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ASYMPTOTIC ANALYSIS OF THE STOKES FLOW WITH VARIABLE VISCOSITY IN A THIN ELASTIC CHANNEL

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Abstract. The non-steady viscous flow in a thin channel with elastic wall is considered. The viscosity is constant everywhere except for some small neighborhood of the origin of the coordinate system, where it is a variable function. The problem contains two small parameters: ε, that is the ratio of the thickness of the channel and its length, and $\delta = \varepsilon^{\gamma}$, $\gamma \geq 3$, that is the "softness of the wall", i.e. its inverse (rigidity) is great. An asymptotic expansion of the solution is constructed and, in particular, the leading term is described. An important new element of this paper is the procedure of construction of the boundary layer in the neighborhood of the origin of the coordinate system, generated by the variable viscosity. The error estimates for the difference of a truncated asymptotic ansatz and the exact solution are obtained. To this end, the existence and uniqueness of the solution are studied and some a priori estimates are proved.

1. **Introduction.** Physical problems involving the interaction of a fluid with a moving or deformable structure have important applications in biomechanics, hydroelasticity, etc. In the last few years, such problems have been studied extensively both from the mathematical and from the numerical viewpoints.

For instance, some results concerning the existence of weak or strong solutions when the domain occupied by the fluid is either fixed or is time-dependent can be found in [3], [4], [5], [8], [10], [15], etc.

An asymptotic analysis of the fluid-structure interaction was developed in [1], [2], [12], [13], [14] and this list is non-exhaustive.

The goal of the present paper is to extend the results obtained in [12], [13], [14] to the case of a viscous fluid with variable viscosity. Below, we give some motivating reasons for this study.

A high level of blood cholesterol, which represents a frequent disease of the modern life, determines the formation of clots in vessels. When a clot becomes great, the vessel is obturated and this represents one of the main causes of death. This

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phenomenon modifies the constant viscosity of the blood, transforming this fluid into a fluid with variable viscosity. In order to cure this disease some special substances may be injected. These substances as well change locally the viscosity of the blood. Finally, the great viscosity in the clot formation zone gives an approximate model for absence of flow in this zone. This idea comes from the fictitious domain method.

There are, of course, many other practical problems involving fluids with variable viscosity. For example, the presence of bacteria in suspension (see [9]) may change locally the viscosity.

We consider a non steady flow of a viscous fluid with variable viscosity through a thin channel with visco-elastic walls. The fluid motion is simulated by the Stokes equations, the walls behavior is described by the Sophie Germain's fourth order in space non-steady state equation for the transversal displacements of the elastic walls (the plate model), while the longitudinal walls displacements are disregarded. The fluid-structure interaction is simulated by the equality of the fluid velocity at the boundary and the time derivative of the walls displacement (the longitudinal velocity is taken equal to zero). This condition is acceptable in the case of small strains.

The problem contains two small parameters: one of them is the ratio ε of the thickness of the channel to its length; the second, δ, is the ratio of the linear density to the stiffness of the wall. Parameter δ is taken as some power of ε, namely, $\delta = \varepsilon^\gamma$, $\gamma \geq 3$. We construct an asymptotic expansion of the solution. The ansatz for a flow with periodic boundary conditions at the ends of the channel (see [12]) is modified and completed by two types of boundary layer correctors: the first type corresponds to the variable viscosity in the neighborhood of the origin of the coordinate system and the second one represents the boundary layer functions, corresponding to the boundary conditions at the ends of the channel. The leading term of the asymptotic solution described.

As usually we should construct the boundary layer corrector exponentially stabilizing to zero at infinities. On the other hand, it is known that the Stokes problem in an infinite strip has a solution with the velocity satisfying this condition of decay at $\pm\infty$ while the pressure stabilizes to some constants. One of these constants (for instance, corresponding to $-\infty$) may be chosen equal to zero, but the other one (corresponding to $+\infty$) may be different from zero. We subtract this constant from the pressure on the half-strip corresponding to positive values of the longitudinal variable. Consequently, the pressure (and so the stress) becomes discontinuous. To compensate this stress gap we will impose the opposite stress gap for the macroscopic pressure.

Since problems corresponding to the two types of correctors are non homogeneous, the results of [6] can not be applied directly. We generalize these results using the arguments of [13].

After constructing the asymptotic solution and establishing its properties, we use the *a priori* estimates obtained before to show that this solution represents a good approximation of the exact solution. The analysis of the leading term of the asymptotic solution allows us to improve the result given by the *a priori* estimates.

2. Physical setting.

Consider a small parameter ε, $\varepsilon = \dfrac{1}{q}$, $q \in \mathbb{N}$ and define the thin rectangle

$$D_\varepsilon = \Big\{(x_1, x_2) \in \mathbb{R}^2 : -\frac{1}{2} < x_1 < \frac{1}{2},\ -\frac{\varepsilon}{2} < x_2 < \frac{\varepsilon}{2}\Big\}.$$

The elastic parts of the boundary of D_ε are defined as follows:

$$\Gamma_\varepsilon^{\pm} = \Big\{(x_1, \pm\frac{\varepsilon}{2}) : -\frac{1}{2} < x_1 < \frac{1}{2}\Big\}.$$

Let μ_0 be a function defined in the strip $\Pi = (-\infty, +\infty) \times (-1/2, 1/2)$, independent of ε and satisfying the following inequality: $\forall \xi \in \Pi,\ 0 < c_1 \le \mu_0(\xi) \le c_2$ with positive constants c_1, c_2. Assume that it is C^1-smooth function in the strip $\bar{\Pi}$, equal to a positive constant μ out of the square $(-1/2, 1/2)^2$. Let us set

$$\mu_\varepsilon(x) = \mu_0\Big(\frac{x}{\varepsilon}\Big).$$

The domain $\big\{|x_1| \le \frac{\varepsilon}{2}\big\}$ represents the region where the viscosity of the fluid becomes variable; out of this zone the viscosity is constant.

The viscous fluid interacts with the elastic boundaries of the vessel, $\Gamma_\varepsilon^{\pm}$. The interaction between the fluid and the elastic boundaries generates the displacements $d_\pm d_\pm(x_1, t)$ of these boundaries in Ox_2 direction. We neglect the longitudinal displacements. The ends of the elastic walls are supposed to be clamped. We study this problem for $t \in [0, T]$, with T an arbitrary positive constant independent of ε and we assume that the walls are not too soft so that the displacements of the boundaries are small enough; so, the fluid flow equations are considered in the initial configuration, that is, the domain D_ε.

The non steady flow in a thin channel with variable viscosity is described by the following coupled system:

$$\begin{cases} \rho_f \dfrac{\partial \mathbf{u}}{\partial t} - 2\mathrm{div}(\mu_\varepsilon D(\mathbf{u})) + \nabla p = \mathbf{f} \text{ in } D_\varepsilon \times (0,T), \\ \mathrm{div}\ \mathbf{u} = 0 \text{ in } D_\varepsilon \times (0,T), \\ \mathbf{u} = \dfrac{\partial d_\pm}{\partial t}\mathbf{e}_2 \text{ on } \Gamma_\varepsilon^{\pm} \times (0,T), \\ \mathbf{u} = \varepsilon^2 \boldsymbol{\psi}^\varepsilon \text{ on } (\partial D_\varepsilon \cap \{x_1 = \pm\frac{1}{2}\}) \times (0,T), \\ \mathbf{u}(x,0) = \mathbf{0} \text{ in } D_\varepsilon, \\ \rho h \dfrac{\partial^2 d_+}{\partial t^2} + \dfrac{h^3 E}{12}\ \dfrac{\partial^4 d_+}{\partial x_1^4} + \nu \dfrac{\partial^5 d_+}{\partial x_1^4 \partial t} = g_+ + p_{\big/ x_2 = \varepsilon/2} \text{ on } \Gamma_\varepsilon^+ \times (0,T), \\ \rho h \dfrac{\partial^2 d_-}{\partial t^2} + \dfrac{h^3 E}{12}\ \dfrac{\partial^4 d_-}{\partial x_1^4} + \nu \dfrac{\partial^5 d_-}{\partial x_1^4 \partial t} = g_- - p_{\big/ x_2 = -\varepsilon/2} \text{ on } \Gamma_\varepsilon^- \times (0,T), \\ d_\pm\big(\pm\dfrac{1}{2}, t\big) = 0 \text{ in } (0,T), \\ \dfrac{\partial d_\pm}{\partial x_1}\big(\pm\dfrac{1}{2}, t\big) = 0 \text{ in } (0,T), \\ d_\pm(x_1, 0) = \dfrac{\partial d_\pm}{\partial t}(x_1, 0) = 0 \text{ in } \big(-\dfrac{1}{2}, \dfrac{1}{2}\big). \end{cases} \tag{1}$$

Here the given data are: $\mathbf{f}-$the exterior force applied to the fluid, $g_\pm\,\mathbf{e}_2-$the exterior forces applied on the elastic boundaries, $\rho_f,\ \rho,\ \nu,\ E-$positive given material constants (ρ_f is the density of the fluid, ρ is the density of the elastic walls, ν is the viscosity coefficient of the wall and E is its Young's modulus), the positive constant h stands for the thickness of the elastic walls. The fluid flow is determined by a small given inflow and outflow velocity described by the function $\varepsilon^2\boldsymbol{\psi}^\varepsilon$ with $\boldsymbol{\psi}^\varepsilon$

defined as follows:

$$\boldsymbol{\psi}^{\varepsilon}(x,t) = \boldsymbol{\psi}^{\varepsilon}(x_2,t) = \boldsymbol{\psi}(\xi_2,t) = \psi(\xi_2,t)\mathbf{e}_1,$$

with $\xi_2 = x_2/\varepsilon$, ψ being a smooth function satisfying

$$\psi = 0 \text{ on } \{\xi_2 = \pm 1/2\} \times (0,T) \cup (-1/2,1/2) \times \{0\}.$$

The unknowns of the system (1) are: the velocity of the fluid, $\mathbf{u}$, the pressure of the fluid, p, and the displacements of the elastic walls, $d_{\pm}$. The fluid flow is described by the non steady Stokes equations. A "viscous" type term, $\nu \dfrac{\partial^5 d_{\pm}}{\partial x_1^4 \partial t}$, is added to the usual forth-order equation for the normal displacement. It corresponds to the visco-elastic behavior of the wall (the so called Kelvin-Voigt model). The coefficient $h^3E/12$ is a great parameter; it will play an important role for our problem. Usually, the Young's modulus, E, takes values between $10^4 Pa$ and $10^6 Pa$. On the other hand, in the blood circulation models we assume that the characteristic longitudinal space scale for vessels is of order of cm and the time scale is of order of seconds. Let us use the SI system of units. This leads us to the necessity of scaling of every derivative in x_1 by the factor 10^2, i. e. the fourth derivative will contain the additional factor 10^8. If h is of order 10^{-3} m or 10^{-2} m, then the coefficient ρh can be taken in the further analysis as a value of order of 1. The coefficient $h^3E/12$ in equations $(1)_{6,7}$ will be replaced (after scaling in x_1) by a coefficient δ^{-1} with the value of δ of order from 10^{-7} to 10^{-4}. If the ratio of the thickness and the length of the vessel ε is of order 10^{-2}, then δ is of order from ε^2 to ε^4. We assume that the "viscous" term is much smaller than the term with the coefficient δ^{-1} and hence the new coefficient denoted also by ν, obtained after scaling in x_1, is $O(1)$. All the coefficients of $(1)_1$ are considered in our asymptotic study as constants of order of 1.

More details concerning (1) can be found, for instance in [12]. Let us mention an important property of the solution: due to the properties of the function ψ, the compatibility condition for the coupled system which describes the physical problem is:

$$0 = \int_{\partial D_\varepsilon} \mathbf{u}(x,t) \cdot \mathbf{n} \, \mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{-1/2}^{1/2} (d_+(x_1,t) - d_-(x_1,t)) \mathrm{d}x_1 \Big).$$

Using next the initial condition for the displacements, from the above relation we get:

$$\int_{-1/2}^{1/2} (d_+(x_1,t) - d_-(x_1,t)) \mathrm{d}x_1 = 0 \text{ for all } t \in [0,T]. \tag{2}$$

This condition states that the global area of the flow domain is preserved.

3. **Variational formulation. Existence, uniqueness, regularity and *a priori* estimates.**

In order to obtain the above properties for the solution of the physical problem, we introduce the variational framework of (1).

To simplify the computations, we consider first (1) with homogeneous boundary conditions on $x_1 = \pm 1/2$, i. e. the problem for $\psi = 0$. Then, the same properties for the solution of (1) follow with the usual technique for non homogeneous problems.

Choosing the regularity: $\mathbf{f} \in L^2(0,T;(L^2(D_\varepsilon))^2)$, $g_+, g_- \in L^2((-1/2,1/2)\times(0,T))$ and introducing the spaces

$$\begin{cases} V^\varepsilon = \{\mathbf{v} \in (H^1(D_\varepsilon))^2 : \operatorname{div}\mathbf{v} = 0,\ \mathbf{v} = \mathbf{0} \text{ on } \partial D_\varepsilon \backslash \Gamma_\varepsilon^\pm,\ v_1 = 0 \text{ on } \Gamma_\varepsilon^\pm\}, \\ B_0 = \{(b_+, b_-) \in (H_0^2(-1/2,1/2))^2 : \displaystyle\int_{-1/2}^{1/2} (b_+(x_1) - b_-(x_1))\mathrm{d}x_1 = 0\} \end{cases}$$

we consider the following variational problem:

$$\begin{cases} \text{Find } (\mathbf{u},\mathbf{d}) \in L^2(0,T;V^\varepsilon) \times H^1(0,T;B_0), \\ \qquad\qquad \text{with } (\mathbf{u}',\mathbf{d}') \in L^2(0,T;(V^\varepsilon)') \times H^1(0,T;(B_0)'), \\ \qquad\qquad \text{which satisfies a. e. in } (0,T): \\ \rho_f \dfrac{d}{dt}\displaystyle\int_{D_\varepsilon} \mathbf{u}\cdot\boldsymbol{\varphi} + 2\int_{D_\varepsilon} \mu_\varepsilon D(\mathbf{u}) : D(\boldsymbol{\varphi}) + \rho h \dfrac{d}{dt}\int_{-1/2}^{1/2} \dfrac{\partial \mathbf{d}}{\partial t}\cdot\mathbf{b} + \dfrac{1}{\delta}\int_{-1/2}^{1/2} \dfrac{\partial^2 \mathbf{d}}{\partial x_1^2}\cdot\dfrac{\partial^2 \mathbf{b}}{\partial x_1^2} \\ +\nu\displaystyle\int_{-1/2}^{1/2} \dfrac{\partial^3 \mathbf{d}}{\partial x_1^2 \partial t}\cdot\dfrac{\partial^2 \mathbf{b}}{\partial x_1^2} = \int_{D_\varepsilon} \mathbf{f}\cdot\boldsymbol{\varphi} + \int_{-1/2}^{1/2} \mathbf{g}\cdot\mathbf{b},\ \forall \boldsymbol{\varphi} \in V^\varepsilon, \mathbf{b} \in B_0, \varphi_2 = b_\pm \text{ on } \Gamma_\varepsilon^\pm, \\ u_2 = \dfrac{\partial d_\pm}{\partial t} \text{ on } \Gamma_\varepsilon^\pm, \\ \mathbf{u}(0) = \mathbf{0}, \mathbf{d}(x_1,0) = \dfrac{\partial \mathbf{d}}{\partial t}(x_1,0) = 0. \end{cases} \tag{3}$$

Here and below, $\mathbf{d} = (d_+, d_-)$, $\mathbf{b} = (b_+, b_-)$, (the couples of functions). The relation of (3) holds a. e. in $(0,T)$. Indeed, let us consider first this relation in the sense of distributions; we see that all the terms except the first and the third belong to $L^2(0,T)$ for arbitrary test functions $\boldsymbol{\varphi}$ and $\mathbf{b}$. So, we can conclude that the time derivatives in the first and the third terms exist and belong to $L^2(0,T)$.

We obtain, following step by step the proof of Theorem 3.1, [12], the following result:

Theorem 3.1. *The variational problem (3) has a unique solution,* $(\mathbf{u},\mathbf{d})$, *with* $(\mathbf{u}',\mathbf{d}'') \in L^2(0,T;(L^2(D_\varepsilon))^2) \times L^2(0,T;(L^2(-1/2,1/2))^2)$.

