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NON-EXISTENCE OF POSITIVE STATIONARY SOLUTIONS FOR A CLASS OF SEMI-LINEAR PDES WITH RANDOM COEFFICIENTS

Jérôme Coville

Equipe BIOSP, INRA Avignon
Domaine Saint Paul, Site Agroparc
84914 Avignon cedex 9, France

Nicolas Dirr

Department of Mathematical Sciences, University of Bath
Bath, BA2 7AY, United Kingdom

Stephan Luckhaus

Mathematisches Institut der Universität Leipzig
PF 100920, Leipzig, Germany

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Abstract. We consider a so-called random obstacle model for the motion of a hypersurface through a field of random obstacles, driven by a constant driving field. The resulting semi-linear parabolic PDE with random coefficients does not admit a global nonnegative stationary solution, which implies that an interface that was flat originally cannot get stationary. The absence of global stationary solutions is shown by proving lower bounds on the growth of stationary solutions on large domains with Dirichlet boundary conditions. Difficulties arise because the random lower order part of the equation cannot be bounded uniformly.

1. Introduction.

We are interested in the behavior of a moving interface Γ in a random medium, where Γ is a graph, i.e. defined as

$$\Gamma(t) := \{(x, y) \in \mathbb{R}^2 : y = u(x, t)\} \tag{1}$$

and the function u evolves according to the following equation:

$$\frac{\partial u}{\partial t} = u_{xx}(x, t) + f(x, u(x, t)) + F \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^+, \tag{2}$$

$$u(x, 0) = 0 \tag{3}$$

where $f \in C^1(\mathbb{R}^2)$ is a random field (i.e. a random variable taking values in $C^1(\mathbb{R}^2)$) which represents a random medium and will be defined more precisely later on. Note that f is not restricted to be either positive or negative. F is a positive constant called "driving field." The objective is to prove that the solution of (2)-(3) does not

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get pinned, i.e. does not converge to a nonnegative stationary solution if F is above a critical value F_c. To this end, we will show that nonnegative stationary solutions on bounded intervals $[-N, N]$ with Dirichlet boundary conditions get large with high probability as $N \to \infty$.

The main contribution of this paper is to show that a *finite* F is sufficient to keep the graph moving, even if it will have to pass through regions where $f(x, u, \omega) \ll -1$, provided the probability of finding such a region is small. As f can become arbitrarily big, one cannot find a deterministic subsolution that keeps moving, and instead probabilistic arguments are needed.

The interest in the model stems from the theoretical analysis of the effective behavior on large scales of models for interface evolution specified at a microscopic scale, which is at the heart of many problems in physics and material science. Of particular interest is the influence of material heterogeneities, which are generally assumed to be random. Mathematically, this leads to studying the limit of evolution equations with rapidly varying random coefficients. In the case of dissipative equations, on which we focus here, the randomness leads to new and interesting effects absent in the case of periodic coefficients, e.g. pinning and de-pinning for obstacles with a strength that cannot be bounded uniformly. If the strong obstacles are sufficiently rare, than the interaction through the Laplacian helps the graph overcome them although the total forcing $f(x, u) + F$ remains negative near the obstacle.

One example we have in mind as motivation are driven elastic systems, for a review of the research in physics and its possible applications we refer to [1]. For a survey of front evolutions in random media, with evolution laws different from the ones considered here, see e.g. the recent monograph [8].

The model (2), restricted to a bounded interval $I \subseteq \mathbb{R}$, is a gradient flow for the random energy

$$\mathcal{F}(u, \omega) = \int_I |\nabla u(x)|^2 dx - \int_I F(x, u(x), \omega) dx,$$

where $\frac{\partial}{\partial u} F = f$ and ω is an element in the underlying probability space.

This energy can be understood as a heuristic approximation for the following more geometric energy: If the hypersurface Σ is the boundary of the set A_Σ then we can define for any bounded $D \subseteq \mathbb{R}^2$ the random energy

$$\mathcal{F}(\Sigma | D) := H^1(\Sigma \cap D) - \int_{D \cap A_\Sigma} f(x, u, \omega) dx du$$

where $(x, u) \in \mathbb{R}^2$ and H^1 denotes the 1-dimensional Hausdorff measure.

A gradient flow of that functional can be found in the following way: We require that the first variation of that functional (with respect to inner variations, i.e. deforming the interface with the flow of a smooth vector field) is proportional to the normal velocity of the interface. This leads to the so-called *forced mean curvature flow*,

$$V_{(x,u)} = \kappa_{(x,u)} + f(x, u, \omega),$$

where $\kappa_{(x,u)}$ denotes the mean curvature of the interface (trace of the second fundamental form) at the point $(x, u) \in \Sigma$, and the scalar $V_{(x,u)}$ is the velocity of the interface in the direction of the inner normal at the point $(x, u) \in \Sigma$.

This geometric evolution law leads to nonlinear degenerate parabolic equations, hence questions concerning the large-scale behaviour of solutions are related to homogenizing such equations with periodic or random coefficients. This is an active field of research (see e.g. [2], [6]) but many difficult problems remain open.

We can approximate forced mean curvature flow as follows: If we suppose that the interface is a graph of a function which is "flat" (no overhangs, small gradients) then we can consider a semi-linear equation as in (2) as heuristic approximation of the evolution by forced mean curvature flow.

This semi-linear model, here called random obstacle model (ROM) because of the precise nature of the random nonlinearity $f(x, u, \omega)$ used in this paper, is a special case of a class of equations sometimes called quenched Edwards-Wilkinson model which, for some choices of the random nonlinearity, is used in physics as a model for overdamped interface evolution in a random environment when "overhangs" can be neglected. For further comments on physical properties and justifications of the model we refer to [1]. In particular, one expects that solutions move with a deterministic effective (large-scale) velocity for F larger than a critical forcing F_*. For F slightly larger than F_*, the relation between the effective velocity and $F - F_*$ is expected to be a power law. (See also [4] for the periodic case.).

While there are important differences between the forced mean curvature flow and the semi-linear model (e.g. forced mean curvature flow can "wrap around" strong obstacles), we expect that the techniques we will develop when studying (2) will prove helpful in investigating more general models for interface evolution. This strategy was successful in the periodic case, where first the semi-linear case was solved ([4]) and then the results could be extended to graphs evolving by forced mean curvature flow ([3]).

One more reason why such models are of mathematical interest is the relation with "singular" homogenization problems, i.e. problems where the ε-equation is of second order (possibly degenerate) and the homogenized equation of first order. Note that the effective velocity $c(\eta)$ of an interface evolving with average slope η can be found by considering

$$\frac{\partial u}{\partial t} = u_{xx}(x, t) + f(x, \eta \cdot x + u(x, t)) + F,$$

i.e. this can be seen as the "cell problem" for

$$\frac{\partial v(y, \tau, \omega)}{\partial \tau} = \varepsilon v_{yy}(y, \tau, \omega) + f(\varepsilon^{-1}y, \varepsilon^{-1}v(y, \tau, \omega), \omega) + F$$

with $\tau = \varepsilon^{-1}t,\ y = \varepsilon^{-1}x$.

The paper is organized as follows. In Section 2 we define the random obstacle model precisely and state our main results.

In Section 3, we introduce an auxiliary model which is more suitable for explicit estimates and whose solutions can be related to solutions of the original equation (2) by the comparison principle for parabolic equation. This auxiliary problem has the property that any of its stationary solutions u solve $u_{xx} = -F$ away from the obstacles and is a convex function on the obstacles. This fact allows us to define a discretization, using that each solution is determined by its values when entering and leaving an obstacle. This yields a discretised path $\bar{v}^\delta : \mathbb{Z} \to \delta\mathbb{Z}$ characterizing each stationary solution.

