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A REVIEW OF CONSERVATION LAWS ON NETWORKS

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Abstract. This paper deals with various applications of conservation laws on networks. In particular we consider the car traffic, described by the Lighthill-Whitham-Richards model and by the Aw-Rascle-Zhang model, the telecommunication case, by using the model introduced by D'Apice-Manzo-Piccoli and, finally, the case of a gas pipeline, modeled by the classical p-system. For each of these models we present a review of some results about Riemann and Cauchy problems in the case of a network, formed by a single vertex with n incoming and m outgoing arcs.

1. **Introduction.** In recent years, partial differential equations on networks attracted lot of attention. One of the main motivation is the wide range of different applications covered by the research theme: vehicular traffic, data networks, irrigation channels, gas pipelines, supply chains, blood circulation and so on (see [2, 5, 6, 9, 10, 11, 16, 21, 25]). A network is simply a finite collection of directed arcs connected together by vertices or nodes. On each arc of the network we consider a system of partial differential equations in conservation form. The aim is to study the Cauchy problem on the whole network, which clearly depends on the solution at vertices. Since we are considering hyperbolic partial differential equations and waves propagate with finite speed, it is completely equivalent to consider just the case of a network composed by a single vertex; see [18, Theorem 4.3.9]. Therefore we just consider a hyperbolic system of conservation laws on a single node, composed by n incoming arcs and m outgoing ones.

To describe the dynamics, it is sufficient to define solutions to Riemann problems at the node, which are Cauchy problems with constant initial conditions on the arcs meeting at the node. The maps providing such solutions are called Riemann solvers. The solution to the general Cauchy problem is constructed via the wave-front tracking technique, which consists in approximating the exact solution by piecewise constant functions. Once we know how to solve Riemann problems in the arcs and at the node, we are able to construct wave-front tracking approximate solutions. The second step consists in proving some compactness estimates (in our case uniform bounds on the total variation of the flux) in order to extract a converging sequence. Finally the limit function is the exact solution. Clearly the solution depends on the choice of the Riemann solver itself.

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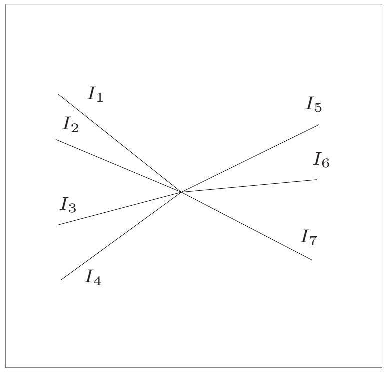


FIGURE 1. A node composed by 4 incoming and 3 outgoing arcs.

In the paper we describe both the scalar case, in particular for traffic and telecommunication networks, and the system case for vehicular traffic and gas pipelines. For the system case the results for the Cauchy problems are just local in the BV and L^∞ norm, while for the scalar case this restriction can be dropped.

The contents of the paper are the following ones. In Section 2 we introduce the basic definitions about systems of conservation laws and the Riemann and Cauchy problems at a node. In Section 3 we describe various applications. More precisely, we treat the case of car traffic in subsection 3.1, of telecommunication networks in subsection 3.2 and of gas pipelines in subsection 3.3.

2. Basic definitions.

Consider a node J with n incoming arcs $I_1, \ldots, I_n$ and m outgoing ones $I_{n+1}, \ldots, I_{n+m}$. We model each incoming arc I_i ($i \in \{1, \ldots, n\}$) of the node with the real interval $I_i = \mathbb{R}^- :=]-\infty, 0]$. Similarly we model each outgoing arc I_j ($j \in \{n+1, \ldots, n+m\}$) of the node with the real interval $I_j = \mathbb{R}^+ := [0, +\infty[$; see Figure 1. On each arc I_l ($l \in \{1, \ldots, n+m\}$) we consider the system of conservation laws

$$(u_l)_t + f(u_l)_x = 0, \tag{1}$$

where $u_l = u_l(t, x) \in \Omega$ is the *conserved quantity*, $f : \Omega \to \mathbb{R}^N$ is the *flux* and Ω is an open and connected subset of $\mathbb{R}^N$. Hence the datum is given by a finite collection of functions u_l defined on $[0, +\infty[\times I_l$. Suppose that the flux f is a smooth function and that the Jacobian matrix $A(u) = Df(u)$ has N real and distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_N(u)$, i.e. the system (1) is strictly hyperbolic; see [4]. Denote also by $l_h(u)$ and $r_h(u)$ ($h \in \{1, \ldots, N\}$) respectively the left and right eigenvalues of $A(u)$, normalized so that

$$|r_h(u)| = 1, \quad l_h(u) \cdot r_k(u) = \delta_{hk}$$

for every $u \in \Omega$, $h, k \in \{1, \ldots, N\}$, where δ_{hk} denotes the Kronecker symbol. Moreover, if $N \geq 2$, then we assume that, for every $h \in \{1, \ldots, N\}$, either $\nabla\lambda_h(u) \cdot r_h(u) = 0$ for every $u \in \Omega$ or $\nabla\lambda_h(u) \cdot r_h(u) > 0$ for every $u \in \Omega$. This means that the h-th characteristic field is respectively linearly degenerate or genuinely nonlinear; see [4].

Definition 2.1. A Riemann problem at the node J is the following Cauchy problem

$$\begin{cases} (u_1)_t + f(u_1)_x = 0, & t > 0,\ x \in I_1, \\ \vdots & \vdots \\ (u_{n+m})_t + f(u_{n+m})_x = 0, & t > 0,\ x \in I_{n+m}, \\ u_1(0,x) = \bar{u}_1, & x \in I_1, \\ \vdots & \vdots \\ u_{n+m}(0,x) = \bar{u}_{n+m}, & x \in I_{n+m}, \end{cases} \tag{2}$$

where $\bar{u}_1, \dots, \bar{u}_{n+m} \in \Omega$ are constant.

Definition 2.2. A solution to the Riemann problem (2) is a vector

$$(u_1(t,x), \dots, u_{n+m}(t,x)),$$

whose components are functions $u_l :]0,+\infty[\times I_l \to \Omega$ satisfying the following properties:

1. for every $i \in \{1, \dots, n\}$, u_i is the restriction to $]0,+\infty[\times I_i$ of the solution to the classical Riemann problem
$$\begin{cases} (u_i)_t + f(u_i)_x = 0, & t > 0,\ x \in \mathbb{R}, \\ u_i(0,x) = \bar{u}_i, & x < 0, \\ u_i(0,x) = u_i(1,0-), & x > 0; \end{cases}$$
2. for every $j \in \{n+1, \dots, n+m\}$, u_j is the restriction to $]0,+\infty[\times I_j$ of the solution to the classical Riemann problem
$$\begin{cases} (u_j)_t + f(u_j)_x = 0, & t > 0,\ x \in \mathbb{R}, \\ u_j(0,x) = \bar{u}_j, & x > 0, \\ u_j(0,x) = u_j(1,0+), & x < 0; \end{cases}$$
3. for every $l \in \{1, \dots, n+m\}$,
$$\lim_{t \to 0^+} u_l(t,\cdot) = \bar{u}_l$$
with respect to the L^1 topology.

