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ON VANISHING VISCOSITY APPROXIMATION OF CONSERVATION LAWS WITH DISCONTINUOUS FLUX

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Abstract. We characterize the vanishing viscosity limit for multi-dimensional conservation laws of the form

$$u_t + \operatorname{div} \mathfrak{f}(x,u) = 0, \quad u|_{t=0} = u_0$$

in the domain $\mathbb{R}^+ \times \mathbb{R}^N$. The flux $\mathfrak{f} = \mathfrak{f}(x,u)$ is assumed locally Lipschitz continuous in the unknown u and piecewise constant in the space variable x; the discontinuities of $\mathfrak{f}(\cdot,u)$ are contained in the union of a locally finite number of sufficiently smooth hypersurfaces of $\mathbb{R}^N$. We define "$\mathcal{G}_{VV}$-entropy solutions" (this formulation is a particular case of the one of [3]); the definition readily implies the uniqueness and the L^1 contraction principle for the $\mathcal{G}_{VV}$-entropy solutions. Our formulation is compatible with the standard vanishing viscosity approximation

$$u^\varepsilon_t + \operatorname{div}\left(\mathfrak{f}(x,u^\varepsilon)\right) = \varepsilon\Delta u^\varepsilon, \quad u^\varepsilon|_{t=0} = u_0, \quad \varepsilon \downarrow 0,$$

of the conservation law. We show that, provided u^ε enjoys an ε-uniform L^∞ bound and the flux $\mathfrak{f}(x,\cdot)$ is non-degenerately nonlinear, vanishing viscosity approximations u^ε converge as $\varepsilon \downarrow 0$ to the unique $\mathcal{G}_{VV}$-entropy solution of the conservation law with discontinuous flux.

Introduction. The study of conservation laws and related degenerate parabolic problems with space-time discontinuous flux has been intense during the last fifteen years. It is stimulated by applications such as sedimentation, porous medium flows in discontinuous media, road traffic models. We refer to [1]–[11], [13, 14], [16]–[18], [24]–[27] and references therein for some of the applications and known results. Notice that only very few studies treat the multidimensional case.

However, most of the interesting phenomena appear already in the model one-dimensional case, with the discontinuity along $\Sigma = \{x = 0\}$:

$$u_t + (\mathfrak{f}(x,u))_x = 0, \qquad \mathfrak{f} : (x,z) \in \mathbb{R}\times\mathbb{R} \mapsto \begin{cases} f^l(z) & x < 0, \\ f^r(z) & x > 0. \end{cases} \tag{1}$$

From the purely mathematical viewpoint, the problem is quite challenging because of the possibility to give various non-equivalent generalizations of Kruzhkov's notion of entropy solution; moreover, different entropy solutions to the same equation may correspond to different applicative contexts. This phenomenon was discovered by

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Adimurthi, Mishra and Veerappa Gowda in [1]. In [7], Bürger, Karlsen and Towers proved well-posedness for (1) for a whole class of different solution notions.

Following [7] and the previous works [5, 6], in [3] we set up a framework that encompasses all the notions of solution to the Cauchy problem for (1) which lead to an L^1-contraction semigroup. An interesting application can be found in [2]. The goal of the present note is to provide a separate description of the important particular case of the standard vanishing viscosity limits for (1) and for corresponding multidimensional problems.

Let us give a brief account on the previous work on the subject. Vanishing viscosity limits for conservation laws with discontinuous flux were studied in many preceding works, including those of Gimse and Risebro [13, 14], of Karlsen, Risebro and Towers [16, 17, 18, 26, 27], of Diehl [8, 9, 10, 11], of Panov [24], and many others. In all these works, some intrinsic "entropy" formulations for (1) were given, for which existence and/or uniqueness of solutions was analyzed. The work [24] contains the most general existence result; see also [17]. Notice that although the definition of solution in [17] is inspired by the vanishing viscosity method, existence can rely upon a justification of convergence of suitably designed numerical schemes (cf. [7] and [3]). The uniqueness issue is most challenging. In [16, 18, 26, 27], the authors give an integral formulation of the Kruzhkov type, with a penalization term supported at the discontinuity hypersurfaces Σ of the flux mapping $(t,x) \mapsto \mathfrak{f}(x,\cdot)$. Then uniqueness is justified under the so-called crossing condition; uniqueness may fail when the crossing condition fails (see [3]). Also for the formulation of [24], in general one cannot hope for uniqueness. Diehl, in the works [8, 9, 10] (see also Gimse and Risebro [13, 14]), obtained an entropy formulation on the interface Σ in terms of restrictions on the one-sided limits on Σ of a weak solution u. This "coupling approach" turns out to be very general, thanks to the strong trace results for entropy solutions (see [23]). The Γ-condition of Diehl [8, 9, 10] was derived from the vanishing viscosity (plus smoothing) standing-waves approach of [12], and expressed in a rather complicated manner. Recently in [11], Diehl reformulated the Γ-condition under a simple form reminiscent of the Oleĭnik entropy conditions (cf. [20]); and he succeeded in proving uniqueness of solutions for this formulation, without requiring the crossing condition of [18]. Consequently, the Γ-condition of Diehl [11] should be recognized as the right admissibility condition for the vanishing viscosity limits for (1). Our contribution can be seen as a justification of existence for the Diehl formulation.[1]

The description we will give of the "vanishing viscosity germ" $\mathcal{G}_{VV}$ (see Definition 1) turns out to be exactly this new form of the Diehl's Γ-condition. This is by no means surprising. Indeed, our analysis also stems from a simplified vanishing viscosity standing-waves analysis (see Proposition 7); then, in order to link the viscosity profiles of Proposition 7(i) to the germ $\mathcal{G}_{VV}$, we take advantage of some hints from the general theory of admissibility germs for (1) (see [3] and the Appendix of the present paper). As soon as the admissibility germ $\mathcal{G}_{VV}$ is identified, we define $\mathcal{G}_{VV}$-entropy solutions intrinsically. To this end, we either prescribe possible one-sided traces of u at the discontinuity surface Σ (cf. Diehl [11]); or, we postulate

[1]In fact, we embed the question of identification of the vanishing viscosity limits into a kind of theory constructed in [3], which also covers different solutions such as those of [1, 7]; with this general point of view, justification of uniqueness is immediate as soon as the properties of the corresponding "admissibility germ" are established. Extension from the model setting (1) to the general multidimensional setting becomes a matter of techniques.

global entropy inequalities not with the Kruzhkov entropies $z \mapsto |z-k|$, $k = \text{const}$, but with "adapted entropies" $z \mapsto |z - c(x)|$, where piecewise constant functions $c(x)$ are defined from the germ $\mathcal{G}_{VV}$. The latter approach follows the idea of Baiti and Jenssen [6], of Audusse and Perthame [5] (cf. the interesting re-interpretation of Panov [25]) and of Bürger, Karlsen and Towers [7].

In our framework, the main restriction on the flux $\mathfrak{f}$ is the one that ensures a uniform L^∞ bound on solutions u^ε of equation (13) below. We make a number of simplifying assumptions, including the Lipschitz continuity and the genuine nonlinearity of $\mathfrak{f}(x,\cdot)$ in the sense $\mathfrak{f}'(x,\cdot) \neq 0$ a.e., the smoothness of the discontinuity surfaces of $\mathfrak{f}(\cdot,u)$ and their independence of t. Most of these assumptions can be bypassed; see [3, 4]. For the sake of simplicity, we treat the case of a sole discontinuity of $\mathfrak{f}(\cdot,u)$ along a hypersurface

$$\Sigma = \left\{ (x_1, x') \in \mathbb{R}^N \;\middle|\; x_1 = \Phi(x') \right\}$$

of $\mathbb{R}^N$ given by the graph of a smooth function $\Phi : \mathbb{R}^{N-1} \to \mathbb{R}$. The case with a locally finite number of smooth discontinuity hypersurfaces (possibly crossing, or piecing together) can be obtained similarly, using partition of unity techniques. Thus, our result applies, e.g., to conservation laws in stratified media, such as those that appear in geological studies.

Let us give the outline of the paper. In Section 1 we give the definitions (which take the form of two equivalent formulations) and state the main results. In Section 2, we motivate the definitions in the one-dimensional case (1). Section 3 contains the proof of uniqueness and of the equivalence of the two main definitions. In Section 4, the existence is shown via convergence analysis of the vanishing viscosity approximations. An appendix summarizes the framework adopted in [3], and contains one longer proof. We refer to [3, 4] for the details and an extensive bibliography.

