NETWORKS AND HETEROGENEOUS MEDIA
©American Institute of Mathematical Sciences
Volume **5**, Number **4**, December **2010**

doi:10.3934/nhm.2010.5.765

pp. **765–782**

GROUNDWATER FLOW IN A FISSURISED POROUS STRATUM

Michiel Bertsch

Istituto per le Applicazioni del Calcolo "Mauro Picone"
Consiglio Nazionale delle Ricerche
Via dei Taurini 19
I-00185 Roma, Italy
Dipartimento di Matematica, Università di Roma "Tor Vergata"
Via della Ricerca Scientifica,
I-00133 Roma, Italy

Carlo Nitsch

Dipartimento di Matematica e Applicazioni "R. Caccioppoli"
Università degli Studi di Napoli "Federico II"
Via Cintia, Monte S. Angelo
I-80126 Napoli, Italy

(Communicated by Juan Luis Vázquez)

Abstract. In [2] Barenblatt e.a. introduced a fluid model for groundwater flow in fissurised porous media. The system consists of two diffusion equations for the groundwater levels in, respectively, the porous bulk and the system of cracks. The equations are coupled by a fluid exchange term. Numerical evidence in [2, 8] suggests that the penetration depth of the fluid increases dramatically due to the presence of cracks and that the smallness of certain parameter values play a key role in this phenomenon. In the present paper we give precise estimates for the penetration depth in terms of the smallness of some of the parameters.

1. Introduction.

In this paper we study the system

$$\begin{cases} u_t = \kappa(u^2)_{xx} - \beta(u^2 - v^2) \\ \varepsilon v_t = (v^2)_{xx} + \beta(u^2 - v^2) \end{cases} \quad \text{in } Q = \mathbb{R}^+ \times (-1, \infty), \tag{1}$$

where $\varepsilon, \kappa, \beta > 0$. The initial and boundary data are:

$$\begin{cases} u(x,-1) = v(x,-1) = 0 & \text{for } x > 0 \\ u(0,t) = v(0,t) = f(t) & \text{for } t > -1, \end{cases} \tag{2}$$

where

$$f \in C^2([-1,\infty)),\ f(-1) = 0,\ f \equiv 0 \text{ in } \mathbb{R}^+,\ f > 0 \text{ in } (-1,0),\ f' \geq 0 \text{ in } (-1,-\tfrac{1}{2}). \tag{3}$$

2000 *Mathematics Subject Classification.* Primary: 35K55; Secondary: 35B30, 35Q35, 35R35, 76S05.

Key words and phrases. Groundwater flow, fissurized porous medium, system of partial differential equations, free boundary.

This work was supported in part by the Director, Office of Science, Computational and Technology Research, U.S. Department of Energy under Contract No. DE-AC02-05CH11231, and by Progetto Gnampa 2009 "Soluzioni di equazioni alle derivate parziali: aspetti analitici e geometrici".

System (1)-(2) was introduced in [2] as a model for one-dimensional groundwater flow in a porous stratum which contains a system of cracks: $u(x,t)$ denotes the fluid level in the porous blocks, $v(x,t)$ the level in the system of cracks. Initially, at $t=-1$, no fluid is present. The function $f(t)$ represents a boundary impulse at $x=0$, concentrated in the time interval $(-1,0)$.

Almost fifty years ago Barenblatt ([1]) introduced the *porous medium equation*

$$h_t = C(h^2)_{xx} \tag{4}$$

to describe the evolution of the fluid level h in a porous medium. Through an explicit example he observed that the spatial region occupied by the fluid expands with *finite speed of propagation*. This hyperbolic-type behavior distinguishes the porous medium equation from the linear heat equation and leads to the existence of *interfaces* or *free boundaries* between the regions where $h>0$ and $h=0$. The porous medium equation became a model equation for a large class of degenerate parabolic equations, since then extensively studied in the literature (see [11] for a review). Much less is known about systems of degenerate parabolic equations: most papers in the literature are dedicated to case studies.

Concerning system (1), numerical evidence in [2, 8] suggests that the speed of propagation dramatically increases in the presence of *small* amounts of cracks, a phenomenon which is most relevant in applications (think of the importance of the penetration depth's size in case of contaminated groundwater). Smallness of at least some of the parameters in (1) seems to play a key role in this phenomenon and it is the purpose of this paper to get more quantitative insight.

More precisely, we are interested in the behavior of the interface for small values of ε and β, keeping κ constant. As we shall see in section 2, problem (1)-(2) has a unique solution (u,v) and the interval where $u>0$ and $v>0$ is expanding with respect to time: there exists an *interface* $r(t)$ such that

$$\operatorname{supp} v(t) = \operatorname{supp} u(t) = [0, r(t)] \text{ for } t \geq -1;\ r \text{ increasing, continuous; } r(-1)=0. \tag{5}$$

To indicate the dependance on ε and β, we write $u_{\varepsilon,\beta}$, $v_{\varepsilon,\beta}$ and $r_{\varepsilon,\beta}$. Our main result is an estimate for $r_{\varepsilon,\beta}$ at time $t=-\frac{1}{2}$, an intermediate value in the interval $(-1,0)$ where $f>0$, for small values of ε and β. It turns out that we have to distinguish the cases $\beta \leq \varepsilon$ and $\beta \geq \varepsilon$:

Theorem 1.1. **(The case $\beta \leq \varepsilon$)** *Let $0<\varepsilon\leq 1/\kappa$ and $0<\beta\leq\varepsilon$. Let $r_{\varepsilon,\beta}(t)$ be the interface defined by (5). Then there exist positive constants C_1 and C_2 depending only on f such that*

$$\frac{C_1}{\sqrt{\varepsilon}} \leq r_{\varepsilon,\beta}(-\tfrac{1}{2}) \leq \frac{C_2}{\sqrt{\varepsilon}}.$$

Theorem 1.2. **(The case $\beta \geq \varepsilon$)** *Let $r_{\varepsilon,\beta}(t)$ be the interface defined by (5). Then there exist positive constants β_0, C_1 and C_2 depending only on f and κ such that*

$$\frac{2}{\sqrt{\beta}} \log\left(1 + C_1\frac{\beta}{\varepsilon}\right) \leq r_{\varepsilon,\beta}(-\tfrac{1}{2}) \leq \frac{2}{\sqrt{\beta}}\left(\log\frac{\beta}{\varepsilon} + C_2\right)$$

if $0<\varepsilon\leq 1/\kappa$, $0<\beta\leq\beta_0$ and $\epsilon\leq\beta$.

In particular $r_{\varepsilon,\beta}(-\frac{1}{2}) \to \infty$ as $\varepsilon,\beta\to 0$.

It turns out to be particularly instructive to consider the *first moments* of u and εv (see also [2]). Observe that the total moment can be easily calculated: for all

$t > -1$

$$\int_0^\infty x(u(x,t) + \varepsilon v(x,t))dx = (\kappa + 1)\int_{-1}^t f^2(\tau)d\tau.$$

Theorem 1.3. *Let $(u_{\varepsilon,\beta}, v_{\varepsilon,\beta})$ be the solution of (1)-(2). For any $-1 \le t \le -\frac{1}{2}$,*

$$\beta = o(\varepsilon) \Rightarrow \begin{cases} \int_0^\infty xu_{\varepsilon,\beta}(x,t)dx \to \kappa\int_{-1}^t f^2(\tau)d\tau \\ \int_0^\infty \varepsilon xv_{\varepsilon,\beta}(x,t)dx \to \int_{-1}^t f^2(\tau)d\tau \end{cases} \quad \text{as } \varepsilon \to 0 \tag{6}$$

and

$$\varepsilon = o(\beta) \Rightarrow \begin{cases} \int_0^\infty xu_{\varepsilon,\beta}(x,t)dx \to (\kappa+1)\int_{-1}^t f^2(\tau)d\tau \\ \int_0^\infty \varepsilon xv_{\varepsilon,\beta}(x,t)dx \to 0 \end{cases} \quad \text{as } \beta \to 0. \tag{7}$$

The interpretation is immediate but helps to understand the dichotomy which we have encountered in Theorems 1.1 and 1.2: if the exchange parameter β is small compared to ε, in the limit of vanishing ε and β the exchange term is too weak to have any effect on the values of the single moments, while if ε is small compared to β "all fluid which rapidly enters the porous medium through the cracks, enters the pores". Since the total amount of fluid which enters the porous medium through the cracks vanishes as $\varepsilon, \beta \to 0$ ($\int_0^\infty \varepsilon v_{\varepsilon,\beta}(x,t)dx \to 0$ as $\varepsilon \to 0$ since $v_{\varepsilon,\beta}$ and $\int_0^\infty \varepsilon xv_{\varepsilon,\beta}(x,t)dx$ are uniformly bounded, see also (11)), the first moments are particularly useful to describe the role to the exchange term.