Proof. Let the right hand side functions $\mathbf{f}$ and $\mathbf{g}$ be zero. Taking the test functions $\boldsymbol{\varphi} = \mathbf{u}$ and $\mathbf{b} = \frac{\partial \mathbf{d}}{\partial t}$ we get the following identity

$$\rho_f \frac{\mathrm{d}}{\mathrm{d}t}\int_{D_\varepsilon} \mathbf{u}^2 + 2\int_{D_\varepsilon} \mu_\varepsilon D(\mathbf{u}) : D(\mathbf{u}) + \rho h \frac{\mathrm{d}}{\mathrm{d}t}\int_{-1/2}^{1/2} \Big(\frac{\partial \mathbf{d}}{\partial t}\Big)^2 + \frac{1}{\delta}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-1/2}^{1/2} \Big(\frac{\partial^2 \mathbf{d}}{\partial x_1^2}\Big)^2 + \frac{\nu}{2}\int_0^1 \Big(\frac{\partial^3 \mathbf{d}}{\partial x_1^2 \partial t}\Big)^2 = 0.$$

Integrating from 0 to t this equality and taking into account the initial conditions, we obtain: $\mathbf{u} = \mathbf{0}$ a. e. in $(0,T)$ and $\mathbf{d} = \mathbf{0}$ a. e. in $(0,T)$. Hence, the problem (3) has a unique solution.

The existence of solution is proved by the Galerkin's method as in Theorem 3.1, [12], replacing the Laplacian by the operator $2\operatorname{div}(\mu_\varepsilon D(.))$, functions $\boldsymbol{\varphi}_j$ by $\boldsymbol{\varphi}_j^\pm$ equal to $\beta_j \mathbf{e_2}$ respectively on $\Gamma_\varepsilon^\pm$ and vanishing on the opposite side of the boundary. □

Applying the De Rham's lemma we prove that there exists a distribution q satisfying the equation

$$\rho_f \frac{\partial \mathbf{u}}{\partial t} - 2\operatorname{div}(\mu_\varepsilon D(\mathbf{u})) - \mathbf{f} = -\nabla q.$$

Multiplying this equation by a test function and integrating we prove that there exists a distribution p, such that it satisfies the equations of (1) in the sense of distributions. Improving the smoothness of solution $\mathbf{u}$ we can prove that:

Corollary 3.1. *There exists a unique function $p \in L^2(0,T;H^1(D_\varepsilon))$ such that $(\mathbf{u},p)$ satisfies $(1)_1$ in $L^2(0,T;(L^2(D_\varepsilon))^2)$.*

As in Theorem 3.1, [12], we obtain the following *a priori* estimates, which will be used in order to establish the error between the exact and the asymptotic solutions.

Corollary 3.2. *Let $(\mathbf{u}, p, d_\pm)$ be the solution of the problem (1) corresponding to the data $\mathbf{f}$, $g_\pm$ from the spaces mentioned above. We introduce the notation: $\mathbf{g} = (g_+, g_-)$. Then the following estimates hold:*

$$\begin{aligned}
&\|\mathbf{u}\|_{L^\infty(0,T;(L^2(D_\varepsilon))^2)} \le C(\|\mathbf{f}\|_{L^2(0,T;(L^2(D_\varepsilon))^2)} + \|\mathbf{g}\|_{L^2(0,T;(L^2(-1/2,1/2))^2)}),\\
&\Big\|\frac{\partial d_\pm}{\partial t}\Big\|_{L^\infty(0,T;L^2(-1/2,1/2))} \le C(\|\mathbf{f}\|_{L^2(0,T;(L^2(D_\varepsilon))^2)} + \|\mathbf{g}\|_{L^2(0,T;(L^2(-1/2,1/2))^2)}),\\
&\Big\|\frac{\partial^2 d_\pm}{\partial x_1^2}\Big\|_{L^\infty(0,T;L^2(-1/2,1/2))} \le C\delta^{1/2}(\|\mathbf{f}\|_{L^2(0,T;(L^2(D_\varepsilon))^2)} + \|\mathbf{g}\|_{L^2(0,T;(L^2(-1/2,1/2))^2)}),\\
&\|D(\mathbf{u})\|_{L^2(0,T;(L^2(D_\varepsilon))^2)} \le C(\|\mathbf{f}\|_{L^2(0,T;(L^2(D_\varepsilon))^2)} + \|\mathbf{g}\|_{L^2(0,T;(L^2(-1/2,1/2))^2)}),\\
&\Big\|\frac{\partial^3 d_\pm}{\partial x_1^2 \partial t}\Big\|_{L^2((-1/2,1/2)\times(0,T))} \le C(\|\mathbf{f}\|_{L^2(0,T;(L^2(D_\varepsilon))^2)} + \|\mathbf{g}\|_{L^2(0,T;(L^2(-1/2,1/2))^2)}),\\
&\|\nabla p\|_{L^2(0,T;(H^{-1}(D_\varepsilon))^2)} \le \frac{C}{\delta^{1/2}}(\|\mathbf{f}\|_{L^2(0,T;(L^2(D_\varepsilon))^2)} + \|\mathbf{g}\|_{L^2(0,T;(L^2(-1/2,1/2))^2)}).
\end{aligned} \tag{4}$$

Remark 3.1. The constant C which appears in the previous inequalities is independent of ε. Since $\mathbf{f}$ is defined on $D_\varepsilon \times (0,T)$, $\|\mathbf{f}\|_{L^2(0,T;(L^2(D_\varepsilon))^2)}$ may depend on ε. The classical formulation (1) and the variational formulation (3) are equivalent in the following sense: Let $(\mathbf{u}, p, \mathbf{d})$ be a classical solution of (1). Then $(\mathbf{u}, \mathbf{d})$ is a solution for (3).

Conversely, let $(\mathbf{u}, \mathbf{d})$ be a solution of (3). Then $(\mathbf{u}, p, \mathbf{d})$ satisfies (1) a. e. in $(0,T)$, with p the unique function introduced in Theorem 3.1.

4. **Asymptotic expansions.** Assume that

$$\begin{cases}
\psi,\, g_\pm \in C^\infty([-1/2,1/2]\times[0,T]),\\
\mathbf{f} = f_1(x_1,\ t)\ \mathbf{e}_1,\ \ f_1 \in C^\infty([-1/2,1/2]\times[0,T]),\\
\exists\, t^* < T \text{ s. t. } f_1(x_1,t) = g(x_1,t) = \psi(\xi_2,t) = 0\\
\qquad\qquad \text{in } [-1/2,1/2]\times[-1/2,1/2]\times[0,t^*].
\end{cases} \tag{5}$$

In the sequel we take $\delta = \varepsilon^\gamma$, with $\gamma \in \mathbb{N},\ \gamma \ge 3$.

The asymptotic solution approximating the periodic flow in an infinite rectangle (see [12]) is modified by using two types of correctors: the first type corresponds to the variable viscosity and the second one represents the boundary layer functions, corresponding to the boundary conditions.

The asymptotic solution for the periodic case is (see [12]):

$$\begin{cases} \mathbf{u}^{(K)}(x_1, \frac{x_2}{\varepsilon}, t) = \sum_{j=0}^{K} \varepsilon^{j+2} u_{1,j}\big(x_1, \frac{x_2}{\varepsilon}, t)\mathbf{e}_1 + \sum_{j=0}^{K} \varepsilon^{j+3} u_{2,j}\big(x_1, \frac{x_2}{\varepsilon}, t)\mathbf{e}_2, \\ p^{(K)}(x_1, \frac{x_2}{\varepsilon}, t) = \sum_{j=0}^{K} \varepsilon^{j+1} p_j\big(x_1, \frac{x_2}{\varepsilon}, t) + \sum_{j=0}^{K} \varepsilon^{j} q_j\big(x_1, t), \\ d_{\pm}^{(K)}(x_1, t) = \sum_{j=0}^{K} \varepsilon^{j+\gamma} d_{\pm\, j}\big(x_1, t). \end{cases} \tag{6}$$

where $u_{i,j}, p_j, q_j, d_{\pm\, j}$ are some smooth functions. This ansatz should be completed with two boundary layer correctors.

a) The first type of correctors is the one introduced in [13] for the non periodic case with constant viscosity to repair the traces of ansatz (6) at the ends of the channel. For $i \in \{-1/2, 1/2\}$ we define:

$$\begin{cases} \mathbf{u}_{bl}^{(K)\, i}\big(\frac{x}{\varepsilon}, t\big) = \sum_{j=0}^{K} \varepsilon^{j+2} \mathbf{u}_j^{(i)}\big(\frac{x}{\varepsilon},\ t\big), \\ p_{bl}^{(K)\, i}\big(\frac{x}{\varepsilon}, t\big) = \sum_{j=0}^{K} \varepsilon^{j+1} p_j^{(i)}\big(\frac{x}{\varepsilon}, t\big), \\ d_{\pm\, bl}^{(K)\, i}\big(\frac{x_1}{\varepsilon}, t\big) = \sum_{j=0}^{K} \varepsilon^{j+\gamma} d_{\pm\, j}^{(i)}\big(\frac{x_1}{\varepsilon}, t\big). \end{cases} \tag{7}$$

Its terms decay exponentially as $|x \pm 1/2|/\varepsilon$ tends to infinity.

b) The second type is introduced to repair the discrepancy generated by the difference between the constant and variable viscosities in the neighborhood of the origin. We define these functions as follows:

$$\begin{cases} \mathbf{u}_{\mu}^{(K)}\big(\frac{x}{\varepsilon}, t\big) = \sum_{j=0}^{K} \varepsilon^{j+2} \mathbf{u}_j^{\mu}\big(\frac{x}{\varepsilon}, t\big), \\ p_{\mu}^{(K)}\big(\frac{x}{\varepsilon}, t\big) = \sum_{j=0}^{K} \varepsilon^{j+1} p_j^{\mu}\big(\frac{x}{\varepsilon}, t\big), \\ d_{\pm\, \mu}^{(K)}\big(\frac{x_1}{\varepsilon}, t\big) = \sum_{j=0}^{K} \varepsilon^{j+\gamma}\ d_{\pm\, j}^{\mu}\big(\frac{x_1}{\varepsilon}, t\big), \end{cases} \tag{8}$$

where the terms tend to zero as $|x|/\varepsilon$ tends to infinity.

So, the asymptotic solution contains the regular part of the solution, the correctors corresponding to the boundary conditions at the ends and the boundary layer functions corresponding to the variations of viscosity:

$$\begin{cases} \hat{\mathbf{u}}^{(K)}(x,t) = \mathbf{u}^{(K)}(x_1, \dfrac{x_2}{\varepsilon}, t) + \mathbf{u}_{bl}^{(K)\,-1/2}\Big(\dfrac{x}{\varepsilon} + \dfrac{1}{2\varepsilon}\mathbf{e}_1, t\Big) \\ \qquad\qquad + \mathbf{u}_{bl}^{(K)\,1/2}\Big(\dfrac{x}{\varepsilon} - \dfrac{1}{2\varepsilon}\mathbf{e}_1, t\Big) + \mathbf{u}_{\mu}^{(K)}\Big(\dfrac{x}{\varepsilon}, t\Big), \\ \hat{p}^{(K)}(x,t) = p^{(K)}(x_1, \dfrac{x_2}{\varepsilon}, t) + p_{bl}^{(K)\,-1/2}\Big(\dfrac{x}{\varepsilon} + \dfrac{1}{2\varepsilon}\mathbf{e}_1, t\Big) \\ \qquad\qquad + p_{bl}^{(K)\,1/2}\Big(\dfrac{x}{\varepsilon} - \dfrac{1}{2\varepsilon}\mathbf{e}_1,\ t\Big) + p_{\mu}^{(K)}\Big(\dfrac{x}{\varepsilon}, t\Big), \\ \hat{d}_{\pm}^{(K)}(x_1,t) = d_{\pm}^{(K)}(x_1, \dfrac{x_2}{\varepsilon}, t) + d_{bl}^{(K)\,-1/2}\Big(\dfrac{x_1 + 1/2}{\varepsilon}, t\Big) \\ \qquad\qquad + d_{bl}^{(K)\,1/2}\Big(\dfrac{x_1 - 1/2}{\varepsilon}, t\Big) + d_{\pm\,\mu}^{(K)}\Big(\dfrac{x_1}{\varepsilon}, t\Big). \end{cases} \tag{9}$$

Substituting these expressions (9) in problem (1), and collecting together the terms of the same order with respect to small parameter ε, we get the chain of problems for the terms of ansatz (9).

In the sequel we introduce the new variable $\xi = \frac{x}{\varepsilon}$. The boundary layer functions corresponding to the viscosity are defined for $\xi_1 \in (-\infty, \infty)$. As we previously said, we should construct the boundary layer corrector exponentially stabilizing to zero at infinities. On the other hand, it is known that the Stokes problem with an exponentially decaying right hand side in an infinite strip has a solution with the velocity satisfying this condition of the stabilization to zero at $\pm\infty$ while the pressure stabilizes to some constants. One of these constants (for instance, corresponding to $-\infty$) may be chosen equal to zero, but the other one (corresponding to $+\infty$) may be different from zero. We will subtract this constant from the pressure on the half-strip corresponding to positive values of the longitudinal variable. Consequently, the pressure (and so the stress) becomes discontinuous. To compensate this stress gap we will impose the opposite stress gap for the macroscopic pressure.

In the sequel, all the functions with the index $^+$ are defined for $x_1 > 0$ (or $\xi_1 > 0$) and those with the index $^-$ are defined for $x_1 < 0$ (or $\xi_1 < 0$). We shall use the notation: $\Pi = \mathbb{R} \times (-1/2, 1/2)$, $\Pi^+ = (0, \infty) \times (-1/2, 1/2), \Pi^- = (-\infty, 0) \times (-1/2, 1/2)$.

4.1. **The leading term.** This subsection is devoted to the description of the leading term of the asymptotic solution (9), for two different cases with respect to $\gamma : \gamma > 3$ and $\gamma = 3$.

We begin the description of the leading term of (9) with the correctors corresponding to the boundary conditions.

The problems for the leading term of the correctors corresponding to the boundary conditions

The problem for the velocity-pressure correctors corresponding to the left end is:

$$\begin{cases} -2\mathrm{div}_\xi(\mu_0 D_\xi(\mathbf{u}_0^{(-1/2)})) + \nabla_\xi p_0^{(-1/2)} = 0 \text{ in } \Pi^+ \times (0,T), \\ \mathrm{div}_\xi \mathbf{u}_0^{(-1/2)} = 0 \text{ in } \Pi^+ \times (0,T), \\ \mathbf{u}_0^{(-1/2)}(\xi_1, \pm 1/2, t) = 0, \\ \mathbf{u}_0^{(-1/2)}(0, \xi_2, t) = \Big(\psi(\xi_2, t) - u_{1,0}^-(-1/2, \xi_2, t)\Big)\mathbf{e}_1, \\ \mathbf{u}_0^{(-1/2)} \to \mathbf{0},\ p_0^{(-1/2)} \to 0 \text{ uniformly, when } \xi_1 \to \infty. \end{cases} \tag{10}$$

The boundary layer pair for the right end, $(\mathbf{u}_0^{(1/2)}, p_0^{(1/2)})$, satisfies a similar problem in $\Pi^- \times (0,T)$.

Now let us describe the problems for the leading term of the regular part of the asymptotic solution and of the correctors corresponding to the variable viscosity.