1. Vanishing viscosity germ, $\mathcal{G}_{VV}$-entropy solutions and well-posedness.

Let $\Phi : \mathbb{R}^{N-1} \to \mathbb{R}$ be a C^2 function. Denote

$$\Omega^l := \mathbb{R}^+ \times \left\{ (x_1, x') \in \mathbb{R}^N \;\middle|\; x_1 < \Phi(x') \right\},$$
$$\Omega^r := \mathbb{R}^+ \times \left\{ (x_1, x') \in \mathbb{R}^N \;\middle|\; x_1 > \Phi(x') \right\},$$

and $\Sigma := \overline{\Omega^l} \cap \overline{\Omega^r}$. For $\sigma \in \Sigma$, denote by $\nu(\sigma)$ the unit vector normal to Σ pointing from Ω^l to Ω^r. We consider fluxes of the form

$$\mathfrak{f} : (x,z) \in \mathbb{R}^N \times \mathbb{R} \mapsto \begin{cases} \mathfrak{f}^l(z) & x \in \Omega^l \\ \mathfrak{f}^r(z) & x \in \Omega^r, \end{cases} \quad \mathfrak{f}^{l,r} \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}), \;\; \left(\mathfrak{f}^{lr}\right)' \neq 0 \text{ a.e.} \tag{2}$$

For $\sigma \in \Sigma$, $f^{l,r}(\sigma;\cdot)$ denotes the normal component $\mathfrak{f}^{l,r}(\cdot)\cdot\nu(\sigma)$ on Σ of $\mathfrak{f}^{l,r}(\cdot)$.

In order to simplify the presentation, we will make appeal to strong one-sided traces[2] of a solution u on Σ.

[2] Let us stress that the existence of strong traces *of a solution* u relies on the genuine nonlinearity assumption on the fluxes $\mathfrak{f}^{l,r}$; nonetheless, our formulation can be adapted to the case of arbitrary fluxes. In the general case, one works with strong traces *of the normal components* $\mathfrak{f}^{l,r}(u)\cdot\nu$ *of the flux*, and the normal components $\mathfrak{q}^{l,r}(u)\cdot\nu$ of the corresponding Kruzhkov entropy fluxes for a solution u. See Panov [23] for the definition of the relevant trace notion, and [3] for the corresponding formulation which bypasses the existence of the traces $\gamma^{l,r}u$ of the solution u itself.

We say that a function $g \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$ admits a right-sided trace $\gamma^r g$ on Σ in the strong sense (that is, in the L^1_{loc} topology), if for all $\xi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^N)$,

$$\lim_{h\downarrow 0} \frac{1}{h} \int_{\mathbb{R}^+} \int_0^h \int_{R^{N-1}} |g\,(t, \sigma+(y_1,0)) - (\gamma^r g)\,(t,\sigma)|\, \xi(\sigma)\, dt dy_1 dx' = 0, \tag{3}$$

where $\sigma = (\Phi(x'), x')$. The definition of the strong left-sided trace $\gamma^l g$ on Σ is analogous, with $h \downarrow 0$ replaced by $h \uparrow 0$ in the above formula. The strong trace $\gamma^0 g$ of g on the set $\{t = 0\}$ is defined similarly (see e.g., [22]). Note that if $q : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and g admits one-sided traces $\gamma^{l,r} g$ on Σ, then $q \circ g := q(\cdot, g(\cdot))$ admits one-sided traces on Σ, and $\Big(\gamma^{l,r}(q\circ g)\Big)(\sigma) = q(\sigma, (\gamma^{l,r} q)(\sigma))$ $\mathcal{H}^N$ a.e. for $\sigma \in \Sigma$.

Now, let us introduce the key object that governs the admissibility of solutions.

Definition 1. For a given couple of functions $f^{l,r} \in C(\mathbb{R})$, we denote by $\mathcal{G}_{VV}$ the set of all couples $(u^l, u^r) \in \mathbb{R}^2$ satisfying

$$\begin{cases} s := f^l\left(u^l\right) = f^r\left(u^r\right) \textit{ and} \\ \qquad \textit{either } u^l = u^r, \\ \qquad \textit{or } u_l < u_r \text{ and there exists a} \\ \qquad\qquad u^o \in \left[u^l, u^r\right] \text{ such that} \begin{cases} f^l(z) \geq s \text{ for all } z \in \left[u^l, u^o\right], \\ \qquad \textbf{and} \\ f^r(z) \geq s \text{ for all } z \in [u^o, u^r], \end{cases} \\ \qquad \textit{or } u_l > u_r \text{ and there exists a} \\ \qquad\qquad u^o \in \left[u^r, u^l\right] \text{ such that} \begin{cases} f^l(z) \leq s \text{ for all } z \in \left[u^o, u^l\right], \\ \qquad \textbf{and} \\ f^r(z) \leq s \text{ for all } z \in [u^r, u^o]. \end{cases} \end{cases} \tag{4}$$

This set is called the *vanishing viscosity germ* associated with the couple (f^l, f^r).

Remark 2. In [11], Diehl reformulated the Γ-condition of [8, 9, 10] under the following form: A couple (u^l, u^r) satisfies the Γ-condition if

$$\begin{aligned} &f^l(u^l) = f^r(u^r) \text{ and there exists } u^o \in \mathrm{ch}(u^l, u^r) \text{ such that} \\ &\qquad (u^r - u^o)\,(f^r(z) - f^r(u^r)) \geq 0 \ \forall z \in \mathrm{ch}\,(u^r, u^o)\,, \\ &\qquad \left(u^o - u^l\right)\left(f^l(z) - f^l(u^l)\right) \geq 0 \ \forall z \in \mathrm{ch}\left(u^l, u^o\right), \end{aligned} \tag{5}$$

where for $a, b \in \mathbb{R}$, $\mathrm{ch}(a,b)$ denotes the convex hull $[\min\{a,b\}, \max\{a,b\}]$. Clearly, (4) coincides with (5). Conditions (4),(5) are reminiscent of the Oleĭnik admissibility condition for the case of convex flux functions $f^{l,r}$ (see [20]) and of the "chord condition" (see e.g., [15] and the pioneering work [12] of Gelfand), since the chord conditions (4) and (5) are derived from the travelling-wave approach [12].

Using the previous notation, we call $\mathcal{G}_{VV}(\sigma)$ the vanishing viscosity germ associated with $f^{l,r}(\sigma;\cdot)$. Now we can define $\mathcal{G}_{VV}$-entropy solutions.

Definition 3. A function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$ is called a $\mathcal{G}_{VV}$-entropy solution of

$$u_t + \operatorname{div} \mathfrak{f}(x, u) = 0 \tag{6}$$

$$u|_{t=0} = u_0 \tag{7}$$

with flux $\mathfrak{f}$ given by (2), if

(i) the restriction of u on $\Omega^{l,r}$ is a Kruzhkov entropy solution of equation (6);

(ii) for $\mathcal{H}^N$-a.e. σ on Σ, the couple of strong traces $\big((\gamma^l u)(\sigma),(\gamma^r u)(\sigma)\big)$ of u on Σ belongs to the vanishing viscosity germ $\mathcal{G}_{VV}(\sigma)$;

(iii) $\mathcal{H}^N$-a.e. on $\{0\}\times\mathbb{R}^N$, the initial trace $\gamma^0 u$ equals u_0.

Note that this definition makes sense. Indeed, condition (i) implies the existence of the initial trace $\gamma^0 u$ (see Panov [22]) and of the boundary traces $\gamma^{l,r}u$ on Σ, because Σ is of class C^1 and $f^{l,r}$ are non-degenerate (see Panov [23]).

Let us give another formulation, which does not involve boundary traces of u. For $c\in\mathbb{R}$,

$$\mathfrak{q}(x;\cdot,c):=\operatorname{sign}(\cdot-c)\big(\mathfrak{f}(x,\cdot)-\mathfrak{f}(x,c)\big)$$

is the entropy flux associated with the Kruzhkov entropy $|\cdot-c|$. We write $q^{l,r}(\sigma;\cdot,c)$ for $\mathfrak{q}^{l,r}(\cdot,c)\cdot\nu(\sigma)$, with the obvious meaning of the superscripts l,r. We will also use $\mathfrak{q}_\pm(x;\cdot,c)$ and $q^{l,r}_\pm(\sigma;\cdot,c)$ which correspond to the semi-Kruzhkov entropies $(\cdot-c)^\pm$. For $(c^l,c^r)\in\mathbb{R}^2$, consider

$$c(x)=c^l\mathbb{1}_{\Omega^l}(x)+c^r\mathbb{1}_{\Omega^r}(x). \tag{8}$$

Definition 4. A function $u\in L^\infty(\mathbb{R}^+\times\mathbb{R}^N)$ is called a $\mathcal{G}_{VV}$-entropy solution of problem (6),(7) with flux $\mathfrak{f}$ given by (2), if, firstly, it is a solution in the sense of distributions; and secondly, for all couples $(c^l,c^r)\in\mathbb{R}^2$ and $c(x)$ given by (8), for all $\xi\in\mathcal{D}(\mathbb{R}^+\times\mathbb{R}^N)$, $\xi\geq 0$, one has

$$\begin{aligned}\int_{\mathbb{R}^+}\int_{\mathbb{R}^N}|u(t,x)-c(x)|\,\xi_t+\mathfrak{q}(x;u(t,x),c(x))\cdot\nabla\xi\,dxdt&\\ -\int_{\mathbb{R}^N}|u_0(x)-c(x)|\,\xi(0,x)\,dx+\int_\Sigma R_{VV}\left(\sigma;\left(c^l,c^r\right)\right)\xi(\sigma)\,d\sigma&\geq 0,\end{aligned} \tag{9}$$

with some "remainder function" $R_{VV}:\Sigma\times\mathbb{R}^2\longrightarrow\mathbb{R}^+$ which is Carathéodory and fulfills

$$\forall\ \left(c^l,c^r\right)\in\mathcal{G}_{VV}(\sigma),\ \lim_{r\downarrow 0}\fint_{B_r(\sigma)\cap\Sigma}R_{VV}(\sigma';(c^l,c^r))\,d\sigma'=0, \tag{10}$$

and

$$\begin{gathered}\forall\,(c^l,c^r)\in\mathbb{R}^2\text{ and }\forall\ \left(a^l,a^r\right)\in\mathcal{G}_{VV}(\sigma)\\ q^r\left(\sigma;a^r,c^r\right)-q^l\left(\sigma;a^l,c^l\right)\leq R_{VV}\left(\sigma;\left(c^l,c^r\right)\right).\end{gathered} \tag{11}$$

In [3], the remainder function R_{VV} is given explicitly; yet the definition does not depend on the choice of R_{VV}, as soon as the properties (10),(11) are fulfilled.