Observe that the information about the interface and first moments can not be obtained from the limit solution for vanishing β and ε. Indeed, since u and v are uniformly bounded (see (11)), it is easy to prove that, independently of ε, u converges to a solution of the porous medium equation as $\beta \to 0$:

Proposition 1. *Let $\kappa > 0$ be constant, let U be the unique solution of the problem*

$$\begin{cases} u_t = \kappa(u^2)_{xx} & \text{in } Q \\ u(0,t) = f(t) & \text{if } t > -1 \\ u(x,-1) = 0 & \text{if } x > 0, \end{cases} \tag{8}$$

and let $(u_{\varepsilon,\beta}, v_{\varepsilon,\beta})$ be the solution of (1)-(2). Then for any $T > -1$

$$u_{\varepsilon,\beta} \to U \text{ uniformly in } \mathbb{R}^+ \times (-1, T] \text{ as } \beta \to 0,$$

uniformly with respect to $\varepsilon \in (0, 1]$.

The paper is organized as follows: in section 2 we prove the existence and uniqueness of a solution and list some preliminary material, in section 3 we prove Theorem 1.3, and in section 4 we prove the estimates for the interface.

2. **Preliminary results.**

In this section we prove that problem (1)-(2) has a unique solution and we list some basic properties of the solution.

Definition 2.1. A pair (u, v) of continuous functions defined on $\overline{Q}$ is a solution of (1)-(2) if for all $T > -1$

(*i*) $u, v \in C([-1, T] : L^p(\mathbb{R}^+)) \cap L^\infty((-1, T) : BV(\mathbb{R}^+))$ for $1 \le p < \infty$

(*ii*) $u^2, v^2 \in L^\infty(-1, T : H^1(\mathbb{R}^+))$

(*iii*) u and v satisfy (2)

(*iv*) for all $\zeta \in H^1(Q_T)$ with $\zeta(0,t) = 0$ for a.e. $t \in (-1,T)$ and $\zeta(x,T) = 0$ for a.e. $x \in \mathbb{R}^+$,

$$\begin{aligned}&\iint_{Q_T} \big(u\zeta_t - \kappa(u^2)_x\zeta_x - \beta(u^2-v^2)\zeta\big)\\ =&\iint_{Q_T}\big(\varepsilon v\zeta_t - (v^2)_x\zeta_x + \beta(u^2-v^2)\zeta\big) = 0,\end{aligned} \tag{9}$$

where $Q_T = \mathbb{R}^+ \times (-1,T]$.

Proposition 2. *Let $\varepsilon, \beta, \kappa > 0$ and let f satisfy (3). Then there exists a unique solution of problem (1)-(2).*

Proof. We approximate the degenerate parabolic system by a nondegenerate one:

$$\begin{cases} u_t = \kappa(u^2)_{xx} - \beta(u^2-v^2) & \text{in } Q^{(M)} = (0,M)\times(-1,\infty)\\ \varepsilon v_t = (v^2)_{xx} + \beta(u^2-v^2) & \text{in } Q^{(M)}\\ u(M,t) = v(M,t) = u(x,-1) = v(x,-1) = \sigma & \text{for } x>0,\ t>-1\\ u(0,t) = v(0,t) = f(t)+\sigma & \text{for } t>-1,\end{cases} \tag{10}$$

where $M, \sigma > 0$. By standard theory, problem (10) has a local (with respect to time) smooth solution, $(u_{\sigma,M}, v_{\sigma,M})$, which can be continued as long as it is bounded. We use the theory of invariant rectangles (see [4]) to get a priori estimates. If $T \le -\frac12$, the rectangle $[\sigma, f(T)+\sigma]^2$ is invariant for $t \in [-1,T]$, and if $T > -\frac12$, the rectangle $[\sigma, \max_{[-1,T]} f + \sigma]^2$ is invariant. Therefore

$$0 < \sigma \le u_{\sigma,M}(x,t), v_{\sigma,M}(x,t) \le \begin{cases} f(t)+\sigma & \text{if } x>0,\ -1<t\le -\frac12\\ \max_{[-1,t]} f + \sigma & \text{if } x>0,\ t>-\frac12,\end{cases} \tag{11}$$

and the solution $(u_{\sigma,M}, v_{\sigma,M})$ exists globally, for all $t > -1$.

Let $\xi(x)$ be a smooth nonnegative function such that $\xi(0) = \xi(M) = 0$, $\xi = 1$ in $(1, M-1)$, and $|\xi'|, |\xi''| < C$ in $(0,M)$ for some constant $C > 0$. Omitting the subscripts σ and M, multiplying the equation of u by ξ^2 and integrating over $Q_T^{(M)} = (0,M)\times(-1,T)$ we obtain

$$\int_0^M (u(\cdot,T)-\sigma)\,\xi^2 = -\beta\iint_{Q_T^{(M)}}(u^2-v^2)\xi^2 - 2\kappa\iint_{Q_T^{(M)}}\xi\xi'(u^2)_x.$$

Adding this to a similar expression for v we find that

$$\begin{aligned}\int_0^M \big((u(\cdot,T)-\sigma)+\varepsilon(v(\cdot,T)-\sigma)\big)\,\xi^2 &= -2\iint_{Q_T^{(M)}}\xi\xi'\left(\kappa u^2+v^2\right)_x\\ &= 2\iint_{Q_T^{(M)}}(\xi\xi')'\left(\kappa u^2+v^2\right) \le CT\end{aligned}$$

for some constant C which, in view of (11), does not depend on σ and M. Hence $u_{\sigma,M}-\sigma$ and $v_{\sigma,M}-\sigma$ are uniformly bounded in $L^\infty(-1,T;L^1(0,M))$.

We set $\psi = (f+\sigma)^2\left(1-\frac{x}{M}\right) + \sigma^2\frac{x}{M}$ and multiply the equation for u by $u^2-\psi$, a function which vanishes at $x=0$ and $x=M$. Integrating by parts over $Q_T^{(M)}$ we obtain

$$\tfrac{1}{3}\int_0^M \left(u^3(\cdot,T)-\sigma^3\right) - \int_0^M \left(\psi(x,T)u(\cdot,T)-\psi(x,0)\sigma\right) + 2\iint_{Q_T^{(M)}} \psi_t u$$
$$= -\kappa \iint_{Q_T^{(M)}} (u^2)_x^2 - \kappa \iint_{Q_T^{(M)}} (u^2)_x \frac{(f+\sigma)^2-\sigma^2}{M} - \beta \iint_{Q_T^{(M)}} (u^2-v^2)\left(u^2-\psi\right).$$

Since $u-\sigma$ and $v-\sigma$ are uniformly bounded in $L^\infty(Q_T^{(M)})$ and $L^\infty(-1,T;L^1(0,M))$, the integrals at the left hand side and the latter one at the right hand side are uniformly bounded. By Hölder's and Young's inequalities, the second term at the right hand side is bounded by $\frac{\kappa}{2}\iint_{Q_T^{(M)}}(u^2)_x^2$ plus a constant which does not depend on σ and M. Hence $\iint_{Q_T^{(M)}}(u^2)_x^2$ is uniformly bounded. The same estimate holds for v.

We can also multiply the equation for u by $(u^2-\psi)_t$ and integrate by parts. Leaving the details to the reader and proceeding similarly for v, we find that $u_{\sigma,M}^2 - \sigma^2$ and $v_{\sigma,M}^2-\sigma^2$ are uniformly bounded in $L^\infty(-1,T:H^1(0,M))$.

The uniform boundedness of $u_{\sigma,M}-\sigma$ and $v_{\sigma,M}-\sigma$ in $L^\infty(-1,T:BV(0,M))$ follows from a standard argument (formally: differentiate the equations with respect to x, multiply by, respectively, $\operatorname{sgn}(u_x)$ and $\operatorname{sgn}(v_x)$, and integrate by parts). Since $u_{\sigma,M}-\sigma$ and $v_{\sigma,M}-\sigma$ are uniformly bounded in $H^1(-1,T:H^{-1}(0,M-1))$, this implies that $u_{\sigma,M}-\sigma$ and $v_{\sigma,M}-\sigma$ are uniformly bounded in $C([-1,T]:L^p(0,M)))$ (see [9]). More precisely, we have enough compactness to pass to the limit $\sigma\to 0$, $M\to\infty$ along subsequences and obtain a solution of problem (1)-(2) (observe that the continuity of u and v follows from properties (i) and (ii) of Definition 2.1).

