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ON THE EFFECTS OF DISCONTINUOUS CAPILLARITIES FOR IMMISCIBLE TWO-PHASE FLOWS IN POROUS MEDIA MADE OF SEVERAL ROCK-TYPES

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Abstract. We consider a simplified model for two-phase flows in one-dimensional heterogeneous porous media made of two different rocks. We focus on the effects induced by the discontinuity of the capillarity field at interface. We first consider a model with capillarity forces within the rocks, stating an existence/uniqueness result. Then we look for the asymptotic problem for vanishing capillarity within the rocks, remaining only on the interface. We show that either the solution to the asymptotic problem is the optimal entropy solution to a scalar conservation law with discontinuous flux, or it admits a non-classical shock at the interface modeling oil-trapping.

1. **Introduction.** We are interested in a simplified model of incompressible immiscible two-phase flows within heterogeneous porous media made of several rock types. We consider a one-dimensional porous medium —represented by $\mathbb{R}$— made of two porous sub-media —represented by $\Omega_1 = \{x < 0\}$ and $\Omega_2 = \{x > 0\}$—. For the sake of simplicity, each sub-domain Ω_1 and Ω_2 is supposed to be homogeneous, i.e. its physical properties depend neither on time nor on space. We will focus on the effects of discontinuities arising at the interface between the different rocks, represented in the sequel by $\{x = 0\}$.

We consider a incompressible immiscible two-phase flow within this medium, driven by gravity/buoyancy forces and by global convection. Such models are particularly used in petrol engineering to predict the motion of oil. The underlying mathematical problem, in the case of homogeneous domains has been widely studied, leading to numerous publications. We refer for example to [4, 7, 16, 22] for detailed informations on such models and their mathematical treatments. The case of domains with "smooth" variations of the data has been studied in [5, 17].

We assume that the fluid is constituted of two immiscible phases, so-called the *oil-phase* and the *water-phase*. Considering the conservation of both phases, we obtain the following equation in each Ω_i:

$$\phi_i \partial_t u + \partial_x \left(q f_i(u) - \lambda_i(u) \left(\partial_x \pi_i(u) - \rho g \right) \right) = 0, \tag{1}$$

where $\phi_i \in (0,1)$ denotes the porosity of Ω_i, u stands for the saturation of the oil-phase —then u is bounded between 0 and 1 and $(1-u)$ is the saturation of the

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water-phase—, q denotes the *total flow-rate*, g is the gravity vector, ρ stands for the difference between the volume masses of both phases, f_i, λ_i and π_i are Lipschitz continuous functions, depending also on the rock, fulfilling for $i \in \{1,2\}$:

(H1) f_i is an increasing Lipschitz continuous function, with $f_i(0) = 0$, $f_i(1) = 1$;
(H2) λ_i is Lipschitz continuous, with $\lambda_i(0) = \lambda_i(1) = 0$, $\lambda_i(s) > 0$ if $s \in (0,1)$;
(H3) π_i is an increasing Lipschitz continuous function;
(H4) q is a non-negative constant.

Remark 1. In fact, q has to be supposed constant only in Sections 3.1 and 3.2. The choice $q \geq 0$ is arbitrary, and can be replaced without any additional difficulty by $q \leq 0$. The assumption **(H4)** can be relaxed in Section 2, and replaced by:

(H4bis) $q \in BV_{\mathrm{loc}}(\mathbb{R}_+)$.

We refer to [12] for this latter point.

In this contribution, we will particularly focus on the effects of the discontinuity arising for $\{x = 0\}$ of the capillary pressure function $x \mapsto \pi(\cdot, x)$, where $\pi(u,x) = \pi_i(u)$ if $x \in \Omega_i$. A first existence/uniqueness result on this topic has been given in [8], while the convergence of a numerical scheme was proven in [19] (see also [20, 21] for different numerical methods). Recently, a new formulation for the transmission conditions between Ω_1 and Ω_2 at the interface has been given (see [10, 15]), allowing to treat a large class of couple (π_1, π_2).

Organization of the paper. In Section 2, we introduce the graphical transmission conditions already mentioned above to couple the equations governing the flow within each rock. The non-linear transmission conditions, whose justification is detailed in [10, 15], lead to a monotone operator. Then, we state an adaptation to the case of an unbounded domain of the main results proven in [11, 15], i.e. the existence and uniqueness of the solution for a suitable definition.

In Section 3, we consider the problem obtained for capillary pressures depending only on space, but not on the saturation, i.e. as

$$\partial_u \pi_i(u) \text{ tends to } 0. \tag{2}$$

In this case, the equation 1 turns to a first order scalar conservation law with a discontinuous flux function. This latter family of equation has been widely studied during the last ten years. In particular, it has been shown by Adimurthi et al. [2] that such equations can produce an infinite number of L^1-contraction semi-groups, but there is a unique *optimal entropy condition* respecting a fundamental property of the entropy solutions for regular flux functions, that is the prohibition of undercompressible shock waves. Kaasschieter [23] proved that in the case of a continuous capillary pressure field, the saturation profile for the limit 2 is the unique optimal entropy solution, computed in [1]. The limit 2 means that the capillary forces vanish within the homogeneous subdomains. In the case of a discontinuous capillary pressure field, the capillary forces still work at the interface, oriented from the large capillary pressure to the small capillary pressure. It will be shown that, under technical assumptions, if both phases move in the same direction or if the capillary forces at the interface and the buoyancy work in the same sense, then the saturation profile is the unique optimal entropy solution. Reversely, if both phases move in the opposite directions and if the capillary forces at the interface are opposed to buoyancy, then a steady undercompressible shock-wave occurs at the interface corresponding to a so-called *connection* in [2, 9] and so-called in this paper *non-classical shock*.