In section 4, we estimate the discrete Laplacian of $\bar{v}^\delta(i)$ against the obstacles that sit above and below $i \in \mathbb{Z}$ and are approached by the path, i.e. $\Delta_d \bar{v}(i) + \bar{F} \leq C\ell_{i,[\bar{v}^\delta(i)[}(\omega)$ where $\bar{F}$ is a constant which can be chosen arbitrarily large. A technical

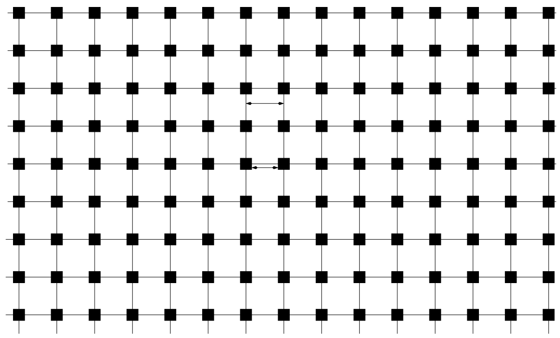

Figure 1. The obstacles

problem is posed by the fact that the path may pass more than one obstacle above the same integer.

In section 5 we estimate the probability of a discrete path being "compatible" with the random environment. This probability can be estimated against an auxiliary random measure on paths:

$$\begin{aligned}\mathbb{P}\left(\left\{\omega:\ u(\omega) \text{ compatible with } \bar{v}^{\delta}(i)\right\}\right) &\leq C^{2N}\mathbb{P}\left(\{\Delta_d \bar{v}^{\delta}(i)\}_{i=-N+1}^{N-1}\right),\\ \mathbb{P}\left(\{\Delta_d \bar{v}^{\delta}(i)\}_{i=-N+1}^{N-1}\right) &:= Z^{-1}e^{-\lambda\sum_{i=-N+1}^{N-1}|\Delta_d\bar{v}^{\delta}(i)+\bar{F}|},\end{aligned}$$

where Z is a normalization (corresponding to the partition function in statistical mechanics).

In section 6 we conclude that the probability of a nonnegative solution of the Dirichlet problem to cross $KN - K|x|$ is $\mathcal{O}(e^{-CN})$. The key observation is that for such a path $N^{-1}\sum_{i=-N+1}^{N-1}\left(\Delta_d\bar{v}^{\delta}(i)+\bar{F}\right)$ must be large, which is very unlikely under the auxiliary (product) probability measure.

Finally, we show by invoking the comparison principle for semi-linear parabolic equations that these results for large N imply non-existence of global nonnegative stationary solutions. This implies that for a solution u of (2), (3) and all $x \in \mathbb{R}$ it holds that $\lim_{t\to\infty} u(t,x,\omega) = +\infty$ almost surely in ω, i.e. the interface cannot be stopped by the obstacles.

2. Results and definitions.

2.1. The random field.

Here, the field f is negative on "obstacles" in $\mathbb{R}^2$ which are random in strength, but positioned on a lattice. More precisely, we make the following assumption:

Definition 2.1 (Obstacles)**.**

1. Let $\mathbb{Z}^* := \mathbb{Z} + 1/2$. We assume that the obstacles lie on a lattice $\mathcal{L} := \mathbb{Z}\times\mathbb{Z}^*$ where for convenience $(b_{ij})_{i\in\mathbb{Z},j\in\mathbb{Z}^*}$ denotes the nodes of this lattice, i.e $b_{i,j} := (i,j)$.
2. Let $\delta \ll 1/2$ and define $Q_\delta(0,0) := [-\delta,\delta]^2$, and $Q_\delta(i,j) := Q_\delta(0,0) + b_{i,j}$. Then the obstacles, i.e. regions where $f < 0$ is possible, are given by the $Q_\delta(i,j)$, see also Figure 1

In order to obtain existence and regularity of the solutions, the nonlinearity $f(x,y)$ should be sufficiently regular, hence in order to define f we have to smooth out the obstacles.

Definition 2.2 (Random field)**.** Let $\phi \in C_c^\infty$ be a nonnegative function such that its support is contained in cube $Q_\delta(0,0)$.

Let $(l(i,j)(\omega))_{(i,j)\in\mathbb{Z}\times\mathbb{Z}^*}$ be a family of independent identically distributed exponential random variables, i.e. there exists $\lambda_0 > 0$ such that for $r \geq 0$

$$\mathbb{P}\{l(i,j)(\omega) > r\} = e^{-\lambda_0 r}.$$

Let Σ be the set of the obstacles, i.e. $\Sigma := \bigcup_{(i,j)\in\mathbb{Z}\times\mathbb{Z}^*}\Big(Q_\delta(b_{i,j})\Big)$, then the field f is defined the following way:

$$f(x,s) = g(x,s) - \sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}^*} l(i,j)\phi((x,s) - b_{i,j})$$

where g is a non-negative function chosen so that the field has mean zero in a suitable sense:

$$g \geq 0 \quad \text{in} \quad \mathbb{R}^2$$
$$\lim_{L\to\infty}(2L)^2\int_{[-L,L]^2} f(x,s)\,dxds = 0$$

Remark 1.

1. As $\mathbb{E}(l(i,j)) = \frac{1}{\lambda}$, the law of large numbers implies that a possible choice of g is
$$g(x,s) = \sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}^*}\frac{1}{\lambda}\phi((x,s) - b_{i,j}).$$
2. The results on non-existence of nonnegative stationary solutions hold for any i.i.d. random variables $l(i,j)$ such that there exists $\lambda_0 > 0$ with
$$\mathbb{P}\{l(i,j)(\omega) > r\} \leq e^{-\lambda_0 r}.$$
3. As we are only interested in the combined effect of $f(x,s)$ and the constant forcing F, the mean zero property of the random nonlinearity is just a normalization.
4. In our analysis, the shape of the obstacles ($supp(\phi)$) plays no role and the results will stand as well if we consider a random field like e.g.
$$f = g(x,s) - \sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}^*} l(i,j)\phi_{i,j}((x,s))$$
where $\phi_{i,j}$ are smooth functions uniformly bounded and such that $supp(\phi_{i,j}) \subset Q_\delta(b_{i,j})$.

2.2. **Results:** We consider the stationary version of (2) with Dirichlet boundary conditions:

$$u_{xx} + f(x,u,\omega) + F = 0 \quad \text{in} \quad [-N+\delta, N-\delta] \tag{4}$$
$$u(-N+\delta) = u(N-\delta) = 0 \tag{5}$$

Theorem 2.3. *Let $u(\omega)$ solve (4, 5). Then there exist $F_0 > 0$, C and K such that for $F > F_0$ and N sufficiently large*

$$\mathbb{P}\left(\{\omega|\, u(x,\omega) \geq (K(N-1) - K|x|)_+ \text{ on } [-N+\delta, N-\delta]\}\right) \geq 1 - Ce^{-\frac{N}{C}},$$

where a_+ denotes the positive part of a real number a.

Corollary 1. *Let $F > F_0$, with F_0 as in Theorem 2.3.*

1. *There is almost surely no global nonnegative stationary solution of (2).*

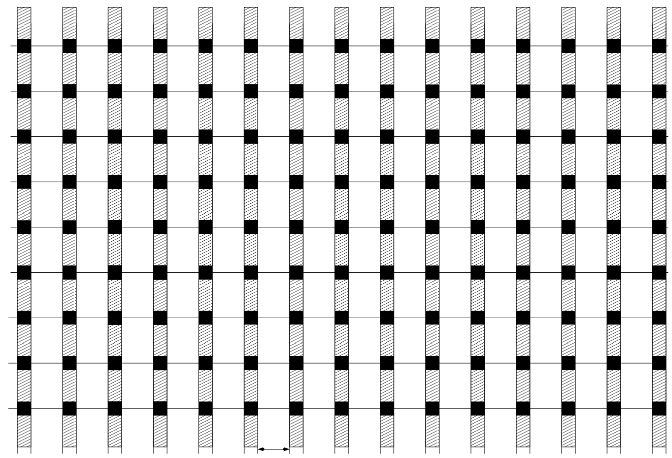

FIGURE 2. Mapping of the obstacles for the auxiliary problem

2. Let u solve (2), (3). Then

$$\lim_{t\to\infty} u(t,x,\omega) = +\infty \quad \text{for all } x \in \mathbb{R}$$

holds with probability one.