It remains to prove the uniqueness of the solution. If (u_1,v_1) and (u_2,v_2) are two solutions, we use $\int_t^T(u_1^2-u_2^2)$ and $\int_t^T(v_1^2-v_2^2)$ as test functions in (9) and add the two corresponding expressions:

$$\iint_{Q_T}\left((u_1^2-u_2^2)(u_1-u_2)+\varepsilon(v_1^2-v_2^2)(v_1-v_2)\right)$$
$$+\frac{1}{2}\int_0^\infty\left(\kappa\left(\int_{-1}^T(u_1^2-u_2^2)_x\right)^2+\left(\int_{-1}^T(v_1^2-v_2^2)_x\right)^2\right)$$
$$=-\beta\iint_{Q_T}\left(\int_t^T\left((u_1^2-u_2^2)-(v_1^2-v_2^2)\right)\right)\left((u_1^2-u_2^2)-(v_1^2-v_2^2)\right)$$
$$=-\frac{\beta}{2}\int_0^\infty\left(\int_{-1}^T\left((u_1^2-u_2^2)-(v_1^2-v_2^2)\right)\right)^2\le 0,$$

and hence $u_1=u_2$ and $v_1=v_2$ in Q_T. □

We now list some basic properties of the solution. We recall that we are particularly interested in values $-1\le t\le -\frac{1}{2}$, for which $f'\ge 0$.

Proposition 3. *Let $\varepsilon,\beta,\kappa>0$, let f satisfy (3) and let (u,v) be the solution of (1)-(2). Then:*

(i) *for all $-1\le t\le -\frac{1}{2}$, u and v are nonincreasing with respect to x*
(ii) *for all $-1\le t\le -\frac{1}{2}$, u and v are nondecreasing with respect to t*
(iii) *if $\varepsilon\kappa\le 1$, $v\ge u$ in $[0,\infty)\times[-1,-\frac{1}{2}]$*

(*iv*) $\operatorname{supp} v(t) = \operatorname{supp} u(t) = [0, r(t)]$ *for all* $t \geq -1$*, where* r *is a nondecreasing and continuous function such that* $r(-1) = 0$

(*v*) *for all* $t > -1$

$$\int_0^\infty x(u(x,t) + \varepsilon v(x,t))dx = (\kappa + 1) \int_{-1}^t f^2(\tau)d\tau.$$

Proof. (*i*): It is enough to prove the result for the solutions $(u_{\sigma,M}, v_{\sigma,M})$ of problem (10). Omitting subscripts we know from (11) that, if $t \leq -\frac{1}{2}$, $u(0,t) = \max u(\cdot,t)$, $v(0,t) = \max v(\cdot,t)$, $u(M,t) = \min u(\cdot,t)$ and $v(M,t) = \min v(\cdot,t)$. Hence $u_x(0,t)$, $v_x(0,t)$, $u_x(0,t)$ and $v_x(0,t)$ are nonnegative. Setting $z = u_x$ and $w = v_x$, the rectangle $(-\infty, 0] \times (-\infty, 0]$ is invariant for the system

$$\begin{cases} z_t = \kappa(2uz)_{xx} - 2\beta(uz - vw) \\ \varepsilon w_t = (2vw)_{xx} + 2\beta(uz - vw). \end{cases} \tag{12}$$

(*ii*): The proof is identical to the previous one: for the solutions of problem (10), $u_t(0,t) \geq 0$, $v_t(0,t) \geq 0$ and $u_t(M,t) = v_t(M,t) = 0$, and setting $z = u_t$ and $w = v_t$, the rectangle $[0,\infty) \times [0,\infty)$ is invariant for system (12).

(*iii*): Setting $z = u_{\sigma,M} - v_{\sigma,M}$ and omitting subscripts,

$$z_t = \kappa((u+v)z)_{xx} + (\kappa\varepsilon - 1)v_t - \beta\,(1+\kappa)\,(u+v)z.$$

Since z vanishes if $x = 0$, $x = M$ or $t = -1$, it follows from part (*ii*) and the maximum principle that $z \leq 0$ in Q if $\kappa\varepsilon - 1 \leq 0$.

(*iv*): It is well-known (see for instance [7]) that nonnegative solutions of the equation $u_t = \kappa\left(u^2\right)_{xx} - \beta u^2$ satisfy the following positivity property:

$$u(x_0, t_0) > 0 \Rightarrow u(x_0, t) > 0 \text{ for } t > t_0.$$

In our case $u_t \geq \kappa\left(u^2\right)_{xx} - \beta u^2$ and $\varepsilon v_t \geq \left(v^2\right)_{xx} - \beta v^2$, and therefore the positivity property holds for both u and v. Hence there exist nondecreasing functions $r_u, r_v : [0,\infty) \to [0,\infty]$ such that for all $t > -1$ and $x > 0$

$$u(x,t) > 0 \Leftrightarrow x < r_u(t), \quad v(x,t) > 0 \Leftrightarrow x < r_v(t).$$

By the continuity of u and v the functions r_u and r_v are continuous from the left:

$$r_u(t) \to r_u(t_0) \text{ and } r_v(t) \to r_v(t_0) \text{ as } t \nearrow t_0. \tag{13}$$

We claim that $r_u(t) = r_v(t)$ for all $t > -1$. Arguing by contradiction we assume that $r_u(t_0) < r_v(t_0)$ for some $t_0 > -1$ (by symmetry, the case $r_u(t_0) > r_v(t_0)$ is identical). By (13) and the monotonicity of r_u, there exists $\delta > 0$ such that

$$u = 0 \text{ and } v > 0 \text{ in } \mathcal{R} := [r_u(t_0), \tfrac{1}{2}(r_u(t_0) + r_v(t_0))] \times [t_0 - \delta, t_0].$$

Hence the equation for u is not satisfied in $\mathcal{R}$ and we have found a contradiction. Setting $r = r_u (= r_v)$, we have to prove that $r(t) < \infty$ for all $t > -1$ and that r is continuous. Since r is continuous from the left and nondecreasing, it is enough to prove that, given $t_0 \geq -1$, for all $r_1 > r(t_0)$ there exists $t_1 > t_0$ such that $r(t_1) \leq r_1$.

Below we briefly sketch the proof, which relies on a technique which was developed in [5, 6] to establish the occurrence of waiting time phenomena in degenerate parabolic equations. In [3] it was already applied to a degenerate parabolic system.

Let $\varphi(x) \in W^{1,\infty}(\mathbb{R}^+)$ such that $\operatorname{supp}\varphi \subseteq [r(t_0), \infty]$. We multiply the equations of $u_{\sigma,M}$ and $v_{\sigma,M}$ in (10) respectively by $(u_{\sigma,M} - \sigma)\varphi^2$ and $(v_{\sigma,M} - \sigma)\varphi^2$, sum the two equations, and integrate by parts over $(0, M) \times (t_0, t_1)$. Passing to the limit $\sigma \to 0$, $M \to \infty$ we eventually get that

$$\frac{1}{2}\int_0^\infty \varphi^2(u^2+\varepsilon v^2)\Big|_{t_1} + \int_{t_0}^{t_1}\int_0^\infty 2\varphi^2(\kappa u u_x^2 + v v_x^2) + \int_{t_0}^{t_1}\int_0^\infty 2\varphi_x\varphi(\kappa u(u^2)_x + v(v^2)_x) \le 0.$$

Hence, by Hölder's and Young's inequalities,

$$\sup_{t\in(t_0,t_1)}\int_0^\infty \varphi^2(u^2+v^2)\Big|_t + \int_{t_0}^{t_1}\int_0^\infty \varphi^2(u u_x^2 + v v_x^2) \le C_1\int_{t_0}^{t_1}\int_{\operatorname{supp}\varphi}(u^3+v^3) \tag{14}$$

for a constant C_1 depending on ε, κ and $||\varphi||_{1,\infty}$.

Without loss of generality we may only consider the right continuity of $r(t)$ at the initial time $t_0 = -1$. Therefore $r(t_0) = r(-1) = 0$.