The equivalence of Definitions 3 and 4 will be shown in Section 3.

Although Definition 4 is not used in the present work, this kind of global entropy formulation would be useful, e.g., for the numerical analysis of the problem (cf. [7, 3, 2]). Indeed, Definition 3 is convenient for the uniqueness proof, but it is not well suited for passage to the limit (cf. the proof of Theorem 5, where the justification of Definition 3(ii) is indirect). On the contrary, it is clear that Definition 4 is stable under the $L^1_{\rm loc}$ convergence of bounded sequences of solutions.

Under the assumptions on Σ and $\mathfrak{f}$ stated above, we prove

Theorem 5. *(i) Assume $u,\hat u$ are $\mathcal{G}_{VV}$-entropy solutions of (6) with initial data $u_0,\hat u_0\in L^\infty(\mathbb{R}^N)$, respectively. Then the following Kato inequality holds: For all $\xi\in\mathcal{D}(\mathbb{R}^+\times\mathbb{R}^N)$, $\xi\geq 0$,*

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^N}\left((u-\hat u)^+\xi_t+\mathfrak{q}_+(x;u,\hat u)\cdot\nabla\xi\right)\,dxdt+\int_{\mathbb{R}^N}(u_0-\hat u_0)^+\,\xi(0,\cdot)\geq 0. \tag{12}$$

(ii) Let $\{u^\varepsilon\}_{\varepsilon>0}$ be an L^∞ bounded sequence of solutions to

$$u^\varepsilon_t + \operatorname{div} \mathfrak{f}\,(x, u^\varepsilon) = \varepsilon \Delta u^\varepsilon \tag{13}$$

with $u^\varepsilon|_{t=0} = u^\varepsilon_0$; let $u^\varepsilon_0 \to u_0$ in $L^1_{loc}(\mathbb{R}^N)$. Then u^ε converges a.e. on $\mathbb{R}^+ \times \mathbb{R}^N$ to the unique $\mathcal{G}_{VV}$-entropy solution of problem (6),(7) as $\varepsilon \downarrow 0$.

It is classical that for locally Lipschitz fluxes $f^{l,r}$, the Kato inequality (12) gives uniqueness, the L^1 contraction and comparison principles.

It is easy to see that in general, $\mathcal{G}_{VV}$-entropy solutions need not exist. For instance, if for some $\sigma \in \Sigma$, the ranges of $\mathfrak{f}^{l,r} \cdot \nu(\sigma)$ do not intersect, the Rankine-Hugoniot condition $f^l(u^l) = f^r(u^r)$ cannot hold for any couple (u^l, u^r). In this case, there is no uniform L^∞ bound on the sequence of viscous approximations u^ε. The L^∞ bound can be enforced through different assumptions; e.g., it is enough to have $\mathfrak{f}^{l,r}(0) = 0_{\mathbb{R}^N} = \mathfrak{f}^{l,r}(1)$ and $0 \le u_0 \le 1$. This is the case for the road traffic and for some porous medium models where u has the meaning of relative density. Let us recapitulate our results for this important particular case.

Corollary 6. *Let $\mathfrak{f}^{l,r}$ be zero at the endpoints of the interval $[0,1]$. Then for all measurable initial datum $u_0 : \mathbb{R}^N \mapsto [0,1]$ there exists a unique $\mathcal{G}_{VV}$-entropy solution $u =: Su_0$ of problem (6),(7).*

The restriction on $L^1(\mathbb{R}^N; [0,1])$ of the map S defined above is an order-preserving semigroup of contractions. Moreover, S is the limit (in the L^1_{loc} topology) of the solution semigroups $S^\varepsilon : u_0 \mapsto u^\varepsilon$ for the vanishing viscosity regularizations (13).

2. **Motivations.** In this section, we limit our attention to the model one-dimensional problem (1). We first perform a standing-wave analysis of the problem, and then relate the result to the description (4) of the vanishing viscosity germ $\mathcal{G}_{VV}$.

Proposition 7.

(i) If (u^l, u^r) belongs to the set $\mathcal{G}^\circ_{VV}$ of couples satisfying $f^l(u^l) = f^r(u^r) =: s$,

$$\begin{cases} \text{either } u^l = u^r; \\ \text{or } u_l < u_r \begin{cases} f^l(z) > s \text{ for all } z \in \big(u^l, u^r\big], \\ \qquad \textbf{or} \\ f^r(z) > s \text{ for all } z \in \big[u^l, u^r\big), \end{cases} \\ \text{or } u_l > u_r \begin{cases} f^l(z) < s \text{ for all } z \in \big[u^r, u^l\big), \\ \qquad \textbf{or} \\ f^r(z) < s \text{ for all } z \in \big(u^r, u^l\big], \end{cases} \end{cases} \tag{14}$$

then there exists a function $W : \mathbb{R} \to \mathbb{R}$ such that $\lim_{\xi\to-\infty} W(\xi) = u^l$, $\lim_{\xi\to+\infty} W(\xi) = u^r$, and $u^\varepsilon(t,x) = W(x/\varepsilon)$ solves (13) in $\mathcal{D}'((0,+\infty)\times\mathbb{R})$.

(ii) The sets $\mathcal{G} = \mathcal{G}^\circ_{VV}$ and $\mathcal{G} = \mathcal{G}_{VV}$ fulfill the "L^1D property"

$$\forall (c^l, c^r), (b^l, b^r) \in \mathcal{G}, \quad q^l(c^l, b^l) \ge q^r(c^r, b^r). \tag{15}$$

(iii) Assume that $\mathcal{G} \subset \mathbb{R}^2$ satisfy (15) and that for all $(a^l, a^r) \in \mathcal{G}$, $f^l(a^l) = f^r(a^r)$. Then the inclusion $\mathcal{G}^\circ_{VV} \subset \mathcal{G}$ implies the inclusion $\mathcal{G} \subset \mathcal{G}_{VV}$. In particular,

$$\begin{gathered} \forall\ \big(c^l, c^r\big) \in \mathcal{G}^\circ_{VV},\ \ \big(f^l\left(a^l\right) = f^r\left(a^r\right)\big)\ \&\ \big(q^l\left(a^l, c^l\right) \ge q^r\left(a^r, c^r\right)\big) \\ \Downarrow \\ \big(a^l, a^r\big) \in \mathcal{G}_{VV}. \end{gathered} \tag{16}$$

Proof (sketched). *(i)* In the case $u^l = u^r$, the standing-wave profile W can be chosen constant on $\mathbb{R}$. The four other cases are symmetric. For instance, in the case $u^l < u^r$ and $f^l(z) > s$ for all $z \in (u^l, u^r]$, the profile W is a continuous function constant (equal to u^r) on $[0, +\infty)$. On the interval $(-\infty, 0]$, W is constructed as the maximal solution of the autonomous ODE $W' = f^l(W) - f^l(u^l)$ with the initial condition $W(0) = u^r$. Indeed, because $f^l(w) - f^l(u^l) = f^l(w) - s > 0$ for $w \in (u^l, u^r]$, the solution W is non-decreasing. Because f^l is assumed Lipschitz continuous and u^l is a stationary solution, W is defined on the whole interval $(-\infty, 0]$, and there exists $d := \lim_{\xi\to-\infty} W(\xi) \in [u^l, u^r]$. In this case, $f^l(d) - s = 0$, which yields $d = u^l$. The result is easy to prove also in the case of merely continuous functions $f^{l,r}$ (see [3]).