4.1.1. *The case $\gamma > 3$.* **a) The problem for the leading term of the regular part of the asymptotic solution**

First, let us specify functions $(q_0, d_{\pm 0})$. Three functions: first $d_0 = d_{+0} + d_{-0}$ and then q_0, d_{+0} are determined by solving the following problems: the Dirichlet problem for the forth order differential equation in $(-1/2, 1/2) \times (0,T)$

$$\begin{cases} \dfrac{\partial^4 d_0}{\partial x_1^4}(x_1,t) = g_-(x_1,t) + g_+(x_1,t), \\ d_0(-1/2,t) = d_0(1/2,t) = 0, \\ \dfrac{\partial d_0}{\partial x_1}(-1/2,t) = \dfrac{\partial d_0}{\partial x_1}(1/2,t) = 0, \end{cases} \tag{11}$$

and the system in $(-1/2, 1/2) \times (0,T)$

$$\begin{cases} \dfrac{\partial q_0}{\partial x_1} = f_1 - 12\mu \displaystyle\int_{-1/2}^{1/2} \psi(\xi_2,t)\mathrm{d}\xi_2, \\ \dfrac{\partial^4 d_{+0}}{\partial x_1^4} = q_0 + g_+, \\ d_{+0}(-1/2,t) = d_{+0}(1/2,t) = 0, \\ \dfrac{\partial d_{+0}}{\partial x_1}(-1/2,t) = \dfrac{\partial d_{+0}}{\partial x_1}(1/2,t) = 0, \\ 2\displaystyle\int_{-1/2}^{1/2} d_{+0}(x_1,t)\mathrm{d}x_1 = \int_{-1/2}^{1/2} d_0(x_1,t)\mathrm{d}x_1. \end{cases} \tag{12}$$

b) The problem for the leading term of the correctors corresponding to the variable viscosity

At the second step let us define the boundary layer correctors $(\mathbf{u}_0^\mu, p_0^{\mu^\pm})$:

$$\begin{cases} -2\mathrm{div}_\xi(\mu_0 D_\xi(\mathbf{u}_0^\mu)) + \nabla_\xi p_0^{\mu^\pm} = -12\Big(\displaystyle\int_{-1/2}^{1/2} \psi(\xi_2,t)\mathrm{d}\xi_2\Big) \\ \qquad \Big((\xi_2 \dfrac{\partial(\mu-\mu_0)}{\partial \xi_2} + (\mu-\mu_0))\mathbf{e}_1 + \xi_2 \dfrac{\partial(\mu-\mu_0)}{\partial \xi_1}\mathbf{e}_2\Big) \text{ in } \Pi^\pm \times (0,T), \\ \mathrm{div}_\xi \mathbf{u}_0^\mu = 0 \text{ in } \Pi \times (0,T), \\ \mathbf{u}_0^\mu(\xi_1, \pm 1/2, t) = 0, \\ [\mathbf{u}_0^\mu]_{/\xi_1=0} = \mathbf{0}, \\ [-p_0^\mu I + 2\mu_0 D_\xi(\mathbf{u}_0^\mu)]_{/\xi_1=0}\mathbf{e}_1 = c_0(t)\mathbf{e}_1, \\ \mathbf{u}_0^\mu \to \mathbf{0} \text{ uniformly, when } \xi_1 \to \pm\infty, \\ p_0^{\mu^+} \to 0 \text{ uniformly, when } \xi_1 \to \infty, \\ p_0^{\mu^-} \to 0 \text{ uniformly, when } \xi_1 \to -\infty. \end{cases} \tag{13}$$

Here c_0 is a function defined in the following way: we consider (13) with condition $(13)_4$ replaced by $[-p_0^\mu I + 2\mu_0 D_\xi(\mathbf{u}_0^\mu)]_{/\xi_1=0}\mathbf{e}_1 = \mathbf{0}$ and with condition $(13)_7$ replaced by $p_0^{\mu^+} \to c_0(t)$ uniformly, when $\xi_1 \to \infty$; let $(\bar{\mathbf{u}}_0^\mu, \bar{p}_0^\mu)$ be solution to this problem (its existence follows from [6], [11]); then we set $\mathbf{u}_0^\mu = \bar{\mathbf{u}}_0^\mu, p_0^{\mu^-} = \bar{p}_0^\mu, p_0^{\mu^+} = \bar{p}_0^\mu - c_0(t)$.

c) The overall leading term of the asymptotic solution

The asymptotic solution can be now written as follows:

$$\begin{cases} \mathbf{u}(x_1,x_2,t) = -6\Big(\int_{-1/2}^{1/2}\psi(\xi_2,t)\mathrm{d}\xi_2\Big)\big(x_2^2 - \frac{\varepsilon^2}{4}\big)\mathbf{e_1} + \varepsilon^2\Big(\mathbf{u}_0^{(-1/2)}(\frac{x}{\varepsilon} + \frac{1}{2\varepsilon}\mathbf{e_1}, t) + \\ \mathbf{u}_0^{(1/2)}(\frac{x}{\varepsilon} - \frac{1}{2\varepsilon}\mathbf{e_1}, t) + \mathbf{u}_0^\mu(\frac{x}{\varepsilon}, t)\Big) + \mathbf{r}_\varepsilon^1,\ (x_1,x_2,t)\in(-1/2,1/2)\times(-\varepsilon/2,\varepsilon/2)\times(0,T) \\ p(x_1,x_2,t) = q_0(x_1,t) + \varepsilon\Big(p_0^{(-1/2)}(\frac{x}{\varepsilon} + \frac{1}{2\varepsilon}\mathbf{e_1}, t) + p_0^{(1/2)}(\frac{x}{\varepsilon} - \frac{1}{2\varepsilon}\mathbf{e_1}, t) \\ \quad + p_0^{\mu^+}(\frac{x}{\varepsilon}, t)\Big) + r_\varepsilon^{2+},\ (x_1,x_2,t) \in (0,1/2)\times(-\varepsilon/2,\varepsilon/2)\times(0,T) \\ p(x_1,x_2,t) = q_0(x_1,t) + \varepsilon\Big(p_0^{(-1/2)}(\frac{x}{\varepsilon} + \frac{1}{2\varepsilon}\mathbf{e_1}, t) + p_0^{(1/2)}(\frac{x}{\varepsilon} - \frac{1}{2\varepsilon}\mathbf{e_1}, t) \\ \quad + p_0^{\mu^-}(\frac{x}{\varepsilon}, t)\Big) + r_\varepsilon^{2-},\ (x_1,x_2,t) \in (-1/2,0)\times(-\varepsilon/2,\varepsilon/2)\times(0,T) \\ d_\pm(x_1,t) = \varepsilon^\gamma\, d_{\pm 0}(x_1,t) + r_{\pm\varepsilon}^3,\ (x_1,t) \in (-1/2,1/2)\times(0,T), \end{cases} \tag{14}$$

where $\mathbf{r}_\varepsilon^1$, $r_\varepsilon^{2\pm}$, $r_{\pm\varepsilon}^3$ are residuals which will be estimated in the last section: $\mathbf{r}_\varepsilon^1 = O(\varepsilon^3)$, $r_\varepsilon^{2\pm} = O(\varepsilon)$ in the norm $L^2((0,T);L^2(\Omega_\varepsilon))$ and $r_{\pm\varepsilon}^3 = O(\varepsilon^{\gamma+1})$ in the norm $L^2((0,T);L^2(-1/2,1/2))$.

Mention that problems (10), (11), (12), (13) have unique solutions, so the leading term of the asymptotic solution is completly determined by solving these problems.

4.1.2. *The case* $\gamma = 3$. In this case, the main term of the regular part of the asymptotic solution is different from that obtained for $\gamma > 3$. We obtain a greater transversal velocity, since in this case $u_{2,0}(x_1, \frac{x_2}{\varepsilon}, t) \neq 0$.

a) The problem for the leading term of the regular part of the asymptotic solution

First, as above, solve the Dirichlet problem (11) for the forth order differential equation in $(-1/2,1/2)\times(0,T)$ and find d_0, then we solve the sixth order parabolic initial boundary problem in $(-1/2,1/2)\times(0,T)$ for the displacement of the upper elastic boundary:

$$\begin{cases} \frac{\partial d_{+0}}{\partial t}(x_1,t) - \frac{1}{24\mu}\frac{\partial^6 d_{+0}}{\partial x_1^6} = -\frac{1}{24\mu}\Big(\frac{\partial^2 g_+}{\partial x_1^2}(x_1,t) + \frac{\partial f_1(x_1,t)}{\partial x_1}\Big) + \frac{1}{2}\frac{\partial d_0}{\partial t}(x_1,t), \\ d_{+0}(-1/2,t) = d_{+0}(1/2,t) = 0, \\ \frac{\partial d_{+0}}{\partial x_1}(-1/2,t) = \frac{\partial d_{+0}}{\partial x_1}(1/2,t) = 0, \\ \Big(\frac{\partial^5 d_{+0}}{\partial x_1^5}\Big)(-1/2,t) = f_1(-1/2,t) + \frac{\partial g_+}{\partial x_1}(-1/2,t) - 12\mu\int_{-1/2}^{1/2}\psi(\xi_2,t)\mathrm{d}\xi_2, \\ 2\int_{-1/2}^{1/2} d_{+0}(x_1,t)\mathrm{d}x_1 = \int_{-1/2}^{1/2} d_0(x_1,t)\mathrm{d}x_1, \\ d_{+0}(x_1,0) = 0. \end{cases} \tag{15}$$

We mention that integrating equation (15)$_1$ in time and space it is easy to see that conditions (15)$_4$-(15)$_5$ imply the following relation

$$\Big(\frac{\partial^5 d_{+0}}{\partial x_1^5}\Big)(1/2,t) = f_1(1/2,t) + \frac{\partial g_+}{\partial x_1}(1/2,t) - 12\mu \int_{-1/2}^{1/2} \psi(\xi_2,t)\mathrm{d}\xi_2.$$

Finally, define $d_{-0} = d_0 - d_{+0}$ and

$$\begin{cases} q_0(x_1,t) = \dfrac{\partial^4 d_{+0}}{\partial x_1^4}(x_1,t) - g_+(x_1,t), \\ u_{1,0}^{\pm}(x_1,\xi_2,t) = \dfrac{1}{2\mu}\Big((\dfrac{\partial q_0}{\partial x_1})^{\pm}(x_1,t) - f_1(x_1,t)\Big)(\xi_2^2 - \dfrac{1}{4}), \\ u_{2,0}(x_1,\xi_2,t) = \dfrac{\partial d_{-0}}{\partial t}(x_1,t) - 6\dfrac{\partial}{\partial t}(d_{+0}(x_1,t) - d_{-0}(x_1,t))\displaystyle\int_{-1/2}^{\xi_2}(\tau^2 - \dfrac{1}{4})\mathrm{d}\tau. \end{cases} \tag{16}$$

b) The problem for the leading term of the correctors corresponding to the variable viscosity

At the second step let us define the boundary layer correctors $(\mathbf{u}_0^{\mu}, p_0^{\mu^{\pm}})$:

$$\begin{cases} -2\mathrm{div}_{\xi}(\mu_0 D_{\xi}(\mathbf{u}_0^{\mu^{\pm}})) + \nabla_{\xi} p_0^{\mu^{\pm}} = \dfrac{1}{\mu}\Big(\Big(\dfrac{\partial q_0}{\partial x_1}\Big)(0,t) - f_1(0,t)\Big) \\ \Big((\xi_2\dfrac{\partial(\mu-\mu_0)}{\partial \xi_2} + (\mu-\mu_0))\mathbf{e}_1 + \xi_2\dfrac{\partial(\mu-\mu_0)}{\partial \xi_1}\mathbf{e}_2\Big) \text{ in } \Pi^{\pm}\times(0,T), \\ \mathrm{div}_{\xi}\mathbf{u}_0^{\mu^{\pm}} = 0 \text{ in } \Pi^{\pm}\times(0,T), \\ \mathbf{u}_0^{\mu^{\pm}}(\xi_1,\pm 1/2,t) = 0, \\ [\mathbf{u}_0^{\mu}]_{/\xi_1=0} = \mathbf{0}, \\ [-p_0^{\mu} I + 2\mu_0 D_{\xi}(\mathbf{u}_0^{\mu})]]_{/\xi_1=0}\mathbf{e}_1 = c_0(t)\mathbf{e}_1, \\ \mathbf{u}_0^{\mu^{+}} \to \mathbf{0} \text{ uniformly, when } \xi_1\to\infty,\ \mathbf{u}_0^{\mu^{-}} \to \mathbf{0} \text{ uniformly, when } \xi_1\to-\infty, \\ p_0^{\mu^{+}} \to 0 \text{ uniformly, when } \xi_1\to\infty,\ p_0^{\mu^{-}} \to 0 \text{ uniformly, when } \xi_1\to-\infty. \end{cases} \tag{17}$$

Here c_0 is defined in the same way as in (13).

c) The overall leading term of the asymptotic solution

The asymptotic solution is given by:

$$\begin{cases} \mathbf{u}(x_1,x_2,t)=\varepsilon^2\Big(u^+_{1,0}(x_1,\dfrac{x_2}{\varepsilon},t)\mathbf{e}_1+\mathbf{u}_0^{(-1/2)}(\dfrac{x}{\varepsilon}+\dfrac{1}{2\varepsilon}\mathbf{e}_1,t) \\ \qquad +\mathbf{u}_0^{(1/2)}(\dfrac{x}{\varepsilon}-\dfrac{1}{2\varepsilon}\mathbf{e}_1,t)+\mathbf{u}_0^{\mu+}(\dfrac{x}{\varepsilon},t)\Big)+\varepsilon^3u_{2,0}(x_1,\dfrac{x_2}{\varepsilon},t)\mathbf{e}_2+\mathbf{R}^{1+}_\varepsilon, \\ \qquad (x_1,x_2,t)\in(0,1/2)\times(-\varepsilon/2,\varepsilon/2)\times(0,T), \\ \mathbf{u}(x_1,x_2,t)=\varepsilon^2\Big(u^-_{1,0}(x_1,\dfrac{x_2}{\varepsilon},t)\mathbf{e}_1+\mathbf{u}_0^{(-1/2)}(\dfrac{x}{\varepsilon}+\dfrac{1}{2\varepsilon}\mathbf{e}_1,t) \\ \qquad +\mathbf{u}_0^{(1/2)}(\dfrac{x}{\varepsilon}-\dfrac{1}{2\varepsilon}\mathbf{e}_1,t)+\mathbf{u}_0^{\mu-}(\dfrac{x}{\varepsilon},t)\Big)+\varepsilon^3u_{2,0}(x_1,\dfrac{x_2}{\varepsilon},t)\mathbf{e}_2+\mathbf{R}^{1-}_\varepsilon, \\ \qquad (x_1,x_2,t)\in(-1/2,0)\times(-\varepsilon/2,\varepsilon/2)\times(0,T), \\ p(x_1,x_2,t)=q_0(x_1,t)+\varepsilon\Big(p_0^{(-1/2)}(\dfrac{x}{\varepsilon}+\dfrac{1}{2\varepsilon}\mathbf{e_1},t)+p_0^{(1/2)}(\dfrac{x}{\varepsilon}-\dfrac{1}{2\varepsilon}\mathbf{e_1},t) \\ \qquad +p_0^{\mu+}(\frac{x}{\varepsilon},t)\Big)+R^{2+}_\varepsilon,\ (x_1,x_2,t)\in(0,1/2)\times(-\varepsilon/2,\varepsilon/2)\times(0,T), \\ p(x_1,x_2,t)=q_0(x_1,t)+\varepsilon\Big(p_0^{(-1/2)}(\dfrac{x}{\varepsilon}+\dfrac{1}{2\varepsilon}\mathbf{e_1},t)+p_0^{(1/2)}(\dfrac{x}{\varepsilon}-\dfrac{1}{2\varepsilon}\mathbf{e_1},t) \\ \qquad +p_0^{\mu-}(\frac{x}{\varepsilon},t)\Big)+R^{2-}_\varepsilon,\ (x_1,x_2,t)\in(-1/2,0)\times(-\varepsilon/2,\varepsilon/2)\times(0,T), \\ d_\pm(x_1,t)=\varepsilon^3\,d_{\pm0}(x_1,t)+R^3_{\pm\varepsilon},\ (x_1,t)\in(0,1/2)\times(0,T). \end{cases} \tag{18}$$

The residuals satisfy the same estimates as in the case $\gamma>3$.

More details for the order of the residuals with respect to some corresponding norms will be established in the last section.

The influence of the variable viscosity in this approximation is represented by the boundary layer term $(\mathbf{u}_0^{\mu^\pm},p_0^{\mu^\pm})$.

5. **Problems, properties of the correctors, the order of solving the problems.** The first subsection of this section is devoted to the jump conditions. The regular part of the asymptotic solution and the correctors corresponding to the viscosity have jumps in 0. In the sequel we shall obtain the relations between these jumps.