Let $\eta \ge 0$. We set

$$\varphi = \varphi_\eta(x) := \left[\begin{array}{ll} 0 & \text{if } x < \eta \\ x-\eta & \text{if } \eta \le x \le \eta+1 \\ 1 & \text{if } x > \eta+1 \end{array}\right.$$

and substitute $\varphi = \varphi_\eta$ into (14). Hence for any $\eta < \xi < \eta + 1$

$$\sup_{t\in(t_0,t_1)}\int_\xi^\infty (u^2+v^2)\Big|_t + \int_{t_0}^{t_1}\int_\xi^\infty (u u_x^2 + v v_x^2) \le \frac{C_1}{(\xi-\eta)^2}\int_{t_0}^{t_1}\int_\eta^\infty (u^3+v^3). \tag{15}$$

By Gagliardo-Nirenberg's inequality there exists a constant K such that

$$\int_\xi^\infty u^3 \le K\left(\int_\xi^\infty u u_x^2\right)^{\frac{5}{11}}\left(\int_\xi^\infty u^2\right)^{\frac{9}{11}}, \tag{16}$$

and v satisfies the same inequality. Combining (15) and (16) and using Hölder's inequality, we conclude that for all $0 < \eta < \xi < 1$

$$\int_{t_0}^{t_1}\int_\xi^\infty (u^3+v^3) \le C_2(t_1-t_0)^{\frac{6}{11}}\frac{1}{(\xi-\eta)^{\frac{28}{11}}}\left(\int_{t_0}^{t_1}\int_\eta^\infty (u^3+v^3)\right)^{\frac{14}{11}}.$$

Now Stampacchia's iteration lemma [10, Lemma 4.1] implies that for any $0 < r_1 < 1$ there exists a $t_1 > t_0 = -1$ such that

$$\int_{t_0}^{t_1}\int_{r_1}^\infty (u^3+v^3) = 0.$$

(v): Let $M > x_0 > 0$. Multiplying the equations for $u_{\sigma,M}$ and $u_{\sigma,M}$ by x, integrating by parts and omitting subscripts, we obtain

$$\begin{aligned}&\int_0^{x_0} x((u(x,T)-\sigma) + \varepsilon(v(x,T)-\sigma))dx \\ &=(\kappa+1)\int_{-1}^T \left((f(t)+\sigma)^2 - u^2(x_0,t)\right)dt + x_0\int_{-1}^T \left(\kappa u^2+v^2\right)_x(x_0,t)dt.\end{aligned} \tag{17}$$

Adding the equations for $u_{\sigma,M}$ and $u_{\sigma,M}$ multiplied by, respectively, $x^2u^2_{\sigma,M}$ and $x^2v^2_{\sigma,M}$, we obtain

$$\begin{aligned}\frac{1}{3}\int_0^M x^2\left((u^3(x,T)-\sigma^3)+\varepsilon(v^3(x,T)-\sigma^3)\right)dx+\iint_{Q_T^{(M)}} x^2\left(\kappa(u^2)^2_x+(v^2)^2_x\right)\\ +\beta\iint_{Q_T^{(M)}} x^2\left(u^2-v^2\right)^2=\kappa\iint_{Q_T^{(M)}} u^4+M^2\sigma^2\int_{-1}^T(\kappa u^2+v^2)_x(M,t)dt\\ -(\kappa+1)M\sigma^4(T+1)\le\kappa\iint_{Q_T^{(M)}} u^4\end{aligned}$$

since $u_{\sigma,M}$ and $u_{\sigma,M}$ have a minimum (with respect to x) at $x=M$. Hence $\iint_{Q_T^{(M)}} x^2\left(\kappa(u^2)^2_x+(v^2)^2_x\right)$ is uniformly bounded and therefore there exists a sequence $(x_{0n},M_n,\sigma_n)\to(\infty,\infty,0)$ such that the terms in (17) which contain x_0 vanish as $n\to\infty$. □

The following two results will be used several times in the paper.

Lemma 2.2. *Let $(\overline{u},\overline{v})$ be the unique solution (in a sense similar to Definition 2.1) of the system*

$$\begin{cases}\overline{u}_t=\kappa(\overline{u}^2)_{xx}+\beta\overline{v}^2 & \text{in } \mathbb{R}^+\times(-1,-\frac{1}{2}]\\ \varepsilon\overline{v}_t=(\overline{v}^2)_{xx} & \text{in } \mathbb{R}^+\times(-1,-\frac{1}{2}]\end{cases}$$

which satisfy the initial-boundary conditions (2). If $\varepsilon\kappa\le 1$, then $\overline{u}\ge u$ and $\overline{v}\ge v$ in $\mathbb{R}^+\times[-1,-\frac{1}{2}]$.

In view of Lemma 3(iii), the proof is a consequence of the comparison principle (applied first to the equation for v and then to the one for u).

Lemma 2.3. *Let $T>-1$ and let $(\underline{u},\underline{v})$ be the unique solution (in a sense similar to Definition 2.1) of the system*

$$\begin{cases}\underline{u}_t=\kappa(\underline{u}^2)_{xx}-\beta(\underline{u}^2-\underline{v}^2) & \text{in } Q_T\\ \varepsilon\underline{v}_t=(\underline{v}^2)_{xx}-\beta\underline{v}^2 & \text{in } Q_T\end{cases}$$

which satisfy the initial-boundary conditions (2). Then $\underline{u}\le u$ and $\underline{v}\le v$ in Q_T.

Again the proof uses the comparison principle (applied first to the equation for v and then to the one for u).

3. **First moments.** In this section we prove Theorem 1.3. We present all calculations in terms of the solution (u,v), but passing to $(u_{\sigma,M},v_{\sigma,M})$ and arguing as in the proof of Proposition 3(v) they can be easily justified.

Multiplying the equation for v by x and integrating by parts we have that

$$\int_0^\infty\varepsilon xv(x,t)dx=\int_{-1}^t f^2(\tau)d\tau-\beta\iint_{Q_t}x(v^2-u^2)dxd\tau. \tag{18}$$

If $-1\le t\le-\frac{1}{2}$, it follows from Proposition 3(iii) that

$$0\le\beta\iint_{Q_t}x(v^2-u^2)dxdt\le\beta\iint_{Q_t}xv^2dxd\tau$$

and $\varepsilon v_t\le(v^2)_{xx}$ in $\mathbb{R}^+\times(-1,-\frac{1}{2}]$. Hence, since f is non decreasing in $(-1,-\frac{1}{2})$,

$$v(x,t)\le V\left(\frac{x\sqrt{\varepsilon}}{\sqrt{t+1}}\right)\text{ if } x>0,-1<t\le-\tfrac{1}{2},$$

where $V(y)$ is the unique solution of

$$(V^2)'' + \tfrac{1}{2}yV' = 0 \text{ in } \mathbb{R}^+, \quad V(0) = f(-\tfrac{1}{2}), \quad V(\infty) = 0.$$

It is well-known that V has compact support: there exists $y^* > 0$ such that $V(y) > 0$ if $y < y^*$ and $V(y) = 0$ if $y \geq y^*$. Therefore, if $\beta = o(\varepsilon)$ as $\varepsilon \to 0$,

$$\begin{aligned}\beta \iint_{Q_t} xv^2 dx d\tau &\leq \beta \iint_{Q_t} xV^2\left(\frac{x\sqrt{\varepsilon}}{\sqrt{\tau+1}}\right) dx d\tau \\ &= \frac{\beta}{\varepsilon} \iint_{(0,y^*)\times(-1,t)} y(\tau+1)V^2(y) dy d\tau \to 0 \text{ as } \varepsilon \to 0,\end{aligned}$$

and (6) follows at once from (18) and Proposition 3(v).

It remains to prove one of the limits in (7). Let $\underline{u}$ and $\underline{v}$ be defined by Lemma 2.3. Since $(\underline{v}^2)_{xx} - \beta\underline{v}^2 = \varepsilon\underline{v}_t \geq 0$ if $t \leq -\frac{1}{2}$, it follows from the maximum principle that $\underline{v}(x,t) \leq f(t)e^{-\frac{1}{2}\sqrt{\beta}x}$ for $x \geq 0$. In particular we find that for $-1 \leq t \leq -\frac{1}{2}$

$$\begin{aligned}\beta \iint_{Q_t} x\underline{v}^2 dx d\tau &= \iint_{Q_t} \left(x(\underline{v}^2)_{xx} - \varepsilon x \underline{v}_\tau\right) dx d\tau = \int_{-1}^t f^2(\tau) d\tau - \varepsilon \int_0^\infty x\underline{v}(x,t) dx \\ &\geq \int_{-1}^t f^2(\tau) d\tau - f(t) \int_0^\infty x e^{-\frac{1}{2}\sqrt{\beta}x} dx \\ &= \int_{-1}^t f^2(\tau) d\tau - \frac{\varepsilon}{\beta} f(t) \int_0^\infty y e^{-\frac{1}{2}y} dy \to \int_{-1}^t f^2(\tau) d\tau \quad \text{as } \beta \to 0\end{aligned}$$

if $\varepsilon = o(\beta)$ as $\beta \to 0$. Hence, by Lemma 2.3,

$$\begin{aligned}\int_0^\infty xu(x,t) dx \geq \int_0^\infty x\underline{u}(x,t) dx &= \iint_{Q_t} \left(\kappa x(\underline{u}^2)_{xx} - \beta x\underline{u}^2 + \beta x \underline{v}^2\right) dx d\tau \\ &\geq (\kappa+1) \int_{-1}^t f^2(\tau) d\tau - \beta \iint_{Q_t} xu^2 dx d\tau + o(1) \quad \text{as } \beta \to 0\end{aligned}$$

and (7) follows at once if we show that

$$\int_0^\infty xu^2(x,t) dx \leq C \quad \text{for } -1 \leq t \leq -\tfrac{1}{2}$$

for some constant C which does not depend on β. Indeed, multiplying the equation for u by xu we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^\infty xu^2(x,t) dx &= \tfrac{2}{3}\kappa f^3(t) - 2\kappa \int_0^\infty xuu_x^2 + \beta \int_0^\infty xu(v^2 - u^2) dx \\ &\leq \tfrac{2}{3}\kappa f^3(t) + \beta f^2(t) \int_0^\infty xu(x,t) dx \leq \tfrac{2}{3}\kappa f^3(t) + \beta(\kappa+1) f^2(t) \int_0^t f^2(\tau) d\tau.\end{aligned}$$

