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HOMOGENIZATION OF THE NEUMANN PROBLEM FOR A QUASILINEAR ELLIPTIC EQUATION IN A PERFORATED DOMAIN

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Abstract. We investigate the Neumann problem for a nonlinear elliptic operator $Au^{(s)} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_i\left(x, \frac{\partial u^{(s)}}{\partial x}\right)\right)$ of Leray-Lions type in the domain $\Omega^{(s)} = \Omega \backslash F^{(s)}$, where Ω is a domain in $\mathbf{R}^n (n \geq 3)$, $F^{(s)}$ is a closed set located in the neighbourhood of a $(n-1)$-dimensional manifold Γ lying inside Ω. We study the asymptotic behaviour of $u^{(s)}$ as $s \to \infty$, when the set $F^{(s)}$ tends to Γ. Under appropriate conditions, we prove that $u^{(s)}$ converges in suitable topologies to a solution of a limit boundary value problem of transmission type, where the transmission conditions contain an additional term.

1. Introduction.

Let Ω be a bounded domain in $\mathbf{R}^n (n \geq 3)$ with a sufficiently smooth boundary $\partial\Omega$. Let $F^{(s)}$ be a closed set in Ω depending on the parameter s running throughout the set of natural numbers. The main assumption on the set $F^{(s)}$ is that as $s \to \infty$, $F^{(s)}$ is located in an arbitrary small neighbourhood of some $(n-1)$-dimensional smooth manifold Γ without boundary which lies inside Ω and partition Ω into two subdomains Ω^+(the interior) and Ω^-(the exterior). In the domain $\Omega^{(s)} = \Omega \backslash F^{(s)}$ we investigate the sequence of solutions $u^{(s)}$ of the boundary value problem

$$Au^{(s)} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_i\left(x, \frac{\partial u^{(s)}}{\partial x}\right)\right) = f, \text{ in } \Omega^{(s)}, \tag{1}$$

$$\frac{\partial u^{(s)}}{\partial \nu_A} =: \sum_{i=1}^{n} a_i\left(x, \frac{\partial u^{(s)}}{\partial x}\right)\cos(\nu, x_i) = 0, \text{ on } \partial F^{(s)}, \tag{2}$$

$$u^{(s)} = 0 \text{ on } \partial\Omega, \tag{3}$$

where ν is the normal to $\partial F^{(s)}$, f is a function defined and compactly supported inside Ω (the support of f does not intersect Γ), $A: \; W_p^1(\mathbf{R}^n) \to W_{p'}^1(\mathbf{R}^n)$ is

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a monotone operator satisfying appropriate conditions to be stated in the next section.

The aim of the present paper is to investigate the behavior of the sequence $u^{(s)}$ of solutions of the problem (1)-(3). Under more precise restrictions on the set $F^{(s)}$, we show that $u^{(s)}$ converges in suitable topologies to a solution of a limit problem that we derive explicitly. This is the philosophy of what is now widely known as Homogenization theory.

The problem (1)-(3) was originally studied by Marchenko and Khruslov in the monograph [32] (see also the long awaited english version [33]).

They considered the linear version of problem (1)-(3) when $a_i(x,\xi) = a_{ij}(x)\xi_j$, $\xi = (\xi_1, ..., \xi_n)$ and $a_{ij}(x) = a_{ji}(x)$, $x \in \Omega$. They proved under appropriate conditions that as $s \to \infty$, the sequence $u^{(s)}$ of solutions of the problem converges in suitable topologies to the solution of a transmission problem

$$\begin{aligned} -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) &= f, \text{ in } \Omega\backslash\Gamma, \\ \left(\frac{\partial u}{\partial \nu_A}\right)_+ - \left(\frac{\partial u}{\partial \nu_A}\right)_- &= 2c(x)(u_+ - u_-) \text{ on } \Gamma, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

$c(x)$ being a function related to the geometry of the sets $F^{(s)}$; $\left(\frac{\partial u}{\partial \nu_A}\right)_\pm$ are the limit values of

$$\frac{\partial u}{\partial \nu_A} = \sum_{i=1}^{n} a_{ij}(x)\cos(\nu, x_i)\frac{\partial u}{\partial x_j}$$

on each side of Γ and ν is the normal to Γ directed toward the interior of Ω^+. These are the equations (3.6)-(3.8) on page 171 in [32].