5.1. **The jump relations.** We shall denote by $[\cdot]$ the jump of a function in $x_1=0$ or $\xi_1=0$. The jump relations are obtained as a consequence of the regularity of the asymptotic solution given below:

$$\begin{cases} [\hat{\mathbf{u}}^{(K)}]_{/x_1=0}=\mathbf{0}, \\ [-\hat{p}^{(K)}I+2\mu_0D_x(\hat{\mathbf{u}}^{(K)})]_{/x_1=0}\mathbf{e}_1=\mathbf{0}, \\ \Big[\dfrac{\partial^a\hat{d}^{(K)}_\pm}{\partial x_1^a}\Big]_{/x_1=0}=0,\ a\in\{0,1,2,3\}. \end{cases} \tag{19}$$

From the previous system we obtain the following jump relations:

- The jump conditions for the velocity

$$\begin{cases} [u_{1,l}]_{/x_1=0}+[u^\mu_{1,l}]_{/\xi_1=0}=0, \\ [u_{2,l-1}]_{/x_1=0}+[u^\mu_{2,l}]_{/\xi_1=0}=0. \end{cases} \tag{20}$$

- The jump condition involving the pressure

$$\Big(-([p_{l-1}]_{/x_1=0}+[q_l]_{/x_1=0}+[p^{\mu}_{l-1}]_{/\xi_1=0})+2\mu_0\Big(\Big[\frac{\partial u_{1,l-2}}{\partial x_1}\Big]_{/x_1=0}+\Big[\frac{\partial u^{\mu}_{1,l-1}}{\partial \xi_1}\Big]_{/\xi_1=0}\Big)\Big)\mathbf{e}_1$$
$$+\mu_0\Big(\Big[\frac{\partial u_{1,l-1}}{\partial \xi_2}\Big]_{/x_1=0}+\Big[\frac{\partial u^{\mu}_{1,l-1}}{\partial \xi_2}\Big]_{/\xi_1=0}+\Big[\frac{\partial u_{2,l-3}}{\partial x_1}\Big]_{/x_1=0}+\Big[\frac{\partial u^{\mu}_{2,l-1}}{\partial \xi_1}\Big]_{/\xi_1=0}\Big)\mathbf{e}_2=\mathbf{0}. \tag{21}$$

- The jump conditions for the displacement

$$\Big[\frac{\partial^a d_{\pm\,l-a}}{\partial x_1^a}\Big]_{/x_1=0}+\Big[\frac{\partial^a d^{\mu}_{\pm\,l}}{\partial \xi_1^a}\Big]_{/\xi_1=0}=0,\ \ a\in\{0,1,2,3\}. \tag{22}$$

5.2. **Problems for $(\mathbf{u}_l^{\pm},p_l^{\pm},q_l^{\pm},d^{\pm}_{\pm\,l})$.** Introducing the asymptotic expansions into (1) and identifying the coefficients of the powers of ε we obtain for $(\mathbf{u}_l^{+},p_l^{+},q_l^{+},d^{+}_{\pm\,l})$ the following problem in $(0,1/2)\times(-1/2,1/2)\times(0,T)$:

$$\begin{cases} -\mu\dfrac{\partial^2 u^+_{1,l}}{\partial \xi_2^2}+\dfrac{\partial q^+_l}{\partial x_1}=f_1\delta_{l0}-\dfrac{\partial p^+_{l-1}}{\partial x_1}+\mu\dfrac{\partial^2 u^+_{1,l-2}}{\partial x_1^2}-\rho_f\dfrac{\partial u^+_{1,l-2}}{\partial t},\\ \dfrac{\partial p^+_l}{\partial \xi_2}=\mu\Big(\dfrac{\partial^2 u^+_{2,l-3}}{\partial x_1^2}+\dfrac{\partial^2 u^+_{2,l-1}}{\partial \xi_2^2}\Big)-\rho_f\dfrac{\partial u^+_{2,l-3}}{\partial t}\\ \dfrac{\partial u^+_{1,l}}{\partial x_1}+\dfrac{\partial u^+_{2,l}}{\partial \xi_2}=0,\\ \mathbf{u}^+_l\Big(x_1,\pm1/2,t\Big)=\dfrac{\partial d^+_{\pm\,l+3-\gamma}}{\partial t}(x_1,t)\mathbf{e}_2,\\ \dfrac{\partial^4 d^+_{+\,l}}{\partial x_1^4}-q^+_l=g_+\delta_{l0}-\rho h\dfrac{\partial^2 d^+_{+\,l-\gamma}}{\partial t^2}-\nu\dfrac{\partial^5 d^+_{+\,l-\gamma}}{\partial x_1^4\partial t}+p^+_{l-1}/_{\xi_2=1/2},\\ \dfrac{\partial^4 d^+_{-\,l}}{\partial x_1^4}+q^+_l=g_-\delta_{l0}-\rho h\dfrac{\partial^2 d^+_{-\,l-\gamma}}{\partial t^2}-\nu\dfrac{\partial^5 d^+_{-\,l-\gamma}}{\partial x_1^4\partial t}-p^+_{l-1}/_{\xi_2=-1/2},\\ d^+_{\pm\,l}(1/2,t)=-d^{(1/2)}_{\pm\,l}(0,\ t),\\ \dfrac{\partial d^+_{\pm\,l}}{\partial x_1}(1/2,t)=-\dfrac{\partial d^{(1/2)}_{\pm\,l+1}}{\partial \xi_1}(0,t). \end{cases} \tag{23}$$

Remark 5.1. We obtain a similar problem for the functions $(\mathbf{u}_l^{-},p_l^{-},q_l^{-},d^{-}_{\pm\,l})$ in $(-1/2,0)\times(-1/2,1/2)\times(0,T)$.

Introducing now the asymptotic expansions into (2), we get

$$\begin{aligned}\int_{-1/2}^{1/2}(d_{+\,l}(x_1,t)-d_{-\,l}(x_1,t))\mathrm{d}x_1&=-\int_{-\infty}^{\infty}(d^{\mu}_{+\,l-1}(\xi_1,t)-d^{\mu}_{-\,l-1}(\xi_1,t))\mathrm{d}\xi_1\\ -\int_{-\infty}^{0}(d^{(1/2)}_{+\,l-1}(\xi_1,t)-d^{(1/2)}_{-\,l-1}(\xi_1,t))\mathrm{d}\xi_1&-\int_{0}^{\infty}(d^{(-1/2)}_{+\,l-1}(\xi_1,t)-d^{(-1/2)}_{-\,l-1}(\xi_1,t))\mathrm{d}\xi_1.\end{aligned} \tag{24}$$

Remark 5.2. The junction conditions for $(\mathbf{u}_l^{+},p_l^{+},q_l^{+},d^{+}_{\pm\,l})$ and $(\mathbf{u}_l^{-},p_l^{-},q_l^{-},d^{-}_{\pm\,l})$ will be obtained below from the compatibility condition (26), derived in the next subsection.

5.3. **Problems for** $(\mathbf{u}_l^{(-1/2)}, p_l^{(-1/2)}, d_{\pm\,l}^{(-1/2)})$, $(\mathbf{u}_l^{(1/2)}, p_l^{(1/2)}, d_{\pm\,l}^{(1/2)})$**.** For the boundary layers corresponding to the left side we obtain two decoupled problems: the first one for $(\mathbf{u}_l^{(-1/2)}, p_l^{(-1/2)})$ and the second one corresponding to the displacements $d_{\pm\,l}^{(-1/2)}$.

The problem:

$$\begin{cases} -2\mathrm{div}_\xi(\mu_0 D_\xi(\mathbf{u}_l^{(-1/2)})) + \nabla_\xi p_l^{(-1/2)} = -\rho_f \dfrac{\partial \mathbf{u}_{l-2}^{(-1/2)}}{\partial t} \text{ in } \Pi^+ \times (0,T), \\ \mathrm{div}_\xi \mathbf{u}_l^{(-1/2)} = 0 \text{ in } \Pi^+ \times (0,T), \\ \mathbf{u}_l^{(-1/2)}(\xi_1, \pm 1/2, t) = \dfrac{\partial d_{\pm\,l+2-\gamma}^{(-1/2)}}{\partial t}(\xi_1,t)\mathbf{e}_2, \\ \mathbf{u}_l^{(-1/2)}(0,\xi_2,t) = \boldsymbol{\psi}(\xi_2,t)\delta_{l0} - u_{1,l}^-(-1/2,\xi_2,t)\mathbf{e}_1 - u_{2,l-1}^-(-1/2,\xi_2,t)\mathbf{e}_2, \\ \mathbf{u}_l^{(-1/2)} \to \mathbf{0},\ p_l^{(-1/2)} \to 0 \text{ uniformly, when } \xi_1 \to \infty, \end{cases} \tag{25}$$

gives the boundary layer correctors for the velocity and for the pressure corresponding to the left end.

Mention that the following compatibility condition holds:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \left(d_{+\,l+2-\gamma}^{(-1/2)} - d_{-\,l+2-\gamma}^{(-1/2)}\right)(\xi_1,t)\mathrm{d}\xi_1 + \int_{-1/2}^{1/2} u_{1,l}^-(-1/2,\xi_2,t)\mathrm{d}\xi_2 \\ -\delta_{l0}\int_{-1/2}^{1/2} \psi(\xi_2,t)\mathrm{d}\xi_2 = 0. \end{aligned} \tag{26}$$

Remark 5.3. Due to the regularity of the boundary layer pressure at infinity, this function is uniquely defined without any additional condition. The compatibility relation (26) generates an additional condition for the problem (23), as we shall see at the end of this section.

The boundary layer correctors for the displacements, corresponding to the left end of the channel, are obtained as solutions of the following problems:

$$\begin{cases} \dfrac{\partial^4 d_{\pm\,l}^{(-1/2)}}{\partial \xi_1^4} = -\rho h \dfrac{\partial^2 d_{\pm\,l-4-\gamma}^{(-1/2)}}{\partial t^2} - \nu \dfrac{\partial^5 d_{\pm\,l-\gamma}^{(-1/2)}}{\partial \xi_1^4 \partial t} \pm p_{l-5}^{(-1/2)}/_{\xi_2=\pm 1/2} \\ \qquad\qquad \text{in } (0,\infty)\times(0,T), \\ \dfrac{\partial^a d_{\pm\,l}^{(-1/2)}}{\partial \xi_1^a} \to 0 \text{ uniformly, when } \xi_1 \to \infty,\ \forall\ a \in \{0,1,2,3\}. \end{cases} \tag{27}$$

In a similar way we introduce the boundary layer correctors corresponding to the right end of the channel. The boundary layers for the velocity and pressure are defined on $\Pi^- \times (0,T)$ and the boundary layers for the displacement are defined on $(-\infty,0)\times(0,T)$.

Remark 5.4. For $\gamma > 3$ all the problems (23), (25) and (27) are steady-state, while for $\gamma = 3$ only problems (25) and (27) are steady-state, the time variable appears in these cases as a parameter. However, all unknowns must satisfy the homogeneous conditions for $t = 0$. These conditions hold due to the hypothesis $(5)_3$.

An important property of the boundary layers in the channel is the property of their exponential stabilization to zero at the infinity. This property insures their presence in some small neighborhood of the ends and of the middle point of the channel. They can be neglected out of the circles of diameter of order $\varepsilon ln(\varepsilon)$, havind the centres at the ends and the middle point. Let us formulate the corresponding theorem.

Theorem 5.1. *For all $l \in \mathbb{N}$, the problem (25) has a unique solution $(\mathbf{u}_l^{(-1/2)}, p_l^{(-1/2)})$ and (27) a unique solution $d_{\pm l}^{(-1/2)}$. There exist α_l, $M_l > 0$, such that for all $R > 1/2$ and for all $t \in [0,T]$*

$$\begin{cases} \|\mathbf{u}_l^{(-1/2)}(t)\|_{(H^1(\Pi^+\cap\{\xi_1>R\}))^2} \leq M_l \exp(-\alpha_l R), \\ \|\nabla p_l^{(-1/2)}(t)\|_{(L^2(\Pi^+\cap\{\xi_1>R\}))^2} \leq M_l \exp(-\alpha_l R), \\ \Big|\dfrac{\partial^m d_{\pm l}^{(-1/2)}}{\partial \xi_1^m}(t)\Big| \leq M_l \exp(-\alpha_l \xi_1),\ \forall\ m \in \mathbb{N},\ \xi_1 > 1. \end{cases} \tag{28}$$

Moreover, the property $p_l^{(-1/2)} \to 0$ when $\xi_1 \to \infty$ yields:

$$|p_l^{(-1/2)}(\xi_1, \xi_2, t)| \leq M_l \exp(-\alpha_l \xi_1),\ \forall\ \xi_1 > 1,\ \xi_2 \in (-1/2, 1/2). \tag{29}$$

Proof. The existence and uniqueness of $\mathbf{u}_l^{(-1/2)}$ and the existence of $p_l^{(-1/2)}$ are obtained in a classical way. The uniqueness of $p_l^{(-1/2)}$ is a consequence of $(25)_5$. The estimates stated by this theorem will be proved recursively with respect to l.

For any $R > 1/2$ we define $\Pi_R^+ = \Pi^+ \cap \{\xi_1 > R\}$.

For $l = 0$ (25) becomes:

$$\begin{cases} -2\mu\Delta_\xi \mathbf{u}_0^{(-1/2)} + \nabla_\xi p_0^{(-1/2)} = 0 \text{ in } \Pi_R^+ \times (0,T), \\ \text{div}_\xi\ \mathbf{u}_0^{(-1/2)} = 0 \text{ in } \Pi_R^+ \times (0,T), \\ \mathbf{u}_0^{(-1/2)}(\xi_1, \pm\dfrac{1}{2}, t) = \mathbf{0} \\ \mathbf{u}_0^{(-1/2)}(0, \xi_2, t) = -u_{1,0}(-1/2, \xi_2, t)\mathbf{e}_1 + \psi(\xi_2, t)\mathbf{e}_1, \\ \mathbf{u}_0^{(-1/2)} \to \mathbf{0},\ p_0^{(-1/2)} \to 0 \text{ uniformly, when } \xi_1 \to \infty, \end{cases} \tag{30}$$

The previous problem is similar to the problems studied in [6], Ch. VI. The difference is that our problem depends on a parameter, the time variable. Hence, we can obtain the exponential decay of the boundary layer correctors as in [6], not with constants α_0, M_0, but with functions depending on t. To obtain (28) for $l = 0$, we follow the ideas of Proposition 4.1, [13]: we prove the boundedness of M_0 and the positiveness of α_0 uniformly with respect to the time variable.

Taking into account the problem satisfied by the boundary layer correctors corresponding to the displacements, (27), it follows that the functions $d_{\pm l}^{(-1/2)}$ are equal to zero at least for $l = 0, 1, ..., 4$.

We suppose that the estimates (28) are satisfied for $0, 1, ..., l-1$ and we prove them for l.

In contrast with (30), in (25) we have, for a general value of l, non homogeneous right hand sides in $(25)_1$ and $(25)_3$. For this reason, the technique of [6] cannot be applied directly. To overcome this difficulty, we proposed in [13] a method based on the construction of several auxiliary functions. We extend this method for the present case, when both the upper and the lower boundaries are elastic.

The first auxiliary function is

$$\begin{aligned}\mathbf{v}_l^{(-1/2)}(\xi_1,\xi_2,t) &= \mathbf{u}_l^{(-1/2)}(\xi_1,\xi_2,t)\\ &-\Big((\xi_2+\frac{1}{2})\frac{\partial d^{(-1/2)}_{+\,l+2-\gamma}}{\partial t}(\xi_1,t)-(\xi_2-\frac{1}{2})\frac{\partial d^{(-1/2)}_{-\,l+2-\gamma}}{\partial t}(\xi_1,t)\Big)\mathbf{e}_2.\end{aligned} \tag{31}$$

In contrast with $\mathbf{u}_l^{(-1/2)}$, this new function satisfies homogeneous boundary conditions, but is not a divergence free function.

We follow the same steps as in [13] introducing the new function

$$\zeta(R,\xi_2,t)=(v_l^{(-1/2)})_1(R,\xi_2,t)-\int_R^\infty \frac{\partial(d^{(-1/2)}_{+\,l+2-\gamma}-d^{(-1/2)}_{-\,l+2-\gamma})}{\partial t}(\xi_1,t)\mathrm{d}\xi_1. \tag{32}$$

Repeating literally the arguments of the proofs of Theorem 4.1 and Proposition 4.1 of [13] we complete the proof of the theorem. □

Similar results can be proved for the boundary layer functions corresponding to the right end.