3. **Blocked path and auxiliary problem.** In this section we define a auxiliary problem that we will constantly use along this paper. We will denote by χ_B the characteristic function of the set B.

Definition 3.1 (Auxiliary field)**.** Let

$$A := \mathbb{R}^2 \setminus \{\bigcup_{i\in\mathbb{Z}} (i-\delta, i+\delta) \times \mathbb{R}\}$$
$$A_\varepsilon := \mathbb{R}^2 \setminus \{\bigcup_{i\in\mathbb{Z}} (i-\delta-\varepsilon, i+\delta+\varepsilon) \times \mathbb{R}\}$$

and define

$$\widetilde{f}(x,s) := -\sum_{(i,j)\in\mathbb{Z}^*\times\mathbb{Z}^*} l(i,j)\phi((x,s)-b_{i,j}).$$

Let us now consider the following auxiliary problem

$$\frac{\partial v}{\partial t} = v_{xx} + \widetilde{f}(x, v(t,x)) + F\chi_A^\varepsilon(x) \tag{6}$$
$$v(0,x) = 0, \tag{7}$$

where χ_A^ε is a smooth function such that $\chi_{A_\varepsilon} \leq \chi_A^\varepsilon \leq \chi_A$. ε is a small parameter which will be fix later on.

To visualize the new random field defined by $\widetilde{g} = \widetilde{f} + F\chi_A^\varepsilon(x)$ see figure 2. Note that it is differentiable in x and s.

Observe that, as the obstacles are negative, $\widetilde{f} + F\chi_A^\varepsilon \leq f + F$. Therefore the comparison principle for the parabolic equation (see section 3.1) implies that solutions of the auxiliary problem remain below solutions of the original problem. Hence existence of a nonnegative stationary solution for the original problem implies existence of one for the auxiliary problem. By contraposition, nonexistence for the auxiliary problem implies nonexistence for the original problem.

Stationary sub/supersolutions can be constructed as piecewise quadratic functions. For any F we can construct the graph of such a solution (also called "paths" to emphasize the analogy with a stochastic process).

Definition 3.2 (blocked path)**.** A graph $(x, v(x))$ is called blocked path if and only if $v \in C^1_{loc}(\mathbb{R})$, and

$$v_{xx} = -F\chi^{\varepsilon}_A(x) \qquad \text{in} \qquad (i+\delta, i+1-\delta), \tag{8}$$

$$v_{xx} = \sum_{j\in\mathbb{Z}^*} l(i,j)(\omega)\phi_{i,j}(x, v(x)) \qquad \text{in} \qquad (i-\delta, i+\delta). \tag{9}$$

where $\phi_{i,j}(x,s) := \phi((x,s) - b_{i,j})$.

Observe that the path for $x \in (i+\delta, i+1-\delta)$ is uniquely determined by the boundary values $v(i+\delta)$ and $v(i+1-\delta)$, because it solves a *linear* elliptic equation there. But note that, for a given realisation of the random field, there may be more than one blocked path, as equations like $u_{xx} = f(x,u)$ do not have unique solutions without further conditions on the nonlinearity.

Remark 2. From Definition 3.2, we see that v is a convex function in $(i-\delta, i+\delta)$ and hence we have

$$v(i+\delta) \geq v(i-\delta) + 2\delta v_x(i-\delta)$$

Let us now define some discrete quantities that we will use throughout the paper.

Definition 3.3. Let $\hat{v}(i)$ and $\bar{v}^{\delta}[i]$ be defined as follows:

$$\hat{v}(i) := v(i-\delta) + 2\delta v_x(i-\delta),$$

$$\bar{v}^{\delta}[i] := \delta\left\lceil \delta^{-1}\hat{v}(i) - \frac{1}{2}\right\rceil = \inf\{j \in \delta\mathbb{Z} \,|\, j \geq \hat{v}(i) - \frac{\delta}{2}\} \in \delta\mathbb{Z}.$$

We will need the following Lemma.

Lemma 3.4. *Let v be as in Definition 3.2 and $\hat{v}$, $\bar{v}^{\delta}$ be in Definition 3.3. Denote by $\bar{w}^{\delta}$ the piecewise linear interpolation of $\bar{v}^{\delta}$, and by w the piecewise linear interpolation of $\hat{v}$. Then $v + \delta/2 \geq \bar{w}^{\delta}$, and $v \geq w$.*

Proof. First, note that convexity of v in $[i-\delta, i+\delta]$ implies that $\hat{v}(i) \leq v(i+\delta)$.

Let

$$I_i := (i-1+\delta, i+\delta)$$

and let the auxiliary function $\hat{w}$ be the solution of

$$\begin{aligned} &\Delta\hat{w} = -F1_{[i-1+\delta, i-\delta]} \quad \text{on} \;\; I_i \\ &\hat{w}(i-1+\delta) = v(i-1+\delta), \quad \hat{w}(i+\delta) = \hat{v}(i). \end{aligned}$$

This function is C^1 on its domain and solves the ODE $\hat{w}_{xx} = -F$ on $(i-1+\delta, i-\delta)$. (Here x is considered as "time"). Suppose $\hat{w}(i-\delta) > v(i-\delta)$. Then $\hat{w}_x(i-\delta) < v_x(i-\delta)$, and integrating the ODE backwards in x we obtain $\hat{w}(i-1+\delta) > v(i-1+\delta)$, a contradiction. Assuming $\hat{w}(i-\delta) < v(i-\delta)$ we obtain a contradiction in a similar way, and we conclude $\hat{w}(i-\delta) = v(i-\delta)$. This implies $\hat{w} = v$ on $[i-1+\delta, i-\delta]$ and (by convexity of v on $[i-\delta, i+\delta]$) $\hat{w} \leq v$ on $[i-1+\delta, i-\delta]$.

Now consider

$$\Delta w = 0 \quad \text{on} \ \ I_i$$
$$w(i-1+\delta) = \hat{v}(i-1), \quad w(i+\delta) = \hat{v}(i)$$

Clearly w is the piecewise linear interpolation of $\hat{v}$.

As $\Delta \hat{w} \leq \Delta w$ and $w \leq \hat{w}$ on ∂I_i, the comparison principle for the Laplace operator implies $\hat{w} \geq w$, so $v \geq \hat{w} \geq w$. The conclusion for $\bar{w}^\delta$ follows immediately. □

3.1. **Existence and uniqueness for parabolic equations.**

Lemma 3.5. *There exists a global classical solution of the parabolic Cauchy problems (2), and (6) with initial conditions which are uniformly bounded and locally C^2. The solutions are unique. If $0 \leq v_0 \leq u_0$, v solves (6) with initial condition v_0, u solves (2) with initial condition u_0, then $v \leq u$.*

Proof. For $M \in \mathbb{N}$, replace $l(i,j)(\omega)$ by $l^M(i,j) := M \wedge l(i,j)$, where $\wedge$ denotes the operation $a \wedge b := \inf\{a, b\}$.The corresponding fields f^M, $\widetilde{f}^M$ are uniformly bounded and uniformly Lipschitz in s. Therefore we can apply the Banach fixed point theorem in L^∞ in order to obtain a local in time solution, which, by local parabolic regularity, is classical. It can be extended as the nonlinearity is uniformly bounded. Hence a global solution $u^M(x,t)$ exists. Note that by the comparison principle u^M is a positive monotonic non-increasing function of M i.e. $u^M > u^N > 0$ for $N > M$, so $u(x,t) := \lim_{M\to\infty} u^M(x,t)$ exists. Applying regularity locally, (where the obstacles are bounded) we obtain that the limit is a classical solution. □

4. **A-priori estimates on the discrete paths.** In this section, we establish some a-priori estimates on $\hat{v}(i)$ and $\overline{v}^\delta[i]$.