Definition 2.3. Given $\Omega \subseteq \mathbb{R}^N$, a Riemann solver at the node J is a function $\mathcal{RS} : \Omega^{n+m} \to \Omega^{n+m}$, which associates to each initial condition the trace at $x = 0$ of a solution to the corresponding Riemann problem at J.

Remark 1. Note that giving the trace at the node J of the solution to a Riemann problem is completely equivalent to give the solution itself.

Note also that conditions 1. and 2. of Definition 2.2 imply that the waves generated by the solution to the Riemann problem (2) have negative speed in incoming arcs and positive speed in outgoing arcs.

According to Remark 1, we introduce, for every $l \in \{1, \dots, n+m\}$, the set Ω_l, composed by all the possible traces at $x = 0$ in the arc I_l.

Definition 2.4. For a given Riemann problem (2) at J we define the following subsets of Ω.

1. For $i \in \{1, \dots, n\}$, the set Ω_i consists of all the elements $\tilde{u} \in \Omega$ such that the classical Riemann problem with initial condition $(\bar{u}_i, \tilde{u})$ is solved with waves with negative speed.

2. For $j \in \{n+1, \dots, n+m\}$, the set Ω_j consists of all the elements $\tilde{u} \in \Omega$ such that the classical Riemann problem with initial condition $(\tilde{u}, \bar{u}_j)$ is solved with waves with positive speed.

Finally, let us consider the following Cauchy problem at J:

$$\begin{cases} (u_1)_t + f(u_1)_x = 0, & t > 0,\ x \in I_1, \\ \vdots & \vdots \\ (u_{n+m})_t + f(u_{n+m})_x = 0, & t > 0,\ x \in I_{n+m}, \\ u_1(0,x) = u_{1,0}, & x \in I_1, \\ \vdots & \vdots \\ u_{n+m}(0,x) = u_{n+m,0}, & x \in I_{n+m}, \end{cases} \tag{3}$$

where, for every $l \in \{1, \dots, n+m\}$, $u_{l,0} \in L^1(I_l)$ is a function with finite total variation.

3. Applications.

Here we present three applications for the model introduced in the previous section: car traffic, telecommunication networks and gas pipelines.

3.1. Car traffic.

Various models in conservation form for car traffic have been introduced in the literature. These models can be grouped in the following way:

First order: This group consists in models composed by a single equation. In this class, there is the (LWR) model proposed by Lighthill and Whitham [26] and independently by Richards [28]. It is described by the equation

$$\rho_t + (\rho v)_x = 0,$$

where ρ is the car density and $v = v(\rho)$ is the average velocity.

Second order: This group consists in models composed by a system of two equations. The most famous model in this class was proposed by Aw and Rascle [1] in 2000 and independently by Zhang [29] in 2002. It is described by the system

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ (\rho(v + p(\rho)))_t + (\rho v(v + p(\rho)))_x = 0, \end{cases}$$

where ρ is the car density, v is the velocity and p plays the role of a pressure. Other second order models are the phase-transition models introduced by R. M. Colombo [8] and by P. Goatin [20]. The Colombo phase-transition model reads

$$\begin{cases} \rho_t + (\rho v_f(\rho))_x = 0, & \text{for free-flow traffic,} \\ \begin{cases} \rho_t + (\rho v_c(\rho, q))_x = 0, \\ q_t + ((q - q^*) v_c(\rho, q))_x = 0 \end{cases} & \text{for congested traffic,} \end{cases}$$

where q is a linearized momentum and $v_f > v_c$ are respectively the velocity in the free-flow and in the congested phase. Instead the Goatin phase-transition model consists in the LWR model for the free-flow traffic and in the Aw-Rascle-Zhang model for congested traffic. Finally various generalizations of phase-transition models have been recently proposed; see [3, 15].

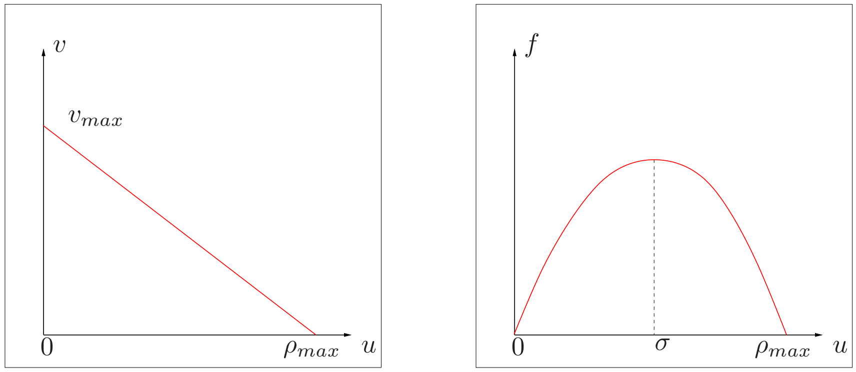


FIGURE 2. A simple example of a velocity function and of the corresponding flux for the LWR model.

Third order: This group consists in models composed by a system of three equations. In this class there is the model proposed by Helbing [22] in 1995. It can be described by the system

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ v_t + v v_x + \frac{1}{\rho}(\rho\theta)_x = \frac{1}{\tau}(v_e(\rho) - v) + \frac{\mu}{\rho} v_{xx}, \\ \theta_t + v\theta_x + 2\theta v_x = 2\frac{\mu}{\rho}(v_x)^2 + \frac{k}{\rho}\theta_{xx} + \frac{2}{\tau}(\theta_e(\rho) - \theta), \end{cases}$$

where θ is a variance.

In this part we deal just the LWR model and the Aw-Rascle-Zhang model on networks. For the Colombo phase-transition model at a node we refer to [12].

3.1.1. *Lighthill-Whitham-Richards model.* This model is based on the conservation of the number of cars. In this setting we have $N = 1$, i.e. the system is scalar, $f(u) = uv$, where v is the average speed of cars, and $\Omega = [0, \rho_{max}]$, where ρ_{max} is the maximum density of cars. The main assumption of this model is that v is a given function depending only on the density in a decreasing way, i.e. $v = v(\rho)$ and $v'(\rho) \leq 0$. Therefore the model can be described by the equation

$$u_t + f(u)_x = 0. \tag{4}$$

We also assume that the flux f is a smooth function such that:

1. $f(0) = f(\rho_{max}) = 0$;
2. f is strictly concave;
3. there exists a unique $\sigma \in [0, \rho_{max}]$ such that $f'(\sigma) = 0$; see Figure 2.

In this setting the sets Ω_i and Ω_j, introduced in Definition 2.4, are explicitly given in the next lemma.