5.4. **Problems for $(\mathbf{u}_l^{\mu^\pm},p_l^{\mu^\pm},d^{\mu^\pm}_{\pm\,l})$.** This section is devoted to the study of the problems satisfied by the boundary layer correctors corresponding to the variable viscosity. Since the problems for $(\mathbf{u}_l^{\mu^\pm},p_l^{\mu^\pm})$ and for $d^{\mu^\pm}_{\pm\,l}$ are not coupled, we can solve them separately. We notice that the situation is different for these two problems: the problems for $d^{\mu^+}_{\pm\,l}$ and for $d^{\mu^-}_{\pm\,l}$ are independent one from the other, with a solution uniquely determined by "the stabilization to zero" conditions at ∞ and $-\infty$, respectively. Their unique solutions $d^{\mu^+}_{\pm\,l}$ and $d^{\mu^-}_{\pm\,l}$, respectively, substituted into (22), will give four jump relations for $d^{\pm}_{\pm\,l}$ and their x_1-derivatives up to the third order. In contrast with this situation, the problems for $(\mathbf{u}_l^{\mu^+},p_l^{\mu^+})$ and $(\mathbf{u}_l^{\mu^-},p_l^{\mu^-})$ can not be solved separately. They are solved together with the jump conditions (20) and (21).

Consider first the problems for the correctors corresponding to the displacements. The functions defined in $(0,\infty)\times(0,T)$ are given by the following obvious result:

Theorem 5.2. *The correctors corresponding to the variable viscosity for the displacement in* $(0,\infty)\times(0,T)$ *are the unique solution of the problem*

$$\begin{cases}\dfrac{\partial^4 d^{\mu^+}_{\pm\,l}}{\partial\xi_1^4}=-\rho h\dfrac{\partial^2 d^{\mu^+}_{\pm\,l-4-\gamma}}{\partial t^2}-\nu\dfrac{\partial^5 d^{\mu^+}_{\pm\,l-\gamma}}{\partial\xi_1^4\partial t}\pm p^{\mu^+}_{l-5}/_{\xi_2=\pm1/2},\\[2mm] \dfrac{\partial^a d^{\mu^+}_{\pm\,l}}{\partial\xi_1^a}\to 0 \text{ uniformly, when } \xi_1\to\infty,\ \forall\ a\in\{0,1,2,3\}.\end{cases} \tag{33}$$

In a similar way we define the functions $d^{\mu^-}_{\pm\,l}$.

The central part of the present paper is to solve the problem for $(\mathbf{u}_l^{\mu^\pm},p_l^{\mu^\pm})$. To this end, we introduce the following notations:

$$\begin{cases}\mathbf{\Psi}_l(\xi_2,t)=-[u_{1,l}]_{/x_1=0}\mathbf{e}_1-[u_{2,l-1}]_{/x_1=0}\mathbf{e}_2,\\ \boldsymbol{\varphi}_l(\xi_2,t)=\mu_0\Big(-2\Big[\dfrac{\partial u_{1,l-1}}{\partial x_1}\Big]_{/x_1=0}\mathbf{e}_1-\Big(\Big[\dfrac{\partial u_{1,l}}{\partial \xi_2}\Big]_{/x_1=0}+\Big[\dfrac{\partial u_{2,l-2}}{\partial x_1}\Big]_{/x_1=0}\Big)\mathbf{e}_2\Big),\\ \mathbf{\Phi}_l(\xi_2,t)=[p_l]_{/x_1=0}\mathbf{e}_1+\boldsymbol{\varphi}_l.\end{cases} \tag{34}$$

The existence and uniqueness of the boundary layer correctors corresponding to the velocity-pressure are obtained in several steps in the next theorem.

Theorem 5.3. *For any $t \in (0,T)$, there exists the unique pairs $(\mathbf{u}_l^{\mu^+}(t), p_l^{\mu^+}(t)) \in (H^1(\Pi^+))^2 \times L^2(\Pi^+)$, $(\mathbf{u}_l^{\mu^-}(t), p_l^{\mu^-}(t)) \in (H^1(\Pi^-))^2 \times L^2(\Pi^-)$ and a real number Q_l such that,*

$$\begin{cases} -2\mathrm{div}_\xi(\mu_0 D_\xi(\mathbf{u}_l^{\mu\pm}(t))) + \nabla_\xi p_l^{\mu^\pm}(t) = \mathbf{A}_l^\pm(t) - \rho_f \dfrac{\partial \mathbf{u}_{l-2}^{\mu^\pm}(t)}{\partial t} \text{ in } \Pi^\pm, \\ \mathrm{div}_\xi \mathbf{u}_l^{\mu^\pm}(t) = 0 \text{ in } \Pi^\pm, \\ \mathbf{u}_l^{\mu^\pm}(\xi_1, \pm 1/2, t) = \dfrac{\partial d^{\mu^\pm}_{\pm\, l+2-\gamma}}{\partial t}(\xi_1, t)\mathbf{e}_2, \\ [\mathbf{u}_l^\mu]_{/\xi_1=0} = \mathbf{\Psi}_l(\xi_2, t), \\ [-p_l^\mu I + 2\mu_0 D_\xi(\mathbf{u}_l^\mu)]_{/\xi_1=0}\mathbf{e}_1 = \mathbf{\Phi}_l(\xi_2, t) + Q_l \mathbf{e}_1, \\ \mathbf{u}_l^{\mu^\pm}(t) \to \mathbf{0}, \text{ uniformly when } \xi_1 \to \pm\infty, \\ p_l^{\mu^\pm}(t) \to 0 \text{ uniformly, when } \xi_1 \to \pm\infty. \end{cases} \tag{35}$$

Moreover, these functions have an exponential decay at ∞ and $-\infty$, respectively.

$\mathbf{A}_l^\pm = \mathbf{A}_l^\pm(\xi_1, \xi_2, t)$ are continuous functions equal to zero for $\xi_1 > 1/2(\xi_1 < -1/2$, respectively), defined in the following way: we set $2\mathrm{div}_x((\mu_\varepsilon - \mu) D_x(\mathbf{u}^{\pm(K)})) = \sum_{l=0}^K \varepsilon^l \mathbf{A}_l^\pm(\xi_1, \xi_2, t) + \varepsilon^{K+1}\mathbf{r}_K$, and we get $\mathbf{A}_l^\pm$ as the terms of the Taylor expansion with respect to $x = \varepsilon\xi$, where $-\varepsilon/2 < x_1 < \varepsilon/2$.

Taking into account the definition of $\mathbf{\Psi}_l$ and $(20)_1$, the compatibility condition for (35) is

$$\int_{-1/2}^{1/2} [u_{1,l}]_{/x_1=0}(\xi_2, t)\mathrm{d}\xi_2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} (d^\mu_{+\, l+2-\gamma} - d^\mu_{-\, l+2-\gamma})(\xi_1, t)\mathrm{d}\xi_1 = 0. \tag{36}$$

Proof. The right hand side of $(35)_1$ belongs to $L^2(\Pi^\pm)$ and exponentially stabilizes to zero. So, the theory of elliptic equations allows us to write the previous system and the following ones in a classical sense.

Denoting by $(\bar{\mathbf{u}}_l^\mu, \bar{p}_l^\mu)$ the pair defined in $\Pi \times (0,T)$ by $(\bar{\mathbf{u}}_l^\mu, \bar{p}_l^\mu)_{/\Pi^\pm \times (0,T)} = (\bar{\mathbf{u}}_l^{\mu^\pm}, \bar{p}_l^{\mu^\pm})$, we consider the following auxiliary problem:

$$\begin{cases} \text{For any } t \in (0,T), \text{find } (\bar{\mathbf{u}}_l^\mu(t), \bar{p}_l^\mu(t)) \text{ such that} \\ -2\mathrm{div}_\xi(\mu_0 D_\xi(\bar{\mathbf{u}}_l^\mu(t))) + \nabla_\xi \bar{p}_l^\mu(t) = \mathbf{A}_l(t) - \rho_f \dfrac{\partial \mathbf{u}_{l-2}^\mu(t)}{\partial t} \text{ in } \Pi, \\ \mathrm{div}_\xi \bar{\mathbf{u}}_l^\mu(t) = 0 \text{ in } \Pi, \\ \bar{\mathbf{u}}_l^\mu(\xi_1, \pm 1/2, t) = \dfrac{\partial d^\mu_{\pm\, l+2-\gamma}}{\partial t}(\xi_1, t)\mathbf{e}_2, \\ [\bar{\mathbf{u}}_l^\mu]_{/\xi_1=0} = \mathbf{\Psi}_l(\xi_2, t), \\ [-\bar{p}_l^\mu I + 2\mu_0 D_\xi(\bar{\mathbf{u}}_l^\mu)]_{/\xi_1=0}\mathbf{e}_1 = \mathbf{\Phi}_l(\xi_2, t), \\ \bar{\mathbf{u}}_l^{\mu^\pm}(t) \to \mathbf{0}, \text{ uniformly, when } \xi_1 \to \pm\infty, \\ \bar{p}_l^{\mu^-}(t) \to 0 \text{ uniformly, when } \xi_1 \to -\infty. \end{cases} \tag{37}$$

satisfying the same compatibility condition.

Let us prove the existence and uniqueness of the solution of (37): For any $t \in (0,T)$, the problem (37) has the unique solution $(\bar{\mathbf{u}}_l^\mu(t), \bar{p}_l^\mu(t))$, with $(\bar{\mathbf{u}}_l^\mu(t), \bar{p}_l^\mu(t)) \in (H^1(\Pi))^2 \times L^2_{loc}(\Pi)$.

In order to compensate the jumps which appear in (37), we prove the following auxiliary result:

Proposition 5.1. *For the functions $\mathbf{\Psi}_l$, $\boldsymbol{\varphi}_l$ given by (34) there exists at least one function $\mathbf{D}_l : \mathbb{R} \times [-1/2, 1/2] \times [0,T] \mapsto \mathbb{R}^2$ which satisfies*

$$\begin{cases} [\mathbf{D}_l]_{/\xi_1=0} = -\mathbf{\Psi}_l(\xi_2,t), \\ [2\mu_0 D_\xi(\mathbf{D}_l)]_{/\xi_1=0}\mathbf{e}_1 = -\boldsymbol{\varphi}_l(\xi_2,t). \end{cases} \tag{38}$$

Proof. We consider the functions

$$\eta_0(\xi_1) = \begin{cases} 1, & \text{if } \xi_1 \in [0,1/4), \\ \dfrac{1}{2}\Big(1+\cos(2\pi(\xi_1 - 1/4))\Big), & \text{if } \xi_1 \in [1/4,3/4], \\ 0, & \text{if } \xi_1 \in (3/4,\infty) \end{cases} \tag{39}$$

and

$$\eta_1(\xi_1) = \xi_1 \eta_0(\xi_1), \tag{40}$$

and we construct $\mathbf{D}_l = D_{1,l}\mathbf{e}_1 + D_{2,l}\mathbf{e}_2$, with the components defined as follows:

$$D_{1,l}(\xi_1,\xi_2,t) = \begin{cases} 0, & \text{if } \xi_1 \le 0, \\ -\eta_0(\xi_1)\Psi_{1,l}(\xi_2,t) - \dfrac{1}{2}\eta_0(\xi_1)\varphi_{1,l}(\xi_2,t)\displaystyle\int_0^{\xi_1} \frac{\mathrm{d}\tau}{\mu_0(\tau,\xi_2)}, & \text{if } \xi_1 > 0, \end{cases} \tag{41}$$

$$D_{2,l}(\xi_1,\xi_2,t) = \begin{cases} 0, & \text{if } \xi_1 \le 0, \\ -\eta_0(\xi_1)\Psi_{2,l} - \eta_0(\xi_1)\varphi_{2,l}\displaystyle\int_0^{\xi_1} \frac{\mathrm{d}\tau}{\mu_0(\tau,\xi_2)} + \eta_1(\xi_1)\frac{\partial \Psi_{1,l}}{\partial \xi_2}, & \text{if } \xi_1 > 0. \end{cases} \tag{42}$$

The assertion of the proposition follows with obvious computations and the proof is achieved. □

The new function defined by (41)-(42) will help us to replace the unknown pair $(\bar{\mathbf{u}}_l^\mu, \bar{p}_l^\mu)$, solution for (37), by another pair, $(\mathbf{v}_l^\mu, \pi_l^\mu)$, solution of a more convenient problem. The components of this pair are defined as follows:

$$\begin{cases} \mathbf{v}_l^\mu(\xi_1,\xi_2,t) = \bar{\mathbf{u}}_l^\mu(\xi_1,\xi_2,t) + \mathbf{D}_l(\xi_1,\xi_2,t), \\ \pi_l^\mu(\xi_1,\xi_2,t) = \begin{cases} \bar{p}_l^{\mu^-}(\xi_1,\xi_2,t), & \text{if } \xi_1 \le 0, \\ \bar{p}_l^{\mu^+}(\xi_1,\xi_2,t) + \eta_0(\xi_1)[p_l]_{/x_1=0}(\xi_2,t), & \text{if } \xi_1 > 0. \end{cases} \end{cases} \tag{43}$$

Remark 5.5. The new pair $(\mathbf{v}_l^\mu, \pi_l^\mu)$ coincides with $(\bar{\mathbf{u}}_l^\mu, \bar{p}_l^\mu)$ in $\Pi^- \times (0,T)$ and satisfies the following jump conditions in $\xi_1 = 0$:

$$\begin{cases} [\mathbf{v}_l^\mu]_{/\xi_1=0} = [\pi_l^\mu]_{/\xi_1=0} = 0, \\ [-\pi_l^\mu I + 2\mu_0 D_\xi(\mathbf{v}_l^\mu)]_{/\xi_1=0}\mathbf{e}_1 = 0. \end{cases} \tag{44}$$

In order to obtain the problem satisfied by $(\mathbf{v}_l^\mu, \pi_l^\mu)$ we introduce the following notations:

$$\begin{cases} \mathbf{F}_l = \mathbf{A}_l - \rho_f \dfrac{\partial \mathbf{u}_{l-2}^\mu}{\partial t} - 2\mathrm{div}\,(\mu_0 D_\xi(\mathbf{D}_l)) + \chi(\Pi^+)\nabla_\xi(\eta_0[p_l]_{/\xi_1=0}), \\ B_l = \mathrm{div}_\xi \mathbf{D}_l, \\ \alpha_{\pm l}(\xi_1, t) = \dfrac{\partial d^{\mu^+}_{\pm l+2-\gamma}}{\partial t}(\xi_1, t) + D_{2,l}(\xi_1, \pm 1/2, t), \end{cases} \tag{45}$$

where $\chi(\Omega)$ represents the characteristic function of the set Ω.

Using the previous notations, the next result is obvious:

Lemma 5.1. *The pair $(\bar{\mathbf{u}}_l^\mu(t), \bar{p}_l^\mu(t))$ is a solution for (37) if and only if the pair $(\mathbf{v}_l^\mu(t), \pi_l^\mu(t))$ satisfies the following problem in Π:*

$$\begin{cases} -2\mathrm{div}_\xi(\mu_0 D_\xi(\mathbf{v}_l^\mu(t))) + \nabla_\xi \pi_l^\mu(t) = \mathbf{F}_l(t), \\ \mathrm{div}_\xi \mathbf{v}_l^\mu(t) = B_l(t), \\ \mathbf{v}_l^\mu(\xi_1, \pm 1/2, t) = \alpha_{\pm l}(\xi_1, t)\mathbf{e}_2 + D_{1,l}(\xi_1, \pm 1/2, t)\mathbf{e}_1, \\ \mathbf{v}_l^\mu(t) \to \mathbf{0}, \text{ uniformly, when } \xi_1 \to \pm\infty, \\ \pi_l^\mu(t) \to 0 \text{ uniformly, when } \xi_1 \to -\infty. \end{cases} \tag{46}$$

with the compatibility condition

$$\int_\Pi B_l(\xi, t)\mathrm{d}\xi = \int_{-\infty}^{\infty} (\alpha_{+l}(\xi_1, t) - \alpha_{-l}(\xi_1, t))\mathrm{d}\xi_1. \tag{47}$$

Proof. The integrals in (47) make sense due to the definition of the functions B_l, $\alpha_{\pm l}$ and to the behavior of the correctors corresponding to the viscosity at $\pm\infty$. The equivalence between the two problems stated in Lemma 5.1 is obtained by standard computations. □

The following important result will give the existence and uniqueness of the pair $(\mathbf{v}_l^\mu, \pi_l^\mu)$ and hence, due to the previous lemma, the same result holds for $(\bar{\mathbf{u}}_l^\mu, \bar{p}_l^\mu)$, too.