First we show a lemma which allows to estimate the discrete Laplacian of $\hat{v}$ at i (which involves $i, i+1$ and $i-1$) by something that depends only on the obstacles above i.

Lemma 4.1. *Let $\hat{v}(i)$ defined as in the previous section, and define the discrete Laplacian as*

$$\Delta_d \hat{v}(i) := \hat{v}(i+1) - 2\hat{v}(i) + \hat{v}(i-1) = \big(\hat{v}(i+1) - \hat{v}(i)\big) - \big(\hat{v}(i) - \hat{v}(i-1)\big)$$

Then

$$-2\delta[v_x(i-1+\delta) - v_x(i-1-\delta)] \leq \Delta_d \hat{v}(i) + \hat{F} \leq (1+2\delta)[v_x(i+\delta) - v_x(i-\delta)].$$

where $F(1-2(\delta+\varepsilon)) \leq \hat{F} \leq (1-2\delta)F$ for the $\varepsilon > 0$ in Def. 3.1.

Note that our discretization, using the tangents, implies that the discrete Laplacian does not necessarily satisfy the same lower bound as the Laplacian of the original path.

Proof. **Step One : Upper Bound**
As a preparation, let us recall some formulas satisfied by v.
Since v satisfies (8), we have for all $i \in \mathbb{Z}$

$$v_x(i+1-\delta) - v_x(i+\delta) = -F \int_{i+\delta}^{i+1-\delta} \chi_A^\varepsilon(x)dx \tag{10}$$

$$v(i+1-\delta) - v(i+\delta) = (1-2\delta)v_x(i+\delta) - F\int_{i+\delta}^{i+1-\delta} \left(\int_{i+\delta}^{s} \chi_A^\varepsilon(x)dx\right) ds \tag{11}$$

Let us define

$$\hat{F} := F \int_{i+\delta}^{i+1-\delta} \chi_A^\varepsilon(x)dx.$$

Observe that since $\chi_A^\varepsilon(x+p) = \chi_A^\varepsilon(x)$ for all integer p, $\hat{F}$ is independent of $i \in \mathbb{Z}$. Moreover

$$F(1-2(\delta+\varepsilon)) \leq \hat{F} \leq (1-2\delta)F$$

since $\chi_{A_\varepsilon}(x) \leq \chi_A^\varepsilon(x) \leq \chi_A(x)$.

Using now (10), the definition of $\hat{v}(i+1)$ and (11) we see that

$$\begin{aligned}\hat{v}(i+1) &= v(i+\delta) + v_x(i+\delta) - F\int_{i+\delta}^{i+1-\delta}\left(\int_{i+\delta}^{s}\chi_A^\varepsilon(x)dx\right)ds \\ &+ 2\delta(v_x(i+1_\delta) - v_x(i+\delta)) \\ &= v(i+\delta) + v_x(i+\delta) - F\int_{i+\delta}^{i+1-\delta}\left(\int_{i+\delta}^{s}\chi_A^\varepsilon(x)dx\right)ds - 2\delta\hat{F}.\end{aligned}$$

Therefore,

$$\hat{v}(i+1)-\hat{v}(i) = v(i+\delta)+v_x(i+\delta)-F\int_{i+\delta}^{i+1-\delta}\left(\int_{i+\delta}^{s}\chi_A^\varepsilon(x)dx\right)ds-2\delta\hat{F}-\hat{v}(i). \quad (12)$$

Observe that since $\chi_A^\varepsilon(x+p) = \chi_A^\varepsilon(x)$ for all integer p we have

$$F\int_{i+\delta}^{i+1-\delta}\left(\int_{i+\delta}^{s}\chi_A^\varepsilon(x)dx\right)ds + 2\delta\hat{F} = F\int_{i-1+\delta}^{i-\delta}\left(\int_{i-1+\delta}^{s}\chi_A^\varepsilon(x)dx\right)ds + 2\delta\hat{F}.$$

Hence, from the definition of the discrete laplacian and using (12) it follows that

$$\Delta_d\hat{v}(i) = v(i+\delta) + v_x(i+\delta) - \hat{v}(i) - v(i-1+\delta) - v_x(i-1+\delta) + \hat{v}(i-1) \quad (13)$$

Using now the definition of $\hat{v}(i)$ and the convexity of v in $(i-\delta, i+\delta)$ for all $i \in \mathbb{Z}$ we see that

$$\begin{aligned}v(i+\delta) + v_x(i+\delta) - \hat{v}(i) &\leq v_x(i+\delta) + 2\delta(v_x(i+\delta) - v_x(i-\delta)) \\ - v(i-1+\delta) + \hat{v}(i-1) &\leq 0.\end{aligned}$$

Hence,

$$\Delta_d\hat{v}(i) \leq (1+2\delta)v_x(i+\delta) - 2\delta v_x(i-\delta) - v_x(i-1+\delta).$$

Using now (10) it follows that

$$\Delta_d\hat{v}(i) \leq (1+2\delta)(v_x(i+\delta) - v_x(i-\delta)) - \hat{F}.$$

Step two: Lower bound

From formula (13) we have

$$\Delta_d\hat{v}(i) = v(i+\delta) - \hat{v}(i) + v_x(i+\delta) - v(i-1+\delta) + \hat{v}(i-1) - v_x(i-1+\delta) \quad (14)$$

Since v is convex in $(i-\delta, i+\delta)$, we have $v(i+\delta)-\hat{v}(i) \geq 0$ and $v_x(i+\delta) \geq v_x(i-\delta)$. Therefore we have

$$\Delta_d\hat{v}(i) \geq v_x(i-\delta) - v_x(i-1+\delta) - v(i-1+\delta) + \hat{v}(i-1). \quad (15)$$

Using now (10), the convexity of $\hat{v}$ in $(i-1-\delta, i-1+\delta)$ and the definition of $\hat{v}(i-1)$ it follows that

$$\Delta_d\hat{v}(i) \geq -\hat{F} - 2\delta[v_x(i-1+\delta) - v_x(i-1-\delta)]. \quad (16)$$

□

Now we proceed to estimate the change of the discrete gradient

$$k(i) := v_x(i+\delta) - v_x(i-\delta)$$

in terms of the obstacle strengths above i. Observe that always $k \geq 0$ by convexity. If the gradients are very steep, the path will pass through several obstacles above the interval $[i-\delta, i+\delta]$. The number of obstacles passed and the time spent in each of them (i.e. the Lebesgue measure of its image under the inverse mapping) can be estimated in terms of $v_x(i-\delta)$ and $v_x(i+\delta)$.

Lemma 4.2. *Let v be a blocked path, $i \in \mathbb{Z}$ and and assume that $k(i) > 0$. Set $M := sup\{|v_x(i-\delta)|; |v_x(i+\delta)|\}$ then we have*

$$k(i) \leq \frac{18\delta}{M} \sum_{\hat{v}(i)-4\delta M \leq j \leq \hat{v}(i)+4\delta M} l(i,j)$$