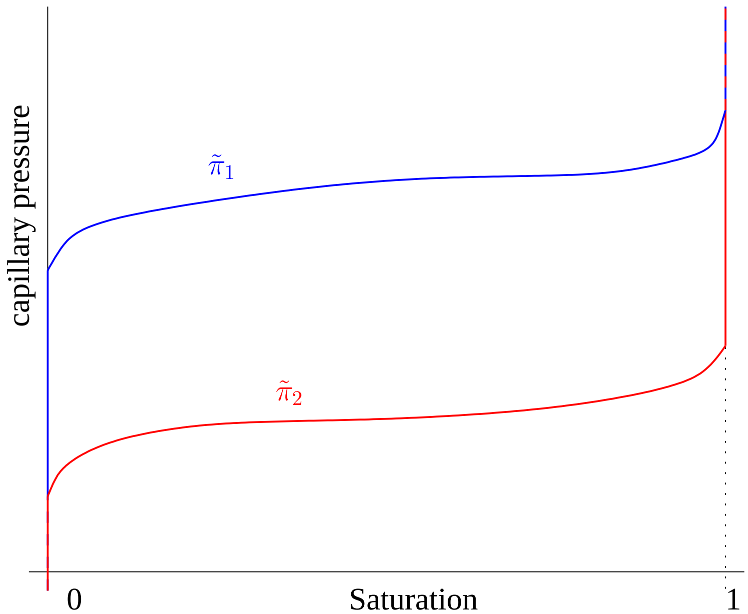

FIGURE 1. The monotone graphs $\tilde{\pi}_i$ ($i \in \{1, 2\}$) are built by adding the semi-axis $(-\infty, \pi_i(0)]$ and $[\pi_i(1), \infty)$ to the graph of the function π_i.

2. **Graphical transmission conditions.** The flow within each sub-domain Ω_i is governed by the equation 1. Coupling conditions have to be imposed on the interface. The first one follows from the conservation of mass, requiring the connection of the fluxes. Denoting by

$$F_i = qf_i(u) - \lambda_i(u) \left(\partial_x \pi_i(u) - \rho g\right), \qquad \text{in } \Omega_i. \tag{3}$$

The solution u must satisfy, in a weak sense,

$$F_1(x = 0^-) = F_2(x = 0^+). \tag{4}$$

The other condition, which consists in requiring the connection of the mobile phases (see [19]), yields the following graphical transmission relation derived in [10, 15]:

$$\tilde{\pi}_1(u_1) \cap \tilde{\pi}_2(u_2) \neq \emptyset, \tag{5}$$

where u_i stands for the trace of $u_{|\Omega_i}$ on $\{x = 0\}$, and the monotone graph $\tilde{\pi}_i$ is defined by

$$\tilde{\pi}_i(s) = \begin{cases} \pi_i(s) & \text{if } s \in (0, 1) \\ (-\infty, \pi_i(0)] & \text{if } s = 0 \\ [\pi_i(1), \infty) & \text{if } s = 1. \end{cases}$$

Let u_0 be an initial data in $L^\infty \cap L^1(\mathbb{R})$, $0 \le u_0 \le 1$ a.e., fulfilling furthermore

$$qf_i(u_0) - \lambda_i(u_0) \left(\partial_x \pi_i(u_0) - \rho g\right) \in L^\infty(\Omega_i), \tag{6}$$

$$\tilde{\pi}_1(u_{0,1}) \cap \tilde{\pi}_2(u_{0,2}) \neq \emptyset, \tag{7}$$

where $u_{0,i}$ stands for the trace on $\{x = 0\}$ of $(u_0)_{|\Omega_i}$.

Definition 2.1. A function u is said to be a *bounded-flux solution* to the problem 1-3-4-5 associated to the initial data u_0 if it fulfills

1. $u \in \mathcal{C}(\mathbb{R}_+; L^1(\mathbb{R}))$ with $0 \le u \le 1$ a.e.;
2. $F_i \in L^\infty(\Omega_i \times \mathbb{R}_+)$;
3. $\tilde{\pi}_1(u_1(t)) \cap \tilde{\pi}_2(u_2(t)) \neq \emptyset$ for a.e. $t \in \mathbb{R}_+$;
4. for all $\psi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$,

$$\int_0^\infty \int_{\mathbb{R}} \phi u \partial_t \psi dx dt + \int_{\mathbb{R}} \phi u_0 \psi(\cdot, 0) dx + \int_0^\infty \int_{\mathbb{R}} F_i \partial_x \psi dx dt = 0. \tag{8}$$

Remark 2. It is worth noticing that the existence of the traces $u_i \in L^\infty(\mathbb{R}_+)$ is provided by the regularity of F_i. Indeed, denoting by φ_i the increasing function defined by $\varphi_i'(s) = \lambda_i(s)\pi_i'(s)$, the $\partial_x \varphi_i(u)$ belongs to $L^\infty(\Omega_i \times \mathbb{R}_+)$, then $\varphi_i(u)$ admits a strong trace on $\{x = 0\}$. Thanks to assumptions **(H2)-(H3)**, φ_i^{-1} is continuous, then u admits also strong traces.