Theorem 5.4. *The problem (46)-(47) has a unique solution.*

Proof. **Uniqueness.** We suppose that (46)-(47) has 2 solutions and let $(\mathbf{v}_l^\mu(t), \pi_l^\mu(t))$ be their difference. It follows that this pair is solution to the homogeneous problem:

$$\begin{cases} -2\mathrm{div}_\xi(\mu_0 D_\xi(\mathbf{v}_l^\mu(t))) + \nabla_\xi \pi_l^\mu(t) = 0, \\ \mathrm{div}_\xi \mathbf{v}_l^\mu(t) = 0, \\ \mathbf{v}_l^\mu(\xi_1, \pm 1/2, t) = 0, \\ \mathbf{v}_l^\mu(t) \to \mathbf{0}, \text{ uniformly, when } \xi_1 \to \pm\infty, \\ \pi_l^\mu(t) \to 0 \text{ uniformly, when } \xi_1 \to -\infty \end{cases} \tag{48}$$

Since the set Π is bounded in $O\xi_2$ direction, we can apply, for any fixed t, Theorem 2.1, p. 17, [16], and the uniqueness result follows.

Existence. In order to replace (46)-(47) by a problem with homogeneous boundary conditions, we introduce the new function

$$\begin{aligned} \mathbf{w}_l^\mu(\xi_1, \xi_2, t) = \mathbf{v}_l^\mu(\xi_1, \xi_2, t) &- (\xi_2 + 1/2)(D_{1,l}(\xi_1, 1/2, t)\mathbf{e}_1 + \alpha_{+l}(\xi_1, t)\mathbf{e}_2) \\ &+ (\xi_2 - 1/2)(D_{1,l}(\xi_1, -1/2, t)\mathbf{e}_1 + \alpha_{-l}(\xi_1, t)\mathbf{e}_2). \end{aligned} \tag{49}$$

Replacing the function $\mathbf{v}_l^\mu$ by its expression given by (49) in (46)-(47) and denoting by $\mathbf{G}_l$ and C_l, respectively, the right hand sides of the first two relations of the problem obtained in this way, we get for the pair $(\mathbf{w}_l^\mu(t), \pi_l^\mu(t))$:

$$\begin{cases} -2\mathrm{div}_\xi(\mu_0 D_\xi(\mathbf{w}_l^\mu(t))) + \nabla_\xi \pi_l^\mu(t) = \mathbf{G}_l(t), \\ \mathrm{div}_\xi \mathbf{w}_l^\mu(t) = C_l(t), \\ \mathbf{w}_l^\mu(\xi_1, \pm 1/2, t) = 0, \\ \mathbf{w}_l^\mu(t) \to \mathbf{0}, \text{ uniformly, when } \xi_1 \to \pm\infty, \\ \pi_l^\mu(t) \to 0 \text{ uniformly, when } \xi_1 \to -\infty. \end{cases} \tag{50}$$

Mention that the compatibility condition

$$\int_\Pi C_l(\xi, t)\mathrm{d}\xi = 0 \tag{51}$$

can be directly checked using (49) and (41).

We consider now the following problem: For any $t \in (0, T)$, find $\boldsymbol{\zeta}_l(t) : \Pi \mapsto \mathbb{R}^2$ satisfying:

$$\begin{cases} \mathrm{div}\boldsymbol{\zeta}_l(t) = C_l(t), \\ \boldsymbol{\zeta}_l(\xi_1, \pm 1/2, t) = 0, \\ \boldsymbol{\zeta}_l(t) \to \mathbf{0}, \text{ uniformly, when } \xi_1 \to \pm\infty. \end{cases} \tag{52}$$

The existence of a solution of the previous problem in $(H_0^1(\Pi))^2$ follows using a result of [6], Ch. III, p. 139.

Finally, we define for $t \in (0, T)$, $\mathbf{W}_l^\mu(t) = \mathbf{w}_l^\mu(t) - \boldsymbol{\zeta}_l(t)$ and we obtain for $(\mathbf{W}_l^\mu(t), \pi_l^\mu(t))$ the problem

$$\begin{cases} -2\mathrm{div}_\xi(\mu_0 D_\xi(\mathbf{W}_l^\mu(t))) + \nabla_\xi \pi_l^\mu(t) = \mathbf{H}_l(t), \\ \mathrm{div}_\xi \mathbf{W}_l^\mu(t) = 0, \\ \mathbf{W}_l^\mu(\xi_1, \pm 1/2, t) = 0, \\ \mathbf{W}_l^\mu(t) \to \mathbf{0}, \text{ uniformly, when } \xi_1 \to \pm\infty, \\ \pi_l^\mu(t) \to 0 \text{ uniformly, when } \xi_1 \to -\infty \end{cases} \tag{53}$$

for which the existence of a solution in $H_0^1(\Pi) \times L^2_{loc}(\Pi)$ can be obtained using again Theorem 2.1, p. 17, [16]. This yields the existence result for (46)-(47), which completes the proof. □

Moreover, taking into account that $\mathbf{W}_l^\mu(t) \to \mathbf{0}$, uniformly, when $\xi_1 \to \infty$ and the behavior of the right hand side of (53) at ∞, we obtain that $\pi_l^\mu(t) \to c_l(t)$ uniformly, when $\xi_1 \to \infty$, with $c_l(t)$ uniquely defined.

Now we define the functions

$$\begin{cases} \mathbf{u}_l^\mu(\xi_1, \xi_2, t) = \bar{\mathbf{u}}_l^\mu(\xi_1, \xi_2, t), \\ p_l^\mu(\xi_1, \xi_2, t) = \begin{cases} \bar{p}_l^{\mu^-}(\xi_1, \xi_2, t), & \text{for } \xi_1 < 0, \\ \bar{p}_l^{\mu^+}(\xi_1, \xi_2, t) - c_l(t), & \text{for } \xi_1 > 0, \end{cases} \end{cases} \tag{54}$$

with $c_l(t)$ introduced previously.

Putting next

$$[q_{l+1}]_{/x_1=0}(t) = c_l(t) = Q_l \tag{55}$$

it follows that $(\mathbf{u}_l^\mu(t), p_l^\mu(t))$ satisfies exactly the system (35), which yields the existence of a solution for (35).

The uniqueness of the solution for (35) is proved by the same technique as in Theorem 5.4. □

Remark 5.6. Taking into account the regularity of the data with respect to the time variable, given by (5), it follows from classical results (see, e. g. [16]) that all the unknown functions have the same regularity. Consequently, the functions $c_l = c_l(t)$ are also smooth functions. Moreover, we obtain for the correctors corresponding to the viscosity the same behavior at $\pm\infty$ as for those corresponding to the boundary conditions, following the steps of Theorem 5.1, i.e. an exponential decay with the constants independent on t.

5.5. **The order of solving the problems.** The last part of this section is devoted to an analysis of the problems for $(\mathbf{u}_l^\pm, p_l^\pm, q_l^\pm, d_{\pm\, l}^\pm)$, $(\mathbf{u}_l^{(-1/2)}, p_l^{(-1/2)}, d_{\pm\, l}^{(-1/2)})$, $(\mathbf{u}_l^{(1/2)}, p_l^{(1/2)}, d_{\pm\, l}^{(1/2)})$, $(\mathbf{u}_l^{\mu^\pm}, p_l^{\mu^\pm}, d_{\pm\, l}^{\mu^\pm})$. We shall present step by step the order of solving the previous problems for different values of γ. All functions with an index$< l$ are supposed to be known from the previous steps.

1. The case $\gamma > 3$. The order of determining the unknown functions in this case is the following:

• We compute $p_l^\pm$ from $(23)_2$ and from the corresponding problem for $x_1 < 0$. We fix a constant by the condition $\int_{-1/2}^{1/2} p_l(\xi_2) d\xi_2 = 0$.

• We integrate twice $(23)_1$ with respect to ξ_2 and we use the boundary conditions $(23)_4$. With a similar computation for $u_{1,l}^-$ it follows

$$u_{1,l}^\pm = \frac{1}{2\mu}\Big(\frac{\partial q_l^\pm}{\partial x_1} - f_1 \delta_{l0}\Big)(\xi_2^2 - 1/4) + U_{1,l-1}^\pm; \tag{56}$$

• We substitute $u_{1,l}^+$ given by (56) into $(23)_3$, integrate with respect to ξ_2 and use the boundary condition $(23)_4$ for $\xi_2 = -1/2$. With a similar computation for $u_{2,l}^-$ we get

$$u_{2,l}^\pm = \frac{\partial d_{-\, l+3-\gamma}^\pm}{\partial t} - \frac{1}{2\mu}\Big(\frac{\partial^2 q_l^\pm}{\partial x_1^2} - \frac{\partial f_1}{\partial x_1}\delta_{l0}\Big)\int_{-1/2}^{\xi_2} (\tau^2 - 1/4) \mathrm{d}\tau + U_{2,l-1}^\pm; \tag{57}$$

• We use the boundary condition $(23)_4$ for $\xi_2 = 1/2$ and, with a similar computation for $x_1 < 0$, we obtain the following differential equation of order 2 for $q_l^\pm$

$$\frac{\partial^2 q_l^\pm}{\partial x_1^2} - \frac{\partial f_1}{\partial x_1}\delta_{l0} = 12\mu \frac{\partial}{\partial t}(d_{+\, l+3-\gamma}^\pm - d_{-\, l+3-\gamma}^\pm) + Q_{l-1}^\pm. \tag{58}$$

For solving these equations we need 4 conditions. Since we cannot obtain 4 conditions for $q_l^\pm$, we shall combine the problems for $q_l^\pm$ with those for $d_{\pm\, l}^\pm$, as we shall explain in the sequel;

• We compute $d_{\pm\, l}^{(-1/2)}), d_{\pm\, l+1}^{(-1/2)}$ by solving (27) for l and for $l+1$; we do the same computation for determining $d_{\pm\, l}^{(1/2)}), d_{\pm\, l+1}^{(1/2)}$ from the corresponding problem;

• We determine $d_{\pm\, l}^{\mu^+}, d_{\pm\, l+1}^{\mu^+}, d_{\pm\, l+2}^{\mu^+}, d_{\pm\, l+3}^{\mu^+}$ from (33) for $l, l+1, l+2, l+3$, respectively; we do the same computation for $d_{\pm\, l}^{\mu^-}, d_{\pm\, l+1}^{\mu^-}, d_{\pm\, l+2}^{\mu^-}, d_{\pm\, l+3}^{\mu^-}$;

• We introduce the notation $d_l^\pm = d_{+l}^\pm + d_{-l}^\pm$. The functions $d_l^\pm$ are uniquely defined as a solution of the system which contains: the differential equation obtained by adding $(23)_5$ and $(23)_6$, the corresponding equation for d_l^-, the 2 boundary conditions obtained from $(23)_{7,8}$, the other 2 boundary conditions in $x_1 = -1/2$ and the 4 jump relations given by (22); then, we replace everywhere $d_{-l}^{\pm}$ by $d_l^{\pm} - d_{+l}^{\pm}$;

• We solve the differential system for $q_l^{\pm}$, $d_{+l}^{\pm}$ which contains the equations (58), the equation $(23)_5$ and the corresponding equation for d_{+l}^- and the following 12 conditions: the jump of q_l in $x_1 = 0$ known from the previous step; the jump of $\frac{\partial q_l}{\partial x_1}$ in $x_1 = 0$ given by introducing (56) into (36), 4 jump relations for d_{+l}, $\frac{\partial d_{+l}}{\partial x_1}$, $\frac{\partial^2 d_{+l}}{\partial x_1^2}$, $\frac{\partial^3 d_{+l}}{\partial x_1^3}$ given by (22), one boundary condition for $\frac{\partial q_l^-}{\partial x_1}(-1/2,t)$ obtained from (56) and (26), 4 boundary conditions given by $(23)_{7,8}$ for d_{+l}^+ and the corresponding conditions in $x_1 = -1/2$ for d_{+l}^- and, finally, (24);

• With $q_l^{\pm}$ determined before, we obtain from (56) and (57) the functions $\mathbf{u}_l^{\pm}$;

• We solve (25) and the corresponding problem for the right side and we obtain $(\mathbf{u}_l^{(-1/2)}, p_l^{(-1/2)})$ and $(\mathbf{u}_l^{(1/2)}, p_l^{(1/2)})$, respectively;

• Finally, we determine $(\mathbf{u}_l^{\mu^{\pm}}, p_l^{\mu^{\pm}})$ by solving (35) following the steps of Theorem 5.3. We determine $[q_{l+1}] = Q_l$.

2. The case γ= 3. The order of determining the unknown functions in this case is the following:

• We compute $p_l^{\pm}$ from $(23)_2$ and from the corresponding problem for $x_1 < 0$;

• With the same computations as in the previous case, we obtain (56);

• In this case, (57) becomes:

$$u_{2,l}^{\pm} = \frac{\partial d_{-l}^{\pm}}{\partial t} - \frac{1}{2\mu}\Big(\frac{\partial^2 q_l^{\pm}}{\partial x_1^2} - \frac{\partial f_1}{\partial x_1}\delta_{l0}\Big)\int_{-1/2}^{\xi_2}(\tau^2 - 1/4)\mathrm{d}\tau + U_{2,l-1}^{\pm}; \tag{59}$$

• For $\gamma = 3$ (58) becomes:

$$\frac{\partial^2 q_l^{\pm}}{\partial x_1^2} - \frac{\partial f_1}{\partial x_1}\delta_{l0} = 12\mu\frac{\partial}{\partial t}(d_{+\,l}^{\pm} - d_{-\,l}^{\pm}) + Q_{l-1}^{\pm}; \tag{60}$$

• We compute $d_{\pm\,l}^{(-1/2)}), d_{\pm\,l+1}^{(-1/2)}$ by solving (27) for l and for $l+1$; we do the same computation for determining $d_{\pm\,l}^{(1/2)}), d_{\pm\,l+1}^{(1/2)}$ from the corresponding problem;

• We determine $d_{\pm\,l}^{\mu^+}, d_{\pm\,l+1}^{\mu^+}, d_{\pm\,l+2}^{\mu^+}, d_{\pm\,l+3}^{\mu^+}$ from (33) for $l, l+1, l+2, l+3$, respectively; we do the same computation for $d_{\pm\,l}^{\mu^-}, d_{\pm\,l+1}^{\mu^-}, d_{\pm\,l+2}^{\mu^-}, d_{\pm\,l+3}^{\mu^-}$;

• As in the previous case, we obtain $d_l^{\pm}$ and we replace $d_{-l}^{\pm}$ by $d_l^{\pm} - d_{+l}^{\pm}$;

• We obtain the problem for $d_{+l}^{\pm}$ in the following way: we differentiate twice $(23)_5$ with respect to x_1 and we use (60); with the same computation for d_{+l}^- we get

$$\frac{\partial d_{+l}^{\pm}}{\partial t} - \frac{1}{12\mu}\frac{\partial^6 d_{+l}^{\pm}}{\partial x_1^6} = D_{+l-1}^{\pm}. \tag{61}$$

We solve these parabolic equations with the initial conditions $d_{+l}^{\pm}(x_1, 0) = 0$ and with the following 12 conditions: 4 jump relations are given by (22); $[q_l]/_{x_1=0}$ and $[\frac{\partial q_l}{\partial x_1}]/_{x_1=0}$, obtained as in the previous case, give another 2 jump conditions for $[\frac{\partial^4 d_{+l}}{\partial x_1^4}]/_{x_1=0}$ and $[\frac{\partial^5 d_{+l}}{\partial x_1^5}]/_{x_1=0}$, respectively; 4 boundary conditions are given by $(23)_{7,8}$ for d_{+l}^+ and the corresponding conditions in $x_1 = -1/2$ for d_{+l}^-; another boundary condition gives $\frac{\partial^5 d_{+l}^-}{\partial x_1^5}(-1/2, t)$ (it is a consequence of (26) which gives, as before, $\frac{\partial q_l^-}{\partial x_1}(-1/2,t)$ and of $(23)_5$ differentiated with respect to x_1); and, finally, (24);

- We determine $q_l^{\pm}$ from $(23)_5$ and from the corresponding problem for $x_1 < 0$;
- We compute $\mathbf{u}_l^{\pm}$ from (56) and (59);
- We determine $(\mathbf{u}_l^{(-1/2)}, p_l^{(-1/2)})$ by solving (25) and $(\mathbf{u}_l^{(1/2)}, p_l^{(1/2)})$ from the problem corresponding to the right side;
- Finally, we determine $(\mathbf{u}_l^{\mu^{\pm}}, p_l^{\mu^{\pm}})$ by following the steps of Theorem 5.3 and $[q_{l+1}] = Q_l$

For $l = 0$, the previous computations give the leading term of the asymptotic solution partially explicitely, partially via the problems described in Subsections 4.1.1 and 4.1.2.

