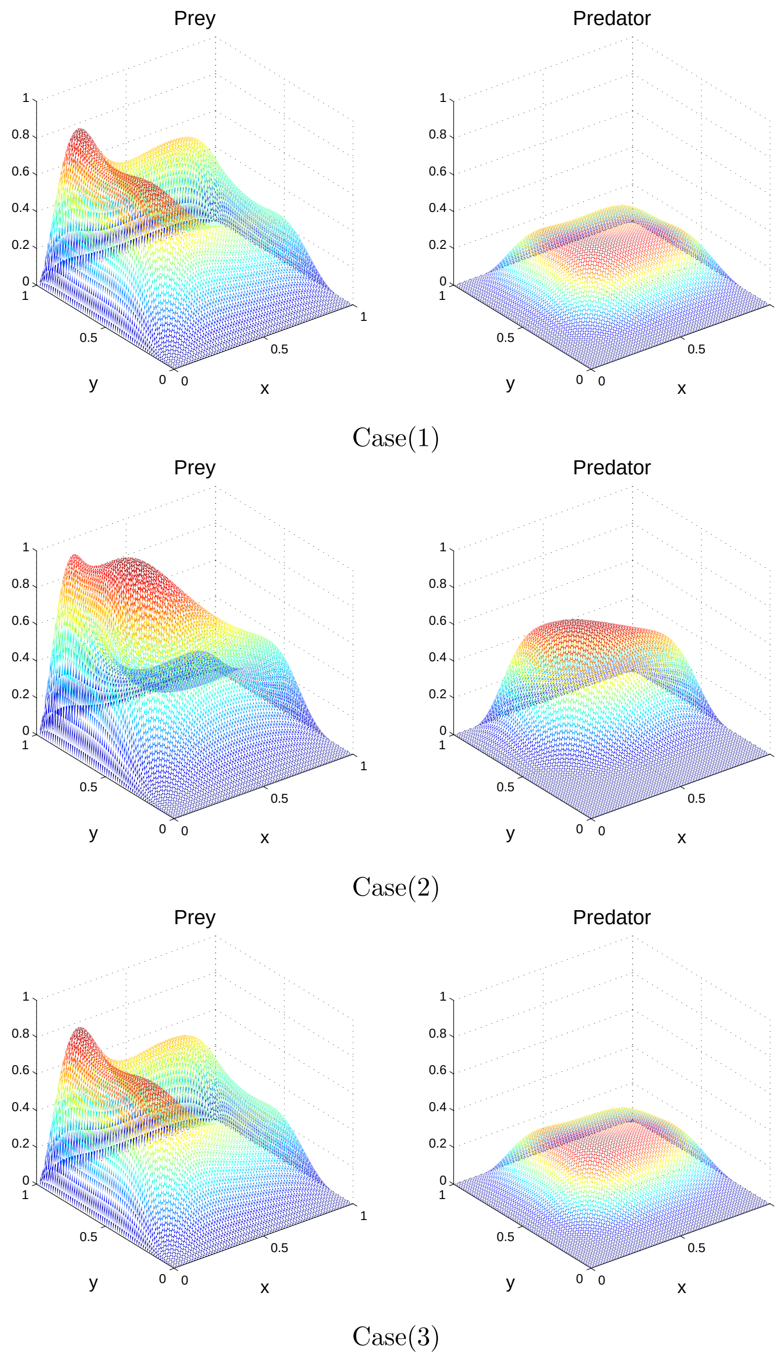


FIGURE 5. Prey-predator interaction with diffusion ($d_u = 0.001, d_v = 0.001$), convection ($K_1 = (0, 0.02), K_2 = (0.02, 0)$). Case(1) without contaminant; Case(2) constant contaminant in all the domain; Case(3) contaminant in some regions of Ω.