Lemma 3.1. *The following statements hold.*

1. If $i \in \{1, \ldots, n\}$, then

$$\Omega_i = \begin{cases} \{\bar{u}_i\} \cup] \max f^{-1}(f(\bar{u}_i)), \rho_{max}], & \text{if } \bar{u}_i < \sigma, \\ [\sigma, \rho_{max}], & \text{if } \bar{u}_i \geq \sigma. \end{cases}$$

2. *If* $j \in \{n+1, \ldots, n+m\}$, *then*

$$\Omega_j = \begin{cases} \{\bar{u}_j\} \cup [0, \min f^{-1}(f(\bar{u}_i))[, & \text{if } \bar{u}_j > \sigma, \\ [0, \sigma], & \text{if } \bar{u}_j \leq \sigma. \end{cases}$$

We now recall the Riemann solver at J, introduced for traffic in [7]. First, we need to define a set $\mathcal{A}$ of matrices to describe the preferences of drivers:

$$\mathcal{A} := \left\{ A = \{a_{ji}\}_{\substack{i=1,\ldots,n \\ j=n+1,\ldots,n+m}} : \begin{array}{c} 0 < a_{ji} < 1 \; \forall i, j \\ \sum_{j=n+1}^{n+m} a_{ji} = 1 \; \forall i \end{array} \right\}. \tag{5}$$

Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. For every $i = 1, \ldots, n$, we denote $H_i = \{e_i\}^\perp$. If $A \in \mathcal{A}$, then we write, for every $j = n+1, \ldots, n+m$, $a_j = (a_{j1}, \ldots, a_{jn}) \in \mathbb{R}^n$ and $H_j = \{a_j\}^\perp$. Let $\mathcal{K}$ be the set of indices $\mathbf{k} = (k_1, ..., k_\ell)$, $1 \leq \ell \leq n-1$, such that $0 \leq k_1 < k_2 < \cdots < k_\ell \leq n+m$ and for every $\mathbf{k} \in \mathcal{K}$ define

$$H_{\mathbf{k}} = \bigcap_{h=1}^{\ell} H_{k_h}.$$

Writing $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$ and following [7] we define the set

$$\mathfrak{N} := \left\{ A \in \mathcal{A} : \mathbf{1} \notin H_{\mathbf{k}}^\perp \text{ for every } \mathbf{k} \in \mathcal{K} \right\}. \tag{6}$$

Notice that, if $n > m$, then $\mathfrak{N} = \emptyset$. The matrices of $\mathfrak{N}$ will give rise to a unique solution to Riemann problems at J.

Remark 2. Each matrix A in $\mathfrak{N}$ or in $\mathcal{A}$ describes the preferences of drivers at the node. Indeed each coefficient a_{ji} of a matrix A represents the percentage of traffic, which passes through the node and goes from the incoming arc I_i to the outgoing arc I_j.

The construction of the Riemann solver, introduced in [7], can be summarized as follows.

1. Fix a matrix $A \in \mathfrak{N}$ and consider the closed, convex and not empty set

$$\Gamma = \left\{ (f(u_1), \cdots, f(u_n)) \in \prod_{i=1}^{n} f(\Omega_i) : A \cdot \begin{pmatrix} f(u_1) \\ \vdots \\ f(u_n) \end{pmatrix} \in \prod_{j=n+1}^{n+m} f(\Omega_j) \right\}. \tag{7}$$

2. Find the point $(\bar{\gamma}_1, \ldots, \bar{\gamma}_n) \in \Gamma$ which maximizes the function

$$E(\gamma_1, \ldots, \gamma_n) = \gamma_1 + \cdots + \gamma_n, \tag{8}$$

and define $(\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{n+m})^T := A \cdot (\bar{\gamma}_1, \ldots, \bar{\gamma}_n)^T$. Since $A \in \mathfrak{N}$, the point $(\bar{\gamma}_1, \ldots, \bar{\gamma}_n)$ is uniquely defined.

3. For every $i \in \{1, \ldots, n\}$, set $\hat{u}_i$ either by $\bar{u}_i$ if $f(\bar{u}_i) = \bar{\gamma}_i$, or by the solution to $f(u) = \bar{\gamma}_i$ such that $\hat{u}_i \geq \sigma$. For every $j \in \{n+1, \ldots, n+m\}$, set $\hat{u}_j$ either by $\bar{u}_j$ if $f(\bar{u}_j) = \bar{\gamma}_j$, or by the solution to $f(u) = \bar{\gamma}_j$ such that $\hat{u}_j \leq \sigma$. Finally, define $\mathcal{RS} : [0, \rho_{max}]^{n+m} \to [0, \rho_{max}]^{n+m}$ by

$$\mathcal{RS}(\bar{u}_1, \ldots, \bar{u}_{n+m}) = (\hat{u}_1, \ldots, \hat{u}_n, \hat{u}_{n+1}, \ldots, \hat{u}_{n+m}). \tag{9}$$

Remark 3. The set Γ, defined in point 1 of the previous construction, describes all the possible traces at the junction, according to both the preference of the drivers and the velocity of the resulting waves. In general, the set Γ contains infinitely many points, i.e. there are infinitely many solution to the Riemann problem (2) satisfying the preference of the drivers and the velocity of the waves. Therefore, we

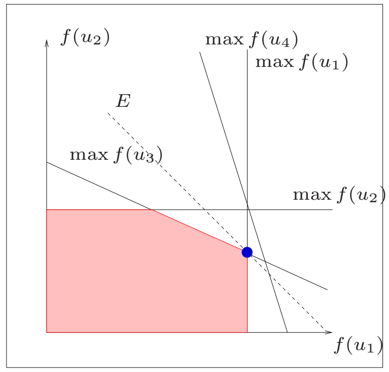


FIGURE 3. The set Γ defined in (7) in the case of a junction with 2 incoming and 2 outgoing roads.

need to impose an additional condition, in order to isolate a unique solution to (2). This rule is described by the maximization procedure in point 2 of the previous construction. Among all the possible solution, we take the one that maximizes the flux at J.

Remark 4. For junctions with 2 incoming roads, the set Γ is two-dimensional; hence it can be easily represented in a graph. In Figure 3, the set Γ is drawn for a junction J with 2 incoming and 2 outgoing roads. The point represents the solution to the Riemann problem.

The following theorem holds; for a proof, based on the wave-front tracking technique, see [19].

Theorem 3.2. *Consider the Cauchy problem (3), the Riemann solver $\mathcal{RS}$ and $T > 0$. Then there exists a weak solution at J $(u_1(t,x), \dots, u_{n+m}(t,x))$, defined for $t \in [0,T]$, such that*

1. *for every $l \in \{1, \dots, n+m\}$, $u_l(0,x) = u_{0,l}(x)$ for a.e. $x \in I_l$;*
2. *for a.e. $t \in [0,T]$,*

$$\mathcal{RS}(u_1(t,0-), \dots, u_{n+m}(t,0+)) = (u_1(t,0-), \dots, u_{n+m}(t,0+)) . \tag{10}$$

Remark 5. Note that Theorem 3.2 gives only existence of solution to the Cauchy problem (3). Uniqueness and continuous dependence of the solution to (3) is an open problem. Instead the Lipschitz continuous dependence does not hold; see [7] for an explicit example.