(ii) One can prove this claim by a tedious case study; see [3]. Let us give an argument that uses the structure of the solutions of (13). Notice that (15) for $\mathcal{G} = \mathcal{G}^{\circ}_{VV}$ can also be deduced from the Kato inequality for solutions of (13). More precisely, let $(c^l, c^r), (b^l, b^r) \in \mathcal{G}^{\circ}_{VV}$. According to (14), let $u^\varepsilon(t, x) := W(x/\varepsilon)$ with $W(-\infty) = c^l$, $W(+\infty) = c^r$; similarly, let $\hat{u}^\varepsilon(t, x) := \hat{W}(x/\varepsilon)$ with $\hat{W}(-\infty) = b^l$, $\hat{W}(+\infty) = b^r$. Then one shows the Kato inequality:

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(|u^\varepsilon - \hat{u}^\varepsilon|\, \xi_t + \mathfrak{q}\left(x; u^\varepsilon, \hat{u}^\varepsilon\right)\, \xi_x + \varepsilon\, |u^\varepsilon - \hat{u}^\varepsilon|\, \xi_{xx}\right) dxdt \geq 0. \tag{17}$$

for all $\xi \in \mathcal{D}((0, \infty) \times \mathbb{R})$, $\xi \geq 0$. Letting $\varepsilon \to 0$, we have

$$u^\varepsilon(t, x) \to c^l \mathbb{1}_{x<0} + c^r \mathbb{1}_{x>0} \text{ and}$$
$$\hat{u}^\varepsilon(t, x) \to b^l \mathbb{1}_{x<0} + b^r \mathbb{1}_{x>0}.$$

Therefore from (17), we readily get

$$\left(q^l\left(c^l, b^l\right) - q^r\left(c^r, b^r\right)\right) \int_{\mathbb{R}^+} \xi(t, 0)\, dt \geq 0$$

by the Green-Gauss theorem. Therefore (15) follows for $\mathcal{G} = \mathcal{G}^{\circ}_{VV}$. Then the L^1D property (15) for $\mathcal{G} = \mathcal{G}_{VV}$ is inferred; indeed, $\mathcal{G}_{VV}$ turns out to be the *closure* of $\mathcal{G}^{\circ}_{VV}$ in the sense defined in [3] (see also the appendix in Section 5), and the closure operation preserves the L^1D property (15). *(iii)* The proof (taken from [3]) is postponed to the appendix in Section 5. ☐

Remark 8. For given $f^{l,r}$ there may exist many different subsets $\mathcal{G}$ of $\mathbb{R}^2$ satisfying the L^1D property (15) and the equalities $\forall (c^l, c^r) \in \mathcal{G}\ f^l(c^l) = f^r(c^r)$ (these equalities encode the Rankine-Hugoniot condition on Σ). Such $\mathcal{G}$ is called *a maximal L^1D admissibility germ* if it possesses no nontrivial extension satisfying the same properties. Any maximal germ leads to a notion of $\mathcal{G}$-entropy solution (see [3]).

Proposition 7 *(ii)* and *(iii)* mean that the germ $\mathcal{G}_{VV}$ is maximal. Proposition 7 *(iii)* also states that $\mathcal{G} = \mathcal{G}^{\circ}_{VV}$ admits a unique maximal extension. This implies, e.g., that in the constraints (10) and (11) of Definition 4, G_{VV} could be replaced with $\mathcal{G}^{\circ}_{VV}$.

In the model case (1), we can simplify Definition 4 by setting, regardless of $\sigma \in \Sigma$,

$$R_{VV}\left(\sigma; \left(c^l, c^r\right)\right) := M \operatorname{dist}\left(\left(c^l, c^r\right), \mathcal{G}^{\circ}_{VV}\right), \tag{18}$$

where dist is the Euclidean distance on $\mathbb{R}^2$ and M is a sufficiently large positive constant.

Now let us explain the notion of a G_{VV}-entropy solution. Both Definitions 3 and 4 state the Kruzhkov entropy inequalities locally, away from the flux discontinuity interface Σ. But they also contain a description of the coupling of $u|_{\Omega^l}$ and $u|_{\Omega^r}$

across Σ. The idea behind Definition 3 lies in the identification of the possible trace couples $(\gamma^l u, \gamma^r u)$ of admissible solutions u. In turn, Definition 4 (with R_{VV} given by (18)) explicitly allows for selected "elementary" weak solutions to (1):

$$c(x) = c^l \mathbb{1}_{\{x<0\}} + c^r \mathbb{1}_{\{x>0\}},$$

which play the role of the constants in the classical Kruzhkov formulation. The definitions are inspired by the idea of "adapted entropies" (cf. Baiti and Jenssen [6], Audusse and Perthame [5], Bürger, Karlsen and Towers [7]).

The selection of the elementary solutions that should be admitted is based upon the vanishing viscosity approach. Proposition 7(*ii*) means that $c(\cdot)$ corresponding to $(c^l, c^r) \in \mathcal{G}^\circ_{VV}$ should be admitted in Definition 4, and the trace couples $(c^l, c^r) \in \mathcal{G}^\circ_{VV}$ should be admitted in Definition 3. Indeed, in this case $c(\cdot)$ is clearly obtained as the limit of the viscous standing-wave profiles u^ε, moreover, we have $\gamma^{l,r}(c(\cdot))(t,0) = c^{l,r}$ for all $t > 0$.

Proposition 7(*ii*) implies the dissipativity property for the coupling of $u|_{\Omega^l}$ and $u|_{\Omega^r}$ across Σ. This property ensures the Kato inequality (12) and yields the uniqueness of $\mathcal{G}_{VV}$-entropy solutions.

Reciprocally, property (16) of Proposition 7(*iii*) constrains the traces $(\gamma^l u, \gamma^r u)$ of an arbitrary function u obtained as limit of viscous approximations u^ε, thus giving rise to Definition 3(*ii*). Indeed, a Kato inequality holds for any pair $u^\varepsilon, \hat{u}^\varepsilon$ of solutions of (13); this inequality, "inherited" at the limit, yields the Kato inequality (12) for any pair of viscous limits $u, \hat{u}$; and the (elementary) solutions $\hat{u}(t,x) = c(x) = c^l \mathbb{1}_{\{x<0\}} + c^r \mathbb{1}_{\{x>0\}}$, $(c^l, c^r) \in \mathcal{G}^\circ_{VV}$, have already been identified as viscous limits. From (12) and (16) we derive that $(\gamma^l u, \gamma^r u)(t) \in \mathcal{G}_{VV}$, for a.e. t.

3. The uniqueness proof and equivalence of definitions.

Throughout this section, we fix a non-negative non-increasing (truncation) function ξ_* in $\mathcal{D}(\mathbb{R}^+)$ satisfying

$$\xi_*(s) = \begin{cases} 1 & s < 1, \\ 0 & s > 2, \end{cases} \quad \text{and we set } \xi_h(x) = \xi_*\left(\frac{|x_1 - \Phi(x')|}{h}\right).$$

Proposition 9. *Definitions 3 and 4 are equivalent.*

Proof I. t is standard (see in particular Panov [22]) that Definition 3(i),(iii) is equivalent to inequalities (9) with $\xi \in \mathcal{D}(\mathbb{R}^+ \times (\mathbb{R}^N \setminus \Sigma))$, $\xi \geq 0$.

For a general $\xi \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^N)$, $\xi \geq 0$, we have $\xi(1 - \xi_h) \in \mathcal{D}(\mathbb{R}^+ \times (\mathbb{R}^N \setminus \Sigma))$. Thus we can focus on the contribution of the truncated test function $\xi\xi_h$ into (9). We only have to show that Definition 3(*ii*) is equivalent to the statement that, for all pairs $(c^l, c^r) \in \mathbb{R}^2$, the inequality

$$\liminf_{h \downarrow 0} \int_{\mathbb{R}^+} \int_{\mathbb{R}^N} \xi \mathfrak{q}(x; u, c(x)) \cdot \nabla \xi_h \, dx dt + \int_\Sigma R_{VV}(\sigma; (c^l, c^r))\, \xi(\sigma)\, d\sigma \geq 0 \qquad (19)$$

holds. The existence of strong traces $\gamma^{l,r}u$ (which follows from [23] and assumption (2)) and the definition of ξ_h allows us to reformulate (19) as

$$\int_\Sigma \left(q^l\left(\gamma^l u, c^l\right) - q^r\left(\gamma^r u, c^r\right) + R_{VV}(\sigma; (c^l, c^r)) \right) \xi(\sigma)\, d\sigma \geq 0, \qquad (20)$$

for all pairs (c^l, c^r).

Now, assume Definition 3(*ii*) holds. As soon as (11) is guaranteed, (20) follows from (11) and Proposition 7(*ii*). Therefore it is sufficient to construct a

Carathéodory function R_{VV} satisfying (10) and (11). In the case of a flat interface Σ, one can take the expression (18). A more subtle choice is

$$R_{VV}\left(\sigma;\left(c^l,c^r\right)\right) \\ = 2\inf_{(b^l,b^r)\in\mathcal{G}_{VV}(\sigma)}\left[\operatorname{Osc}\left(f^l\left(\sigma;\cdot\right)\,;\,c^l,b^l\right)+\operatorname{Osc}\left(f^r\left(\sigma;\cdot\right)\,;\,c^r,b^r\right)\right],$$

where $\operatorname{Osc}(g;c,b)$ denotes the oscillation of the function g on the segment with endpoints b,c. We refer to [3] for the details concerning the choice of $R_{VV}(\cdot;(c^l,c^r))$.