Finally, in case(3), as in the previous cases we can observe clearly the effect of the diffusion and the velocity of convection of the two populations. For this simulation we have the same distribution of the contaminant in the regions R_1, R_2, R_3 and R_4 of the domain, in R_1: $C_1 = 0, C_2 \neq 0, C_3 \neq 0$, in R_2: $C_1 \neq 0, C_2 \neq 0, C_3 \neq 0$, in R_3: $C_1 = 0, C_2 = 0, C_3 = 0$, and in R_4: $C_1 \neq 0, C_2 = 0, C_3 \neq 0$. At first the predators are attracted to the coincidence domain, which grows due to the diffusion, while we can see an evasion of the prey population toward the left lateral edges of the horizontal band because of the distribution of the toxicant. Then we can observe that the displacement of the populations is given by the direction of its respective speed of convection and the two populations escape out of the contaminated area. The preys try to evade the predators.

We can observed that the non-locality of the diffusion alters substantially the behavior of the populations in the sense that there is more dispersion of the prey population and the opposite occurs with the predator population, they localize in some region and the population decreases. This can be observed in Case 2 of the Figure 7.

In Figure 6, we have non-local diffusion, we can observed the predator prey interaction for 4500 time step. In case(1), $C_1 = 0, C_2 = 0$, and $C_3 = 0$ for all the domain Ω, we can observe clearly the effect of the diffusive spatial dispersion of the two populations. At first the predators are attracted to the coincidence domain, which grows due to the diffusion, while we can see an evasion of the prey population toward the lateral edges of the horizontal band. Simultaneously this horizontal band grows with diffusion in the ortogonal direction. At the final stage, we can observed that the prey population diffuses almost over all the domain. In case(2), $C_1 \neq 0, C_2 \neq 0$, and $C_3 \neq 0$, as in the previous case, we can observe the effect of the diffusive spatial dispersion of the two populations and in this case we have a constant contaminant in all the domain. And due to the contaminant the diffusion is a little slower. At first the predators are attracted to the coincidence domain, which grows due to the diffusion but it grows slower than the prey population. While we can see an evasion of the prey population toward the lateral edges of the horizontal band. Simultaneously this horizontal band grows with diffusion in the ortogonal direction but evadind the center of the domain which is the coincidence domain with predators. In case(3), we can observe the effect of the diffusive spatial dispersion of the two populations and the contaminant. As we said the contaminant is not in the whole domain, it is only in some regions,in R_1: $C_1 = 0, C_2 \neq 0, C_3 \neq 0$, in R_2: $C_1 \neq 0, C_2 \neq 0, C_3 \neq 0$, in R_3: $C_1 = 0, C_2 = 0, C_3 = 0$, and in R_4: $C_1 \neq 0, C_2 = 0, C_3 \neq 0$. At first the predators are attracted to the coincidence domain, which grows due to the diffusion, while we can see an evasion of the prey population toward the left lateral edge of the horizontal band where there is not contaminant. Then we can observed that both populations go out to the region with contaminant. As the time passes the toxicant in the environment disappears and for this reason the preys can escape of the predators and go to the right side of the domain. Finally, the preys diffuse almost over the whole domain.

In Figure 7, we have non-local diffusion and convection. In case(1), $C_1 = 0, C_2 = 0$, and $C_3 = 0$ for all the domain Ω, we can observe clearly the effect of the diffusion and the velocity of convection of the two populations. At first the predators are attracted to the coincidence domain, which grows due to the diffusion, while we can see an evasion of the prey population toward the lateral edges of the horizontal band, its evasion is to the left side more than the right one. Then we can observe that the