3.1.2. *Aw-Rascle-Zhang model.* For this model we take $N = 2$, i.e. a system with two equations, $u = \begin{pmatrix} \rho \\ y \end{pmatrix}$, $\Omega = \left\{(\rho, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : \rho^{\gamma+1} \le y \le \rho\right\}$ and $f(u) = \begin{pmatrix} y - \rho^{\gamma+1} \\ \frac{y^2}{\rho} - y\rho^{\gamma} \end{pmatrix}$, where $\gamma > 0$ is a constant; see Figure 4. We denote by f_1 and f_2 respectively the first and second component of the flux. The system can be written in the form

$$\begin{cases} \partial_t \rho + \partial_x (y - \rho^{\gamma+1}) = 0, \\ \partial_t y + \partial_x (\frac{y^2}{\rho} - y\rho^{\gamma}) = 0. \end{cases} \tag{11}$$

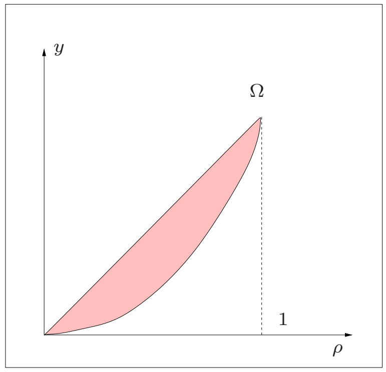


FIGURE 4. The set Ω for the Aw-Rascle-Zhang model.

Here ρ is the density of the cars and $y = \rho v + \rho^{\gamma+1}$ is a "generalized" momentum (v is the velocity of the cars). The following proposition holds; for a proof see [17].

Proposition 1. *In the set Ω, the system (11) is hyperbolic. Moreover, in $\Omega \setminus \{(0,0)\}$, it is strictly hyperbolic. The eigenvalues of the Jacobian matrix of the flux are*

$$\lambda_1(\rho, y) = \frac{y}{\rho} - (\gamma+1)\rho^\gamma, \qquad \lambda_2(\rho, y) = \frac{y}{\rho} - \rho^\gamma. \tag{12}$$

The first characteristic field is genuinely nonlinear, while the second one is linearly degenerate. Finally, the Lax curves of the first family are lines passing through the origin, while the Lax curves of the second family passing through (ρ_0, y_0) are given by

$$y = \frac{y_0}{\rho_0}\rho - \rho^{\gamma+1} - \rho_0^\gamma \rho. \tag{13}$$

Before considering the Riemann problem (2) for the Aw-Rascle-Zhang model, we describe the sets Ω_l of Definition 2.4.

Lemma 3.3. *Let $(\bar\rho_i, \bar y_i) \neq (0,0)$ be the initial condition in an incoming road I_i ($i \in \{1, \ldots, n\}$). A state $(\hat\rho_i, \hat y_i) \in \Omega_i$ belongs to the curve of the first family through $(\bar\rho_i, \bar y_i)$. More precisely, we have the following cases:*

1. *$\bar y_i > (\gamma+1)\bar\rho_i^{\gamma+1}$. There exists a unique point $(\check\rho_i, \check y_i)$ such that $\check y_i < (\gamma+1)\check\rho_i^{\gamma+1}$ and $f_1(\bar\rho_i, \bar y_i) = f_1(\check\rho_i, \check y_i)$. Then*
$$\Omega_i = \{(\bar\rho_i, \bar y_i)\} \cup \left\{(\rho, y) \in \Omega : \rho > \check\rho_i, y = \frac{\bar y_i}{\bar\rho_i}\rho\right\};$$
see Figure 5.
2. *$\bar y_i \leq (\gamma+1)\bar\rho_i^{\gamma+1}$. Then*
$$\Omega_i = \left\{(\rho, y) \in \Omega : y \leq (\gamma+1)\rho^{\gamma+1},\ y = \frac{\bar y_i}{\bar\rho_i}\rho\right\};$$
see Figure 6.

If instead $(\bar\rho_i, \bar y_i) = (0,0)$ then $\Omega_i = \{(0,0)\}$.

Lemma 3.4. *We have the following possibilities.*

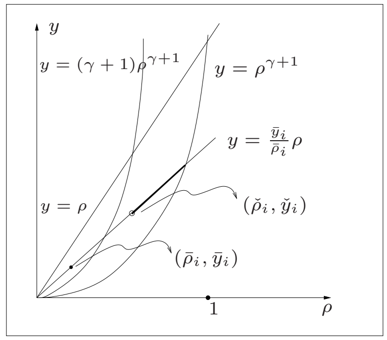


FIGURE 5. Graph of the set Ω_i for the Aw-Rascle-Zhang model: first case.

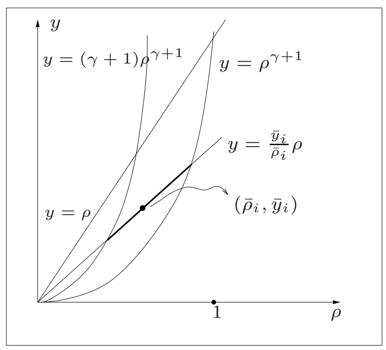


FIGURE 6. Graph of the set Ω_i for the Aw-Rascle-Zhang model: second case.

1. $\bar{y}_j < \bar{\rho}_j^{\gamma+1} - \frac{\gamma}{\gamma+1}\bar{\rho}_j$. *There exists a two-dimensional subset* $\mathcal{D}$ *of*

$$\left\{(\rho, y) \in \Omega : y \geq (\gamma+1)\rho^{\gamma+1}\right\}$$

such that

$$\Omega_j = D \cup \left\{(\rho, y) \in \Omega : y = \frac{\bar{y}_j}{\bar{\rho}_j}\rho + \rho^{\gamma+1} - \bar{\rho}_j^{\gamma}\rho\right\};$$

see Figure 7.

2. $\bar{y}_j \geq \bar{\rho}_j^{\gamma+1} - \frac{\gamma}{\gamma+1}\bar{\rho}_j$. *We have*

$$\Omega_j = \left\{(\rho, y) \in \Omega : y \geq (\gamma+1)\rho^{\gamma+1}\right\};$$

see Figure 8.

In [17], three different Riemann solvers, denoted by $\mathcal{RS}_1$, $\mathcal{RS}_2$ and $\mathcal{RS}_3$, were introduced. They differ only in the outgoing roads. More precisely, $\mathcal{RS}_1$ and $\mathcal{RS}_2$ select respectively the solutions, which maximize the speed and the density of cars in outgoing roads, while $\mathcal{RS}_3$ produces the solution minimizing the total variation.

The construction of these Riemann solvers can be summarized in the following way.

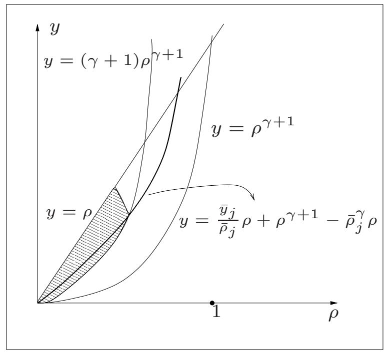


FIGURE 7. The set Ω_l for outgoing roads: first case.