Later contributions on closely related problems covering also Ginzburg-Landau equations were made Khruslov, Berlyand and their co-workers [27], [28], [29], [8], [9], [10], [11]. It should be noted that the work of Marchenko and Suzikov [34] seems to be the first dealing with Neumann problems for elliptic equations domains with fine grained boundaries. The main feature of the above cited works is the absence of periodicity condition on the geometry of the domains $\Omega^{(s)}$.

In the present work we shall be concerned with the most difficult case of nonlinear elliptic equations of Leray-Lions type. The difficulties reside not only in the more complicated nature of the equations but also in the tools that are fundamentally nonlinear, notably nonlinear potential analysis. Our main result is Theorem 1 which coincide with the classical result of Marchenko and Khruslov [32] when $p = 2$. Some results of the paper were announced in [45].

There is a great wealth of results concerning the homogenization of Neumann and mixed boundary value problems in domains with the geometry closely related to the one we consider here. They deal in many instances with the linear case again and in periodically perforated domains. A wide range of results have been obtained in [4], [5], [6], [15], [16], [48], [49], [50], [51], [52], [40], [41], just to cite a few. Some extension of these results to higher-order equations may be found for instance in [53]. We also note the work of Del Vecchio [24] which treats the so-called Thick Neumann sieve problem in the linear setting.

The case of perforated surfaces which is a particular case of the geometry considered here was investigated [2], [3], [23], [25], [26], [37], [38], [39].

Let us note that Dirichlet problems in perturbed domains have been extensively studied in comparison with Neumann problems. A great deal of information on the state of affairs in Homogenization theory since its inception in the mid sixties can be found collected in the following key works [7], [18], [32], [34], [55], [56] and the references therein. Another approach dealing with gamma-convergence was developed by the celebrated Italian school of Calculus of Variations.

The plan of the paper is as follows. In section 2, we introduce some notations, formulate the conditions on problem (1)-(3) and the main result. In section 3, we prove some keys auxiliary results and in section 4, we prove our main result. Finally we construct an explicit example involving the p-Laplacian which illustrates the abstract conditions that are requested from the set $\Omega^{(s)}$. It should be noted that such examples are in short supply even in the linear case.

2. **Preliminaries.** We shall use the following well-known Lebesgue and Sobolev spaces $L_p(\cdot)$, $W_p^1(\cdot)$, $\overset{o}{W}{}_p^1(\cdot)$, $(p \geq 1)$. We denote by $W_{p'}^{-1}(\cdot)$ the dual of $\overset{o}{W}{}_p^1(\cdot)$ where p' is the Hölder conjugate of p, i.e, $p^{-1} + p'^{-1} = 1$. If ξ is a vector we denote its Euclidean norm by $|\xi|$. We denote by C all generic constants independent of s and depending only on the data.

We assume for simplicity that $p \geq 2$ and that the equation (1) is the Euler-Lagrange equation for the functional

$$I(v) = \int_{\Omega^{(s)}} \left[A_i\left(x, \frac{\partial v}{\partial x}\right) \frac{\partial v}{\partial x_i} - fv \right] dx,$$

where the functions $A_i(x,\xi)$, $\xi = (\xi_1, ..., \xi_n)$, are Caratheodory and satisfy