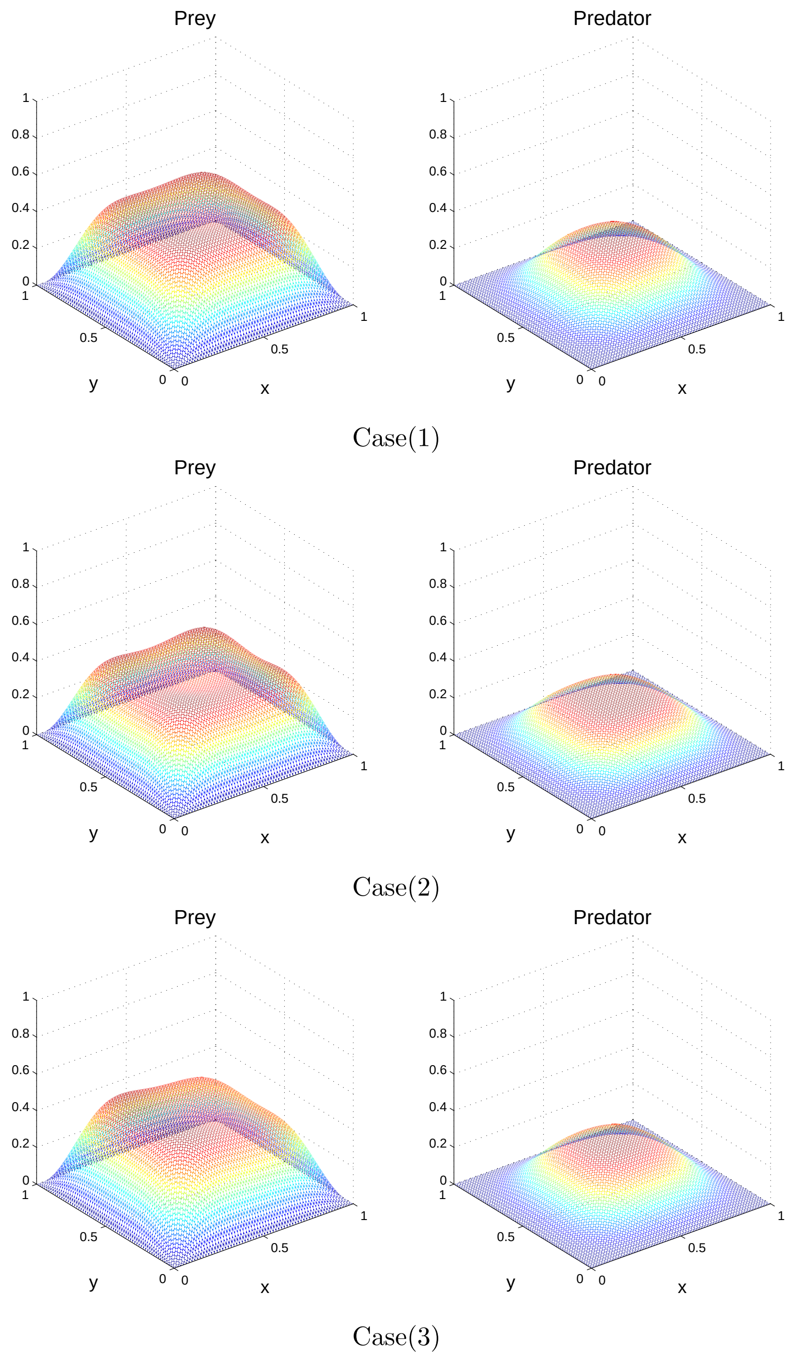
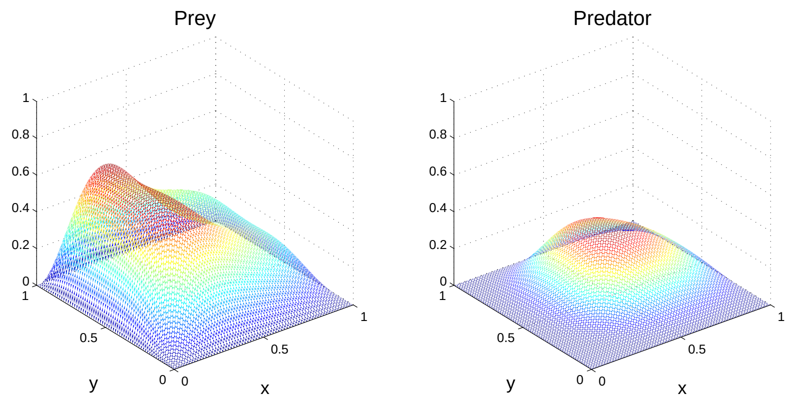
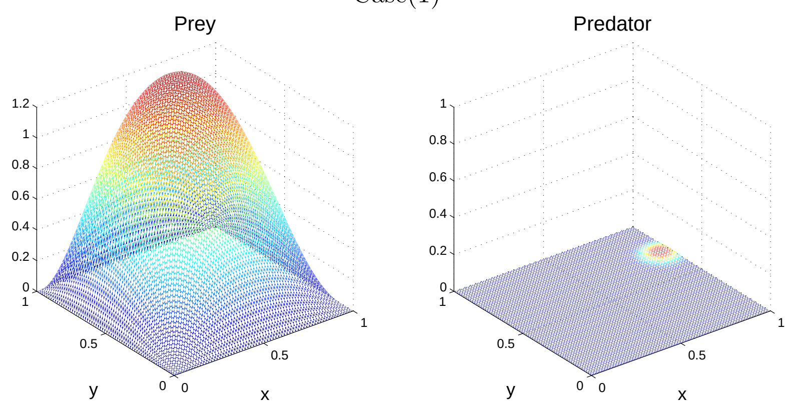