6. **Error estimates.**

In the last section we justify the construction of the asymptotic solution by showing that this solution represents a good approximation for the exact solution, i. e. the error between them is small enough.

The system satisfied by the asymptotic solution of order K, $(\hat{\mathbf{u}}^{(K)}, \hat{p}^{(K)}, \hat{d}_{\pm}^{(K)})$, is as follows:

$$\begin{cases} \rho_f \dfrac{\partial \hat{\mathbf{u}}^{(K)}}{\partial t} - 2\text{div}(\mu_\varepsilon D(\hat{\mathbf{u}}^{(K)})) + \nabla \hat{p}^{(K)} = f_1 \mathbf{e}_1 + \varepsilon^{K+1}\mathbf{E}_K \text{ in } D_\varepsilon \times (0,T), \\ \text{div } \hat{\mathbf{u}}^{(K)} = 0 \text{ in } D_\varepsilon \times (0,T), \\ \hat{\mathbf{u}}^{(K)}(x_1, \pm\dfrac{\varepsilon}{2}, t) = \dfrac{\partial \hat{d}_{\pm}^{(K)}}{\partial t}(x_1,t)\mathbf{e}_2 + \varepsilon^{K+3}\mathbf{B}_{\pm K}(x_1,t) \text{ in } (-1/2,1/2)\times(0,T), \\ \hat{\mathbf{u}}^{(K)}(\pm 1/2, x_2, t) = \varepsilon^2 \psi^\varepsilon(x_2,t)\mathbf{e}_1 + \mathbf{R}_K^{\pm 1/2}(x_2,t) \text{ in} \Big(-\dfrac{\varepsilon}{2}, \dfrac{\varepsilon}{2}\Big) \times (0,T), \\ \hat{\mathbf{u}}^{(K)}(x,0) = \mathbf{0} \text{ in } D_\varepsilon, \\ \rho h \dfrac{\partial^2 \hat{d}_{\pm}^{(K)}}{\partial t^2} + \dfrac{1}{\varepsilon^\gamma}\dfrac{\partial^4 \hat{d}_{\pm}^{(K)}}{\partial x_1^4} + \nu \dfrac{\partial^5 \hat{d}_{\pm}^{(K)}}{\partial x_1^4 \partial t} = g_{\pm} \pm \hat{p}^{(K)}/_{x_2=\pm\varepsilon/2} + G_{\pm K} \text{ on } \Gamma_\varepsilon^{\pm}\times(0,T), \\ \hat{d}_{\pm}^{(K)}(\pm 1/2, t) = S_{\pm K}^{\pm 1/2}(t) \text{ in } (0,T), \\ \dfrac{\partial \hat{d}_{\pm}^{(K)}}{\partial x_1}(\pm 1/2, t) = T_{\pm K}^{\pm 1/2}(t) \text{ in } (0,T), \\ \hat{d}_{\pm}^{(K)}(x_1, 0) = \dfrac{\partial \hat{d}_{\pm}^{(K)}}{\partial t}(x_1,0) = 0 \text{ in } (-1/2,1/2), \end{cases} \tag{62}$$

where $\varepsilon^{K+1}\mathbf{E}_K, \varepsilon^{K+3}\mathbf{B}_{\pm K}, \mathbf{R}_K^{\pm 1/2}(x_2,t), G_{\pm K}, S_{\pm K}^{\pm 1/2}, T_{\pm K}^{\pm 1/2}$ are the residuals which appear when we substitute the expressions of $\hat{\mathbf{u}}^{(K)}$, $\hat{p}^{(K)}$, $\hat{d}_{\pm}^{(K)}$ into the equations and conditions of system (1). Since the expressions of these residuals are rather complicated, but not too technical to obtain, let us give below only the estimates for their norms in the corresponding spaces; these estimates will be used to derive finally the error estimate between the exact and the asymptotic solution.

In order to use the *a priori* estimates (4), we must check that the functions $\hat{\mathbf{u}}^{(K)}$, $\hat{d}_{\pm}^{(K)}$ satisfy the same boundary conditions as $\mathbf{u}$, $d_{\pm}$ on $x_1 = \pm 1/2$ and on $x_2 = \pm\varepsilon/2$. We notice that the boundary conditions for $\hat{\mathbf{u}}^{(K)}$ and $\hat{d}_{\pm}^{(K)}$ on $x_1 = \pm 1/2$

and on $x_2 = \pm\varepsilon/2$ are unfortunately different from those for $\mathbf{u}$ and $d_\pm$, respectively. That is why we define new auxiliary functions $\hat{\mathbf{U}}^{(K)}$ and $\hat{D}_\pm^{(K)}$ which satisfy the same boundary conditions as $\mathbf{u}$ and $d_\pm$, respectively, on $x_1 = \pm 1/2$ and on $x_2 = \pm\varepsilon/2$.

6.1. **Construction of $\hat{D}_\pm^{(K)}$.** The goal of this subsection is to construct two functions $\hat{D}_\pm^{(K)}$ satisfying:

$$\begin{cases} \hat{D}_\pm^{(K)}(\pm 1/2, t) = \dfrac{\partial \hat{D}_\pm^{(K)}}{\partial x_1}(\pm 1/2, t) = 0, \\ \hat{D}_\pm^{(K)}(x_1, 0) = \dfrac{\partial \hat{D}_\pm^{(K)}}{\partial t}(x_1, 0) = 0, \\ \displaystyle\int_{-1/2}^{1/2} \frac{\partial}{\partial t}\Big(\hat{D}_+^{(K)}(x_1,t) - \hat{D}_-^{(K)}(x_1,t)\Big)\mathrm{d}x_1 = 0. \end{cases} \tag{63}$$

We introduced the third relation to satisfy the compatibility condition for the problem which will be set below for $\hat{\mathbf{U}}^{(K)}$, $\hat{p}^{(K)}$, $\hat{D}_\pm^{(K)}$.

Let us introduce the polynomial functions $\tilde{d}_\pm^{(K)}: [-1/2, 1/2] \times [0, T] \mapsto \mathbb{R}$, $\tilde{d}_\pm^{(K)} = a_4^{\pm K}(t)x_1^4 + a_3^{\pm K}(t)x_1^3 + a_2^{\pm K}(t)x_1^2 + a_1^{\pm K}(t)x_1 + a_0^{\pm K}(t)$, with $\tilde{d}_\pm^{(K)}(\pm 1/2, t) = S_{\pm K}^{\pm 1/2}(t)$, $\dfrac{\partial \tilde{d}_\pm^{(K)}}{\partial x_1}(\pm 1/2, t) = T_{\pm K}^{\pm 1/2}(t)$, and

$$\int_{-1/2}^{1/2}\frac{\partial}{\partial t}\Big(\tilde{d}_+^{(K)}(x_1,t) - \tilde{d}_-^{(K)}(x_1,t)\Big)\mathrm{d}x_1 = \int_{-1/2}^{1/2}\frac{\partial}{\partial t}\Big(\hat{d}_+^{(K)}(x_1,t) - \hat{d}_-^{(K)}(x_1,t)\Big)\mathrm{d}x_1 =: W_K(t),$$

where W_K is given by the compatibility condition for problem (62):

$$W_K(t) = -\int_{-\varepsilon/2}^{\varepsilon/2}\Big(\mathbf{R}_K^{1/2}(x_2,t) - \mathbf{R}_K^{-1/2}(x_2,t)\Big)\cdot \mathbf{e}_1 \mathrm{d}x_2$$
$$-\varepsilon^{K+3}\int_{-1/2}^{1/2}\Big(\mathbf{B}_{+K}(x_1,t) - \mathbf{B}_{-K}(x_1,t)\Big)\cdot\mathbf{e}_2\mathrm{d}x_1.$$

It is easy to prove that there exist some functions $\tilde{d}_\pm^{(K)}$ with the above properties. Moreover, the initial conditions $\tilde{d}_\pm^{(K)}(x_1, 0) = \dfrac{\partial \tilde{d}_\pm^{(K)}}{\partial t}(x_1, 0) = 0$ are satisfied due to the hypothesis $(5)_3$. We define the functions

$$\hat{D}_\pm^{(K)} = \hat{d}_\pm^{(K)} - \tilde{d}_\pm^{(K)}, \tag{64}$$

which satisfy (63).

6.2. **Construction of $\hat{\mathbf{U}}^{(K)}$.** For $t \in [0, T]$ we consider the following problem:

Find $\mathbf{U}_\varepsilon^{(K)} : \bar{D}_\varepsilon \times [0, T] \mapsto \mathbb{R}^2$ such that:

$$\begin{cases} \mathbf{U}_\varepsilon^{(K)}(t) \in (H^1(D_\varepsilon))^2, \\ \operatorname{div} \mathbf{U}_\varepsilon^{(K)}(t) = 0 \text{ in } D_\varepsilon, \\ \mathbf{U}_\varepsilon^{(K)}(x_1, \pm\dfrac{\varepsilon}{2}, t) = \dfrac{\partial \tilde{d}_\pm^{(K)}}{\partial t}(x_1, t)\mathbf{e}_2 + \varepsilon^{K+3}\mathbf{B}_{\pm K}(x_1, t), \\ \mathbf{U}_\varepsilon^{(K)}(\pm 1/2, x_2, t) = \mathbf{R}_K^{\pm 1/2}(x_2, t). \end{cases} \tag{65}$$

We can prove that

Proposition 6.1. *If the compatibility condition is satisfied, then the problem (65) has at least a solution, such that,*

$$\|\mathbf{U}_\varepsilon^{(K)}(t)\|_{H^1(D_\varepsilon))^2} = O(\varepsilon^{K+3/2}). \tag{66}$$

Proof. The existence of a solution of (65) is a consequence of the choice of $\tilde{d}^{(K)}$. We define $\boldsymbol{\eta}_\varepsilon^{(K)} : S \times (0,T) \mapsto \mathbb{R}^2$, where $S = (-1/2, 1/2)^2$, and $\boldsymbol{\eta}_\varepsilon^{(K)}(y_1, y_2, t) = (U_\varepsilon^{(K)})_1(x_1, x_2, t)\mathbf{e}_1 + \dfrac{1}{\varepsilon}(U_\varepsilon^{(K)})_2(x_1, x_2, t)\mathbf{e}_2$, with $(y_1, y_2) = (x_1, \dfrac{x_2}{\varepsilon})$.

Obvious computations lead to the following problem for $\boldsymbol{\eta}_\varepsilon^{(K)}(t)$:

$$\begin{cases} \operatorname{div}_y \boldsymbol{\eta}_\varepsilon^{(K)}(t) = 0 \text{ in } S, \\ \boldsymbol{\eta}_\varepsilon^{(K)}(y_1, \pm 1/2, t) = \varepsilon^{K+3} B_{\pm\, K,1}(y_1, t)\mathbf{e}_1 + \dfrac{1}{\varepsilon}\Big(\dfrac{\partial \tilde{d}_\pm^{(K)}}{\partial t}(y_1, t) + \varepsilon^{K+3} B_{\pm\, K,2}(y_1, t)\Big)\mathbf{e}_2, \\ \boldsymbol{\eta}_\varepsilon^{(K)}(\pm 1/2, y_2, t) = \displaystyle\sum_{l=0}^{K} \varepsilon^{l+2} u_{1,l}^{(1/2)}(\frac{-1}{\varepsilon}, y_2, t)\mathbf{e}_1 \\ \qquad\qquad + \Big(\displaystyle\sum_{l=0}^{K} \varepsilon^{l+1} u_{2,l}^{(1/2)}(\frac{-1}{\varepsilon}, y_2, t) + \varepsilon^{K+2} u_{2,K}(\frac{-1}{2}, y_2, t)\Big)\mathbf{e}_2. \end{cases}$$

As in [7] we can prove that there exists a function $\boldsymbol{\eta}_\varepsilon^{(K)}(t) \in (H^1(S))^2$ so that $\|\boldsymbol{\eta}_\varepsilon^{(K)}(t)\|_{(H^1(S))^2} \le C(S)\|\boldsymbol{\eta}_\varepsilon^{(K)}(t)\|_{(H^{1/2}(\partial S))^2}$, with $C(S)$ independent on t.

Using the properties of the boundary layer correctors (their exponential decay) we check that

$$\|\boldsymbol{\eta}_\varepsilon^{(K)}(t)\|_{(H^1(S))^2} = O(\varepsilon^{K+2}). \tag{67}$$

Standard computations give $\|\mathbf{U}_\varepsilon^{(K)}(t)\|_{(H^1(D_\varepsilon))^2} \le \dfrac{1}{\varepsilon^{1/2}}\|\boldsymbol{\eta}_\varepsilon^{(K)}(t)\|_{(H^1(S))^2}$ and combining these two inequalities the proof is achieved. □

The function

$$\hat{\mathbf{U}}^{(K)} = \hat{\mathbf{u}}^{(K)} - \mathbf{U}_\varepsilon^{(K)}, \tag{68}$$

satisfies the same type of boundary conditions as $\mathbf{u}$ on $x_1 = \pm 1/2$ and on $x_2 = \pm\varepsilon/2$. Moreover, $\hat{\mathbf{U}}^{(K)}(x, 0) = \mathbf{0}$, due to $(5)_3$. The problem for the new functions

$\hat{\mathbf{U}}^{(K)},\ \hat{p}^{(K)},\ \hat{D}_{\pm}^{(K)}$ is an obvious consequence of (62), (64) and (68):

$$\begin{cases} \rho_f \dfrac{\partial \hat{\mathbf{U}}^{(K)}}{\partial t} - 2\mathrm{div}(\mu_\varepsilon D(\hat{\mathbf{U}}^{(K)})) + \nabla \hat{p}^{(K)} \\ \qquad = f_1 \mathbf{e}_1 + \varepsilon^{K+1}\mathbf{E}_K - \rho_f \dfrac{\partial \mathbf{U}_\varepsilon^{(K)}}{\partial t} + 2\mathrm{div}(\mu_\varepsilon D(\mathbf{U}_\varepsilon^{(K)})) \text{ in } D_\varepsilon \times (0,T), \\ \mathrm{div}\, \hat{\mathbf{U}}^{(K)} = 0 \text{ in } D_\varepsilon \times (0,T), \\ \hat{\mathbf{U}}^{(K)}(x_1, \pm\dfrac{\varepsilon}{2}, t) = \dfrac{\partial \hat{D}_{\pm}^{(K)}}{\partial t}(x_1,t)\mathbf{e}_2 \text{ in } (-1/2,1/2)\times(0,T), \\ \hat{\mathbf{U}}^{(K)}(\pm 1/2, x_2, t) = \varepsilon^2 \psi^\varepsilon(x_2,t)\mathbf{e}_1, \text{ in } (-\varepsilon/2,\varepsilon/2)\times(0,T), \\ \hat{\mathbf{U}}^{(K)}(x,0) = \mathbf{0} \text{ in } D_\varepsilon, \\ \rho h \dfrac{\partial^2 \hat{D}_{\pm}^{(K)}}{\partial t^2} + \dfrac{1}{\varepsilon^\gamma}\dfrac{\partial^4 \hat{D}_{\pm}^{(K)}}{\partial x_1^4} + \nu \dfrac{\partial^5 \hat{D}_{\pm}^{(K)}}{\partial x_1^4 \partial t} = g_\pm + G_{\pm K} \\ \pm \hat{p}^{(K)}/_{x_2 = \pm\varepsilon/2} - \rho h \dfrac{\partial^2 \tilde{d}_{\pm}^{(K)}}{\partial t^2} - \dfrac{1}{\varepsilon^\gamma}\dfrac{\partial^4 \tilde{d}_{\pm}^{(K)}}{\partial x_1^4} - \nu \dfrac{\partial^5 \tilde{d}_{\pm}^{(K)}}{\partial x_1^4 \partial t} \text{ in } (-1/2,1/2)\times(0,T), \\ \hat{D}_{\pm}^{(K)}(\pm 1/2, t) = \dfrac{\partial \hat{D}_{\pm}^{(K)}}{\partial x_1}(\pm 1/2, t) = 0, \text{ in } (0,T), \\ \hat{D}_{\pm}^{(K)}(x_1, 0) = \dfrac{\partial \hat{D}_{\pm}^{(K)}}{\partial t}(x_1, 0) = 0 \text{ in } (-1/2,1/2). \end{cases} \tag{69}$$

6.3. **Error estimates.** The next lemma gives an error estimate between the exact solution of problem (1) and the asymptotic solution of order K, given by (9). It is now a direct consequence of the *a priori* estimates (4).