Proof. Step 1: As v is convex on $[i-\delta, i+\delta]$, the gradient is monotone, hence $|v_x(x)| \leq M$ for all $x \in I(i) := [i-\delta, i+\delta]$. As a consequence, we have on $I(i)$

$$v(i) - \delta M \leq v(x) \leq v(i) + \delta M,$$

As $|\hat{v}(i) - v(i-\delta)| \leq 2\delta M$, $|v(i) - v(i-\delta)| \leq \delta M$, we obtain

$$|v(x) - \hat{v}(i)| \leq 4\delta M \quad \text{on}[i-\delta, i+\delta].$$

Step 2. Define the time spent by the path in the j-th obstacle above i as

$$S_j := \big|\{x : \ v(x) \in [j-\delta, j+\delta]\}\big|,$$

where $|A|$ denotes the Lebesgue measure of the set A and $j \in Z^* = 1/2 + \mathbb{Z}$. Note that by convexity v_x changes sign at most once, hence each S_j is the union of at most two intervals, moreover $S_j = \emptyset$ if $|j - \hat{v}(i)| > 4\delta M$ Hence, as for $x \in I(i)$ $v_{xx}(x) \leq l(i,j)$ on obstacle j and zero else,

$$v_x(i+\delta) - v_x(i-\delta) \leq \sum_{\hat{v}(i)-4\delta M \leq j \leq \hat{v}(i)+4\delta M} l(i,j) S_j.$$

where $j \in \mathbb{Z}^*$

Step 3. Note that $k \leq 2M$. As the gradient is monotone on $I(i)$, there exists a $\hat{\tau}$ such that $|v_x(\hat{\tau})| = M - k/3$ and $|v_x(x)| \geq M - k/3 \geq M/3 \geq 0$ on $\hat{I}(i)$, where

$$\hat{I}(i) = \begin{cases} [\hat{\tau}, i+\delta] & \text{if } M = |v_x(i+\delta)| \\ [i-\delta, \hat{\tau}] & \text{if } M = |v_x(i-\delta)|. \end{cases}$$

As the gradient does not change sign on $\hat{I}(i)$, the sets $\hat{S}_j := S_j \cap \hat{I}(i)$ are intervals. Moreover,

$$|\hat{S}_j| \leq \frac{2\delta}{M/3} = \frac{6\delta}{M}$$

as $|v_x| \geq M/3$ on $\hat{I}(i)$. Hence

$$\frac{k}{3} = M - v_x(\hat{\tau}) \leq \sum_{\hat{v}(i)-4\delta M \leq j \leq \hat{v}(i)+4\delta M} l(i,j)\hat{S}_j \leq \frac{6\delta}{M} \sum_{\hat{v}(i)-4\delta M \leq j \leq \hat{v}(i)+4\delta M} l(i,j)$$

and the result follows. ◻

Remark 3. Note that in the case where $k(i) \geq 1$ then the corresponding $M(i) \geq \frac{1}{2}$. Indeed, by definition of $M(i)$ and $k(i)$ we have $2M(i) \geq |v_x(i-\delta)| + |v_x(i+\delta)| \geq v_x(i+\delta) - v_x(i-\delta) = k(i) \geq 1$, i.e. $M(i) \geq 1/2$.

Combining now Lemmas 4.1 and 4.2 we deduce the following estimates, which allow to estimate the discrete Laplacian of the blocking path $(\bar{v}^\delta[j])_{j\in[-N,N]\cap\mathbb{Z}}$ at a site i against a normalized sum of random variables.

Lemma 4.3. *Let v be a blocked path. Then for all $i \in [-N+\delta, N-\delta]\cap\mathbb{Z}$ there exists $M(i), M(i-1) > \frac{1}{2}$ such that following holds*

$$\Delta_d \bar{v}(i) + \bar{F} \leq (1+2\delta)\left[\frac{360\delta^2}{2\delta(4M(i)+\frac{1}{2})}\sum_{\bar{v}(i)-\delta(4M(i)+\frac{1}{2})\leq j\leq \bar{v}(i)+\delta(4M(i)+\frac{1}{2})} l(i,j)(\omega)\right]$$

$$\geq -2\delta\left[\frac{360\delta^2}{2\delta(4M(i-1)+\frac{1}{2})}\sum_{\bar{v}(i-1)-\delta(4M(i-1)+\frac{1}{2})\leq j\leq \bar{v}(i-1)+\delta(4M(i-1)+\frac{1}{2})} l(i-1,j)(\omega)\right],$$

where $\bar{F} := \hat{F} - (1+2\delta)$.

Proof. Let us first start with the proof of the upper bound. Observe first that

$$\hat{v}(i) - \frac{\delta}{2} \leq \bar{v}^\delta[i] \leq \hat{v}(i) + \frac{\delta}{2},$$

which implies that

$$\Delta_d \hat{v}^\delta[i] - 2\delta \leq \Delta_d \bar{v}^\delta[i] \leq \Delta_d \hat{v}^\delta[i] + 2\delta.$$

Therefore using Lemma 4.1 we have

$$\Delta_d \bar{v}^\delta[i] \leq (1+2\delta)k(i) - \hat{F} + 2\delta. \tag{17}$$

with $k(i) > 0$. By Lemma 4.2 and Remark 3, for $k(i) \geq 1$ there exists $M(i) \geq \frac{1}{2}$ so that

$$k(i) \leq \frac{18\delta}{M(i)}\sum_{\hat{v}(i)-4\delta M(i)\leq j\leq \hat{v}(i)+4\delta M(i)} l(i,j)(\omega).$$

So we easily see that

$$k(i) \leq \frac{18\delta^2(4M(i)+\frac{1}{2})}{M(i)(4M(i)+\frac{1}{2})\delta}\sum_{\bar{v}^\delta[i]-(4M(i)+\frac{1}{2})\delta\leq j\leq \bar{v}^\delta[i]+(4M(i)+\frac{1}{2})\delta} l(i,j)(\omega). \tag{18}$$

Therefore, since $M(i) > \frac{1}{2}$ we have

$$k(i) \leq \frac{180\delta^2}{(4M(i)+\frac{1}{2})\delta}\sum_{\bar{v}^\delta[i]-(4M(i)+\frac{1}{2})\delta\leq j\leq \bar{v}^\delta[i]+(4M(i)+\frac{1}{2})\delta} l(i,j)(\omega). \tag{19}$$

Hence, for all $k(i) \geq 0$, we have

$$k(i) \leq 1 + \frac{180\delta^2}{(4M(i)+\frac{1}{2})\delta}\sum_{\bar{v}^\delta[i]-(4M(i)+\frac{1}{2})\delta\leq j\leq \bar{v}^\delta[i]+(4M(i)+\frac{1}{2})\delta} l(i,j)(\omega).$$

and the estimate follows . The lower bound is treated in a similar way. ∎

5. **Probabilistic estimates.** We first recall a standard fact for the Laplace transform of independent exponential random variables and random variables with distribution function bounded by an exponential.

Lemma 5.1. *1. Let $\{X_i\}_{i\in N}$ be independent identically distributed random variables such that for a parameter λ_0 and a constant $C>0$*

$$\mathbb{P}[X_0 > r] \leq Ce^{-\lambda_0 r}. \tag{20}$$

Then we have for any $\lambda < \lambda_0$ and $L \in \mathbb{N}$, $L \geq 2$,

$$\mathbb{E}\left[e^{\lambda X_1}\right] \leq C\frac{\lambda_0}{\lambda_0 - \lambda} \tag{21}$$

$$\mathbb{E}\left[e^{\lambda \sum_{i=1}^{L} X_i}\right] \leq C^L\left(\frac{\lambda_0}{\lambda_0-\lambda}\right)^L \tag{22}$$

$$\mathbb{E}\left[e^{\lambda\left(\frac{1}{L}\sum_{i=1}^{L} X_i\right)}\right] \leq C^L e^{\lambda\frac{4\ln(4/3)\lambda}{3\lambda_0}} \quad \text{for } L \geq 2, \lambda \in (2/3\lambda_0, \lambda_0) \tag{23}$$

2. Let $\{X_i\}_{i\in N}$ be independent exponential random variables with parameter $\lambda_0 > 0$. Then (21)-(22) hold as equalities with $C=1$, while (23) holds as inequality with $C=1$.