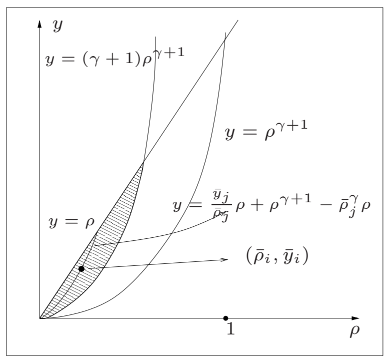


FIGURE 8. The set Ω_l for outgoing roads: second case.

1. Fix a matrix $A \in \mathfrak{N}$ and consider the closed, convex and not empty set Γ

$$\left\{ (f_1(u_1), \cdots, f_1(u_n)) \in \prod_{i=1}^{n} f_1(\Omega_i) : A \cdot \begin{pmatrix} f_1(u_1) \\ \vdots \\ f_1(u_n) \end{pmatrix} \in \prod_{j=n+1}^{n+m} f_1(\Omega_j) \right\}. \quad (14)$$

2. Find the point $(\bar{\gamma}_1, \dots, \bar{\gamma}_n) \in \Omega$ which maximizes the function

$$E(\gamma_1, \dots, \gamma_n) = \gamma_1 + \cdots + \gamma_n, \quad (15)$$

   and define $(\bar{\gamma}_{n+1}, \dots, \bar{\gamma}_{n+m})^T := A \cdot (\bar{\gamma}_1, \dots, \bar{\gamma}_n)^T$. Since $A \in \mathfrak{N}$, the point $(\bar{\gamma}_1, \dots, \bar{\gamma}_n)$ is uniquely defined.
3. For every $i \in \{1, \dots, n\}$, set $\hat{u}_i$ either by $\bar{u}_i$ if $f_1(\bar{u}_i) = \bar{\gamma}_i$, or by the solution to $f_1(u) = \bar{\gamma}_i$ such that $\hat{u}_i \in \Omega_i$.
4. **Riemann solver $\mathcal{RS}_1$.** For every $j \in \{n+1, \dots, n+m\}$, set $\hat{u}_j \in \Omega_j$ by the solution to $f_1(u) = \bar{\gamma}_j$ which maximizes the speed $v = \frac{f_2(u)}{y}$ of the cars at J. Finally, define $\mathcal{RS}_1 : \Omega^{n+m} \to \Omega^{n+m}$ by

$$\mathcal{RS}_1(\bar{u}_1, \dots, \bar{u}_{n+m}) = (\hat{u}_1, \dots, \hat{u}_n, \hat{u}_{n+1}, \dots, \hat{u}_{n+m}). \quad (16)$$

5. **Riemann solver $\mathcal{RS}_2$.** For every $j \in \{n+1, \dots, n+m\}$, set $\hat{u}_j \in \Omega_j$ by the solution to $f_1(u) = \bar{\gamma}_j$ which maximizes the density ρ of the cars at J.

Finally, define $\mathcal{RS}_2 : \Omega^{n+m} \to \Omega^{n+m}$ by

$$\mathcal{RS}_2(\bar{u}_1, \ldots, \bar{u}_{n+m}) = (\hat{u}_1, \ldots, \hat{u}_n, \hat{u}_{n+1}, \ldots, \hat{u}_{n+m}). \tag{17}$$

6. **Riemann solver $\mathcal{RS}_3$.** For every $j \in \{n+1, \ldots, n+m\}$, set $\hat{u}_j \in \Omega_j$ by the solution to $f_1(u) = \bar{\gamma}_j$ which minimizes the total variation of the solution. Finally, define $\mathcal{RS}_3 : \Omega^{n+m} \to \Omega^{n+m}$ by

$$\mathcal{RS}_3(\bar{u}_1, \ldots, \bar{u}_{n+m}) = (\hat{u}_1, \ldots, \hat{u}_n, \hat{u}_{n+1}, \ldots, \hat{u}_{n+m}). \tag{18}$$

Remark 6. Note that both Riemann solvers $\mathcal{RS}_1$, $\mathcal{RS}_2$ and $\mathcal{RS}_3$ conserve the number of cars passing through the junction, but not the generalized momentum y. In the literature there are other different choices for solving Riemann problems at a node in the case of the Aw-Rascle-Zhang model. In particular in [23, 24], the authors provide two Riemann solvers, which conserve both the number of cars passing through the junction and the generalized momentum y.

We can prove the following theorem; for a proof see [17].

Theorem 3.5. *Consider the Cauchy problem (3) and a Riemann solver $\mathcal{RS}$ among $\mathcal{RS}_1$, $\mathcal{RS}_2$ and $\mathcal{RS}_3$. Assume that the initial condition of (3) is a small perturbation in BV of a constant solution to (3). Then there exists a unique weak solution at J $(u_1(t,x), \ldots, u_{n+m}(t,x))$ such that*

1. *for every $l \in \{1, \ldots, n+m\}$, $u_l(0,x) = u_{0,l}(x)$ for a.e. $x \in I_l$;*
2. *for a.e. $t > 0$,*

$$\mathcal{RS}(u_1(t,0-), \ldots, u_{n+m}(t,0+)) = (u_1(t,0-), \ldots, u_{n+m}(t,0+)). \tag{19}$$

Moreover the solution depends in a Lipschitz continuous way on the initial condition.

3.2. **Telecommunication networks.** The model for telecommunication on networks was introduced by C. D'Apice, R. Manzo and B. Piccoli [16] in 2006. It is based on the conservation equation for the packets of data on each arc of the network. Each node sends packets to the following one a first time, then packets which are lost in this process are sent a second time and so on. The important point is that each packet is sent until it reaches next node, thus, looking at macroscopic level, it is assumed that packets are conserved. Since the packet transmission velocity on the line is assumed constant, we can derive an average transmission velocity among nodes considering the amount of packets that may be lost. More precisely, assigning a loss probability as function of the density, it is possible to compute a velocity function and thus a flux function.

In this setting we have $N = 1$, i.e. it is a scalar system, $f(u) = uv$, where v is the average speed of packets, u is the packet density, and $\Omega = [0, \rho_{max}]$, where ρ_{max} is the maximum capacity of a transmission line. As in Subsection 3.1.1, we assume that the flux f is a smooth function such that:

1. $f(0) = f(\rho_{max}) = 0$;
2. f is strictly concave;
3. there exists a unique $\sigma \in [0, \rho_{max}]$ such that $f'(\sigma) = 0$.

Therefore the macroscopic dynamic consists of a single conservation law:

$$\rho_t + f(\rho)_x = 0. \tag{20}$$

As in Subsection 3.1, the sets Ω_i and Ω_j, introduced in Definition 2.4, are explicitly given in Lemma 3.1. The Riemann solver at the node J, introduced in [16], was constructed according to the rule: *packets are sent to outgoing lines in order to* *maximize the flux through the node*. The Riemann solver at J depends on a vector $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{n+m})$ such that $\theta_l > 0$ for every $l \in \{1, \dots, n+m\}$ and

$$\sum_{i=1}^{n} \theta_i = \sum_{j=n+1}^{n+m} \theta_j = 1.$$

It can be summarized as follows.