4. **The penetration depth.** In this section we prove Theorems 1.1 and 1.2. The proof of Theorem 1.1 is immediate: setting $s = \sqrt{\varepsilon}x$, the lower bound follows from Lemma 2.3:

$$\underline{v}_t = (\underline{v}^2)_{ss} - \frac{\beta}{\varepsilon}\underline{v}^2 \geq (\underline{v}^2)_{ss} - \underline{v}^2$$

and $\sqrt{\varepsilon} r_{\varepsilon,\beta}(-\frac{1}{2}) \geq s_0$, where s_0 is the interface at time $t = -\frac{1}{2}$ of the solution of

$$\begin{cases} \varepsilon v_t = (v^2)_{ss} - v^2 \\ v(0,t) = f(t), \ v(s,-1) = 0 \end{cases} \quad \text{for } s > 0, t > -1.$$

Similarly, the upper bound follows from Lemma 2.2: $\overline{v}_t = (\overline{v}^2)_{ss}$.

It remains to prove Theorem 1.2. Without loss of generality we may assume that

$$\beta > 2\varepsilon \tag{19}$$

instead of $\beta \geq \varepsilon$. Indeed, if $\varepsilon \leq \beta \leq 2\varepsilon$, Theorems 1.1 and 1.2 are equivalent and Theorem 1.1 is still valid (as follows at once from its proof).

Introducing the independent variables

$$y = x\sqrt{\beta} \text{ and } \tau = \frac{\beta}{\varepsilon}(t + t_0), \tag{20}$$

system (1) becomes

$$\begin{cases} u_\tau = \varepsilon\left(\kappa(u^2)_{yy} - u^2 + v^2\right) \\ v_\tau = (v^2)_{yy} + u^2 - v^2 \end{cases} \quad \text{in } \mathbb{R}^+ \times \left(\frac{\beta(t_0 - 1)}{\varepsilon}, \infty\right). \tag{21}$$

Setting

$$u = e^{-y/2}q \quad \text{and} \quad v = e^{-y/2}w, \tag{22}$$

we have that

$$\begin{cases} q_\tau = e^{-y/2}\varepsilon\left(\kappa(q^2)_{yy} - 2\kappa(q^2)_y + (\kappa - 1)q^2 + w^2\right) \\ w_\tau = e^{-y/2}\left((w^2)_{yy} - 2(w^2)_y + q^2\right) \end{cases} \quad y > 0, \tau > \frac{\beta(t_0 - 1)}{\varepsilon}. \tag{23}$$

Setting

$$z = \int_0^y e^{s/4} ds = 4e^{y/4} - 4 \quad \Leftrightarrow \quad e^{y/4} = (z + 4)/4,$$

we end up with the following system for q and w as functions of z and τ:

$$\begin{cases} q_\tau = \varepsilon\left(\kappa(q^2)_{zz} - \frac{7\kappa}{z+4}(q^2)_z + \frac{16(\kappa-1)}{(z+4)^2}q^2 + \frac{16}{(z+4)^2}w^2\right) \\ w_\tau = (w^2)_{zz} - \frac{7}{z+4}(w^2)_z + \frac{16}{(z+4)^2}q^2 \end{cases} \text{if } z > 0, \tau > \frac{\beta(t_0 - 1)}{\varepsilon}. \tag{24}$$

The proof of Theorem 1.2 is based on the construction of comparison functions for $w(z, \tau)$. The proof of the lower bound for $r_{\varepsilon,\beta}(-\frac{1}{2})$ is straightforward. The proof of upper bound is more complicated and rather lengthy.

4.1. **Theorem 1.2: Lower bound for** $r_{\varepsilon,\beta}(-\frac{1}{2})$**.** From the equation for $w(z, \tau)$ one easily obtains the lower bound for $r_{\varepsilon,\beta}(t)$. Let $t_0 = \frac{3}{4}$, so $-\frac{3}{4} \leq t \leq -\frac{1}{2}$ if $0 \leq \tau \leq \frac{\beta}{4\varepsilon}$. Let $\underline{w}$ be the solution of

$$\begin{cases} w_\tau = (w^2)_{zz} & \text{for } 0 < \tau \leq \frac{\beta}{4\varepsilon} \\ w(z, 0) = 0 & \text{for } z > 0 \\ w(0, \tau) = f(-3/4) & \text{for } 0 < \tau \leq \frac{\beta}{4\varepsilon}. \end{cases}$$

Since $\underline{w}_z \leq 0$, $\underline{w}$ is a subsolution: $w \geq \underline{w}$ in $\mathbb{R}^+ \times [0, \frac{\beta}{4\varepsilon}]$. In addition $\underline{w}$ is selfsimilar:

$$\underline{w}(z, \tau) = g\left(\frac{z}{4\sqrt{\tau}}\right) = g\left(\frac{e^{y/4} - 1}{\sqrt{\tau}}\right) = g\left(\frac{e^{\frac{x\sqrt{\beta}}{4}} - 1}{\sqrt{\frac{\beta}{\varepsilon}(t + \frac{3}{4})}}\right). \tag{25}$$

Since g has compact support, say $[0, \alpha_0]$, the interface of $\underline{w}$ is given by $z = 4\alpha_0\sqrt{\tau}$. This yields a lower bound for the interface $r_{\varepsilon,\beta}(t)$ of the original problem at $t = -\frac{1}{2}$:

$$r_{\varepsilon,\beta}(-\tfrac{1}{2}) \geq \frac{4}{\sqrt{\beta}} \log\left(1 + \frac{\alpha_0}{2}\sqrt{\frac{\beta}{\varepsilon}}\right) \geq \frac{2}{\sqrt{\beta}} \log\left(1 + \frac{\alpha_0^2}{4}\frac{\beta}{\varepsilon}\right). \tag{26}$$

4.2. **Theorem 1.2: The upper bound for $r_{\varepsilon,\beta}(-\frac{1}{2})$.** Let y, τ, q and w be defined by (20) and (22). Considering q and w as functions of y and t, we have that:

Lemma 4.1. *There exists a constant C such that for any $y > 0$, $-1 < t < -\frac{1}{2}$, $0 < \varepsilon \leq 1/\kappa$ and $\beta > 0$ sufficiently small,*

(i) $w^2(y,t) \leq f^2(t) + \int_0^y q^2(s,t)ds$;

(ii) $q(y,t) \leq f(t) + C\beta(t+1)$;

(iii) $\int\limits_0^\infty \zeta^2(y)e^{\lambda y}q(y,t)dy \leq \dfrac{C\beta}{1-2\lambda}$, *where $0 \leq \lambda < \frac{1}{2}$ and ζ is a smooth function such that $\zeta(0) = 0$, $\zeta' \geq 0$ in $(0,1)$ and $\zeta(y) = 1$ for $y > 1$.*

Proof. (i): Since $w_t \geq 0$ if $t \leq -\frac{1}{2}$, it follows from (23) that

$$(w^2)_{yy} - 2(w^2)_y + q^2 \geq 0.$$

Hence $w^2(y,t) \leq \ell(y)$, where ℓ is the unique solution of

$$\begin{cases} \ell'' - 2\ell' = -q^2(y,t) & \text{for } y > 0 \\ \ell(0) = f(t), \quad \ell(\infty) = 0. \end{cases}$$

The desired result follows easily from the explicit formula for ℓ:

$$e^{-2y}\ell'(y) = \int_y^\infty e^{-2s}q^2(s,t)ds,$$

and, since $e^{-y}q^2 = u^2$ is decreasing in y,

$$\begin{aligned} \ell(y) - f^2(t) &= \int_0^y e^{2s}\int_s^\infty e^{-2\xi}q^2(\xi,t)d\xi ds \\ &\leq \int_0^y e^s q^2(s,t)\int_s^\infty e^{-\xi}d\xi ds = \int_0^y q^2(s,t)ds. \end{aligned}$$