Reciprocally, assume (20) with R_{VV} satisfying (10) and (11). Letting $\xi|_\Sigma$ concentrate at a Lebesgue point σ of $\gamma^{l,r}u$, with the help of (10) we find that

$$\text{for all } (c^l,c^r)\in\mathcal{G}_{VV}(\sigma)\quad q^l((\gamma^l u)(\sigma),c^l)-q^r((\gamma^r u)(\sigma),c^r)\geq 0.$$

By (16) we conclude that
$((\gamma^l u)(\sigma),(\gamma^r u)(\sigma))\in\mathcal{G}_{VV}(\sigma)$. □

Proof of Theorem 5(i). We use Definition 3. From (*i*) and (*iii*), by the standard Kruzhkov doubling of variables technique we obtain the Kato inequality (12) with $\xi\in D(\mathbb{R}^+\times(\mathbb{R}^N\setminus\Sigma))$. As in the previous proof, using the truncation ξ_h, we see that it is sufficient to prove that

$$\liminf_{h\downarrow 0}\int_{\mathbb{R}^+}\int_{\mathbb{R}^N}\xi\,\mathfrak{q}_+(x;u,\hat u)\cdot\nabla\xi_h\,dxdt\geq 0. \tag{21}$$

The definition of ξ_h and the existence of the strong traces $\gamma^{l,r}u$ and $\gamma^{l,r}\hat u$ allow to rewrite (21) as $q^l_+(\gamma^l u,\gamma^l\hat u)\geq q^r_+(\gamma^r u,\gamma^r\hat u)$, $\mathcal{H}^N$-a.e. on Σ. This inequality is easily checked from Definition 3(*ii*) and the L^1D property (15) of $\mathcal{G}_{VV}(\sigma)$, $\sigma\in\Sigma$. □

4. **Convergence of the vanishing viscosity method.** In the model case (1), the outline of the proof is given at the end of Section 2. In the general case, we also exploit the Kato inequality for solutions u^ε and $\hat u^\varepsilon$, but we have to deal with solutions to the nonhomogeneous equation (13). A blow-up technique yields the conclusion.

Proof of Theorem 5(ii). First, the L^∞ bound assumed on u^ε and the genuine nonlinearity assumption in (2) allow us to use the precompactness results of Lions, Perthame and Tadmor [19] or of Panov [21, 24] in the domains $\Omega^{l,r}$. Hence, up to extraction of a convergent sequence, u^ε converges a.e. to some $u\in L^\infty(\mathbb{R}^+\times\mathbb{R}^N)$. Moreover, u fulfills Definition 3(*i*), (*iii*); it is also a solution of (6) in the sense of distributions, so that the Rankine-Hugoniot condition on Σ holds. As soon as we prove that u satisfies Definition 3(*ii*), by the uniqueness result of Theorem 5(*i*) we get the convergence $u^\varepsilon\to u$ as $\varepsilon\downarrow 0$

As mentioned in the introduction, Definition 3(i) and the flux non-degeneracy in (2) ensure the existence of the strong traces $\gamma^{l,r}u$ on Σ. Let $\sigma_o=(t_o,x_o)$ be a common Lebesgue point of $\gamma^{l,r}u$. The Rankine-Hugoniot condition for u implies

$$\mathfrak{f}^l\left(\left(\gamma^l u\right)(\sigma_o)\right)\cdot\nu\left(\sigma_o\right)=\mathfrak{f}^r_o\left(\left(\gamma^r u\right)(\sigma_o)\right)\cdot\nu\left(\sigma_o\right).$$

In order to conclude the proof, we only have to justify that

$$\mathfrak{q}^l\left(\left(\gamma^l u\right)(\sigma_o)\,,\,c^l\right)\cdot\nu\left(\sigma_o\right)\geq\mathfrak{q}^r\left(\left(\gamma^r u\right)(\sigma_o)\,,\,c^r\right)\cdot\nu\left(\sigma_o\right), \tag{22}$$

for all pairs $(c^l,c^r)\in\mathcal{G}^\circ_{VV}(\sigma_o)$. Indeed, (22) and property (16) would yield

$$\left(\left(\gamma^l u\right)(\sigma_o)\,,\,\left(\gamma^l u\right)(\sigma_o)\right)\in\mathcal{G}_{VV}(\sigma_o).$$

Recall that $f^{l,r}(\sigma;\cdot)$ denotes $\mathfrak{f}^{l,r}(\cdot)\cdot\nu(\sigma)$; we will also write $f_o^{l,r}(\cdot)$ for $f^{l,r}(\sigma_o;\cdot)$. Translating and rotating the axes, we can (at least, locally) reduce the situation to

$$x_o = 0, \quad \Phi(0) = 0 \text{ and } \nabla\Phi(0) = 0, \tag{23}$$

so that $\{(t, x_1, x') \,|\, x_1 = 0\}$ is the tangent plane to Σ at the point $\sigma_o = (t, 0)$. By Proposition 7, there exists a solution to the one-dimensional problem

$$\left(\mathfrak{f}_o\left(x_1, W\right)\right)_{x_1} = W_{x_1x_1}, \quad W(-\infty) = c^l,\ W(+\infty) = c^r \tag{24}$$

(W is the standing-wave profile corresponding to the model problem (1) with $\mathfrak{f}_o(x_1,\cdot) = f_o^l(\cdot)1\!\!1_{\{x_1<0\}} + f_o^r(\cdot)1\!\!1_{\{x_1>0\}}$). The properties of W include

$$\begin{gathered} W \in C(\mathbb{R})\cap W^{2,\infty}(\mathbb{R}\backslash\{0\}), \\ W' \in L^1(\mathbb{R})\cap L^\infty(\mathbb{R}),\ \ W''|_{\mathbb{R}\backslash\{0\}} \in L^1(\mathbb{R}\backslash\{0\}), \\ W'(0^+) - W'(0^-) = f_o^r(W(0)) - f_o^l(W(0)). \end{gathered} \tag{25}$$

Consider the approximate solutions w^ε, $\varepsilon > 0$, to equation (13) and their limit w:

$$\begin{aligned} w^\varepsilon(t,x) &:= W\Big(\frac{x_1 - \Phi(x')}{\varepsilon}\Big), \\ w(t,x) &:= \lim_{\varepsilon\downarrow 0} w^\varepsilon(t,x) = c^l 1\!\!1_{\Omega^l} + c^r 1\!\!1_{\Omega^r}. \end{aligned} \tag{26}$$

A straightforward calculation using the pointwise formulation of (24) and the jump condition in (25) shows that the function w^ε verifies the equation

$$w_t^\varepsilon + \operatorname{div}\mathfrak{f}(x, w^\varepsilon) = r^\varepsilon + \varepsilon\Delta w^\varepsilon \tag{27}$$

(in the sense of distributions) with source term $r^\varepsilon = r_1^\varepsilon + r_2^\varepsilon + r_3^\varepsilon + r_4^\varepsilon + r_5^\varepsilon$, where

$$\begin{gathered} r_1^\varepsilon = -\frac{1}{\varepsilon} W'(\xi)\ \mathfrak{f}'(x, W(\xi)) \cdot \nabla_{(x_1,x')}\Phi(x'), \\ r_2^\varepsilon = -\frac{1}{\varepsilon}\left|\nabla\Phi(x')\right|^2\ W''(\xi), \quad r_3^\varepsilon = \Delta\Phi(x')\ W'(\xi), \end{gathered}$$

and the terms r_4^ε, r_5^ε are measures supported on Σ and acting on $\varphi \in C(\mathbb{R}^{N+1})$ by

$$\langle r_4^\varepsilon, \varphi\rangle := -\left(W'\left(0^+\right) - W'\left(0^-\right)\right)\int_{\mathbb{R}^+}\int_{\mathbb{R}^{N-1}} \left|\nabla\Phi(x')\right|^2 \varphi\left(t, \Phi(x'), x'\right)\,dtdx',$$

$$\langle r_5^\varepsilon, \varphi\rangle := \int_{\mathbb{R}^+}\int_{\mathbb{R}^{N-1}} \left(\mathfrak{f}^r(W(0)) - \mathfrak{f}^l(W(0))\right)\cdot\nabla_{(x_1,x')}\Phi(x')\,\varphi\left(t, \Phi(x'), x'\right)\,dtdx'.$$

In the expressions which define $r_1^\varepsilon, \ldots, r_3^\varepsilon$, we mean that $\xi = \xi(x_1, x') = (x_1 - \Phi(x'))/\varepsilon$. The functions W, W' and W'' are evaluated pointwise, for $\xi \neq 0$; $\nabla_{(x_1,x')}\Phi(x')$ denotes the N-dimensional vector $(0, \nabla\Phi(x'))$; and $\mathfrak{f}'$ is the a.e. defined derivative in z of $\mathfrak{f}(x,z)$. Note that the product $W'(\xi)\mathfrak{f}'(x, W(\xi))$ makes sense.