A. for all $x \in \Omega$, $t \in \mathbf{R}$ and ξ

$$A_i(x, t\xi) = |t|^{p-2} t A_i(x,\xi), \tag{4}$$

B. there exist two positive constants c_1 and c_2 such that for all ξ, $\eta \in \mathbf{R}^n$ with $\eta = (\eta_1, ..., \eta_n)$,

$$\sum_{i=1}^{n} (A_i(x,\xi) - A_i(x,\eta))(\xi_i - \eta_i) \geq c_1 |\xi - \eta|^p, \tag{5}$$

$$|A_i(x,\xi) - A_i(x,\eta)| \leq c_2 \left(|\xi|^{p-2} + |\eta|^{p-2}\right) |\xi - \eta|. \tag{6}$$

Therefore

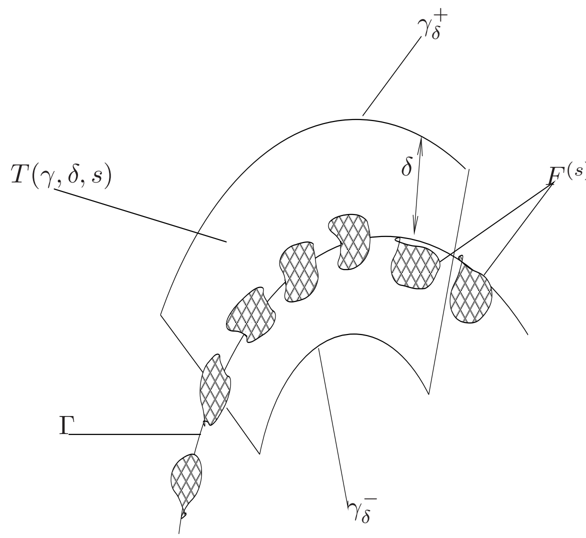
$$a_i(x,\xi) = \sum_{k=1}^{n} \frac{\partial A_k(x,\xi)}{\partial \xi_i} \xi_k + A_i(x,\xi). \tag{7}$$

Hence any minimizer of the functional I in $W_p^1\left(\Omega^{(s)}\right) \cap \overset{o}{W}{}_p^1(\Omega)$ which satisfies the boundary condition (2)-(3) is a weak solution of (1)-(3), the existence of which under the above conditions is classical and can be found in [30] (Chap. 10), [31]. We note that $A_i(x,\xi) = |\xi|^{p-2}\xi_i$ corresponds to the case of the p-Laplacian.

We introduce some notations. Let γ be an arbitrary open set on Γ and let $T(\gamma,\delta)$ be a layer of thickness 2δ centered around γ. We denote by $\gamma_\delta^\pm$ the bases of the layer $T(\gamma,\delta)$, i.e., the surfaces located at the different sides of γ at distance δ. We set $T(\gamma,\delta,s) = T(\gamma,\delta) \backslash F^{(s)}$.

We denote by $W(\gamma,\delta,s)$ the class of functions from $W_p^1(T(\gamma,\delta,s))$ taking values one and zero on γ_δ^+ and γ_δ^- respectively, i.e.,

$$W(\gamma,\delta,s) = \left\{ v \in W_p^1(T(\gamma,\delta,s)) : \ v(x) = 1 \text{ on } \gamma_\delta^+, \, v(x) = 0 \text{ on } \gamma_\delta^- \right\}.$$



The main characteristic of influence of the sets $F^{(s)}$ is expressed in term of the following functions of sets

$$C_A(\gamma,\delta,s) = \inf_{\varphi^{(s)}} \int_{T(\gamma,\delta,s)} \sum_{i=1}^{n} A_i\left(x, \frac{\partial \varphi^{(s)}}{\partial x}\right) \frac{\partial \varphi^{(s)}}{\partial x_i} dx, \tag{8}$$

where infimum is taken over the functions $\varphi^{(s)} \in W(\gamma,\delta,s)$. These quantities are referred to as A-conductivity of the set $T(\gamma,\delta,s)$, following Mazya [36] who seems to have been the first to introduce the concept in [35]. The following properties of A-conductivity hold:

1. (a) $C_A(\gamma_1,\delta,s) \le C_A(\gamma_2,\delta,s)$ if $\gamma_1 \subseteq \gamma_2$.
   (b) $C_A(\gamma,\delta_1,s) \le C_A(\gamma,\delta_2,s)$ if $\delta_1 \ge \delta_2$.
   (c) $C_A(\gamma_1 \cup \gamma_2,\delta,s) \ge C_A(\gamma_1,\delta,s) + C_A(\gamma_2,\delta,s)$ if $\gamma_1 \cap \gamma_2$.
   (d) $C_A(\gamma,\delta,s_1) \le C_A(\gamma,\delta,s_2)$ if $F^{(s_2)} \cap T(\gamma,\delta) \subseteq F^{(s_1)} \cap T(\gamma,\delta)$.

We refer to [36] (Section 4.1) for proofs. Closely related concepts in the framework of Dirichlet problems of monotone type in varying domains were introduced recently in [22].

Consider the class of functions $W_p^1(\Omega^+ \cup \Omega^-) = W_p^1(\Omega^+) \times W_p^1(\Omega^-)$ with the norm

$$\|u\|_{W_p^1(\Omega^+\cup\Omega^-)} =: \|u\|_{W_p^1(\Omega^+)} + \|u\|_{W_p^1(\Omega^-)}$$

and the class of functions $W_p^1\left(\Omega^{(s)+} \cup \Omega^{(s)-}\right) = W_p^1\left(\Omega^{(s)+}\right) \times W_p^1\left(\Omega^{(s)-}\right)$ with norm

$$\|u\|_{W_p^1\left(\Omega^{(s)+}\cup\Omega^{(s)-}\right)} =: \|u\|_{W_p^1\left(\Omega^{(s)+}\right)} + \|u\|_{W_p^1\left(\Omega^{(s)-}\right)}$$

where $\Omega^{(s)\pm} =: \Omega^{(s)} \cap \Omega^{\pm}$.

Now we are in the position to formulate our main result.

Theorem 1. *Assume that the above assumptions on problem (1)-(3) are satisfied and $f \in W_{p'}^{-1}(\Omega)$. As $s \to \infty$, we require that*

a) the set $F^{(s)}$ lies in an arbitrary small neighbourhood of the manifold $\Gamma \in \Omega$, *b) for any portion $\gamma \in \Gamma$, there exist the limits*

$$\lim_{\delta\to\infty}\varliminf_{s\to\infty} C_A(\gamma,\delta,s) = \lim_{\delta\to\infty}\varlimsup_{s\to\infty} C_A(\gamma,\delta,s) = \int_\gamma c(x)\,d\Gamma, \tag{9}$$

where c is a non negative, measurable function on Γ.

Then the sequence of solutions $u^{(s)}$ of problem (1)-(3) converges weakly in $W_p^1(\Omega^- \cup \Omega^+)$ and strongly in $W_q^1(\Omega_+ \cup \Omega_-)$, $1 < q < p$ to a function u which is a solution of the transmission problem

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(a_i\left(x, \frac{\partial u}{\partial x}\right)\right) = f, \quad in\ \Omega, \tag{10}$$

$$\left(\frac{\partial u}{\partial \nu_A}\right)_+ - \left(\frac{\partial u}{\partial \nu_A}\right)_- = pc(x)\,|u_+ - u_-|^{p-2}(u_+ - u_-) \quad on\ \Gamma, \tag{11}$$

$$u = 0 \ on\ \partial\Omega, \tag{12}$$

where the sign "+" and "-" indicate the boundary values of the function on the different sides of Γ, $\dfrac{\partial u}{\partial \nu_A}$ is the derivative along the normal to Γ in the direction corresponding to "+".