As already stressed, u represents the saturation in oil of the fluid, then it has naturally to stay bounded between 0 and 1, as required in the first point of Definition 2.1. The denomination *bounded-flux solution* clearly comes from the second point. The connection of the capillary pressures 5 is required by the third point. The equations 1, the connection of the fluxes 4 and the respect of the initial value are required in a weak sense by the formulation 8.

We can now state the main result of the current section.

Theorem 2.2 ([11, 15]). *Let u_0 fulfill 6-7, then under assumptions **(H1)-(H2)-(H3)-(H4)**, there exists a unique bounded-flux solution u corresponding to the initial data u_0. Moreover, if v_0 denotes another initial data fulfilling 6-7, and if v denotes the bounded-flux solution associated to v_0, then the following contraction/comparison principle holds: for all $t \in \mathbb{R}_+$,*

$$\int_{\mathbb{R}} \phi(x) \left(u(x,t) - v(x,t)\right)^\pm dx \le \int_{\mathbb{R}} \phi(x) \left(u_0(x) - v_0(x)\right)^\pm dx, \tag{9}$$

where a^+ (resp. a^-) denotes the positive (resp. negative) part of $a \in \mathbb{R}$ and $\phi(x) = \phi_i$ if $x \in \Omega_i$.

Theorem 2.2 is a straightforward generalization to the case of unbounded domains of the results presented in [11, 15]. The fact that the flux belongs to L^∞ is required to deal with the interface during the uniqueness proof, based on the *doubling variable technique.* In order to obtain the needed L^∞-estimate on the flux, the problem has to be reduced to the one-dimensional case. In that latter case, the flux satisfies formally a linear parabolic equation with discontinuous coefficients, and thus the maximum principle [15]. This point can also be carried out by considering a monotone finite volume scheme, and the underlying monotone scheme satisfied by the discrete fluxes [11].

Remark 3. A very simple density argument would allow us to extend the L^1-contraction semi-group — then also the existence/uniqueness frame— to initial data in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, but this would lead to an abstract definition for the solution that we will avoid here, but that is clarified in [11, 15].

Remark 4. Changing u by $(1-u)$ in the problem does not change its nature, then we can extend the existence uniqueness frame for large data, i.e. for data such that $(1-u)$ belongs to $\mathcal{C}(\mathbb{R}_+; L^1(\mathbb{R}))$.

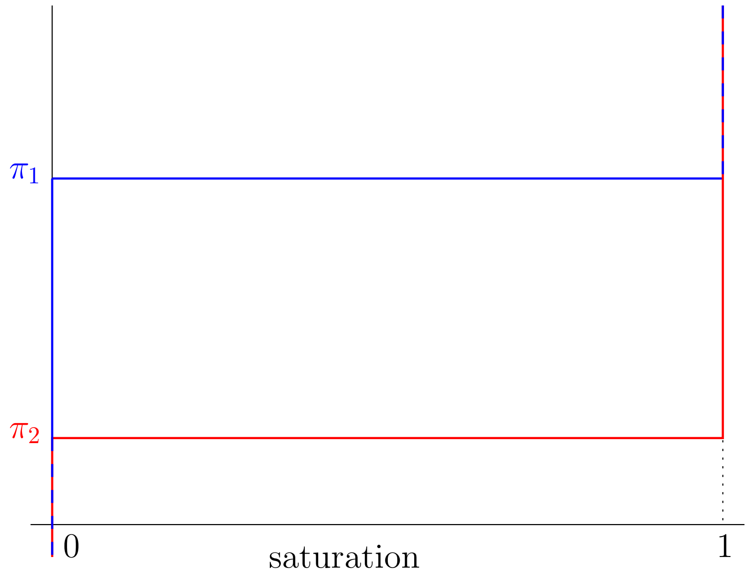

FIGURE 2. Capillary pressure graphs for capillary pressure depending only on space.

3. **Capillary pressure depending only on space.** The graphical connection of the capillary pressure allows us to consider any increasing Lipschitz continuous functions π_i in the model. Particularly, we can choose $\pi_i^\varepsilon(u) = P_i + \varepsilon u$, where P_i is a fixed real value. We then look for the asymptotic problem as ε tends to 0.

Notation. In the sequel, we will denote by u_ε the unique bounded flux solution to the problem 1-3-4-5 associated to the initial data u_0 —which is supposed to fulfill 6-7—, where π_i has been replaced by π_i^ε.