Taking a smooth approximation $H_\alpha(u^\varepsilon - w^\varepsilon)$ of $\operatorname{sign}(u^\varepsilon - w^\varepsilon)$ for the test function in the difference of equations (13) and (27), as $\alpha \downarrow 0$ we deduce the following Kato inequality: For all non-negative test functions $\varphi \in \mathcal{D}(\mathbb{R}^{N+1})$ supported in a neighbourhood of σ_o,

$$\begin{aligned} -\int_{\mathbb{R}^{N+1}} &\left(|u^\varepsilon - w^\varepsilon|\,\varphi_t + \mathfrak{q}\left(x; u^\varepsilon, w^\varepsilon\right)\cdot\nabla\varphi + \varepsilon\,|u^\varepsilon - w^\varepsilon|\,\Delta\varphi\right)\,dxdt \\ &\leq \int_{\mathbb{R}^{N+1}} \left(|r_1^\varepsilon| + |r_2^\varepsilon| + M\,|\Delta\Phi(x')|\right)\,\varphi\,dx_1dx'dt \\ &\qquad + M\int_{(t,x')\in\mathbb{R}^N}\Big(\left|\nabla\Phi(x')\right|^2 + \left|\nabla\Phi(x')\right|\Big)\ \varphi(t, \Phi(x'), x')\,dtdx', \end{aligned} \tag{28}$$

where

$$M = \max\left\{\left|W'(0^+) - W'(0^-)\right|, \left|\mathfrak{f}^r(W(0)) - \mathfrak{f}^l(W(0))\right|, \|W'\|_\infty\right\}.$$

In the sequel, M denotes a generic constant depending on the profile W and on $\sup|\mathfrak{f}'|$ on the segment with endpoints $c^{l,r}$. Because we have $dx_1 dx' = \varepsilon d\xi dx'$ in the sense of measures, the integrability properties in (25) yield $\forall\varphi \in \mathcal{D}(\mathbb{R}^{N+1})$, $\varphi \geq 0$,

$$\begin{aligned}
&\int_{R^{N+1}} \left(|r_1^\varepsilon| + |r_2^\varepsilon|\right) \varphi(t, x_1, x')\, dx_1 dx' dt \\
&\qquad \leq \int_{\mathbb{R}^{N+1}} \left(|W'(\xi)|\, |\nabla\Phi(x')| + |W''(\xi)|\, |\nabla\Phi(x')|^2\right) \varphi(t, \xi, x')\, dt d\xi dx' \qquad (29)\\
&\qquad \leq M \int_{(t,x')\in\mathbb{R}^N} \left(|\nabla\Phi(x')| + |\nabla\Phi(x')|^2\right) \max \varphi(t, \cdot, x')\, dt dx'.
\end{aligned}$$

Now we fix a test function of the form $\varphi(t,x) := \psi(t,x')\xi_h(x)$, where ξ_h was introduced in Section 3. Keeping h and ψ fixed, we let $\varepsilon \downarrow 0$ in (28). Using the uniform in ε bound (29) and the definitions of u and w, we infer that

$$\begin{aligned}
&-\int_{\mathbb{R}^{N+1}} \left(|u - w|\left(\psi\xi_h\right)_t + \mathfrak{q}(x; u, w)\cdot\nabla\left(\psi\xi_h\right)\right) dx dt \\
&\qquad\qquad \leq M \int_{\mathbb{R}^N} \left(|\nabla\Phi| + |\nabla\Phi|^2 + h\,|\Delta\Phi|\right) \psi\, dt dx'.
\end{aligned}$$

Now replace ψ by a nonnegative test function $\psi_\delta \in \mathcal{D}(\mathbb{R}^N)$ with integral equal to one, supported in a δ-neighbourhood of σ_o (here, we mean that Σ is parametrized by $(t,x') \in \mathbb{R}^N$). As $h \downarrow 0$ and then $\delta \downarrow 0$, the right-hand side of the above inequality vanishes, due to the normalization (23). As to the left-hand side, it converges to

$$\begin{aligned}
&-\lim_{\delta\downarrow 0}\int_\Sigma \left(\mathfrak{q}^l\left(\left(\gamma^l u\right)(\sigma), c^l\right) - \mathfrak{q}^r\left(\left(\gamma^r u\right)(\sigma), c^r\right)\right)\cdot\nu(\sigma)\,\psi_\delta(\sigma)\, dt dx' \\
&\qquad\qquad = -\left(\mathfrak{q}^l\left(\left(\gamma^l u\right)(\sigma_o), c^l\right) - \mathfrak{q}^r\left(\left(\gamma^r u\right)(\sigma_o), c^r\right)\right)\cdot\nu\left(\sigma_o\right).
\end{aligned}$$

This establishes (22) and concludes the proof. □

Proof of Corollary 6 (sketched). Existence for (13) with $u_0 \in L^2(\mathbb{R}^N)$ can be obtained by the classical Galerkin method. Uniqueness and, more generally, the comparison principle and the L^1 contraction property for solutions u^ε of (13) are also classical (cf. (28) in the above proof). Then the comparison principle allows to drop the restriction on u_0 for the existence of a solution u^ε to (13).

Finally, because we assume that $\mathfrak{f}^{l,r}(0) = 0 = \mathfrak{f}^{l,r}(1)$ and $0 \leq u_0 \leq 1$, the comparison principle yields $0 \leq u^\varepsilon \leq 1$. This justifies Corollary 6. □

5. Appendix: Theory of germs and the maximality of $\mathcal{G}_{VV}$.

Here we justify Proposition 7*(iii)*. To this end, let us first give a general definition of an L^1D germ and of the closure operation on germs. In relation with the left- and right-side fluxes f^l and f^r in (1) and the associated Kruzhkov fluxes

$$q^{l,r}(z,k) = \operatorname{sign}(z-k)\left(f^{l,r}(z) - f^{l,r}(k)\right),$$

we introduce the following definitions:

Definition 10. A *right* (respectively, *left*) *contact shock* is a couple of real values (u^r, u_+) (resp., (u_-, u^l)) such that the function

$$u(x) = u^r \mathbb{1}_{x<0} + u_+\mathbb{1}_{x>0} \ \ (\text{resp., } u(x) = u_-\mathbb{1}_{x<0} + u^l\mathbb{1}_{x>0})$$

is a stationary Kruzhkov-admissible shock for the conservation law $u_t + (g(u))_x = 0$ with the flux $g = f^r$ (resp., $g = f^l$).

Definition 11 (**Germs; closed, complete, maximal and definite germs).**

- Any set $\mathcal{G}$ of couples $(c^l, c^r) \in \mathbb{R} \times \mathbb{R}$ satisfying the Rankine-Hugoniot relation
$$f^l(c^l) = f^r(c^r) \tag{30}$$
and the L^1-dissipativity relation (15) is called an L^1D *admissibility germ* (a *germ*, for short) associated with the couple of fluxes (f^l, f^r).
- The *closure* of a germ $\mathcal{G}$ is the smallest set $\overline{\mathcal{G}}$ containing $\mathcal{G}$ such that $\overline{\mathcal{G}}$ is topologically closed, and moreover, for all couples $(c^l, c^r) \in \overline{\mathcal{G}}$, $\overline{\mathcal{G}}$ also contains all couples (c_-, c_+) such that (c_-, c^l) is a left contact shock, (c^r, c^+) is a right contact shock.
- A germ $\mathcal{G}$ is called *closed*, if $\overline{\mathcal{G}} = \mathcal{G}$.
- A germ $\mathcal{G}$ is called *complete*[3], if all Riemann problem for (1) admits a self-similar solution u such that $(\gamma^l u, \gamma^r u) \in \mathcal{G}$, where $\gamma^l u$, resp.$\gamma^r u$, is the limit of u as $x \to 0^-$, resp.as $x \to 0^+$.
- We say that $\mathcal{G}'$ is an *extension* of a germ $\mathcal{G}$ if $\mathcal{G} \subset \mathcal{G}'$ and $\mathcal{G}'$ still satisfies the L^1-dissipation property (15) and the Rankine-Hugoniot condition (30).
- A germ $\mathcal{G}$ is called *maximal*, if it does not admit a nontrivial extension.
- A germ $\mathcal{G}$ is called *definite*, it it admits only one maximal extension.

In relation with definite and maximal germs, consider one more definition.

Definition 12 (**dual of a germ).** Let $\mathcal{G}$ be a germ. The dual of $\mathcal{G}$ is the set
$$\mathcal{G}^* := \left\{ \begin{array}{c} \left(b^l, b^r\right) \in \mathbb{R} \times \mathbb{R} \;\middle|\; f^l\left(b^l\right) = f^r\left(b^r\right) \text{ and} \\ \forall \left(c^l, c^r\right) \in \mathcal{G},\; q^l\left(c^l, b^l\right) \geq q^r\left(c^r, b^r\right) \end{array} \right\}. \tag{31}$$

We pause to give an example illustrating these definitions. Let
$$f^l(u) = 3u(1-u) \text{ and } f^r(u) = 4u(1-u).$$
For u^r and u^l in $[0,1]$ the right and left contact shocks are given by
$$u_+ = \begin{cases} 1-u^r \text{ or } u^r & \text{for } u^r \in [0,1/2], \\ u^r & \text{for } u^r \in [1/2,1]. \end{cases}$$
$$u_- = \begin{cases} u^l & \text{for } u^l \in [0,1/2], \\ 1-u^l \text{ or } u^l & \text{for } u^l \in [1/2,1]. \end{cases}$$
The Rankine-Hugoniot condition implies that any couple (c^l, c^r) in a germ must satisfy
$$c^r = \frac{1}{2}\left(1 \pm \sqrt{1 - 3c^l\left(1-c^l\right)}\right) =: h^{\pm}(c^l). \tag{32}$$
In addition, for every two couples (b^l, b^r), (c^l, c^r) in a germ, the L^1D condition (15) implies that
$$\begin{cases} \text{either } f^{l,r}\left(b^{l,r}\right) = f^{l,r}\left(c^{l,r}\right) \\ \text{or } b^l < c^l \implies b^r < c^r. \end{cases}$$

[3]The definition in [3] of a complete germ is slightly different; it autorizes left- and right- contact shocks in the solutions of a Riemann problem. Contrarily to [3], the definition of the present paper implies that a complete germ is closed; this is not always convenient.