1. Define

$$\Gamma_{inc} = \sum_{i=1}^{n} \sup f(\Omega_i), \quad \Gamma_{out} = \sum_{j=n+1}^{n+m} \sup f(\Omega_j),$$

   and

$$\Gamma = \min \{\Gamma_{inc}, \Gamma_{out}\} .$$

2. Introduce the closed, convex and not empty sets

$$\begin{aligned} I &= \left\{ (\gamma_1, \dots, \gamma_n) \in \prod_{i=1}^{n} f(\Omega_i) \colon \sum_{i=1}^{n} \gamma_i = \Gamma \right\} \\ J &= \left\{ (\gamma_{n+1}, \dots, \gamma_{n+m}) \in \prod_{j=n+1}^{n+m} f(\Omega_j) \colon \sum_{j=n+1}^{n+m} \gamma_j = \Gamma \right\} . \end{aligned}$$

3. Denote with $(\bar\gamma_1, \dots, \bar\gamma_n)$ the orthogonal projection on the convex set I of the point $(\Gamma\theta_1, \dots, \Gamma\theta_n)$ and with $(\bar\gamma_{n+1}, \dots, \bar\gamma_{n+m})$ the orthogonal projection on the convex set J of the point $(\Gamma\theta_{n+1}, \dots, \Gamma\theta_{n+m})$.
4. For every $i \in \{1, \dots, n\}$, set $\hat u_i$ either by $\bar u_i$ if $f(\bar u_i) = \bar\gamma_i$, or by the solution to $f(u) = \bar\gamma_i$ such that $\hat u_i \geq \sigma$. For every $j \in \{n+1, \dots, n+m\}$, set $\hat u_j$ either by $\bar u_j$ if $f(\bar u_j) = \bar\gamma_j$, or by the solution to $f(u) = \bar\gamma_j$ such that $\hat u_j \leq \sigma$. Finally, define $\mathcal{RS} : [0, \rho_{max}]^{n+m} \to [0, \rho_{max}]^{n+m}$ by

$$\mathcal{RS}(\bar u_1, \dots, \bar u_{n+m}) = (\hat u_1, \dots, \hat u_n, \hat u_{n+1}, \dots, \hat u_{n+m}) . \tag{21}$$

We can prove the following theorem; for a proof see [19].

Theorem 3.6. *Consider the Cauchy problem (3) and the Riemann solver $\mathcal{RS}$. Then there exists a unique solution at J $(u_1(t,x), \dots, u_{n+m}(t,x))$ such that*

1. *for every $l \in \{1, \dots, n+m\}$, $u_l(0,x) = u_{0,l}(x)$ for a.e. $x \in I_l$;*
2. *for a.e. $t > 0$,*

$$\mathcal{RS}(u_1(t,0-), \dots, u_{n+m}(t,0+)) = (u_1(t,0-), \dots, u_{n+m}(t,0+)) . \tag{22}$$

Moreover the solution depends in a Lipschitz continuous way with respect to the initial condition in the L^1 norm.

Remark 7. Theorem 3.6 gives existence and well posedness of solution to the Cauchy problem (3) for the Riemann solver at J introduced in this section. Recall instead that, for the case of traffic, only an existence result for solutions to the Cauchy problem holds. This fact is due essentially to the estimates for waves interacting with J. Indeed, in the case of a telecommunication network, if a wave interacts with J, then the total variation of the flux of the solution does not increase.

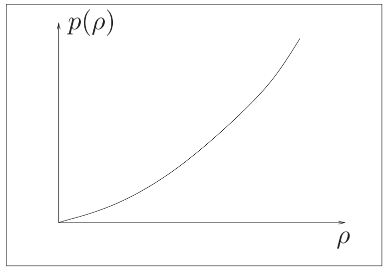


FIGURE 9. The pressure law $p(\rho) = k\rho^\gamma$ for the p-system

3.3. **Gas pipelines.** To describe the evolution of the gas in a pipe we use the p-system. We take $N = 2$, i.e. a system with two equations, $u = \begin{pmatrix} \rho \\ q \end{pmatrix}$, $\Omega = \{(\rho, q) \in \mathbb{R} \times \mathbb{R} : \rho > 0\}$ and $f(u) = \begin{pmatrix} q \\ \frac{q^2}{\rho} + p(\rho) \end{pmatrix}$, where the pressure p is a given increasing, smooth and convex function. A typical example is the γ-pressure law $p(\rho) = k\rho^\gamma$ with $k > 0$ and $\gamma \geq 1$; see Figure 9. Here ρ and q represents respectively the density and the linear momentum of the gas. Therefore the p-system can be written in the form

$$\begin{cases} \partial_t \rho + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{\rho} + p(\rho) \right) = 0, \end{cases} \qquad \begin{array}{rcl} t & \in & [0, +\infty[\,, \\ x & \in & I_l. \end{array} \tag{23}$$

For a later use, we introduce the *dynamic pressure*, i.e. the flow of the linear momentum

$$P(u) = \frac{q^2}{\rho} + p(\rho) \tag{24}$$

and the entropy flow

$$F(u) = q \left(\frac{q^2}{2\rho^2} + \int_1^\rho \frac{p'(r)}{r} dr \right). \tag{25}$$

Proposition 2. *In the set Ω, system (23) is strictly hyperbolic. The eigenvalues of the Jacobian matrix of the flux are*

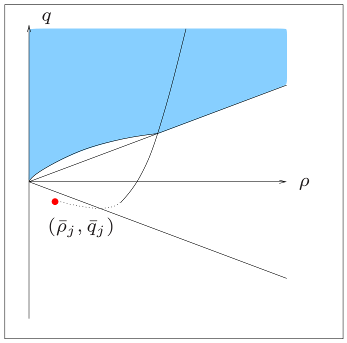
$$\lambda_1(u) = \frac{q}{\rho} - \sqrt{p'(\rho)}, \qquad \lambda_2(u) = \frac{q}{\rho} + \sqrt{p'(\rho)}. \tag{26}$$

The two characteristic speeds are both genuinely nonlinear. Finally, in the (ρ, q) plane, the Lax curves of the first family are concave, while the Lax curves of the second family are convex.

Differently from the previous cases, we do not distinguish between incoming and outgoing arcs. Therefore we assume that all the arcs are modeled by the real interval $\mathbb{R}^+$, i.e. $n = 0$.