We denote by G_i the Lipschitz continuous function defined by

$$G_i(u) = qf_i(u) + \lambda_i(u)\rho g.$$

(H5) For $i \in \{1,2\}$, there exists $b_i \in [0,1]$ such that G_i is increasing on $[0, b_i]$ and decreasing on $[b_i, 1]$.

Assume for the moment that u_ε converges towards a function u in $L^1(\mathbb{R} \times \mathbb{R}_+)$ as $\varepsilon \to 0$, admitting strong traces u_i on the interface. Then u is a weak solution of

$$\begin{cases} \phi_i \partial_t u + \partial_x G_i(u) = 0 & \text{in } \Omega_i, \\ G_1(u_1) = G_2(u_2) & \text{at the interface } \{x = 0\}, \\ u_{|t=0} = u_0. \end{cases} \tag{10}$$

It is not difficult to check that $u_{|\Omega_i}$ satisfies a classical entropy criterion, i.e. for all $\kappa \in [0,1]$,

$$\phi_i \partial_t (u - \kappa)^{\pm} + \partial_x \left(H_{i\pm}(u,\kappa)\right) \leq 0 \quad \text{ in } \mathcal{D}'(\Omega_i \times \mathbb{R}_+), \tag{11}$$

where

$$H_{i+}(u,\kappa) = H_{i-}(\kappa,u) = \begin{cases} G_i(u) - G_i(\kappa) & \text{if } u \geq \kappa, \\ 0 & \text{otherwise.} \end{cases}$$

The equations 11 does not provide enough informations about the behavior of the solution at the interface. In particular, it is shown in [2] that any undercompressive steady shock wave at the interface is allowed by 11, leading to infinitely many L^1-contraction semi-groups. An additional condition has to be derived at the interface to select the convenient contraction semi-group. Let κ_{opt} be the piecewise constant function defined by (see Figure 3):

- if $f_1(b_1) \le f_2(b_2)$, then

$$\kappa_{\text{opt}}(x) = \begin{cases} \theta_1 = b_1 \text{ if } x < 0, \\ \theta_2 = \min\{\nu \mid f_2(\nu) = f_1(b_1)\} \text{ if } x > 0; \end{cases}$$

- if $f_1(b_1) \ge f_2(b_2)$, then

$$\kappa_{\text{opt}}(x) = \begin{cases} \theta_1 = \max\{\nu \mid f_1(\nu) = f_2(b_2)\} \text{ if } x < 0, \\ \theta_2 = b_2 \text{ if } x > 0. \end{cases}$$

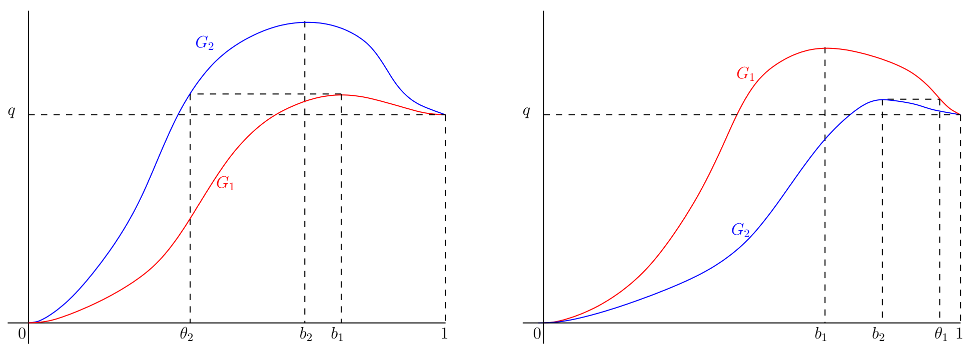

FIGURE 3. Example of functions G_i satisfying **(H5)** and of the so-called *optimal connections* $\kappa_{\text{opt}}(x)$ (see [2, 9]).

We can now define the notion of *optimal entropy solution* to the problem 10, which is the unique solution prohibiting all the undercompressive waves (see [2, 9]). This solution is the limit of the saturation profile for vanishing capillarity in the case of a continuous capillary pressure field (see [1, 23]).

Definition 3.1. A function u is said to be an *optimal entropy solution* to 10 if it satisfies

1. $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$, $0 \le u \le 1$ a.e.;
2. $\forall \psi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_+)$,

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} \phi u \partial_t \psi dx dt + \int_{\mathbb{R}} \phi u_0 \psi(\cdot, 0) dx \\ + \int_0^\infty \sum_{i \in \{1,2\}} \int_{\Omega_i} G_i(u) \partial_x \psi dx dt = 0; \end{aligned} \tag{12}$$

3. $\forall \kappa \in [0,1]$, $\forall \psi \in \mathcal{D}^+(\Omega_i \times \mathbb{R}_+)$,

$$\begin{aligned} \int_0^\infty \int_{\Omega_i} \phi_i (u - \kappa)^\pm \partial_t \psi + \int_{\Omega_i} \phi_i (u_0 - \kappa)^\pm \psi(\cdot, 0) dx \\ + \int_0^\infty \int_{\Omega_i} H_{i\pm}(u, \kappa) \partial_x \psi dx dt \ge 0; \end{aligned} \tag{13}$$