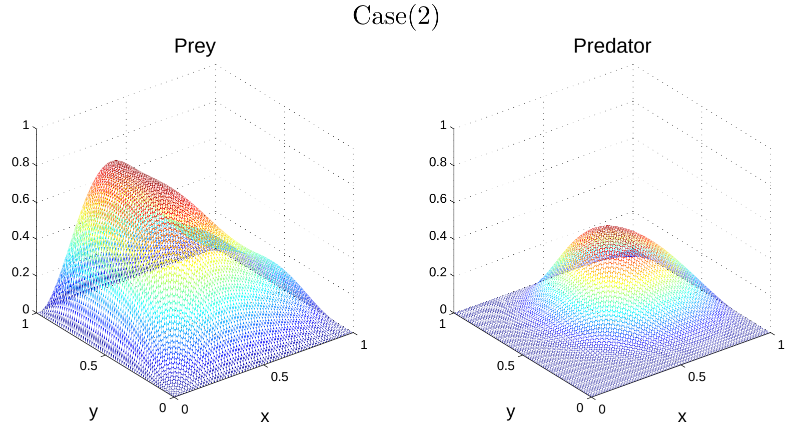
FIGURE 6. Prey-predator interaction with nonlocal diffusion, after 4500 time step. Case(1) without contaminant; Case(2) constant contaminant in all the domain; Case(3) contaminant in some regions of Ω.



Case(1)

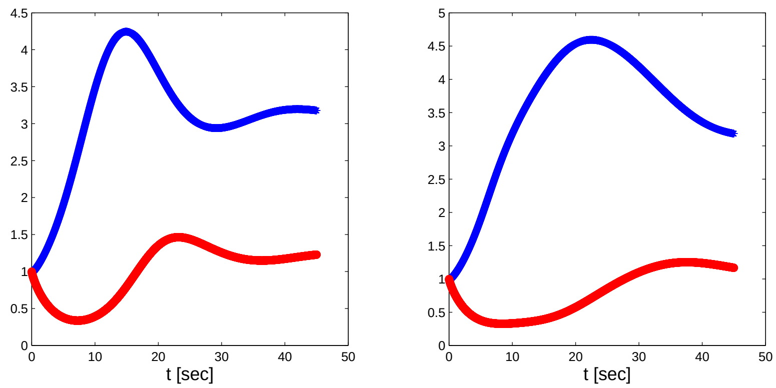


Case(2)

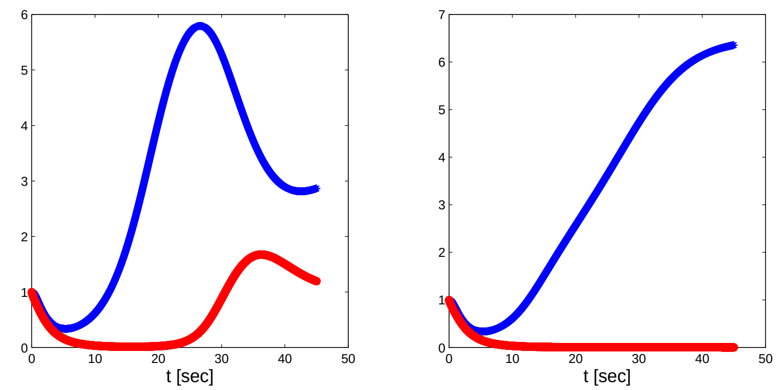


Case(3)