Proof. We first show 2. The first equality is standard, the second follows by using independence. For the third, note that by concavity of $\ln(1-x)$ on $[0,3/4]$

$$\ln(1-x) \geq \frac{4}{3}x\ln(3/4) \text{ for } x \in \left[0, \frac{3}{4}\right]$$

Using independence and this concavity estimate with $x = \lambda_0/(\lambda L)$

$$\mathbb{E}\left[e^{\lambda\frac{1}{L}\sum_{i=1}^{L} X_i}\right] = \left(\frac{\lambda_0}{\lambda_0 - \frac{\lambda}{L}}\right)^L = e^{-L\ln\left(1-\frac{\lambda}{\lambda_0 L}\right)} \leq e^{\ln(4/3)\frac{4\lambda}{3\lambda_0}}.$$

In order to show 2., it is sufficient to prove the first inequality, the others then follow as in the previous case. For (21) note that the expectation of a random variable is the Riemann-Stieltjes integral with the distribution function as integrator. Now integrate by parts and use that the integrand $e^{\lambda x}$ is monotone. □

Remark 4. Observe that the above estimate on the Laplace transform of S_L is independent of L.

Let us define $\widetilde{S}_M$ by

$$\widetilde{S}_M(\omega)(i,j) := \sum_{-M\leq j-l\leq M} l(i,l).$$

The we have the following Corollary:

Corollary 2. *For any discrete function $j(i): \mathbb{Z} \to \mathbb{Z}$, the random variables*

$$\{\widetilde{S}_M(\omega)(i,j(i))\}_{i\in\mathbb{Z}}$$

are independent and identically distributed. Moreover, there exist constants $C, \hat{\lambda}$ which depend only on λ_0 such that

$$\mathbb{P}\left(\widetilde{S}_M(\omega)(i,j(i)) > r\right) \leq e^{C-\hat{\lambda}r}$$

Proof. The first assertion is obvious. The second is a consequence of (21) and (23) and the exponential Chebyshev inequality with a parameter $\lambda \in (2/3\lambda_0, \lambda_0)$. □

Let us now estimate the probability of a blocked path with boundary conditions on $[-N, N]$ to be compatible with the $l(i,j)$.

Definition 5.2. (blocked Dirichlet path) Let $v(-N+\delta) = v(N-\delta) = 0$. Moreover, let v solve (8) for $-N \le i \le N-1$, and let v solve (9) for $-N+1 \le i \le N-1$. Extend v to $[-N-\delta, N+\delta]$ by

$$v(x) = v_x(-N+\delta)(x+N-\delta) \quad \text{on} \quad [-N-\delta, -N+\delta]$$

and

$$v(x) = v_x(N-\delta)(x-N+\delta) \quad \text{on} \quad [N-\delta, N+\delta].$$

Remark 5.

1. Note that this path solves (9) for $-N \le i \le N$ if we set $l(i,j) = 0$ for $i = -N$ or $i = N$.
2. If $v \ge 0$ on $[-N+\delta, N-\delta]$, then

$$0 \ge v(x) \ge -2\delta F N \quad \text{for} \quad x \in [-N-\delta, -N+\delta] \cup [N-\delta, N+\delta].$$

Definition 5.3. Let $\bar{v}^\delta : [-N, N] \cap \mathbb{Z} \to \delta\mathbb{Z}$ be a discrete path. We call the path *compatible* with a random obstacle configuration if there exists a (not necessarily unique) path as in Definition 5.2 which is mapped to $\bar{v}^\delta$ under the discretization defined in Def. 3.3.

Note that the discrete path is fixed. Whether it is compatible or not depends on the configuration of the random field.

Lemma 5.4. *Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and let $l(i,j)(\omega)$ be i.i.d. exponential random variables with parameter $\lambda_0 > 0$ and let $\bar{v}^\delta$ be a discrete path with fixed boundary conditions*

$$\bar{v}^\delta(-N+\delta) = 0, \; \bar{v}^\delta(N) = b \text{ for some } b \in [-FN, FN].$$

Then there exist constants $\hat{C}(\delta, \lambda_0)$, $\lambda_1(\delta, \lambda_0)$ independent of b such that we have for F sufficiently large

$$\mathbb{P}[\bar{v}^\delta \text{ compatible}, \; \bar{v}^\delta(N) = b] := \mathbb{P}_b[\bar{v}^\delta \text{ compatible}] \le e^{N\hat{C}} e^{-\lambda_1 \sum_{-N+1}^{N-1} |\Delta_d \bar{v}^\delta(i) + \bar{F}|}.$$

with $\bar{F}$ as in Lemma 4.3

The previous estimates bounds the probability of the random obstacle configurations such that a *fixed* discrete path is compatible with the random environment. In order to prove that the probability that *there exists* some compatible nonnegative path is small, we would have to sum over all possible paths, each weighted with the right hand side of the previous estimate. It is complicated to bound these sums, because the number of possible discrete paths grows faster than exponentially in N. Fortunately, most of them are extremely unlikely to be compatible. In order to quantify this, we define an auxiliary probability measure on discrete paths.

Definition 5.5.

$$\begin{aligned} \widetilde{\mathbb{P}}_b[\Delta_d \bar{v}^\delta] &:= \frac{1}{Z^{2N-1}} e^{-\lambda_1 \sum_{-N+1}^{N-1} |\Delta_d \bar{v}^\delta(i) + \bar{F}|}, \\ Z &:= \sum_{k=-\infty}^{\infty} e^{-\lambda_1 |\delta k + \bar{F}|} \end{aligned}$$

The normalization constant is obtained by summing over all possible discrete paths for fixed boundary conditions. This is equivalent to summing over all discrete Laplacians. Note that Z is bounded from above and below by constants independent of F.

Note that the law of the positive and the negative part of $\Delta_d \bar{v}^\delta(i) + \bar{F}$ under $\widetilde{P}$ is that of (discretized) independent exponential random variables. In particular, probabilities of sums of the discrete Laplacians have certain exponential moments and can be estimated by large deviation techniques.

Corollary 3. *With $\widetilde{P}$ as in Def. 5.5, there exists $N_0(\lambda_0, \delta)$ such that*

$$\mathbb{P}_b[\bar{v}^\delta \text{ compatible}] \le e^{\widetilde{C}N} \widetilde{\mathbb{P}}[\Delta_d \bar{v}^\delta]$$

for $N > N_0$.

Corollary 3. We suppose that Lemma 5.4 holds. Then

$$\begin{aligned} P_b[\bar{v}^\delta \text{ compatible}] &\le e^{N\hat{C}} e^{-\lambda_1 \sum_{-N+1}^{N-1} |\Delta_d \bar{v}^\delta(i) + \bar{F}|} \\ &= e^{N\hat{C}} \left(Z^2\right)^N Z^{-1} \frac{1}{Z^{2N-1}} e^{-\lambda_1 \sum_{-N+1}^{N-1} |\Delta_d \bar{v}^\delta(i) + \bar{F}|} \\ &\le e^{N\widetilde{C}} \widetilde{\mathbb{P}}[\Delta_d \bar{v}^\delta] \end{aligned}$$

for N sufficiently large. Here we can choose e.g.

$$\widetilde{C} = 2\hat{C} + 2\ln(Z).$$

□

Lemma 5.4. In order to simplify notation we write

$$S_{\bar{v}^\delta}(\omega)(i) := \widetilde{S}_{M(\bar{v}^\delta)}(\omega)(i, \bar{v}^\delta(i)).$$

We write the absolute value as sum of positive and negative part.