4. $\forall \psi \in \mathcal{D}^+(\mathbb{R}\times\mathbb{R}_+)$,

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} \phi(u-\kappa_{\rm opt})^{\pm}\partial_t\psi\, dxdt + \int_{\mathbb{R}} \phi(u_0-\kappa_{\rm opt})^{\pm}\psi(\cdot,0)dx \\ + \int_0^\infty \sum_{i\in\{1,2\}} \int_{\mathbb{R}} H_{i\pm}(u,\kappa_{\rm opt})\partial_x\psi dxdt \geq 0.\end{aligned} \tag{14}$$

Such an optimal entropy solution is unique as soon as it admits strong traces on the interface (see [9]). The existence of strong traces is provided under assumption **(H5)** by a result of Panov [26]. Denoting by u_i the trace on $\{x=0\}$ of $u_{|_{\Omega_i\times\mathbb{R}_+}}$, the relation 12 ensures that

$$G_1(u_1) = G_2(u_2). \tag{15}$$

Theorem 3.2 ([9]). *Let $u_0 \in L^\infty(\mathbb{R})$ with $0\leq u_0\leq 1$ a.e., then, under assumptions **(H1)-(H2)-(H4)-(H5)**, there exists a unique optimal entropy solution u to the problem 10 in the sense of Definition 3.1. Moreover, if $v_0\in L^\infty(\mathbb{R})$ with $0\leq v_0\leq 1$ a.e. and if one denotes by v the unique corresponding optimal entropy solution, then the following comparison principle holds: for a.e. $t\geq 0$, $\forall R\geq 0$,*

$$\int_{-R}^{R} \phi(u(x,t)-v(x,t))^{\pm}dx \leq \int_{-R-L_Gt}^{R+L_Gt} \phi(u_0(x)-v_0(x))^{\pm}dx.$$

In the following, we will discuss the convergence —or not— of the function u_ε towards the optimal entropy solution. Roughly speaking, it will be seen that, under additional technical assumptions, if both phases move in the same direction, or if the capillary forces are oriented in the same sense as the gravity, then u_ε converges towards the optimal entropy solution. Reversely, if both phases move in opposite directions, and if the capillary forces and the gravity forces work in the same sense, then a stationary *non-classical shock* —i.e. a steady undercompressive shock wave— occurs at the interface.

In the following, without loss of generality, we suppose that $\rho g > 0$, i.e. the gravity works in the sense of increasing x.

3.1. **Gravity and capillarity working in the same sense.** As previously, we define $\pi_i^\varepsilon(u) = P_i + \varepsilon u$. In this section, whose results' proofs are detailed in [13], we suppose that $P_1 > P_2$, i.e. that the capillary forces are also oriented in the sense of increasing x. In the case where $\varepsilon < P_1 - P_2$, the graphical transmission condition $\tilde{\pi}_1^\varepsilon\left((u_\varepsilon)_1\right)\cap\tilde{\pi}_2^\varepsilon\left((u_\varepsilon)_2\right)\neq\emptyset$ turns to the very simple relation:

$$(u_\varepsilon)_1 = 0 \quad \text{or} \quad (u_\varepsilon)_2 = 1. \tag{16}$$

In order to study the problem, we have to make the following assumptions.

(H6) The gravity and the capillarity work in the same sense at the interface, that is in our case $P_1 > P_2$.

Theorem 3.3 ([13]). *Let $u_0\in\mathcal{C}_c^\infty(\mathbb{R}^\star)$, $0\leq u_0\leq 1$. Then under assumptions **(H1)-(H2)-(H4)-(H5)-(H6)**, u_ε converges in $L^1(\mathbb{R}\times\mathbb{R}_+)$ towards the unique entropy solution to the problem 10 corresponding to initial data u_0 in the sense of definition 3.1.*

Remark 5. The L^1-contraction principle 9, holding both for the bounded-flux solution and for the entropy solution, allows us to extend this result of convergence using density arguments to any $u_0\in L^1(\mathbb{R})$, $0\leq u_0\leq 1$ a.e., using the abstract notion of solution pointed out in Remark 3.

The proof of Theorem 3.3 can be split in two distinct parts. In a first time, we have to prove that u_ε converges in L^1. This is performed in [13] using a compactness argument based on BV estimates. Then it remains to check that the limit u is an entropy solution, i.e. that it fulfills the entropy formulations 13 and 14. The idea used in [13] is the study of the steady solutions to the problem 1-3-4-5, which are bounded-flux solutions. As ε tends to 0, they tend to piece-wise constants functions, $\hat{\kappa}(x)$, to which the limit u can be compared. The entropy formulations follow.

Remark 6. The entropy solution u admits strong traces u_i, $i \in \{1,2\}$, on the interface $\{x=0\}$ thanks either to a BV-estimate if the initial data is smooth, or to the fact that **(H5)** implies that G_i is not constant on any non-degenerate interval of $[0,1]$, allowing to use the result of Panov [26]. Nevertheless, those traces do not satisfy "$u_1 = 0$ or $u_2 = 1$" in general (see e.g. [6]).