(ii): By (23) and part (i) of this lemma,

$$\begin{aligned} q_t =& \beta e^{-\frac{y}{2}}\left(\kappa(q^2)_{yy} - 2\kappa(q^2)_y + (\kappa-1)q^2 + w^2\right) \\ \leq& \beta e^{-\frac{y}{2}}\left(\kappa(q^2)_{yy} - 2\kappa(q^2)_y + (\kappa-1)q^2 + f^2(t) + \int_0^y q^2(s,t)ds\right). \end{aligned}$$

We fix $-1 < T \leq -\frac{1}{2}$ and look for a supersolution $\overline{q}(t)$ for $-1 < t \leq T$ which does not depend on y (one easily checks that the maximum principle continues to hold even though the equation contains an integral term). We require that $\overline{q}(0) = f(T)$ and

$$\overline{q}\,' \geq \beta\left(\kappa - 1 + \max_{\mathbb{R}^+}\{ye^{-\frac{y}{2}}\}\right)\overline{q}^2 + f^2(t).$$

Since $\overline{q} \geq f(T) \geq f(t)$ we may define $\overline{q}$ by

$$\begin{cases} \overline{q}\,' = \beta\left(\kappa + 2e^{-1}\right)\overline{q}^2 & \text{for } -1 < t \leq T \\ \overline{q}(-1) = f(T). \end{cases}$$

If $\beta > 0$ is sufficiently small, $\overline{q}$ is well defined in $[-1, T]$, and for such values of β we have that $\overline{q}(T) \leq f(T) + C\beta(T+1)$.

(iii): Multiplying the equation for $q(y,t)$ by $\zeta^2(y)e^{\lambda y}/\beta$ and integrating by parts over $\mathbb{R}^+$ we obtain from part (i) of this lemma that

$$\begin{aligned}
&\frac{1}{\beta}\frac{d}{dt}\int_0^\infty \zeta^2(y)e^{\lambda y}q(y,t)dy \\
&\qquad \le \int_0^\infty \left(\kappa\left(\zeta^2 e^{\left(\lambda-\frac12\right)y}\right)'' + 2\kappa\left(\zeta^2 e^{\left(\lambda-\frac12\right)y}\right)' + (\kappa-1)\zeta^2 e^{\left(\lambda-\frac12\right)y}\right) q^2 \\
&\qquad\quad + f^2(t)\int_0^\infty \zeta^2 e^{\left(\lambda-\frac12\right)y} + \int_0^\infty \zeta^2 e^{\left(\lambda-\frac12\right)y}\int_0^y q^2(s,t)ds \\
&\qquad = \left(\kappa - 1 + \kappa\left(\lambda-\tfrac12\right)^2 + 2\kappa\left(\lambda-\tfrac12\right)\right)\int_0^\infty \zeta^2 e^{\left(\lambda-\frac12\right)y}q^2 \\
&\qquad\quad + \int_0^\infty \left(4\kappa\left(\lambda-\tfrac12\right)\zeta\zeta' + 2\kappa\left(\zeta'\right)^2 + 2\kappa\zeta\zeta''\right)e^{\left(\lambda-\frac12\right)y}q^2 \\
&\qquad\quad + f^2(t)\int_0^\infty \zeta^2 e^{\left(\lambda-\frac12\right)y} + \int_0^\infty\left(\int_y^\infty \zeta^2(s)e^{\left(\lambda-\frac12\right)s}ds\right)q^2(y,t).
\end{aligned}$$

Since

$$\int_y^\infty \zeta^2(s)e^{\left(\lambda-\frac12\right)s}ds \le \frac{1}{\frac12-\lambda}e^{\left(\lambda-\frac12\right)y}$$

and q is uniformly bounded (by part (ii) of this lemma),

$$\frac{d}{dt}\int_0^\infty \zeta^2(y)e^{\lambda y}q(y,t)dy \le A\beta + \frac{B\beta}{\frac12-\lambda}\int_0^\infty e^{\left(\lambda-\frac12\right)y}q^2(y,t)dy \le \frac{C\beta}{1-2\lambda}.$$

□

Lemma 4.1 implies the following upperbounds for the functions q and w:

Proposition 4. *For all $y_0 > 0$ and $0 \le \lambda < \frac12$ there exists a constant $C = C(\lambda; y_0)$ such that for all β sufficiently small*

$$q(y,t) \le C\beta e^{-\lambda y} \quad \textit{for } y \ge y_0,\ -1 < t \le -\tfrac12 \textit{ and } 0 < \varepsilon \le 1/\kappa. \tag{27}$$

In addition

$$\limsup_{\beta\to 0} w(y,t) \le f^2(t) \quad \textit{uniformly in } (y,t,\varepsilon) \in \mathbb{R}^+ \times \left(-1,-\tfrac12\right) \times (0, 1/\kappa]. \tag{28}$$

Proof. By part (iii) of Lemma 4.1

$$\int_{y_0/2}^\infty e^{\lambda y}q(y,t) \le C\beta \quad \left(-1 < t \le -\tfrac12\right)$$

for some constant C depending on y_0 and λ. Hence

$$\int_{y_0/2}^\infty e^{\left(\lambda+\frac12\right)y}u(y,t) \le C\beta \quad \left(-1 < t \le -\tfrac12\right)$$

and since $u_y \le 0$ this implies that

$$u(y,t)\int_{y_0/2}^y e^{\left(\lambda+\frac12\right)s}ds \le C\beta.$$

Hence

$$u(y,t) \le \frac{\left(\lambda+\frac12\right)C\beta}{e^{\left(\lambda+\frac12\right)y} - e^{\left(\lambda+\frac12\right)y_0/2}} \le C_1\beta e^{-\left(\lambda+\frac12\right)y} \quad \text{if } y > y_0,$$

and multiplying by $e^{\frac12 y}$ we obtain (27).

It follows from (27) and the boundedness of q that

$$\int_0^\infty q^2(y,t)dy \to 0 \quad \text{as } \beta \to 0.$$

Hence part (i) of Lemma 4.1 implies (28). □

Now we are ready to construct a supersolution for w with compact support. We set

$$\tau = \frac{\beta}{\varepsilon}(t+1), \quad z = 4\left(e^{\frac{y}{4}} - 1\right)$$

and look for a supersolution of selfsimilar type:

$$\overline{w}(z,\tau) = h\left(\frac{z+4}{\sqrt{\tau}}\right) \quad \text{in } Q(\beta) := \left\{(z,\tau) : z \geq 4\left(\sqrt{\tau}-1\right),\ 1 \leq \tau \leq \frac{\beta}{2\varepsilon}\right\}.$$

In view of (24) it is natural to define the parabolic operator

$$\mathcal{L}(w) := -w_\tau + (w^2)_{zz} - \frac{7}{z+4}(w^2)_z + \frac{16}{(z+4)^2}q^2.$$

Then

$$\tau\mathcal{L}(\overline{w}) = (h^2)'' - \frac{7}{s}(h^2)' + \frac{1}{2}sh' + \frac{16}{s^2}q^2(z,\tau),$$

where we have set $s = (z+4)/\sqrt{\tau}$. By (27), with $\lambda = \frac{1}{4}$,

$$q^2(y,\tau) \leq C\beta^2 e^{-\frac{y}{2}} = \frac{C\beta^2}{s^2\tau} \leq \frac{C\beta^2}{s^2} \quad \text{if } \tau \geq 1.$$

On the other hand, $q \leq w$, and therefore we would like to define h as the solution of the problem

$$\begin{cases} (h^2)'' - \frac{7}{s}(h^2)' + \frac{1}{2}sh' + \dfrac{1}{s^2}\min\left\{16h^2(s), \dfrac{C\beta^2}{s^2}\right\} = 0 & \text{for } s \geq 4 \\ h(4) = \gamma \end{cases} \tag{29}$$

for a suitable choice of $\gamma > 0$.