By Lemma 4.3 we get that there exist universal positive constants C_0 such that the fixed discrete path $\bar{v}^\delta$ is compatible only if

$$\begin{aligned} \omega &\in \left(\bigcap_{i=-N+1}^{N-2} \left(A_{\bar{v}^\delta,+}(i) \cap A_{\bar{v}^\delta,-}(i) \right) \right) \cap A_{\bar{v}^\delta,+}(N-1) \cap A_{\bar{v}^\delta,-}(-N+1) \\ A_{\bar{v}^\delta,+}(i) &:= \left\{ \omega : \ C_0 \left(\Delta_d \bar{v}^\delta(i) + \bar{F} \right)_+ \le S_{\bar{v}^\delta}(\omega)(i) \right\} \\ A_{\bar{v}^\delta,-}(i) &:= \left\{ \omega : \ C_0 \left(\Delta_d \bar{v}^\delta(i+1) + \bar{F} \right)_- \le S_{\bar{v}^\delta}(\omega)(i) \right\} \\ B_{\bar{v}^\delta}(i) &:= \left(A_{\bar{v}^\delta,+}(i) \cap A_{\bar{v}^\delta,-}(i) \right). \end{aligned}$$

Note that

$$B_{\bar{v}^\delta}(i) \subseteq \left\{ S_{\bar{v}^\delta}(i) \ge \frac{C_0}{2} \left(\Delta_d \bar{v}^\delta(i+1) + \bar{F} \right)_- + \frac{C_0}{2} \left(\Delta_d \bar{v}^\delta(i) + \bar{F} \right)_+ \right\}$$

and we estimate with the help of Corollary 2 for $i \in \{-N+1, \ldots, N-2\}$

$$\mathbb{P}(B_{\bar{v}^\delta}(i)) \le e^{\hat{C} - \frac{\widehat{\lambda}_1 \delta}{C_0} \left(\left(\Delta_d \bar{v}^\delta(i) + \bar{F} \right)_+ + \left(\Delta_d \bar{v}^\delta(i+1) + \bar{F} \right)_- \right)}$$

for constants $\hat{C}$ and $\widehat{\lambda}_1$ depending only on λ_0 but not on F.

Moreover, for $i = N-1$ we obtain

$$\mathbb{P}(A_{\bar{v}^\delta,+}(N-1)) \le e^{\hat{C} - \frac{\widehat{\lambda}_1 \delta}{C_0} \left(\Delta_d \bar{v}^\delta(N-1) + \bar{F} \right)_+}$$

and for $i = -N+1$ we obtain

$$\mathbb{P}(A_{\bar{v}^\delta,-}(-N+1)) \le e^{\hat{C} - \frac{\hat{\lambda}_1 \delta}{C_0}\left(\Delta_d \bar{v}^\delta(-N+1) + \bar{F}\right)_-}$$

The events $B_{\bar{v}^\delta}(i)$ are independent for different i, hence

$$\begin{aligned} \mathbb{P}_b[\bar{v}^\delta \text{ compatible}] &\le \mathbb{P}(A_{\bar{v}^\delta,-}(-N+1))\mathbb{P}(A_{\bar{v}^\delta,+}(N-1)) \prod_{i=-N+1}^{N-2} \mathbb{P}(B_{\bar{v}^\delta}(i)) \\ &\le e^{N\hat{C}} e^{-\frac{\hat{\lambda}_1 \delta}{C_0} \sum_{-N+1}^{N-1} |\Delta_d \bar{v}^\delta(i) + \bar{F}|}. \end{aligned}$$

The claim follows now by choosing $\lambda_1 = \frac{\hat{\lambda}_1 \delta}{C_0}$.

□

Remark 6. Note that the 1-1-correspondence between second derivatives and paths with Dirichlet boundary conditions allows us to express each path uniquely through its discrete Laplacians and thus estimate its probability with the help of the previous lemma.

As a consequence the discrete Laplacians on average much larger than $-F$ are extremely unlikely. We will show that nonnegative paths that cross the "triangle" $KN - K|x|$ require such unlikely values of the discrete Laplacian.

6. **Final argumentation.**

6.1. **Some formulas on discrete path and comparison of two paths.**

In this section, we recall some well known formulas for discrete paths and their discrete derivatives. The proofs are straightforward computations and therefore omitted.

Let us first recall some basic formulas satisfied by a discrete path z defined in $\mathbb{Z} \times \mathbb{R}$.

Lemma 6.1. *Let* $\nabla^l z[\ell+1] := z[\ell+1] - z[\ell]$ *and* $\nabla^r z[\ell+1] := z[\ell+1] - z[\ell+2]$. *Then for* $\ell \in \mathbb{Z}$ *we have*

(i)

$$\nabla^l z[\ell+1] = \Delta_d z[\ell] + \nabla^l z[\ell] = \sum_{i=1}^{\ell} \Delta_d z[i] + \nabla^l z[1]$$

$$\nabla^l z[\ell+1] = \Delta_d z[\ell] + \nabla^l z[\ell] = \sum_{i=k}^{\ell} \Delta_d z[i] + \nabla^l z[k].$$

(ii)

$$z[\ell+1] - z[0] = \sum_{i=1}^{\ell} \sum_{j=1}^{i} \Delta_d z[j] + (\ell+1)\nabla^l z[1].$$

$$z[\ell+1] - z[k] = \sum_{i=k+1}^{\ell+1} (z[i] - z[i-1]) = \sum_{i=k+1}^{\ell} \sum_{j=k+1}^{i} \Delta_d z[j] + (\ell+1-k)\nabla^l z[k+1].$$

(iii)

$$\nabla^r z[0] = \Delta_d z[1] + \nabla^r z[1] = \sum_{i=1}^{\ell} \Delta_d z[i] + \nabla^r z[\ell],$$

$$\nabla^r z[k] = \Delta_d z[k+1] + \nabla^r z[k+1] = \sum_{i=k+1}^{\ell} \Delta_d z[i] + \nabla^r z[\ell].$$

(iv)

$$z[0] - z[\ell+1] = \sum_{i=0}^{\ell-1} \sum_{j=i+1}^{\ell} \Delta_d z[j] + (\ell+1)\nabla^r z[\ell]$$

$$z[k] - z[\ell+1] = \sum_{i=k}^{\ell} (z[i] - z[i+1]) = \sum_{i=k}^{\ell-1} \sum_{j=i+1}^{\ell} \Delta_d z[j] + (\ell+1-k)\nabla^r z[\ell].$$

(v)

$$\nabla^l z[\ell+1] = -\nabla^r z[\ell]$$

Let us now define what we mean by "crossing."

Definition 6.2. Let z_1 and z_2 be two given paths in $\mathbb{Z} \times \mathbb{R}$. We say that z_1 cross z_2 if and only if there exists $i \in \mathbb{Z}$ such that $z_1[i] \geq z_2[i]$ and $z_1[i+1] \leq z_2[i+1]$.

We will apply this to the discrete path $\bar{v}^\delta$ and the triangle $z_K(i) := NK - K|i|$.

6.2. **Proof of Theorem 2.3.** First we state a trivial fact for discrete sums.

Lemma 6.3. *Let a_j be nonnegative numbers, then*

$$\sum_{i=1}^{N} \sum_{j=i}^{N} a_j = \sum_{i=1}^{N} j a_j \leq N \sum_{i=1}^{N} a_j \tag{24}$$

We will show that paths that remain nonnegative but cross the triangle z_k require values of the average discrete Laplacian which are very unlikely under $\widetilde{P}$. In order to do so, we distinguish cases: Either the path is above the triangle near one of the two endpoints of the interval $[-N, N]$ and crosses at the interior, or it crosses at N or $-N$. In both cases, this implies information on the gradient. Note that the nonnegativity of the original subsolution does not imply the nonnegativity of the discretized path, but only that the discretized path is larger than $-\delta FN$, δ times the minimal possible gradient. In particular, it implies that the terminal value b of the discretized path is in $[-\delta FN, 0]$.

Notation: As only discrete paths appear in the following estimates, we will write $v[i]$ for $\bar{v}^\delta[i]$ In order to simplify notation.