In particular, in a germ you cannot "jump decreasingly (in the sense that $b^l > b^r$) through 1/2 more than once". Furthermore, this decreasing jump must occur at the maximal allowed value of the flux at the jump.

Hence, an example of a germ is the set

$$\mathcal{G} = \left\{b^l \in [0,1/4],\ b^r = h^-\left(b^l\right)\right\} \cup \left\{b^l \in [5/6,1],\ b^r = h^+\left(b^l\right)\right\}.$$

By adding all contact shocks to $\mathcal{G}$ we obtain its closure,

$$\overline{\mathcal{G}} = \left\{b^l \in [0,1/4],\ b^r = h^-\left(b^l\right)\right\} \cup \left\{b^l \in [5/6,1],\ b^r = h^+\left(b^l\right)\right\} \cup \left\{b^l \in [0,1/4],\ b^r = h^+\left(b^l\right)\right\}.$$

Consider the Riemann problem with left state 3/8 and right state $h^-(3/8)$. This couple is not in $\overline{\mathcal{G}}$, and if we wish to find a self similar solution with traces in $\overline{\mathcal{G}}$, we must first jump by a shock with negative speed to a value $c^l \in [5/6,1]$. If the solution is to have traces in $\overline{\mathcal{G}}$ then the trace from the right must be $h^+(c^l)$. It is however impossible to connect $h^+(c^l)$ with $h^-(3/8)$ by a Kruzhkov-admissible solution having waves of non-negative speeds. Thus $\overline{\mathcal{G}}$ is not complete.

For $\kappa \in [1/4,1/2]$ we can define a family of maximal extensions to $\mathcal{G}$ by

$$\mathcal{G}_\kappa = \left\{\left(b^l, h^-\left(b^l\right)\right)\ \middle|\ 0 \le b_l \le \kappa\right\} \cup \left\{\left(b^l, h^+\left(b^l\right)\right)\ \middle|\ 0 \le b_l \le \kappa\right\} \cup \left\{\left(1-\kappa, h^-(1-\kappa)\right)\right\} \cup \left\{\left(b^l, h^+\left(b^l\right)\right)\ \middle|\ 1-\kappa \le b^l \le 1\right\}.$$

Each of these extensions "jumps decreasingly through 1/2" once, and limits the maximal flux through $x = 0$ to $f^l(\kappa)$. Since $\mathcal{G}$ has several maximal extensions, it is not definite, see Proposition 14(*iii*) below. Regarding the dual $\mathcal{G}^*$, by Proposition 13(*v*), it will not be a germ. For each κ, $\mathcal{G}_\kappa = \overline{\mathcal{G}_\kappa}$ and we have added precisely the decreasing jump which makes it complete. makes it complete. Hence, by Proposition 15(*ii*), $\mathcal{G}^*_\kappa = \mathcal{G}_\kappa$.

The dual of $\overline{\mathcal{G}}$ (and also the dual of $\mathcal{G}$) is formed by the addition of all points which satisfy the Rankine-Hugoniot condition and the L^1D condition with respect to points in $\overline{\mathcal{G}}$ (and in $\mathcal{G}$). Hence the dual is given by

$$\overline{\mathcal{G}}^* = \mathcal{G}^* = \left\{b^l \in [0,3/4],\ b^r = h^-\left(b^l\right)\right\} \cup \left\{b^l \in [0,1],\ b^r = h^+\left(b^l\right)\right\}.$$

Observe that in accordance with Proposition 13(*ii*) $\mathcal{G}^* = \cup_{\kappa\in[1/4,1/2]}\mathcal{G}_\kappa$. These sets are depicted in Figure 1.

We refer to [3] for details, further examples and for the proofs of the below relations between different properties of $\mathcal{G}$, $\overline{\mathcal{G}}$, $\mathcal{G}^*$. These propositions can be helpful in order to determine whether a given subset $\mathcal{G}$ of $\mathbb{R}^2$ is a germ, and in order to describe the properties of a given germ $\mathcal{G}$.

Proposition 13 (**dual germ, maximality and definiteness**). *Let $\mathcal{G}$ be a subset of $\mathbb{R}^2$; let $\mathcal{G}^*$ be defined by (31).*

- (*i*) *One has $\mathcal{G} \subset \mathcal{G}^*$ if and only if $\mathcal{G}$ is a germ.*
- (*ii*) *Assume $\mathcal{G}$ is a germ. Then $\mathcal{G}^*$ is the union of all extensions of $\mathcal{G}$. In particular, if $\mathcal{G}$ is a definite germ, then $\mathcal{G}^*$ is the unique maximal extension of $\mathcal{G}$.*
- (*iii*) *One has $\mathcal{G}^* = \mathcal{G}$ if and only if $\mathcal{G}$ is a maximal germ.*
- (*iv*) *If $\mathcal{G}$ is a definite germ, then $(\mathcal{G}^*)^* = \mathcal{G}^*$.*
- (*v*) *If $\mathcal{G}^*$ is a germ, then $\mathcal{G}$ is definite.*

Proposition 14 (**closure and closed germs**).

- (*i*) *Assume $\mathcal{G}$ is a germ. Then its closure $\overline{\mathcal{G}}$ is also a germ (thus, $\overline{\mathcal{G}}$ is an extension of $\mathcal{G}$); furthermore, $\overline{\mathcal{G}} \subset \mathcal{G}^*$ and $(\overline{\mathcal{G}})^* = \mathcal{G}^*$.*

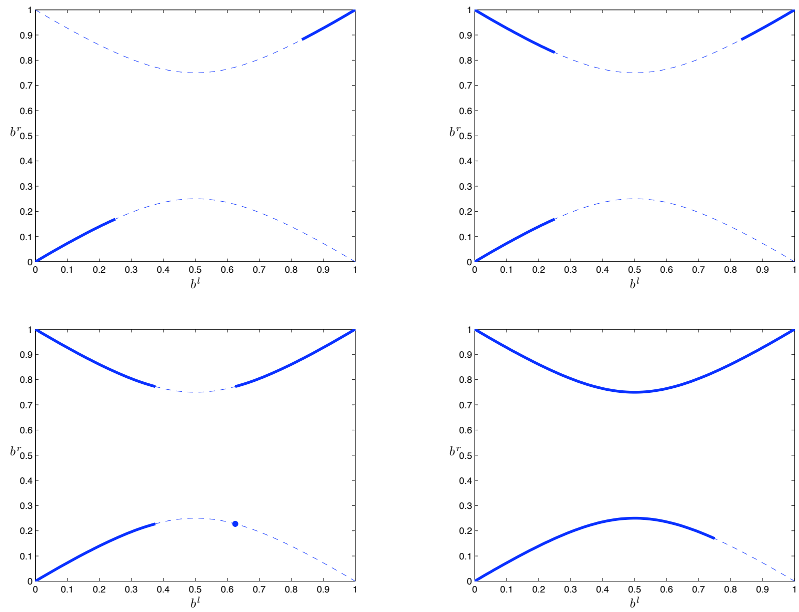

FIGURE 1. The Rankine-Hugoniot condition shown as broken curves, the other sets as solid curves in the (b^l, b^r) plane. Top left: the germ $\mathcal{G}$, top right: the closure $\overline{\mathcal{G}}$, bottom left: the extension $\mathcal{G}_{3/8}$, bottom right: the dual $\mathcal{G}^*$.

(ii) If $\mathcal{G}$ is a maximal germ, then it is closed.

(iii) Any maximal extension of $\mathcal{G}$ contains $\overline{\mathcal{G}}$. In particular, $\mathcal{G}$ is a definite germ if and only if $\overline{\mathcal{G}}$ is a definite germ.

(iv) Let $\mathcal{G}$ be a definite germ. Then $\mathcal{G}$-entropy and $\overline{\mathcal{G}}$-entropy solutions coincide.

Proposition 15 (complete germs).

(i) Assume $\mathcal{G}$ is a complete germ. Then $\mathcal{G}$ is a maximal (and thus closed) germ.

(ii) Assume $\mathcal{G}$ is a germ such that $\overline{\mathcal{G}}$ is complete. Then $\mathcal{G}$ is definite, and $\mathcal{G}^* = \overline{\mathcal{G}}$.

Remark 16. Notice that in case *(ii)* of Proposition 15, $\mathcal{G}$ is a definite germ, and $\mathcal{G}^*$ is maximal and complete. Such germs are expected to lead to a well-posedness theory for $\mathcal{G}$-entropy solutions. The germ $\mathcal{G}_{VV}$ of Definition 1 is one example of a maximal germ; it is complete, e.g., under the assumptions of Corollary 6.