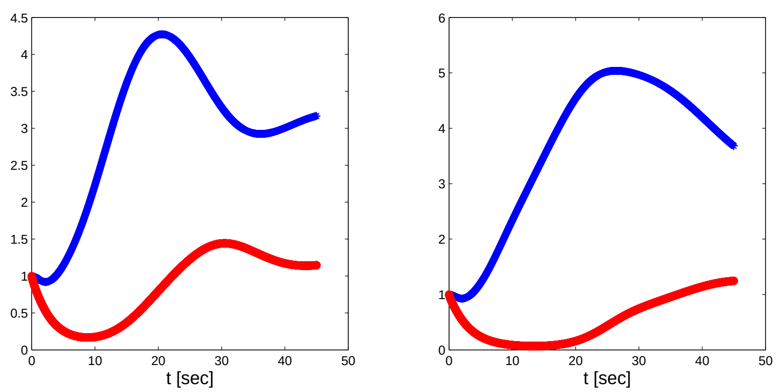
FIGURE 7. Prey-predator interaction with nonlocal diffusion, and convection ($K_1 = (0, 0.02), K_2 = (0.02, 0)$). after 4500 time step. Case(1) without contaminant; Case(2) constant contaminant in all the domain; Case(3) contaminant in some regions of Ω.



Case(1)



Case(2)



Case(3)

FIGURE 8. Behavior of the nonlocal diffusion. Case(1) without contaminant; Case(2) constant contaminant in all the domain; Case(3) contaminant in some regions of Ω. On the left side: populations with diffusion terms and without convection terms; on the right side: population with diffusion and convection terms.

displacement of the populations is given by the direction of its respective speed of convection. The preys try to evade the predators. In case(2), $C_1 \neq 0, C_2 \neq 0$, and $C_3 \neq 0$, as in the previous case we can observe clearly the effect of the diffusion and the velocity of convection of the two populations, and some little changes because of the contaminant in all the domain. At first the predators are attracted to the coincidence domain, which grows due to the diffusion, while we can see an evasion of the prey population toward the lateral edges of the horizontal band. Then we can observe that the displacement of the populations is given by the direction of its respective speed of convection. The preys try to evade the predators. We can observed that the prey populations increase and the predator population decrease and concentrate in a small region of the domain. Finally, in case(3), as in the previous cases we can observe clearly the effect of the diffusion and the velocity of convection of the two populations. For this simulation we have the same distribution of the contaminant in the regions R_1, R_2, R_3 and R_4 of the domain given before, in R_1: $C_1 = 0, C_2 \neq 0, C_3 \neq 0$, in R_2: $C_1 \neq 0, C_2 \neq 0, C_3 \neq 0$, in R_3: $C_1 = 0, C_2 = 0, C_3 = 0$, and in R_4: $C_1 \neq 0, C_2 = 0, C_3 \neq 0$. At first the predators are attracted to the coincidence domain, which grows due to the diffusion, while we can see an evasion of the prey population toward the left lateral edges of the horizontal band because of the distribution of the toxicant. Then we can observe that the displacement of the populations is given by the direction of its respective speed of convection and the two populations escape out of the contaminated area. The preys try to evade the predators.

Aditionally, in Figure 8 is possible to see the graphics of the non-local diffusion which are proportional to the total population. The pictures on the left side are only with diffusion and the others on the right side are with diffusion and convection. As it is observed in the figure the non-local diffusion has in general an oscillatory behavior. Besides, while the non-local diffusion of the preys (blue) tends to increase as the time passes, the non-local diffusion of the predator (red) tends to decrease reaching almost void values in some cases (see Case(2) in Figure 8).

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REFERENCES

[1] A. S. Ackleh and L. Ke, *Existence-uniqueness and long time behavior for a class on nonlocal nonlinear parabolic evolution equations*, Proc. Amer. Math. Soc., **128** (2000), 3483–3492.

[2] V. Anaya, M. Bendahmane and M. Seplveda, *Mathematical and numerical analysis for reaction-diffusion systems modeling the spread of early tumors*, Bol. Soc. Esp. Mat. Apl., (2009), 55–62.