Proof. If $-\nabla^r v[-N] \leq K$, then by Lemma 6.1

$$v[0] - v[-N] = \sum_{i=-N+1}^{-1} \sum_{j=-N+1}^{i} \Delta_d v[j] - N\nabla^r v[-N].$$

Since $v[-N] = 0$ and rewriting the double sum the right way, it follows that

$$-FN \leq v[0] \leq NK + \sum_{i=-N+1}^{-1} (-i)(\Delta_d v[i]).$$

After adding and subtracting $\bar{F}$ in each term in the summation

$$-FN \leq NK + \sum_{i=-N+1}^{-1} (-i)(\Delta_d v[i] + \bar{F}) - \bar{F}\frac{N(N-1)}{2}.$$

so, invoking (24) it follows that

$$\bar{F}\frac{N(N-1)}{2} - (F+K)N \leq 2(N-1) \sum_{i=-N+1}^{N-1} (\Delta_d v[i] + \bar{F})_+.$$

By definition of $\bar{F}$, we have

$$\bar{F} \geq F(1 - 2(\delta + \varepsilon)) - (1 + 2\delta).$$

Therefore for ε small, says $\varepsilon \leq \delta$ and F such that $F \geq 2\frac{1+2\delta}{1-8\delta}$ we achieve

$$\bar{F} \geq \frac{F}{2}.$$

Whence

$$F\frac{N(N-1)}{4} - (F+K)N \leq 2(N-1) \sum_{i=-N+1}^{N-1} (\Delta_d v[i] + \bar{F})_+.$$

This implies that for N large and K fixed

$$\frac{1}{2(N-1)} \sum_{i=-N+1}^{N-1} (\Delta_d v[i] + \bar{F})_+ \geq \frac{1-2\delta}{8}F.$$

As the $(\Delta_d v[i] + \bar{F})_+$ are independent random variables under the auxiliary probability measure $\widetilde{\mathbb{P}}$ defined in Def. 5.5 which have exponential moments bounded as in (21), we can derive an upper bound for the large deviations principle: (For the basic form of the large deviations principle needed, see e.g [5] Ch. 5.11) Let

$$\mathcal{I}(F) = \frac{F}{\mu} - 1 + \ln\left(\frac{\mu}{F}\right),$$

where $\mu := \lambda_0^{-1}$ with λ_0 as in Lemma 5.1. (I.e. for exponential random variables μ is the expectation of $(\Delta_d v[i] + \bar{F})_+$ under $\widetilde{\mathbb{P}}$. Note that μ is decreasing in λ_0.) Then, by the large deviations principle, for any $\eta > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$

$$\widetilde{\mathbb{P}}\left(\frac{1}{2(N-1)} \sum_{i=-N+1}^{N-1} (\Delta_d v[i] + \bar{F})_+ \geq \frac{(1-2\delta)}{8}F\right) \leq e^{-N\left(C + \mathcal{I}\left(\frac{(1-2\delta)F}{8}\right) - \eta\right)}.$$

where C is the constant in the bound (21). ($C = 1$ for exponential random variables.) Now choose F sufficiently large such that

$$e^{\widetilde{C}C - \mathcal{I}\left(\frac{(1-2\delta)F}{8}\right)} < 1,$$

where the constants are defined in Lemma 5.4.

Then there exists a constant C_3 depending on λ_0 and δ such that for N sufficiently large

$$\mathbb{P}(\text{case } 1) \leq e^{-C_3 N}.$$

The case $\nabla^l v[N] \geq -K$ is done in a similar way.

Second case: $-\nabla^r v[-N] > K, \nabla^l v[N] < -K$. This implies that the path has to cross the triangle inside the interval $[-N, N]$. Suppose the path crosses z_K on

$[-N,0]$, the other case is follows by symmetry. Then there exists N_1, $-N < N_1 < 0$, such that $-\nabla^r v[N_1] \le K$ and $v[N_1] \le KN$. Then by Lemma 6.1

$$v[N] - v[N_1] = \sum_{i=N_1+1}^{N-1} \sum_{j=N_1+1}^{i} \Delta_d v[j] - (N - N_1)\nabla^r v[N_1],$$

so

$$-FN \le v[N] \le 2KN+KN+\sum_{i=N_1+1}^{N-1} \sum_{j=N_1+1}^{i} (\Delta_d v[j]+\bar{F})-\bar{F}\frac{(N-N_1)(N-N_1-1)}{2},$$

which implies

$$\bar{F}\frac{N(N-1)}{2}-(F+3K)N \le \sum_{i=N_1+1}^{N-1} \sum_{j=N_1+1}^{i} (\Delta_d v[j]+\bar{F}) \le 2(N-1)\sum_{i=-N+1}^{N-1} (\Delta_d(v[i])+\bar{F})_+,$$

i.e. for N sufficiently large

$$\frac{1}{2(N-1)} \sum_{i=-N+1}^{N-1} (\Delta_d v[i] + \bar{F})_+ \ge \frac{(1-2\delta)F}{4}.$$

Now we can repeat the probabilistic argument from the first case.

Finally, we sum over all possible $-FN$values of the the terminal condition b. This sum grows linearly in N, hence using the exponential decay of the probabilities we obtain that there exists $C_4(\delta, \lambda_0)$ and $F_0(\delta, \lambda_0)$ such that for $F > F_0$

$$\mathbb{P}\big(\omega : \ \bar{v}^\delta \text{ compatible and } \bar{v}^\delta \text{ crosses } z_K\big) \le e^{-C_4 N}.$$

Now we conclude with Lemma 3.4. □

6.3. **Proof of corollary.**

Proof. Define v^N as the solution of the initial-boundary value problem

$$\begin{aligned} \frac{\partial v^N}{\partial t} &= v^N_{xx}(x,t) + \widetilde{f}(x, v^N(x,t)) + F \quad \text{in} \quad (-N+\delta, N-\delta), \\ v^N(-N,t) &= u = v^N(N,t) = 0 \\ v^N(x,0) &= 0, \end{aligned}$$

and let $u(x,t)$ solve 2. The comparison principle for parabolic equations implies that $v^N(x,t) \le u(x,t)$ for $x \in [-N-\delta, N+\delta]$, $t > 0$.

Moreover, $v^N(x,t) \nearrow v^N_{\text{stat}}(x)$ as $t \to \infty$, where $v^N_{\text{stat}}(x)$ is a stationary solution of the Dirichlet problem.

Note that $\partial_t v^N(x,t) \ge 0$ as $\partial_t v^N(x,0) \ge 0$, and the time derivative $w := \partial_t v^N$ solves

$$\partial_t w = \Delta w + V(x) w,$$

where the potential $V(x) = \frac{\partial f}{\partial u}(x, v^N(x,t))$ is bounded on compact subsets of $\mathbb{R}^N$. (Note that ω is a fixed parameter here. $v^N \le FN^2 1_{[-N,N]}$,so only obstacles within $[-N,N] \times FN^2$ can occur, but these are bounded for ω fixed.)

Now a linear parabolic PDE with sufficiently regular potential $V(x)$ and nonnegative initial condition remains nonnegative: $\widetilde{w} = e^{-t\|V\|_\infty} w$ solves

$$\partial_t \widetilde{w} = \Delta \widetilde{w} + \widetilde{V}(x) w, \quad \widetilde{V} \le 0$$

with initial condition $\widetilde{w} \ge 0$. So the classical parabolic comparison principle (see e.g. [7]) implies $\widetilde{w} \ge 0$.

By Thm. 2.3 and the first Borel-Cantelli Lemma (see e.g. [5]),

$$\mathbb{P}\left(\omega : v^N_{\text{stat}}(0) \leq KN \text{ for infinitely many } N\right) = 0,$$

so there exist almost surely arbitrarily large N such that

$$\liminf_{t\to\infty} u(0,t) \geq \lim_{t\to\infty} v^N(0,t) = v^N_{\text{stat}}(0) \geq KN,$$

which implies

$$\liminf_{t\to\infty} u(0,t,\omega) = +\infty \tag{25}$$

with probability 1. By the comparison principle, this contradicts the existence of a global nonnegative stationary solution.

Moreover, by arguments as in Lemma 3.4, (25) holds for $x \in [-1,1]$. As the distribution of the obstacles is invariant under translations in x-direction, (25) holds for $x \in \mathbb{R}$. □

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E-mail address: `jerome.coville@avignon.inra.fr`
E-mail address: `N.Dirr@maths.bath.ac.uk`
E-mail address: `luckhaus@mis.mpg.de`