Proposition 5. *Let $\beta \geq 0$ and $\gamma > 0$. Then there exists a compactly supported solution of Problem (29) which is nondecreasing with respect to β, and, for each $\beta \geq 0$, $h \to \infty$ in $C_{loc}([0,\infty))$ as $\gamma \to \infty$.*

Before proving Proposition 5, we finish the proof of Theorem 1.2. Let $h_{(\beta)}$ be the solution of Problem (29) and let (w,q) be the solution of the system which we are interested in (of course (w,q) will depend on β and ε). By the latter part of Proposition 5, we can choose γ so large that for all $0 < \varepsilon < 1/\kappa$ and sufficiently small $\beta > 0$ satisfying $\beta > 2\varepsilon$

$$w(z,1) \leq h_{(0)}(z+4) \quad \text{for } z > 0.$$

Here we have used the uniform boundedness (with respect to β and ε) of the support of $w(z,1)$ which follows from the fact that, for all β and ε, $w(z,\tau)$ is smaller than the solution of (see (24) and use that $q \leq w$)

$$\begin{cases} w_\tau = (w^2)_{zz} - \frac{7}{z+4}(w^2)_z + \frac{16}{(z+4)^2}w^2 & \text{in } \mathbb{R}^+ \times (0,1] \\ w(0,\tau) = f(1/2) & \text{for } 0 < \tau \leq 1 \\ w(z,0) = 0 & \text{for } z > 0. \end{cases}$$

By the monotonicity of $h_{(\beta)}$ with respect to β,

$$w(z,1) \leq h_{(\beta)}(z+4) \quad \text{for } z > 0.$$

Hence we can apply the comparison principle to $w(z,t)$ and $h_{(\beta)}\left(\frac{z+4}{\sqrt{\tau}}\right)$ in $Q(\beta)$ if γ is so large that

$$w(z,\tau) \le \gamma \quad \text{for all } z>0,\ 0<\tau\le \frac{\beta}{2\varepsilon}.$$

In view of (28) this is clearly possible and therefore

$$w(z,t) \le h_{(\beta)}\left(\frac{z+4}{\sqrt{\tau}}\right) \quad \text{in } Q(\beta).$$

Let $[0, s_{(\beta)}]$ be the support of $h_{(\beta)}$. By the monotonicity of $h_{(\beta)}$ with respect to β we have that $s_{(\beta)} \le s_{(1)}$ and proceeding as in (25) and (26) we find the desired upperbound of $r_{\varepsilon,\beta}\left(\frac{1}{2}\right)$.

In order to prove Proposition 5 we use the following shooting problem:

$$\begin{cases} h \in C([4,\infty)) \cap C^2(\{s \ge 4; h(s)>0\}) \\ h(4)=\gamma,\ h'(4)=\alpha \\ (h^2)'' - \frac{7}{s}(h^2)' + \frac{1}{2}sh' + \dfrac{1}{s^2}\min\left\{16h^2(s), \dfrac{C\beta^2}{s^2}\right\} = 0 \text{ if } s>4 \text{ and } h(s)>0 \\ \text{if } h \text{ hits } 0 \text{ for a finite value of } s, \text{ it is extended by } 0, \end{cases} \tag{30}$$

where $\alpha \in \mathbb{R}$ is the shooting parameter.

Lemma 4.2. *Let h be the solution of the shooting problem (30).*

(i) If $h'(s_1) \le 0$ and $h(s_1)>0$ for some $s_1 \ge 4$, then $h'(s)<0$ for $s>s_1$ as long as $h(s)>0$.

(ii) Let

$$\alpha_\gamma := -\frac{\gamma}{2\left(\sqrt{1+\gamma}-1\right)} \quad \text{and} \quad s_\gamma := -4\alpha_\gamma = 2\left(\sqrt{1+\gamma}+1\right). \tag{31}$$

Let $4 \le s_1 < s_\gamma$. If

$$h(s_1) = \gamma + \alpha_\gamma(s_1-4) \quad \text{and} \quad h'(s_1) \le \alpha_\gamma, \tag{32}$$

then h is concave for $s>s_1$ as long as it remains positive, h hits 0 in a point $s_0<s_\gamma$, and

$$\limsup_{s\to s_0} h'(s) < -\frac{1}{4}s_0. \tag{33}$$

Proof. Part (i) follows at once from the equation:

$$hh'' = -h'\left(\frac{1}{4}s + h' - \frac{7}{s}h\right) - \frac{1}{2s^2}\min\left\{16h^2(s), \frac{C\beta^2}{s^2}\right\}. \tag{34}$$

(ii). By (34), $h''(s)<0$ for $s \ge s_1$ as long as $h(s)>0$ and $h'(s) \le -\frac{1}{4}s$. We consider the segment in the plane which connects the points $(4,\gamma)$ and $(s_\gamma, 0)$, where $s_\gamma>4$ is chosen such that

$$(-\alpha_\gamma :=)\frac{\gamma}{s_\gamma - 4} = \frac{1}{4}s_\gamma,$$

i.e. α_γ and s_γ are given by (31). Condition (32) means that $(s_1, h(s_1))$ is a point of this segment and that $h'(s_1)$ is bounded by the (negative) slope α_γ of the segment. Hence $h'(s)+\frac{1}{4}s<0$ and h is concave for $s>s_1$ as long as $h(s)>0$. In particular h hits 0 in a point $s_0<s_\gamma$, and $h'(s_0)<\alpha_\gamma = -\frac{1}{4}s_\gamma < -\frac{1}{4}s_0$. □

Lemma 4.3. *Let h be a solution of the shooting problem* (30) *which hits $h = 0$ at $s_0 > 0$, such that* $\lim_{s\to s_0} h(s)h'(s) = 0$ *and* $\limsup_{s\to s_0} h'(s) < 0$. *Then*

$$\lim_{s\to s_0^-} h'(s) = -\frac{1}{4}s_0.$$

Proof. Since $(h^2)' = 0$ at the interface s_0, we may use De l'Hôpital's Rule:

$$\lim_{s\to s_0^-} h'(s) = \lim_{s\to s_0^-} \frac{h'h}{h} = \lim_{s\to s_0^-} \frac{(h^2)''}{2h'} = \lim_{s\to s_0^-} \left(\frac{7}{s}h - \frac{1}{4}s - \frac{8h^2}{s^2h'}\right) = -\frac{1}{4}s_0.$$

□

Lemma 4.4. *Let $\beta \geq 0$.*

(i) Let h be a solution of the shooting problem (30) *and let $\alpha_\gamma < 0$ be defined by* (31). *If $\alpha \leq \alpha_\gamma$, then h is not a solution of problem* (29).

(ii) Let h_γ be a solution of problem (29). *Then $h_\gamma \to \infty$ in $C_{loc}([0,\infty))$ as $\gamma \to \infty$.*

Proof. (i) By part (ii) of Lemma 4.2 h hits zero in a point s_0 where $\limsup_{s\to s_0} h'(s) < -\frac{1}{4}s_0$. If h were also a solution of problem (29), then $(h^2)'(s_0) = 0$ and Lemma 4.3 leads to contradiction.

(ii). Arguing as in the proof of part (i), it follows from part (ii) of Lemma 4.2 that

$$h_\gamma(s) > \gamma + \alpha_\gamma(s-4) \quad \text{for } 0 \leq s \leq s_\gamma.$$

In view of (31) $\alpha_\gamma, s_\gamma \to \infty$ as $\gamma \to \infty$ and the result follows at once. □

Lemma 4.5. *Let $\gamma > 0$ and $\beta \geq 0$ be fixed. Then there exists $C_\gamma > 0$ such that for any solution h of the shooting problem* (30) *which hits 0 at a finite value of s, say s_0,*

$$\int_4^{s_0} h(s)ds \leq C_\gamma.$$

Proof. If $\alpha \leq \alpha_\gamma$ the result follows at once from part (ii) of Lemma 4.2.

If $\alpha > \alpha_\gamma$ we integrate the equation in (30) by parts:

$$\begin{aligned}(h^2)'(s_0) =& 2\alpha\gamma - \frac{7}{4}\gamma^2 + 7\int_4^{s_0} \frac{1}{s^2}h^2(s)ds + 2\gamma + \frac{1}{2}\int_4^{s_0} h(s)ds \\ &- \int_4^{s_0} \frac{1}{s^2}\min\left\{16h^2(s), \frac{C\beta^2}{s^2}\right\} ds.\end{aligned}$$

Since $(h^2)'(s_0) \leq 0$ this implies that

$$\frac{1}{2}\int_4^{s_0} h(s)ds \leq -2\alpha\gamma + \frac{7}{4}\gamma^2 - 2\gamma + C\beta^2\int_4^\infty \frac{1}{s^4}ds < \frac{7}{4}\gamma^2 - 2\alpha_\gamma\gamma + \frac{C\beta^2}{192}.$$

□

Lemma 4.6. *The shooting problem* (30) *does not possess solutions h such that*

$$h(s) > 0 \text{ for all } s > 0 \quad \text{and} \quad h(s) \to 0 \text{ as } s \to \infty. \tag{35}$$

Proof. We argue by contradiction and suppose that there exists a solution h which satisfies (35). Then there exists a sequence $s_k \to \infty$ such that $(h^2)'(s_k) \to 0$ as $k \to \infty$. We claim that

$$(h^2)'(s) \to 0 \quad \text{as } s \to \infty. \tag{36}$$

To see this, it is enough to prove that $(h^2)'(s)$ has a limit as $s \to \infty$. By part (i) of Lemma 4.2, $h'(s) < 0$ for s large enough; therefore, for s large enough,

$$(h^2)''(s) = -\frac{1}{2} s h'(s) \left(1 - \frac{28}{s^2} h(s)\right) - \frac{16}{s^2} h^2(s) \geq -C_1$$

for some constant C_1. Since $(h^2)' \in L^1(s, \infty)$, $\lim_{s\to\infty} (h^2)'(s)$ exists and we have proved (36).