As it is stressed in [13], under assumption **(H5)**, the optimal entropy solution is the unique function u satisfying the family of inequalities 13 which maximizes the oil flux through the interface. Indeed, denote by $u_i \in L^\infty(\mathbb{R}_+)$ the trace on $\{x=0\}$ of $u_{|\Omega_i \times \mathbb{R}_+}$, then thanks to Adimurthi et al. [1], the flux $F(u_1,u_2)$ at the interface is given by

$$F(u_1,u_2) = \min\left\{\mathrm{God}_1(u_1,0)\ ,\mathrm{God}_2(1,u_2)\right\} \tag{17}$$

where God_i is the exact Riemann solver corresponding to the function G_i, that is

$$\mathrm{God}_i(a,b) = \begin{cases} \displaystyle\min_{s\in[a,b]} G_i(s) & \text{if} \quad 0 \le a \le b \le 1, \\ \displaystyle\max_{s\in[b,a]} G_i(s) & \text{if} \quad 0 \le b \le a \le 1. \end{cases}$$

Let $\tilde{u}$ be another solution satisfying 13 and the Rankine-Hugoniot relation

$$G_1(\tilde{u}_1) = G_2(\tilde{u}_2),$$

where $\tilde{u}_i$ stands for the trace on $\{x=0\}$ of $\tilde{u}_{|\Omega_i \times \mathbb{R}_+}$ Then since $\mathrm{God_i}(a,a) = G_i(a)$ and since $\mathrm{God_i}$ is non-decreasing with respect to its first argument and non-increasing with respect to its second argument, one has

$$G_1(\tilde{u}_1) \le \mathrm{God}_1(\tilde{u}_1,0) \quad \text{and} \quad G_2(\tilde{u}_2) \le \mathrm{God}_2(1,\tilde{u}_2).$$

In particular, the oil flux at the interface satisfies

$$G_1(\tilde{u}_1) = G_2(\tilde{u}_2) \le \min\left\{\mathrm{God}_1(\tilde{u}_1,0)\ ,\mathrm{God}_2(1,\tilde{u}_2)\right\}. \tag{18}$$

The relation 18 ensures that the optimal entropy solution to the Riemann problem corresponding to a left state $\tilde{u}_1$ and a right state $\tilde{u}_2$ provides a larger flux that the weak solution $\tilde{u}$. This particularly allows us to claim that u is the solution to 13 which satisfies 15 and maximizes the flux at the interface.

3.2. **Gravity opposed to capillarity.** In this section, we consider the case where $P_1 < P_2$, where the capillarity works at the interface in the sense of decreasing x. In this case, for $\varepsilon < P_2 - P_1$, the graphical condition $\tilde{\pi}_1^\varepsilon\left((u_\varepsilon)_1\right) \cap \tilde{\pi}_2^\varepsilon\left((u_\varepsilon)_2\right) \neq \emptyset$ turns to the relation

$$(u_\varepsilon)_1 = 1 \quad \text{or} \quad (u_\varepsilon)_2 = 0. \tag{19}$$

Contrary to what occurs in Section 3.1, stated in Remark 6, we will show that, under the technical assumptions **(H7)-(H8)**, the relation 19 is preserved in the limit as $\varepsilon \to 0$ if both phases flow in opposite directions.

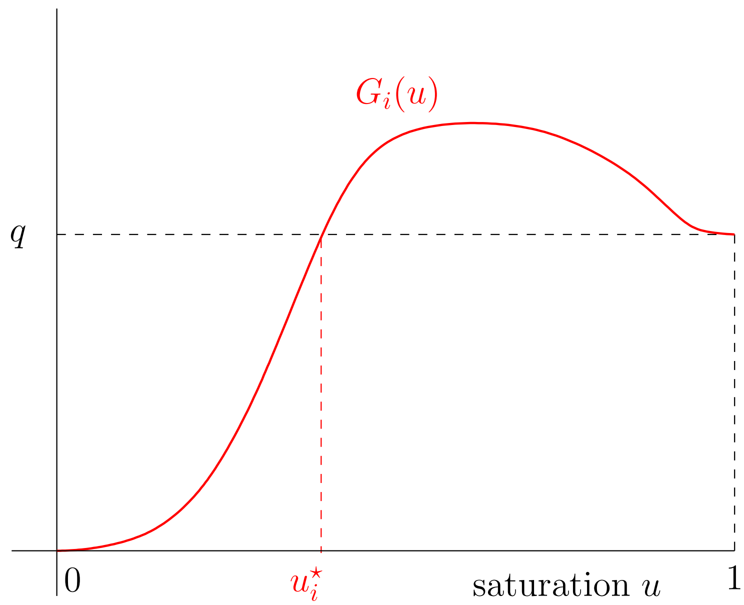

FIGURE 4. An example of function G_i fulfilling the assumptions **(H5)**. The represented value $u_i^\star$ defined in **(H8)** separates the values of the saturation where both phases flow in the same direction and the values of the saturation where both phases flow in opposite directions (cf. Remark 7).

Assumptions. We detail now the assumptions needed for the current frame.

(H7) The gravity and the capillarity work in the opposite senses at the interface, that is in our case $P_2 > P_1$.