Remark 2. The convergence of $F^{(s)}$ to Γ is in terms of Hausdorff distance. We recall that given a metric space $(\mathcal{X}, \rho)$ with the distance function $\rho(\cdot,\cdot)$, the Hausdorff distance between two non empty subsets X, Y of $\mathcal{X}$ is the quantity

$$\rho_H(X,Y) = \max\left\{\sup_{x\in X}\inf_{y\in Y}\rho(x,y), \sup_{y\in Y}\inf_{x\in X}\rho(x,y)\right\}.$$

In our case ρ is of course the Euclidean distance in $\mathbf{R}^n$.

Remark 3. We note that when $p = 2$, our results coincide with those of Marchenko and Khruslov [33], [32]. Unlike the Dirichlet case where the additional term in the limit problem appears in the equation (see [17], [33], [32], [21], [44], [46], [55]), the limit problem here contains an additional term in the transmission conditions. We note that in the limit of some Dirichlet problems in domains in fine grained boundaries where the grains have a surface distribution, the additional term might also appear in the transmission conditions; we refer to [34] (linear case), [54], [55] (nonlinear case), [1] (Navier Stokes) where such questions were considered (see also [42], [43], for the case of systems of quasilinear elliptic equations) Precise expressions of the function $c(x)$ can be found in many of the papers mentioned here; in particular when the perforations are spherical.

We collect in the next section some results needed for the proof of the theorem.

3. **Auxiliary results.** We start with

Lemma 4. *Let $u \in W_p^1(T(\gamma,\delta,s))$, $U^\pm = vrai \max_{x\in\gamma_\delta^\pm} u(x)$, $u^\pm = vrai \min_{x\in\gamma_\delta^\pm} u(x)$. Then*

$$\int_{T(\gamma,\delta,s)} \sum_{i=1}^n A_i\left(x, \frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x_i}dx \geq a^p C_A(\gamma,\delta,s), \tag{13}$$

where

$$a = \begin{cases} u^+ - U^-, & if\ u^+ > U^-, \\ u^- - U^+, & if\ u^- > U^+, \end{cases}$$

and $a = 0$ otherwise.

Proof. Assume that $U^+ > U^-$ and consider the truncated function

$$\bar{u}(x) = \begin{cases} u^+ \text{ if } u(x) > u^+, \\ u(x) \text{ if } U^- \leq u(x) \leq u^+, \\ U^- \text{ if } u(x) < U^-. \end{cases}$$

We show that $\bar{u} \in W_p^1(T(\gamma, \delta, s))$. Let G_j be an increasing sequence of domains in $T(\gamma, \delta, s)$ which exhausts $T(\gamma, \delta, s)$, i.e., $T(\gamma, \delta, s) = \cup_j G_j$. Then there exists a sequence of piecewise linear functions $\{u_j^l\}_{l=1,2,...}$ such that

$$\lim_{l\to\infty} \left|\left|u_j^l - u\right|\right|_{W_p^1(G_j)} = 0. \tag{14}$$

We define a new truncated function

$$\bar{u}_j^l(x) = \begin{cases} u_j^{l+} \text{ if } u_j^l(x) \geq u_j^{l+}, \\ u_j^l \text{ if } U_j^{l-} < u_j^l(x) < u_j^{l+}, \\ U_j^{l-} \text{ if } u_j^l(x) \leq U_j^{l-}. \end{cases}$$

Since u_j^l are piecewise linear in G_j it follows that $\bar{u}_j^l \in W_p^1(G_j)$. Straightforward computations show that

$$\int_{G_j} \sum_{i=1}^n A_i\left(x, \frac{\partial \bar{u}_j^l}{\partial x}\right) \frac{\partial \bar{u}_j^l}{\partial x_i} dx \leq \int_{G_j} \sum_{i=1}^n A_i\left(x, \frac{\partial u_j^l}{\partial x}\right) \frac{\partial u_j^l}{\partial x_i} dx, \tag{15}$$

since $A_i(x,\xi)\xi_i \geq 0$, for all $\xi = (\xi_1, .., \xi_n)$. By Lemma 3.2 from [[30], Chap. 2], we have

$$\lim_{l\to\infty} \left|\left|\bar{u}_j^l - \bar{u}\right|\right|_{W_p^1(G_j)} = 0. \tag{16}$$