[3] V. Anaya, M. Bendahmane and M. Sepúlveda, *A numerical analysis of a reaction-diffusion system modelling the dynamics of growth tumors*, Math. Models Methods Appl. Sci., **20** (2010), 731–756.

[4] B. Ainseba, M. Bendahmane and A. Noussair, *A reaction-diffusion system modeling predator-prey with prey-taxis*, Nonlinear Anal. Real World Appl., **128** (2008), 2086–2105.

[5] L. Bai and K. Wang, *A diffusive stage-structured model in a polluted environment*, Nonlinear Anal. Real World Appl., **7** (2006), 96–108.

[6] M. Bendahmane, K. H. Karlsen and J. M. Urbano, *On a two-sidedly degenerate chemotaxis model with volume-filling effect*, Math. Models Methods Appl. Sci., **17** (2007), 783–804.

[7] M. Bendahmane and M. Sepúlveda, *Convergence of a finite volume scheme for nonlocal reaction-diffusion systems modelling an epidemic disease*, Discrete Contin. Dyn. Syst. Ser. B, 11 (2009), 823–853.

[8] M. Chipot and B. Lovat, *Some remarks on nonlocal elliptic and parabolic problem*, Nonlinear Anal., **30** (1997), 4619–4627.

[9] B. Dubey and J. Hussain, *Modelling the interaction of two biological species in a polluted environment*, J. Math. Anal. Appl., **246** (2000), 58–79.

[10] B. Dubey and J. Hussain, *Models for the effect of environmental pollution on forestry resources with time delay*, Nonlinear Anal. Real World Appl., **5** (2004), 549–570.

[11] R. Eymard, Th. Gallouët and R. Herbin, "Finite Volume Methods. Handbook of Numerical Analysis," vol. **VII**, North-Holland, Amsterdam, 2000.

[12] R. Eymard, D. Hilhorst and M. Vohralík, *A combined finite volume-nonconforming/mixed-hybrid finite element scheme for degenerate parabolic problems*, Numer. Math., **105** (2006), 73–131.

[13] H. I. Freedman and J. B. Shukla, *Models for the effects of toxicant in single-species and predator-prey systems*, J. Math. Biol., **30** (1991), 15–30.

[14] T. G. Hallam, C. E. Clark and R. R. Lassider, *Effects of toxicants on populations: A qualitative approach I. Equilibrium environment exposured*, Ecol. Model, **18** (1983), 291–304.

[15] T. G. Hallam, C. E. Clark and G. S Jordan, *Effects of toxicants on populations: A qualitative approach II. First order kinetics*, J. Math. Biol., **18** (1983), 25–37.

[16] T. G. Hallam and J. T. De Luna, *Effects of toxicants on populations: A qualitative approach III. Environment and food chains pathways*, J. Theor. Biol., **109** (1984), 11–29.

[17] J.-L. Lions, "Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires," Dunod, 1969.

[18] C. A. Raposo, M. Sepúlveda, O. Vera, D. Carvalho Pereira and M. Lima Santos, *Solution and asymptotic behavior for a nonlocal coupled system of reaction-diffusion*, Acta Appl. Math. **102** (2008), 37–56.

[19] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl., **146** (1987), 65–96.

[20] J. B. Shukla and B. Dubey, *Simultaneous effect of two toxicants on biological species: A mathematical model*, J. Biol. Syst., **4** (1996), 109–130.

[21] R. Temam, "Navier-Stokes Equations, Theory and Numerical Analysis," 3rd revised edition, North-Holland, Amsterdam, reprinted in the AMS Chelsea series, AMS, Providence, 2001.

[22] M. Vohralik, "Numerical Methods for Nonlinear Elliptic and Parabolic Equations. Application to Flow Problems in Porous and Fractured Media," Ph.D. dissertation, Université de Paris-Sud & Czech Technical University, Prague, 2004.

[23] X. Yang, Z. Jin and Y. Xue, *Weak average persistence and extinction of a predator-prey system in a polluted environment with impulsive toxicant input*, Chaos Solitons Fractals, **31** (2007), 726–735.

[24] K. Yosida, "Functional Analysis and its Applications," New York, Springer-Verlag, 1971.

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