(H8) Let φ_i be the increasing function defined by $\varphi_i(u) = \int_0^u \lambda_i(s)ds$. There exist $R > 0$, $\alpha > 0$ and $m \in (0,1)$ such that

$$f_1 \circ \varphi_1^{-1}(s) \geq q + R(\varphi_1(1) - s)^m \quad \text{if } s \in [\varphi_1(1) - \alpha, \varphi_1(1)].$$

Moreover, denoting by

$$u_i^\star = \min\{\nu \in [0,1] \mid G_i(u_i^\star) = q\},$$

it follows from assumptions **(H1)-(H2)-(H4)-(H5)** that

$$G_i(s) \leq q \quad \text{if} \quad s \leq u_i^\star, \tag{20}$$

$$G_i(s) > q \quad \text{if} \quad s \in (u_i^\star, 1). \tag{21}$$

Remark 7. If $u \in [0, u_i^\star]$, then the oil flow-rate $G_i(u)$ has the same sign as the water flow-rate $q - G_i(u)$. Thus both phases move in the same direction. Reversely, if $u \in (u_i^\star, 1)$, then both phases move in opposite directions (see Figure 4).

Both phases moving in the same direction. As stated in Remark 7, the case where both phases move in the same direction corresponds to small saturations, i.e.

$$0 \leq u \leq u_i^\star \quad \text{a.e. in } \Omega_i \times \mathbb{R}^+. \tag{22}$$

Suppose that $0 \le u_0 \le u_i^\star$, then the L^∞-estimate 22 holds also at the limit (but not for u_ε for positive ε). It follows from **(H5)** that G_i can be supposed to be monotone. In particular, one has $0 \le u_i(t) \le u_i^\star$ for a.e. $t \ge 0$. Since G_i is increasing on $[0, u_i^\star]$, the discontinuity at the interface between u_1 and u_2 cannot be undercompressive. In this case, it has been shown in [14] that the solution u coincides with the unique optimal entropy solution u to the problem 10.

Theorem 3.4 ([14]). *Let $u_0 \in \mathcal{C}_c^\infty(\mathbb{R}^\star \times \mathbb{R}_+)$, with $0 \le u_0(x) \le u_i^\star$ for $x \in \Omega_i$. Then under assumptions **(H1)-(H2)-(H4)-(H7)**, u_ε converges in $L^1(\mathbb{R} \times \mathbb{R}_+)$ towards the unique entropy solution to the problem 10 in the sense of Definition 3.1.*

Remark 8. Here again, the convergence result can be extended to any initial data in $L^1(\mathbb{R})$ using the abstract notion of solution for u_ε.

Both phases moving in opposite directions. Contrary to what precedes, we will now consider the case of large initial data. Note that 19 and Remark 4 allow us to consider the bounded flux solution associated to large initial data for small ε.

Definition 3.5. A function u is said to be a *non-classical solution* to 10 if it fulfills

1. $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$, $0 \le u \le 1$ a.e.;
2. for both $i \in \{1, 2\}$, for all $\psi \in \mathcal{D}^+(\overline{\Omega}_i \times \mathbb{R}_+)$, $\forall \kappa \in [0, 1]$,

$$\begin{aligned} &\int_0^\infty\int_{\Omega_i} \phi_i \,(u-\kappa)^\pm \,\partial_t \psi dx dt + \int_{\Omega_i} \phi_i (u_0 - \kappa)^\pm \psi(\cdot, 0) dx \\ &+ \int_0^\infty\int_{\Omega_i} H_{i\pm}(u,\kappa)\partial_x \psi dx dt + L_{G_i}(\gamma_i - \kappa)^\pm \int_0^T \psi(0, \cdot) dt \ge 0, \end{aligned} \tag{23}$$

   where $\gamma_1 = 1$,$\gamma_2 = u_2^\star$ and L_{G_i} is any Lipschitz constant of G_i.

As it appears in the formulation 23, where the formulations in both Ω_i are disjoined, a *non-classical solution* can be seen as the apposition of the entropy solutions of two distinct initial boundary value problems in both Ω_i

$$\left\{ \begin{array}{ll} \phi_i \partial_t u + \partial_x G_i(u) = 0 & \text{in } \Omega_i \times \mathbb{R}_+, \\ u_{|t=0} = u_0 & \text{in } \Omega_i, \\ u_{|x=0} = \gamma_i & \text{in } \mathbb{R}_+. \end{array} \right. \tag{24}$$

Thanks to [24, 25], the problem 24 admits a unique solution in Ω_i, fulfilling a L^1-contraction principle, then we can directly claim that there exists a unique *non-classical solution* to the problem 10 in the sense of Definition 3.5, and that a L^1-contraction principle holds true.

Suppose that the trace conditions on the interface are fulfilled in a strong sense, as it is the case if u_0 satisfies the conditions 25 stated below (see [14]). Then, because of the discontinuity between 1 and $u_2^\star$ occurring at the interface $\{x = 0\}$, u does not satisfy the entropy formulation 14. This non-entropy satisfying discontinuity is said to be a *non-classical shock*, since it is an undercompressive shock wave, violating this way the fundamental property of the entropy solutions.