Next using inequality (6), we get

$$\begin{aligned} & \int_{G_j} \left| A_i\left(x, \frac{\partial \bar{u}_j^l}{\partial x}\right) \frac{\partial \bar{u}_j^l}{\partial x_i} - A_i\left(x, \frac{\partial \bar{u}}{\partial x}\right) \frac{\partial \bar{u}}{\partial x_i} \right| dx \\ \leq & \int_{G_j} \left| A_i\left(x, \frac{\partial \bar{u}_j^l}{\partial x}\right) \frac{\partial \bar{u}_j^l}{\partial x_i} - A_i\left(x, \frac{\partial \bar{u}}{\partial x}\right) \frac{\partial \bar{u}_j^l}{\partial x_i} \right| dx \\ & + \int_{G_j} \left| A_i\left(x, \frac{\partial \bar{u}}{\partial x}\right) \frac{\partial \bar{u}_j^l}{\partial x_i} - A_i\left(x, \frac{\partial \bar{u}}{\partial x}\right) \frac{\partial \bar{u}}{\partial x_i} \right| dx \\ \leq & \int_{G_j} \left(\left|\frac{\partial \bar{u}_j^l}{\partial x}\right|^{p-2} + \left|\frac{\partial \bar{u}}{\partial x}\right|^{p-2} \right) \left|\frac{\partial\left(\bar{u}_j^l - \bar{u}\right)}{\partial x_i}\right| \left|\frac{\partial \bar{u}_j^l}{\partial x_i}\right| dx \\ & + \int_{G_j} \left|\frac{\partial \bar{u}}{\partial x}\right|^{p-1} \left|\frac{\partial\left(\bar{u}_j^l - \bar{u}\right)}{\partial x_i}\right| dx. \end{aligned}$$

Applying Hölder's inequality to both integrals in the right-hand side of this inequality and appealing to (16) we get

$$\lim_{l\to\infty} \int_{G_j} A_i\left(x, \frac{\partial \bar{u}_j^l}{\partial x}\right) \frac{\partial \bar{u}_j^l}{\partial x_i} dx = \int_{G_j} \sum_{i=1}^n A_i\left(x, \frac{\partial \bar{u}}{\partial x}\right) \frac{\partial \bar{u}}{\partial x_i} dx.$$

Analogously using (14), we have

$$\lim_{l\to\infty} \int_{G_j} \sum_{i=1}^n A_i\left(x, \frac{\partial u_j^l}{\partial x}\right) \frac{\partial u_j^l}{\partial x_i} dx = \int_{G_j} \sum_{i=1}^n A_i\left(x, \frac{\partial u}{\partial x}\right) \frac{\partial u}{\partial x_i} dx.$$

Hence passing to the limit in both sides of (15) and using (5), we get

$$\int_{G_j}\left|\frac{\partial \bar{u}}{\partial x}\right|^p dx \leq \int_{G_j}\left|\frac{\partial u}{\partial x}\right|^p dx.$$

Summing over j, we get

$$\int_{T(\gamma,\delta,s)}\left|\frac{\partial \bar{u}}{\partial x}\right|^p dx < \infty.$$

Thus $\bar{u} \in W_p^1\left(T\left(\gamma,\delta,s\right)\right)$.