Now we can apply De l'Hôpital's Rule to

$$\lim_{s\to\infty} \frac{(h^2)'(s) - \int_s^\infty \min\left\{\frac{16h^2(t)}{t^2}, \frac{C\beta^2}{t^4}\right\} dt}{h(s)}$$

$$= \lim_{s\to\infty} \frac{\frac{7}{s}(h^2)' - \frac{1}{2} s h'}{h'} = \lim_{s\to\infty} \left(\frac{14h(s)}{s} - \frac{s}{2}\right) = -\infty.$$

On the other hand

$$\frac{1}{h(s)} \int_s^\infty \min\left\{\frac{16h^2(t)}{t^2}, \frac{C\beta^2}{t^4}\right\} dt \leq \frac{1}{h(s)} \int_s^\infty \frac{16h^2(t)}{t^2} dt \leq h(s) \int_s^\infty \frac{16}{t^2} dt \to 0$$

as $s \to \infty$. Hence we have found that

$$\lim_{s\to 0} h'(s) = \lim_{s\to\infty} \frac{(h^2)'(s)}{2h(s)} = -\infty,$$

which is a contradiction with (36). □

Lemma 4.7. *For any $\gamma > 0$ and $\beta \geq 0$ there exist solutions of the shooting problem (30) which are positive for all $s > 0$.*

Proof. We claim that for any given $s_1 \geq 4$ the solution h is non decreasing in $[s_1, \infty)$ if $h(s_1)$ and $h'(s_1)$ are large enough. If the claim holds, the desired result follows at once, since for any given γ, γ_1 and α_1 there exist $\delta > 0$ and $\alpha > 0$ such that $h(4+\delta) > \gamma_1$ and $h'(4+\delta) > \alpha_1$.

Let $h(s_1) \geq A s_1^2$ for some positive constant A to be determined below and let $h'(s_0) > 0$. Then there exists a maximal interval $[s_1, s_2)$ (where $s_2 \in (s_1, \infty]$) such that

$$h(s_1) \geq A s_1^2 \text{ and } h'(s_0) > 0 \quad \text{in } [s_1, s_2). \tag{37}$$

We have to prove that $s_2 = \infty$ if we choose $h(s_1)$ and $h'(s_1)$ sufficiently large.

Using the equation for h^2 (see (30)), it follows from (37) that

$$(h^2)'' \geq \left(7 - \frac{1}{4A}\right) \frac{1}{s} (h^2)' - \frac{C\beta^2}{s^4} \quad \text{in } (s_1, s_2).$$

Hence

$$\left(s^{-\left(7-\frac{1}{4A}\right)} (h^2)'\right)' \geq -C\beta^2 s^{-11-\frac{1}{4A}},$$

and integration in (s_1, s) leads to the inequality

$$(h^2)'(s) \geq \left(\frac{s}{s_1}\right)^{7-\frac{1}{4A}} \left((h^2)'(s_1) - \frac{C\beta^2}{\left(10 + \frac{1}{4A}\right) s^3}\right).$$

Choosing A such that $7 - \frac{1}{4A} \geq 2$, i.e. $A \geq \frac{1}{20}$, a second integration implies that we can choose $h(s_1)$ and $h'(s_1)$ so large that $h(s) \geq 2As^2$ and $h'(s) \geq 1$ in (s_1, s_2). Hence $s_2 = \infty$. □

Proof of Propotition 5. Let

$$\mathcal{A}_\gamma := \{\alpha \in \mathbb{R}\colon \text{the solution } h \text{ of the shooting problem (30) hits zero at finite } s\}.$$

Set $\alpha_0 = \sup \mathcal{A}_\gamma$. By Lemma 4.7, $\alpha_0 < \infty$.

We claim that $\alpha_0 \in \mathcal{A}_\gamma$. We consider a sequence $\alpha_n \nearrow \alpha_0$, with $\alpha_n \in \mathcal{A}$, and the corresponding solutions h_n and h_0 of problem (30) (if α_0 is an isolated element of $\mathcal{A}$ the claim is trivial). By Lemma 4.5 $\int_4^\infty h_n(s)ds \le C$, and, since $h_n \to h_0$ in $C_{loc}([4,\infty))$, also $\int_4^\infty h_0(s)ds \le C$. Hence $h_0(s) \to 0$ as $s \to \infty$ and it follows from Lemma 4.6 that $\alpha_0 \in \mathcal{A}$.

Now, let s_0 be the value of s at which h_0 hits 0. We must show that

$$(h_0^2)'(s_0) = 0.$$

Taking this time a sequence $\alpha_n \searrow \alpha_0$, $h_n(s) > 0$ for all $s \ge 4$ and it remains to prove that, in a neighborhood of s_0, $h_n^2 \to h_0^2$ in C^1. Since $h_n \to h_0$ in $C_{loc}([4,\infty))$ it is enough to prove that $(h_n^2)''$ is bounded in a neighborhood of s_0 uniformly with respect to n.

For all $\bar{h} > 0$ there exist $\bar{\delta} > 0$ and an integer $\bar{n}$ such that $h_n(s) \le \bar{h}$ whenever $n > \bar{n}$ and $s \in [s_0 - \delta, 2s_0]$. We choose $\bar{h} \le \frac{s_0^2}{112}$ and $\delta \le \frac{s_0}{2}$, then

$$(h_n^2)'' = -\frac{1}{2} s h_n'(s)\left(1 - \frac{28}{s^2} h_n(s)\right) - \frac{16}{s^2} h_n(s)^2 \ge -C$$

for all $s \in [s_0 - \delta, 2s_0]$ uniformly with respect to n.

We claim that $h_n' \ge -\frac{3}{4}s$ in $[s_0-\delta, 2s_0]$: if $h_n'(\bar{s}) < -\frac{3}{4}\bar{s}$ for some $\bar{s} \in [s_0-\delta, 2s_0]$, then

$$h_n(s)h_n''(s) \le -h_n'(s)\left(\frac{1}{4}s + h_n'(s)\right) \le 0 \qquad \text{in } [\bar{s}, 3s_0],$$

and h_n hits 0 in some point $s_1 \le 2s_0 + \frac{s_0}{84}$, in contradiction with the strict positivity of h_n. The lower bound for h_n' and the equation for h_n^2 (see (30)) yield $(h_n^2)'' \le C$ in $[s_0 - \delta, 2s_0]$, uniformly in n. □

Acknowledgement. The authors deeply thank Professor G.I. Barenblatt for bringing the problem to their attention and for stimulating discussions.

REFERENCES

[1] G. I. Barenblatt, *On some unsteady motions in a liquid or a gas in a porous medium*, Prikladnaja Matematika i Mechanika, **16** (1952), 67–78.
[2] G. I. Barenblatt, E. A. Ingerman, H. Shvets and J. L. Vázquez, *Very intense pulse in the gorundwater flow in fissurised-porous stratum*, PNAS, **97** (2000), 1366–1369.
[3] M. Bertsch, R. Dal Passo and C. Nitsch, *A system of degenerate parabolic nonlinear pde's: A new free boundary problem*, Interfaces Free Bound, **7** (2005), 255–276.
[4] K. N. Chuen, C. C. Conley and J. A. Smoller, *Positively invariant regions for systems of nonlinear diffusion equations*, Indiana Univ. Math. J., **26** (1977), 373–392.
[5] R. Dal Passo, L. Giacomelli and G. Grün, *A waiting time phenomena for thin film equations*, Ann. Scuola Norm. Sup. Pisa (4), **30** (2001), 437–463.
[6] R. Dal Passo, L. Giacomelli and G. Grün, "Waiting Time Phenomena for Degenerate Parabolic Equations – A Unifying Approach," in "Geometric Analysis and Nonlinear Partial Differential Equations" (S. Hildebrant and H. Karcher, eds.), Springer-Verlag, (2003), 637–648.
[7] R. Kersner, *Nonlinear heat conduction with absorption: Space localization and extinction in finite time*, SIAM J. Appl. Math., **43** (1983), 1274–1285.
[8] Y. Shvets, "Problems of Flooding in Porous and Fissured Porous Rock," Ph.D. thesis, University of California, Berkeley, 2005, `http://gradworks.umi.com/31/87/3187151.html`.
[9] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl. (4), **146** (1987), 65–96.

[10] G. Stampacchia, "Équationes Elliptiques Du Second Ordre à Coefficients Discontinus," Les presses de l'université de Montréal, 1966.
[11] J. L. Vázquez, "The Porous Medium Equation. Mathematical Theory," Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2007.

Received February 2010 for publication.

E-mail address: m.bertsch@iac.cnr.it
E-mail address: c.nitsch@unina.it