In terms of the above definitions, the statement of Proposition 7*(iii)* exactly means that $\mathcal{G}^o_{VV}$ is a definite germ of which $\mathcal{G}_{VV}$ is the dual; in particular, $\mathcal{G}_{VV}$ is a maximal germ. The below proof is based on the property that $\mathcal{G}_{VV}$ coincides with the closure $\overline{\mathcal{G}^o_{VV}}$ of G^o_{VV}. Let us point out that the difference between a germ and its closure is responsible for the apparent distinction between the pioneering "minimal jump" admissibility condition of Gimse and Risebro [13, 14] and the Γ-condition given by Diehl in [8, 9, 10, 11]. The set of trace values determined by the two conditions has the same closure; according to Proposition 15, this distinction does not change the germ-based notion of entropy solution.

Proof of Proposition 7(iii). In the first step, we show that $\mathcal{G}_{VV} \subset \overline{\mathcal{G}^o_{VV}}$. By Proposition 7*(ii)*, since $\mathcal{G}^o_{VV}$ is a germ, $\overline{\mathcal{G}^o_{VV}}$ is also a germ by Proposition 14*(i)*.

Then in the second step, we show that $\mathcal{G}_{VV}$ contains the dual of $\mathcal{G}^o_{VV}$. According to Propositions 13 and 14, this yields the reciprocal inclusion $\mathcal{G}_{VV} \supset \overline{\mathcal{G}^o_{VV}}$ and then the maximality of the germ $\mathcal{G}_{VV} = \overline{\mathcal{G}^o_{VV}}$. We will repeatedly use, without mentioning it, the continuity of $f^{l,r}$.

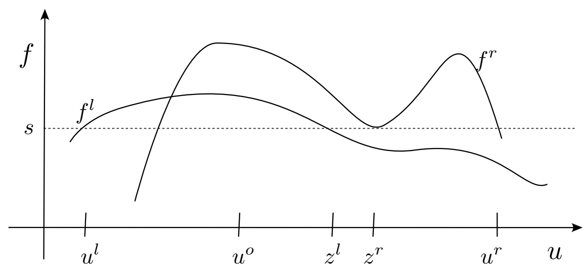

FIGURE 2. The case where $u^l < u^r$ and $f^l - s$ has a zero in $[u_o, u_r)$.

Step 1: Let $(u^l, u^r) \in \mathcal{G}_{VV}$. If $u^l = u^r$, then $(u^l, u^r) \in \mathcal{G}^o_{VV} \subset \overline{\mathcal{G}^o_{VV}}$. The other cases are symmetric; let us treat the one where $u^l < u^r$.

Take s and u^o as introduced in (4). If both $f^l > s$ on the interval $(u^l, u^r]$ or $f^r > s$ on the interval $[u^l, u^r)$, then $(u^l, u^r) \in \mathcal{G}^o_{VV}$. Hence we assume that both f^l and f^r take the value s somewhere in the interval (u^l, u^r).

Consider the function $f^l(\cdot) - s$ on the interval $[u^o, u^r]$. If it has a zero, set $z^l := \min\{z \in [u^o, u^r] \mid f^l(z) = s\}$, then $z^r := \min\{z \in [z^l, u^r] \mid f^r(z) = s\}$ (since $f^r(u^r) = s$, z^r is well defined). See Figure 2.

By construction and by (4), we have

- $u^l \le z^l$, $f^l(u^l) = f^l(z^l) = s$, and $f^l \ge s$ on $[u^l, z^l]$;
- $z^l \le z^r$, $f^l(z^l) = f^r(z^r) = s$, and $f^r > s$ on the interval (z^l, z^r);
- $z^r \le u^r$, $f^r(z^r) = f^r(u^r) = s$, and $f^r \ge s$ on $[z^r, u^r]$.

This means that (u^l, z^l) (resp., (z^r, u^r)) is a left-contact shock (resp., a right-contact shock), and $(z^l, z^r) \in \mathcal{G}^o_{VV}$.

Next consider the situation where $f^l(\cdot) - s$ has no zero on the interval $[u^o, u^r)$, but $f^l(u^r) = s$. In this case (u^l, u^r) is a left-contact shock and $(u^l, u^r) \in \overline{\mathcal{G}^o_{VV}}$.

In all the cases, by the definition of the closure we conclude that $(u^l, u^r) \in \overline{\mathcal{G}^o_{VV}}$.

Step 2: Recall that $\mathcal{G}^o_{VV} \subset \mathcal{G}_{VV}$. Therefore it suffices to show that if $f^l(u^l) = f^r(u^r)$, and $(u^l, u^r) \notin \mathcal{G}_{VV}$, then there exists $(c^l, c^r) \in \mathcal{G}^o_{VV}$ such that

$$q^l(u^l, c^l) < q^r(u^r, c^r), \tag{33}$$

and thus $(u^l, u^r) \notin \left(\mathcal{G}^o_{VV}\right)^*$. Set $s := f^{l,r}(u^{l,r})$. As before, it suffices to consider the case $u^l < u^r$. Define

$$z^l := \sup\left\{z \in \left[u^l, u^r\right] \;\middle|\; f^l \ge s \text{ on } \left[u^l, z\right]\right\},$$
$$z^r := \inf\left\{z \in \left[u^l, u^r\right] \;\middle|\; f^r \ge s \text{ on } \left[z, u^r\right]\right\}.$$

If we had $z^l \ge z^r$, then (4) would hold with $u^o = z^l$, so that (u^l, u^r) would lie in $\mathcal{G}_{VV}$. Thus $z^l < z^r$. Now, there are three cases to be investigated:

(a) f^l and f^r have a crossing point z^o in the interval (z^l, z^r) such that $f^{l,r}(z^o) < s$;

(b) f^l and f^r have a crossing point z^o in the interval (z^l, z^r) such that $f^{l,r}(z^o) \geq s$;
(c) either $f^l < f^r$ on the interval (z^l, z^r), or $f^r < f^l$ on the interval (z^l, z^r).

See Figure 3

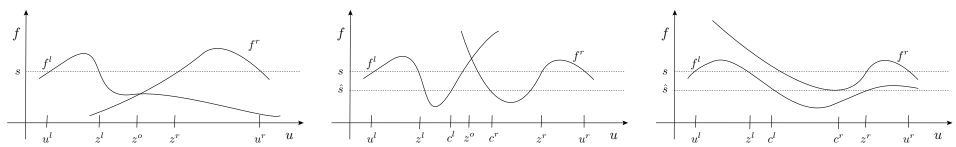

FIGURE 3. Left: case (a), middle: case (b), right: case (c).

In the case (a), setting $(c^l, c^r) := (z^o, z^o)$ leads to (33), because $u^l < c^l = c^r < u^r$, and $f^{l,r}(u^{l,r}) > f^{l,r}(c^{l,r})$. Note that $(c^l, c^r) = (z^o, z^o) \in \mathcal{G}^{\circ}_{VV}$.

In the case (b), by definition of $z^{l,r}$, there exists $\hat{s} < s$ such that $\hat{s} \in f^l([z^l, z^o]) \cap f^r([z^o, z^r])$. In this case, set $c^l := \max\{z \in [z^l, z^o] \,|\, f^l(z) = \hat{s}\}$, $c^r := \min\{z \in [z^o, z^r] \,|\, f^r(z) = \hat{s}\}$. We then have (33) for the same reasons as in the case (a). In addition, $(c^l, c^r) \in \mathcal{G}^{\circ}_{VV}$, because $c^l < c^r$ and $f^l > \hat{s}$ on $(c^l, z^o]$, $f^r > \hat{s}$ on $[z^o, c^r)$.

The two situations covered by case (c) are similar. Consider the case where $f^l < f^r$ on (z^l, z^r). Choose c^r as the point in $[z^l, z^r]$ where f^r attains its minimum value over $[z^l, z^r]$ and which is the closest one to z^l, i.e.,

$$c^r = \min \left\{ \operatorname{minarg}_{[z^l, z^r]} f^r(z) \right\}.$$

By definition of z^r, $\hat{s} := f^r(c^r) < s$. Because $f^r(z^l) \geq f^l(z^l) = s > \hat{s}$, we have $c^r > z^l$. In turn, this yields $f^l(c^r) < f^r(c^r) = \hat{s}$. Since $f^l(z^l) = s > \hat{s}$, there exists c^l in the interval (z^l, z^r) such that $f^l(c^l) = \hat{s}$. The couple (c^l, c^r) fulfills (4). In addition, by the definition of c^r we have $f^r \geq \hat{s}$ on $[z^l, z^r] \supset [c^l, c^r]$; thus $(c^l, c^r) \in \mathcal{G}^{\circ}_{VV}$.

In all cases, we have constructed $(c^l, c^r) \in \mathcal{G}^{\circ}_{VV}$ with property (33). The contradiction shows that $(\mathcal{G}^{\circ}_{VV})^* \subset \mathcal{G}_{VV}$ and thus concludes the proof. $\square$

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