Lemma 3.7. *Define the following sets*

$$\begin{aligned} \mathcal{R}_1 &= \begin{cases} \left\{ \left(\rho, L_2^-(\rho; \bar\rho_j, \bar q_j)\right) : \begin{array}{c} \rho > \bar\rho_j \\ L_2^-(\rho; \bar\rho_j, \bar q_j) > \bar q_j \end{array} \right\} & \text{if } \lambda_2(\bar\rho_j, \bar q_j) < 0 \\ \left\{ \left(\rho, L_2^-(\rho; \bar\rho_j, \bar q_j)\right) : \lambda_2 \left(\rho, L_2^-(\rho; \bar\rho_j, \bar q_j)\right) > 0 \right\} & \text{if } \lambda_2(\bar\rho_j, \bar q_j) \geq 0 \end{cases} \\ \mathcal{R}_2 &= \left\{ (\rho, q) \colon \lambda_1(\rho, q) \geq 0,\, q > q^+(\rho; \bar\rho_j, \bar q_j) \right\} \end{aligned}$$



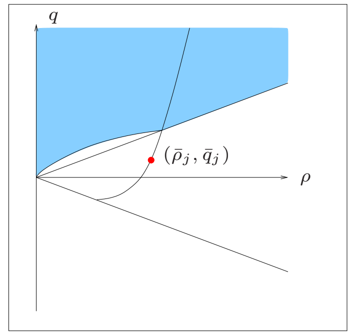


FIGURE 10. The sets Ω_j in the cases $\lambda_2(\bar\rho_j,\bar q_j) < 0$ (left) and $\lambda_2(\bar\rho_j,\bar q_j) \geq 0$ (right)

where $L_2^-(\rho;\bar\rho_j,\bar q_j)$ denotes the reverse Lax curve of second family through the point $(\bar\rho_j,\bar q_j)$ and $q^+(\rho;\bar\rho_j,\bar q_j)$ is a suitable function such that $\lambda_1(\rho,q^+(\rho;\bar\rho_j,\bar q_j)) \geq 0$ for every $\rho > 0$. It holds that

$$\Omega_j = \mathcal{R}_1 \cup \mathcal{R}_2; \tag{27}$$

see Figure 10.

The Riemann solver $\mathcal{RS}$, introduced in [9], can be summarized in the following way.

1. Find $(\hat u_1,\dots,\hat u_{n+m})$ belonging to $\Omega_1\times\cdots\times\Omega_m$ such that the following conditions hold.
   (a) The mass is conserved at J, i.e.

$$\sum_{j=1}^{m}\hat q_j = 0. \tag{28}$$

   (b) The linear momentum is conserved at J, i.e.

$$P(\hat u_l) = P(\hat u_j) \tag{29}$$

   for every $l,j\in\{1,\dots,m\}$.
   (c) Entropy may not decrease, i.e.

$$\sum_{j=1}^{n}F(\hat u_j) \leq 0. \tag{30}$$

2. Define $\mathcal{RS}:\Omega^m\to\Omega^m$ by

$$\mathcal{RS}(\bar u_1,\dots,\bar u_n) = (\hat u_1,\dots,\hat u_m)\,. \tag{31}$$

Remark 8. Note that, in general, there exist infinitely many $(\hat u_1,\dots,\hat u_{n+m})$ satisfying the point 1. of the previous construction. If we want uniqueness, we need to restrict to the subsonic case, i.e. to the case where $\lambda_1 < 0$ and $\lambda_2 > 0$.

Remark 9. In [2], the authors proposed a different Riemann solver at the node J. The main difference is that at the place of the condition regarding the conservation of the linear momentum (29) there is the condition

$$p(\hat\rho_j) = p(\hat\rho_l) \tag{32}$$

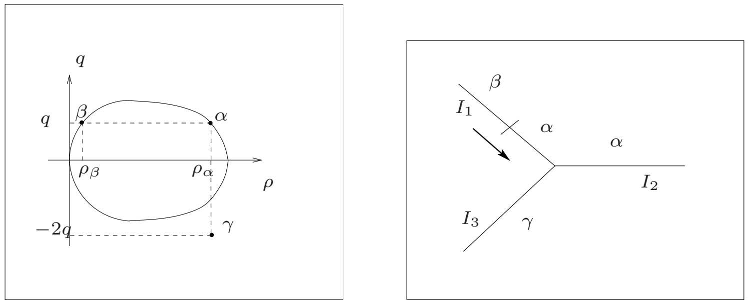


FIGURE 11. The states α, β and γ of Remark 9 (left) and the initial configuration (right)

for every $j, l \in \{1, \ldots, m\}$. With condition (32), the solution to the Riemann problem (2) and to the Cauchy problem (3), in general, does not depend in a continuous way with respect the initial condition.

Indeed, consider a node with $m = 3$ arcs. Define the states $\alpha \equiv (\rho_\alpha, q)$, $\beta \equiv (\rho_\beta, q)$ and $\gamma \equiv (\rho_\gamma, -2q)$ such that

$$q > 0\,, \quad \rho_\beta < \rho_\alpha = \rho_\gamma\,, \quad P(\alpha) = P(\beta)\,, \quad p(\alpha) = p(\gamma) \tag{33}$$

see Figure 11.

Introduce the initial datum ($\varepsilon > 0$)

$$(\rho_{1,0}, q_{1,0})(x) = \left\{ \begin{array}{l} \beta \text{ if } x \in [\varepsilon, +\infty[\\ \alpha \text{ if } x \in [0, \varepsilon[\end{array} \right. \qquad \begin{array}{rcl} (\rho_{2,0}, q_{2,0})(x) & = & \alpha \\ (\rho_{3,0}, q_{3,0})(x) & = & \gamma\,. \end{array} \tag{34}$$

Note that the triple (α, α, γ) is a stationary solution for the Riemann problem introduced in [2], while the triple (β, α, γ) is not. Therefore, when ε tends to 0, the initial condition tends to (β, α, γ) in L^1, but in the solution some waves appear.

Remark 10. The geometry of the node does not influence the solution of the Riemann problem. In [10] a geometric Riemann solver was introduced. In that paper the position and the size of the tubes play a role in the construction of the solution. More precisely the rule (29) was substituted by another one, which prescribes the conservation of the linear momentum only along some directions, determined by the position and size of the tubes.

Remark 11. In [13], the authors proposed another solution at a node between two pipes with different sections. They approximate the node by a sequence of tubes with sections $a_n(x)$, which vary in a piecewise regular way and such that $\lim_n a_n(0)$ is a Dirac measure, and then they consider the corresponding solutions in these regular cases; see also [27]. Finally the condition at the node is obtained by passing to the limit as $n \to +\infty$. A similar procedure is used also for the complete 3×3 Euler system in [14].

We have the following result.

Proposition 3. *Consider the Riemann problem (2). Let* $(\tilde{u}_1, \ldots, \tilde{u}_m)$ *satisfy*

$$\lambda_1(\tilde{\rho}_l, \tilde{q}_l) < 0 < \lambda_2(\tilde{\rho}_l, \tilde{q}_l) \tag{35}$$

for every $l \in \{1, \ldots, m\}$ *and the conditions (28), (29) and (30). There exists* $\delta > 0$ *such that, if*

$$\|(\tilde{u}_1, \ldots, \tilde{u}_{n+m}) - (\bar{u}_1, \ldots, \bar{u}_{n+m})\| < \delta, \tag{36}$$

then locally there exists a unique $(\hat{u}_1, \ldots, \hat{u}_{n+m})$ *in* $\Omega_1 \times \cdots \times \Omega_{n+m}$ *satisfying the point 1. of the construction of the Riemann solver.*

For a proof see [9]. The main result in this setting is the following theorem about the Cauchy problem (3).