We state now a convergence result which, roughly speaking claims that, under technical assumptions, if both phases move in opposite directions, and if the gravity and the capillarity work in opposite directions, then u_ε converges towards the unique non-classical solution u.

We require that u_0 is constant equal to 1 on a small interval at the upstream side of the interface, and constant equal to $u_2^\star$ at the downstream side, i.e.

$$\left\{ \begin{array}{l} u_0 \in \mathcal{C}^\infty(\overline{\Omega}_i), \text{ with } (1-u_0) \text{ compactly supported}, u_i^\star \le u_0 \le 1, \\ \text{there exists } \eta > 0 \text{ s.t. } (u_0)_{|_{[-\eta,0]}} = 1, (u_0)_{|_{[0,\eta]}} = u_2^\star. \end{array} \right. \tag{25}$$

In [14], using assumption **(H8)**, some particular sub- and super-solutions are built in order to show that, for ε sufficiently small, $u_\varepsilon \equiv 1$ on the small interval $\left[-\frac{\eta}{2}, 0\right]$. Then the limit u admits also a strong trace $u_1 = 1$ at the interface. Since $u < 1$ on the interval $[0, \eta]$, we deduce from the Rankine-Hugoniot relation 15 and from assumption **(H5)** that $u_2 = u_2^\star$.

Theorem 3.6 ([14]). *Assume that **(H1)-(H2)-(H4)-(H5)-(H7)-(H8)** hold. Let u_0 be an initial data satisfying 25. Then u_ε converges almost everywhere in $\mathbb{R} \times \mathbb{R}_+$ towards the unique* non-classical solution *u to the problem 10 in the sense of Definition 3.5.*

Remark 9. Here again, a convergence result for all initial data $u_0 \in L^\infty$ such that $(1-u_0) \in L^1$ can be derived using a density argument.

Solution of type $(1, u_2^\star)$. In this paragraph, in order to characterize the saturation profile when the capillary forces and buoyancy work in opposite directions in a unified way, we use the notion of solution of type $(1, u_2^\star)$ introduced in [2, 9] and defined below. Denote by $\kappa_{\rm nc}$ the piecewise constant function defined by

$$\kappa_{\rm nc}(x) = \left\{ \begin{array}{lll} 1 & \text{if} & x < 0, \\ u_2^\star & \text{if} & x > 0. \end{array} \right.$$

Definition 3.7. A function u is said to be a *solution of type* $(1, u_2^\star)$ to the problem 10 if it belongs to $L^\infty(\mathbb{R} \times \mathbb{R}_+)$ with $0 \le u \le 1$ a.e., if it satisfies the weak formulation 12, the inner entropy inequalities 13 and the following formulation: $\forall \psi \in \mathcal{D}^+(\mathbb{R} \times \mathbb{R}_+)$,

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} \phi (u - \kappa_{\rm nc})^\pm \partial_t \psi \, dx dt + \int_{\mathbb{R}} \phi (u_0 - \kappa_{\rm nc})^\pm \psi(\cdot, 0) dx \\ + \int_0^\infty \sum_{i \in \{1,2\}} \int_{\mathbb{R}} H_{i\pm}(u, \kappa_{\rm nc}) \partial_x \psi dx dt \ge 0. \end{aligned} \tag{26}$$

Theorem 3.8 ([9, 14]). *Let $u_0 \in L^1(\mathbb{R})$ be a smooth function with $u_0(0^-) = 1$ or $u_0(0^+) = 0$, then, under assumptions **(H1)-(H2)-(H4)-(H5)-(H7)-(H8)**, u_ε converges almost everywhere in $\mathbb{R} \times \mathbb{R}_+$ towards the unique solution of type $(1, u_2^\star)$ in the sense of Definition 3.7.*

Oil trapping at the interface. Assume that $q = 0$, then $\gamma_2 = 0$. The boundary conditions prescribed in 24 are fulfilled in a strong sense, i.e. u admits strong traces on both sides of the interface equal to γ_i. The flux at the interface is then equal to $f_1(1) = f_2(0) = 0$. This means that the oil present in Ω_1 cannot reach Ω_2, and that it remains trapped by the rock discontinuity.

Suppose now that $q > 0$. Thanks to conservation of mass, some oil has to overpass the interface. The solution of type $(1, u_2^\star)$ is the one that minimizes this quantity. Indeed, it is easy to check that $u_{|\Omega_1}$ is the unique entropy solution to

$$\left\{ \begin{array}{ll} \phi_1 \partial_t u + \partial_x G_1(u) = 0 & \text{in } \Omega_1 \times \mathbb{R}_+, \\ u_{|_{t=0}} = u_0 & \text{in } \Omega_1, \\ u_{|_{x=0}} = 1 & \text{in } \mathbb{R}_+. \end{array} \right. \tag{27}$$

Since the dependance of the solution u of 27 with respect to the boundary condition prescribed on the interface is monotone, u maximizes the quantity of oil remaining in Ω_1 thus it minimizes the quantity of oil that overpasses the interface.

It is also worth noticing that 27 implies that the oil-flux at the interface is lower or equal to q. So the problem can be seen as a locally constrained conservation law in the sense of [3, 18].

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