Lemma 6.1. *Let* $(\hat{\mathbf{u}}^{(K)},\ \hat{p}^{(K)},\ \hat{d}_{\pm}^{(K)})$ *be the asymptotic solution given by (9) and* $(\mathbf{u},\ p,\ d_\pm)$ *the exact solution of problem (1). Then the following estimates hold:*

$$\begin{cases} \|\mathbf{u} - \hat{\mathbf{u}}^{(K)}\|_{L^\infty(0,T;(L^2(D_\varepsilon))^2)} = \begin{cases} O(\varepsilon^{3/2}) \text{ for } K \in \{0,1,2,3\}, \\ O(\varepsilon^{\min\{K-5/2, K+4-\gamma\}}) \text{ for } K \geq 4 \end{cases} \\ \|D(\mathbf{u} - \hat{\mathbf{u}}^{(K)})\|_{L^2(0,T;(L^2(D_\varepsilon))^4)} = \begin{cases} O(\varepsilon^{3/2}) \text{ for } K \in \{0,1,2,3\}, \\ O(\varepsilon^{\min\{K-5/2, K+4-\gamma\}}) \text{ for } K \geq 4 \end{cases} \\ \left\| \dfrac{\partial}{\partial t}(d_\pm - \hat{d}_{\pm}^{(K)}) \right\|_{L^\infty(0,T;L^2(-1/2,1/2))} = \begin{cases} O(\varepsilon^{3/2}) \text{ for } K \in \{0,1,2,3\}, \\ O(\varepsilon^{\min\{K-5/2, K+4-\gamma\}}) \text{ for } K \geq 4 \end{cases} \\ \left\| \dfrac{\partial^2}{\partial x_1^2}(d_\pm - \hat{d}_{\pm}^{(K)}) \right\|_{L^\infty(0,T;L^2(-1/2,1/2))} = \begin{cases} O(\varepsilon^{3/2+\gamma/2}) \text{ for } K \in \{0,1,2,3\}, \\ O(\varepsilon^{\min\{K-5/2, K+4-\gamma\}+\gamma/2}),\ K \geq 4 \end{cases} \\ \left\| \dfrac{\partial^3}{\partial x_1^2 \partial t}(d_\pm - \hat{d}_{\pm}^{(K)}) \right\|_{L^2((-1/2,1/2)\times(0,T))} = \begin{cases} O(\varepsilon^{3/2}) \text{ for } K \in \{0,1,2,3\}, \\ O(\varepsilon^{\min\{K-5/2, K+4-\gamma\}}) \text{ for } K \geq 4 \end{cases} \\ \|\nabla(p - \hat{p}^{(K)})\|_{L^2(0,T;(H^{-1}(D_\varepsilon))^2)} = \begin{cases} O(\varepsilon^{5/2-\gamma/2}) \text{ for } K \in \{0,1,2,3\}, \\ O(\varepsilon^{\min\{K-3/2, K+5-\gamma\}-\gamma/2}) \text{ for } K \geq 4 \end{cases} \end{cases} \tag{70}$$

Proof. To obtain these estimates, we use (4) for $(\mathbf{u}, d_\pm)$ and $(\hat{\mathbf{U}}^{(K)}, \hat{D}_{\pm}^{(K)})$ which satisfy the same boundary conditions on $x_1 = \pm 1/2$ and on $x_2 = \pm\varepsilon/2$. Taking into account the problems (1) and (69) and using the *a priori* estimates (4), we get the estimate for $\min\{\|\varepsilon^{K+1}\mathbf{E}_K - \rho_f \dfrac{\partial \mathbf{U}_\varepsilon^{(K)}}{\partial t} + 2\mathrm{div}(\mu_\varepsilon D(\mathbf{U}_\varepsilon^{(K)}))\|_{L^2(0,T;(L^2(D_\varepsilon))^2)}$, $\|G_{\pm\,K} - \rho h \frac{\partial^2 \tilde{d}_\pm^{(K)}}{\partial t^2} - \frac{1}{\varepsilon^\gamma}\frac{\partial^4 \tilde{d}_\pm^{(K)}}{\partial x_1^4} - \nu \frac{\partial^5 \tilde{d}_\pm^{(K)}}{\partial x_1^4 \partial t}\|_{L^2((-1/2,1/2)\times(0,T))}\}$. From all the terms appearing above, the estimates will be given by $G_{\pm\,K}$ and $\dfrac{1}{\varepsilon^\gamma}\dfrac{\partial^4 \tilde{d}_\pm^{(K)}}{\partial x_1^4}$, the other terms having greater orders than these.

Let us get, for instance, the estimate $(70)_2$. From (68) and (66) it follows that

$$\|D(\mathbf{u} - \hat{\mathbf{u}}^{(K)})\|_{L^2(0,T;(L^2(D_\varepsilon))^4)} \le \|D(\mathbf{u} - \hat{\mathbf{U}}^{(K)})\|_{L^2(0,T;(L^2(D_\varepsilon))^4)} + O(\varepsilon^{K+3/2}).$$

Taking into account the definition of $G_{\pm\,K}$, we get (see [13])

$$\|G_{\pm\,K}\|_{L^2((-1/2,1/2)\times(0,T))} = \begin{cases} O(\varepsilon^{3/2}) \text{ for } K \in \{0,1,2,3\}, \\ O(\varepsilon^{\min\{K-5/2,K+4-\gamma\}}) \text{ for } K \ge 4. \end{cases} \tag{71}$$

Moreover, from the definition of $\tilde{d}_\pm^{(K)}$, it follows that

$$\Big\|\frac{\partial^4 \tilde{d}_\pm^{(K)}}{\partial x_1^4}\Big\|_{L^2((-1/2,1/2)\times(0,T))} = O(\varepsilon^{\min\{K+4,K+\gamma\}}), \tag{72}$$

for $K > 4$, while for $K \le 4$, $\tilde{d}_\pm^{(K)} = 0$, since the coefficients of $\tilde{d}_\pm^{(K)}$ are defined by means of the correctors $d_{\pm l}^{(-1/2)}$ and $d_{\pm l}^{(1/2)}$ which are equal to 0 for $l \le 4$. Here we take into account (27) and the corresponding problem for the right end of the channel.

The other estimates of (70) are proved with the same technique. □

In order to improve the estimates (70), we analyse the order of the leading term of the asymptotic solution with respect to different norms. For estimating the second term of the right hand side of (9), we introduce the notation $\Omega_\varepsilon = (0,1)\times(-\varepsilon/2,\varepsilon/2)$. It is obvious that $(x_1,x_2) \in D_\varepsilon$ iff $(x_1 + 1/2, x_2) \in \Omega_\varepsilon$.

Proposition 6.2. *For the leading term of the correctors corresponding to the boundary conditions for the left end, the following estimates hold:*

$$\begin{cases} \|\mathbf{u}_{bl}^{(0)-1/2}\|_{L^\infty(0,T;(L^2(\Omega_\varepsilon))^2)} = O(\varepsilon^3), \\ \|\nabla_x \mathbf{u}_{bl}^{(0)-1/2}\|_{L^2(0,T;(L^2(\Omega_\varepsilon))^2)} = O(\varepsilon^2), \\ \|\nabla_x p_{bl}^{(0)-1/2}\|_{L^2(0,T;(H^{-1}(\Omega_\varepsilon))^2)} = O(\varepsilon^2), \\ \|\dfrac{\partial}{\partial t} d_{\pm bl}^{(0)-1/2}\|_{L^\infty(0,T;L^2(0,1))} = O(\varepsilon^{\gamma+5+1/2}), \\ \|\dfrac{\partial^2}{\partial x_1^2} d_{\pm bl}^{(0)-1/2}\|_{L^\infty(0,T;L^2(0,1))} = O(\varepsilon^{\gamma+3+1/2}). \end{cases} \tag{73}$$

Proof. $(73)_1$ is a consequence of $(7)_1$ and of Theorem 5.1, which lead to the following computation:

$$\int_{\Omega_\varepsilon} |\mathbf{u}_{bl}^{(0)-1/2}(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, t)|^2 \mathrm{d}x_1 \mathrm{d}x_2 \le \varepsilon^4 \cdot \varepsilon^2 \int_0^\infty \int_{-1/2}^{1/2} |\mathbf{u}_0^{(-1/2)}(\zeta_1,\zeta_2,t)|^2 \mathrm{d}\zeta_1 \mathrm{d}\zeta_2 \le C\varepsilon^6.$$

In a similar way we obtain $(73)_2$:

$$\int_{\Omega_\varepsilon} |\nabla_x \mathbf{u}_{bl}^{(0)-1/2}(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, t)|^2 \mathrm{d}x_1 \mathrm{d}x_2 \le \varepsilon^4 \int_0^\infty \int_{-1/2}^{1/2} |\nabla_\zeta \mathbf{u}_0^{(-1/2)}(\zeta_1,\zeta_2,t)|^2 \mathrm{d}\zeta_1 \mathrm{d}\zeta_2 \le C\varepsilon^4.$$

For obtaining (73)$_3$ we proceed as follows:

$$\int_0^T \|\nabla_x p_{bl}^{(0)-1/2}\|^2_{(H^{-1}(\Omega_\varepsilon))^2}\mathrm{d}t = \varepsilon^2 \int_0^T \|\frac{\partial p_0^{(-1/2)}}{\partial x_i}\mathbf{e}_i\|^2_{(H^{-1}(\Omega_\varepsilon))^2}\mathrm{d}t$$

$$= \varepsilon^2\int_0^T\Big(\sup\Big\{\frac{\Big|\int_{\Omega_\varepsilon} p_0^{(-1/2)}\big(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon},t\big)\frac{\partial \varphi_i}{\partial x_i}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2\Big|}{\|\nabla_x\boldsymbol{\varphi}\|_{(L^2(\Omega_\varepsilon))^2}} : \boldsymbol{\varphi}\in (H_0^1(\Omega_\varepsilon))^2, \boldsymbol{\varphi}\neq \mathbf{0}\Big\}\Big)^2\mathrm{d}t$$

$$\le C_1\varepsilon^2\int_0^T\Big(\int_{\Omega_\varepsilon}\big(p_0^{(-1/2)}\big(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon},t\big)\big)^2\mathrm{d}x_1\mathrm{d}x_2\Big)\mathrm{d}t$$

$$\le C_1\varepsilon^4\int_0^T\Big(\int_0^\infty\int_{-1/2}^{1/2}\big(p_0^{(-1/2)}(\zeta_1,\zeta_2,t)\big)^2\mathrm{d}\zeta_1\mathrm{d}\zeta_2\Big)\mathrm{d}t \le C_1\varepsilon^4.$$

Finally, (73)$_{4,5}$ are a consequence of (27) which gives $d_{\pm_l}^{(-1/2)} = 0$ for $l = 0, 1, ..., 4$ and of Theorem 5.1.

The same estimates are obtained for the correctors corresponding to the right end. □

Remark 6.1. The norms chosen for the estimates (73) are the same as in (70).

Corollary 6.1. *The leading term of the correctors corresponding to the variable viscosity satisfies estimates of the same type as (73).*

Proof. Since the asymptotic expansions (7) and (8) start with the same powers of ε, the assertion is obtained as a consequence of the behavior at $\pm\infty$ of the correctors corresponding to the variable viscosity, given in Theorem 5.3. □

Consider now the leading term of the regular part of the asymptotic solution.

Proposition 6.3. *The components of the leading term of the regular part of the asymptotic solution satisfy the estimates:*

$$\begin{cases} \|\mathbf{u}^{(0)}\|_{L^\infty(0,T;(L^2(D_\varepsilon))^2)} = O(\varepsilon^{5/2}), \\ \|\nabla_x\mathbf{u}^{(0)}\|_{L^2(0,T;(L^2(D_\varepsilon))^2)} = O(\varepsilon^{3/2}), \\ \|\nabla_x p^{(0)}\|_{L^2(0,T;(H^{-1}(D_\varepsilon))^2)} = O(\varepsilon^{1/2}), \\ \|\frac{\partial}{\partial t}d_\pm^{(0)}\|_{L^\infty(0,T;L^2(-1/2,1/2))} = O(\varepsilon^\gamma), \\ \|\frac{\partial^2}{\partial x_1^2}d_\pm^{(0)}\|_{L^\infty(0,T;L^2(-1/2,1/2))} = O(\varepsilon^\gamma). \end{cases} \tag{74}$$

Proof. Taking into account the expressions of the leading term of the regular part of the asymptotic solution given by (14) for $\gamma > 3$ and by (18) for $\gamma = 3$ we proceed as in Proposition 6.1 and the estimates (74) are proved. □

Finally, combining the previous computations, we obtain for the leading term of the asymptotic solution the following estimates:

$$\begin{cases} \|\hat{\mathbf{u}}^{(0)}\|_{L^\infty(0,T;(L^2(D_\varepsilon))^2)} = O(\varepsilon^{5/2}), \\ \|\nabla_x \hat{\mathbf{u}}^{(0)}\|_{L^2(0,T;(L^2(D_\varepsilon))^2)} = O(\varepsilon^{3/2}), \\ \|\nabla_x \hat{p^{(0)}}\|_{L^2(0,T;(H^{-1}(D_\varepsilon))^2)} = O(\varepsilon^{1/2}), \\ \|\frac{\partial}{\partial t}\hat{d}_\pm^{(0)}\|_{L^\infty(0,T;L^2(-1/2,1/2))} = O(\varepsilon^\gamma), \\ \|\frac{\partial^2}{\partial x_1^2}\hat{d}_\pm^{(0)}\|_{L^\infty(0,T;L^2(-1/2,1/2))} = O(\varepsilon^\gamma). \end{cases} \tag{75}$$

The previous computations allow us to improve the estimates given by Lemma 6.1 in the following sense:

Theorem 6.1. *Let* $(\hat{\mathbf{u}}^{(j)},\ \hat{p}^{(j)},\ \hat{d}_\pm^{(j)})$ *be the asymptotic solution of order* j *and* $(\mathbf{u},\ p,\ d_\pm)$ *the exact solution of problem (1). Then the following estimates hold:*

$$\begin{cases} \|\mathbf{u} - \hat{\mathbf{u}}^{(j)}\|_{L^\infty(0,T;(L^2(D_\varepsilon))^2)} = O(\varepsilon^{j+7/2}), \\ \|\nabla(\mathbf{u} - \hat{\mathbf{u}}^{(j)})\|_{L^2(0,T;(L^2(D_\varepsilon))^4)} = O(\varepsilon^{j+5/2}), \\ \|\frac{\partial}{\partial t}(d_\pm - \hat{d}_\pm^{(j)})\|_{L^\infty(0,T;L^2(-1/2,1/2))} = O(\varepsilon^{j+\gamma+1}), \\ \|\frac{\partial^2}{\partial x_1^2}(d_\pm - \hat{d}_\pm^{(j)})\|_{L^\infty(0,T;L^2(-1/2,1/2))} = O(\varepsilon^{j+\gamma+1}), \\ \|\nabla(p - \hat{p}^{(j)})\|_{L^2(0,T;(H^{-1}(D_\varepsilon))^2)} = O(\varepsilon^{j+3/2}). \end{cases} \tag{76}$$

Proof. Let us prove the first estimate of (76). Let $j \geq 0$ be a fixed integer and $K >> j$. Then, from $(70)_1$ and $(75)_1$ we get:

$$\|\mathbf{u}-\hat{\mathbf{u}}^{(j)}\|_{L^\infty(0,T;(L^2(D_\varepsilon))^2)} \leq \|\mathbf{u}-\hat{\mathbf{u}}^{(K)}\|_{L^\infty(0,T;(L^2(D_\varepsilon))^2)} + \|\hat{\mathbf{u}}^{(K)}-\hat{\mathbf{u}}^{(j)}\|_{L^\infty(0,T;(L^2(D_\varepsilon))^2)}$$
$$= O(\varepsilon^{\min\{K-5/2, K+4-\gamma\}}) + O(\varepsilon^{j+7/2}) = O(\varepsilon^{j+7/2}).$$

The other estimates of (76) are obtained in a similar way and the proof is achieved. ☐

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