Theorem 3.8. *Let* $(\tilde{u}_1, \ldots, \tilde{u}_m)$ *satisfy*

$$\lambda_1(\tilde{\rho}_l, \tilde{q}_l) < 0 < \lambda_2(\tilde{\rho}_l, \tilde{q}_l) \tag{37}$$

for every $l \in \{1, \ldots, m\}$ *and the conditions (28), (29) and (30).*

Then, there exist positive constants δ, L *and a map* $S\colon [0, +\infty[\, \times \mathcal{D} \to \mathcal{D}$, *with the following properties:*

1. $\mathcal{D} \subseteq \Omega^m$.
2. *For* $(u_1, \ldots, u_m) \in \mathcal{D}$, $S_0(u_1, \ldots, u_m) = (u_1, \ldots, u_m)$ *and for* $s, t \geq 0$,
$$S_s S_t(u_1, \ldots, u_m) = S_{s+t}(u_1, \ldots, u_{n+m}).$$
3. *For* $(u_1, \ldots, u_m), (u'_1, \ldots, u'_m) \in \mathcal{D}$ *and* $s, t \geq 0$,
$$\begin{aligned} &\|S_t(u_1, \ldots, u_m) - S_s(u'_1, \ldots, u'_m)\|_{L^1} \\ &\leq L \cdot (\|(u_1, \ldots, u_m) - (u'_1, \ldots, u'_m)\|_{L^1} + |t - s|). \end{aligned}$$
4. *If* $(u_1, \ldots, u_m) \in \mathcal{D}$ *is piecewise constant, then for* $t > 0$ *sufficiently small,* $S_t(u_1, \ldots, u_m)$ *coincides with the juxtaposition of the solutions to Riemann problems centered at the points of jumps or at the junction.*
5. *For a.e.* $t > 0$,
$$\mathcal{RS}(S_t(u_1, \ldots, u_m)) = S_t(u_1, \ldots, u_m). \tag{38}$$

The proof of the Theorem is contained in [11].

REFERENCES

[1] A. Aw and M. Rascle, *Resurrection of "second order" models of traffic flow*, SIAM J. Appl. Math., **60** (2000), 916–938 (electronic).

[2] M. K. Banda, M. Herty and A. Klar, *Gas flow in pipeline networks*, Netw. Heterog. Media, **1** (2006), 41–56 (electronic).

[3] S. Blandin, D. Work, P. Goatin, B. Piccoli and A. Bayen, *A general phase transition model for vehicular traffic*, Preprint, 2009.

[4] A. Bressan, "Hyperbolic Systems of Conservation Laws," volume 20 of Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[5] G. Bretti, C. D'Apice, R. Manzo and B. Piccoli, *A continuum-discrete model for supply chains dynamics*, Netw. Heterog. Media, **2** (2007), 661–694 (electronic).

[6] S. Čanić and E. H. Kim, *Mathematical analysis of the quasilinear effects in a hyperbolic model blood flow through compliant axi-symmetric vessels*, Math. Methods Appl. Sci., **26** (2003), 1161–1186.

[7] G. M. Coclite, M. Garavello and B. Piccoli, *Traffic flow on a road network*, SIAM J. Math. Anal., **36** (2005), 1862–1886 (electronic).

[8] R. M. Colombo, *Hyperbolic phase transitions in traffic flow*, SIAM J. Appl. Math., **63** (2002), 708–721 (electronic).

[9] R. M. Colombo and M. Garavello, *A well posed Riemann problem for the p-system at a junction*, Netw. Heterog. Media, **1** (2006), 495–511 (electronic).

[10] R. M. Colombo and M. Garavello, *On the p-system at a junction*, In "Control methods in PDE-dynamical systems, volume **426** of Contemp. Math.," pages 193–217. Amer. Math. Soc., Providence, RI, 2007.

[11] R. M. Colombo and M. Garavello, *On the Cauchy problem for the p-system at a junction*, SIAM J. Math. Anal., **39** (2008), 1456–1471.

[12] R. M. Colombo, P. Goatin and B. Piccoli, *Road networks with phase transitions*, J. Hyperbolic Differ. Equ., **7** (2010), 85–106.

[13] R. M. Colombo and F. Marcellini, *Smooth and discontinuous junctions in the p-system*, J. Math. Anal. Appl., **361** (2010), 440–456.

[14] R. M. Colombo and F. Marcellini, *Coupling conditions for the* 3×3 *euler system*, Netw. Heterog. Media, to appear.

[15] R. M. Colombo, F. Marcellini and M. Rascle, *A 2-phase traffic model based on a speed bound*, SIAM J. Math. Anal., to appear.

[16] C. D'apice, R. Manzo and B. Piccoli, *Packet flow on telecommunication networks*, SIAM J. Math. Anal., **38** (2006), 717–740 (electronic).

[17] M. Garavello and B. Piccoli, *Traffic flow on a road network using the Aw-Rascle model*, Comm. Partial Differential Equations, **31** (2006), 243–275.

[18] M. Garavello and B. Piccoli, "Traffic Flow on Networks, volume **1** of AIMS Series on Applied Mathematics," American Institute of Mathematical Sciences (AIMS), Springfield, MO, 2006.

[19] M. Garavello and B. Piccoli, *Conservation laws on complex networks*, Ann. H. Poincaré, doi:10.1016/j.anihpc.2009.04.001, **26** (2009), 1925–1951.

[20] P. Goatin, *The Aw-Rascle vehicular traffic flow model with phase transitions*, Math. Comput. Modelling, **44** (2006), 287–303.

[21] S. Göttlich, M. Herty and A. Klar, *Modelling and optimization of supply chains on complex networks*, Commun. Math. Sci., **4** (2006), 315–330.

[22] D. Helbing, *Improved fluid-dynamic model for vehicular traffic*, Phys. Rev. E, **51** (1995), 3164–3169.

[23] M. Herty, S. Moutari and M. Rascle, *Optimization criteria for modelling intersections of vehicular traffic flow*, Netw. Heterog. Media, **1** (2006), 275–294 (electronic).

[24] M. Herty and M. Rascle, *Coupling conditions for a class of second-order models for traffic flow*, SIAM J. Math. Anal., **38** (2006), 595–616 (electronic).

[25] G. Leugering and E. J. P. G. Schmidt, *On the modelling and stabilization of flows in networks of open canals*, SIAM J. Control Optim., **41** (2002), 164–180 (electronic).

[26] M. J. Lighthill and G. B. Whitham, *On kinematic waves. II. A theory of traffic flow on long crowded roads*, Proc. Roy. Soc. London. Ser. A., **229** (1955), 317–345.

[27] T. P. Liu, *Nonlinear stability and instability of transonic flows through a nozzle*, Comm. Math. Phys., **83** (1982), 243–260.

[28] P. I. Richards, *Shock waves on the highway*, Operations Res., **4** (1956), 42–51.

[29] H. M. Zhang, *A non-equilibrium traffic model devoid of gas-like behavior*, Transportation Research Part B, **236** (2002), 275–290.

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