Next we consider the function

$$v\left(x\right) = \frac{\bar{u} - U^-}{u^+ - U^-} = \begin{cases} 1, \text{ if } u\left(x\right) \geq u^+, \\ \dfrac{u - U^+}{u^+ - U^-}, \text{ if } U^- < u\left(x\right) < u^+, \\ 0, \text{ if } u\left(x\right) \leq U^-. \end{cases}$$

It is easy to see that $v \in W_p^1\left(T\left(\gamma,\delta,s\right)\right)$. Let $\bar{v}$ be defined by

$$\bar{v} = \begin{cases} v^+, \text{ if } v\left(x\right) \geq v^+, \\ v\left(x\right), \text{ if } V^- < v\left(x\right) < v^+, \\ V^-, \text{ if } v\left(x\right) \leq V^-, \end{cases}$$

where $V^{\pm} = vrai \max\limits_{\gamma_\delta^{\pm}} v\left(x\right)$, $v^{\pm} = vrai \min\limits_{\gamma_\delta^{\pm}} v\left(x\right)$. Due to the homogeneity of A_i, we have

$$\int_{T(\gamma,\delta,s)}\sum_{i=1}^n A_i\left(x,\frac{\partial \bar{v}}{\partial x}\right)\frac{\partial \bar{v}}{\partial x_i}dx = \frac{1}{\left(u^+ - U^-\right)^p}\int_{T(\gamma,\delta,s)}\sum_{i=1}^n A_i\left(x,\frac{\partial \bar{u}}{\partial x}\right)\frac{\partial \bar{u}}{\partial x_i}dx.$$

From this relation and the definition of $C_A\left(\gamma,\delta,s\right)$, we get

$$\int_{T(\gamma,\delta,s)}\sum_{i=1}^n A_i\left(x,\frac{\partial \bar{u}}{\partial x}\right)\frac{\partial \bar{u}}{\partial x_i}dx \geq \left(u^+ - U^-\right)^p C_A\left(\gamma,\delta,s\right),$$

since as an easy verification shows $\bar{v} \in W_p^1\left(T\left(\gamma,\delta,s\right)\right)$, $\bar{v}\left(x\right) = 1$ on γ_δ^+ and $\bar{v}\left(x\right) = 0$ on γ_δ^-. We have

$$\int_{T(\gamma,\delta,s)}\sum_{i=1}^n A_i\left(x,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x_i}dx \geq \int_{T(\gamma,\delta,s)}\sum_{i=1}^n A_i\left(x,\frac{\partial \bar{u}}{\partial x}\right)\frac{\partial \bar{u}}{\partial x_i}dx.$$

Thus from the previous inequality we deduce that (13) follows for $U^+ > U^-$. Similar arguments lead to the same conclusion for $U^+ < U^-$. This completes the proof of the lemma. □

We assumed that Γ is a manifold without boundary which divides the domain Ω into two subdomains Ω^+ the interior and Ω^- the exterior with respect to Γ. Let $T\left(\Gamma,\delta\right)$ be a layer of thickness 2δ centered around the manifold Γ. Let $T\left(\Gamma,\delta,s\right) = T\left(\Gamma,\delta\right)\backslash F^{(s)}$. We consider the functional

$$\Phi_\delta^{(s)}\left(\psi^{(s)}\right) = \int_{T(\Gamma,\delta,s)}\sum_{i=1}^n A_i\left(x,\frac{\partial \psi^{(s)}}{\partial x}\right)\frac{\partial \psi^{(s)}}{\partial x_i}dx,$$

over the set $\tilde{W}$ of functions from $W_p^1\left(T\left(\Gamma,\delta,s\right)\right)$ taking on the surfaces Γ_δ^+, Γ_δ^- bounding the layer $T\left(\Gamma,\delta\right)$ the values of $u\left(x\right) \in W_p^1\left(\Omega^+ \cup \Omega^-\right)$. It is a well known fact (see e.g. [30] (Chap. 5)) that under the growth conditions on A_i, there exists at least a function $u^{(s)}$ minimizing $\Phi_\delta^{(s)}$, i.e,

$$\Phi_\delta^{(s)}\left(u^{(s)}\right) = \inf_{\psi^{(s)}\in\tilde{W}} \Phi_\delta^{(s)}\left(\psi^{(s)}\right).$$

Of key importance in this work is

Theorem 5. *Assume that the conditions of Theorem 1 are satisfied. Then for any function $u \in W_p^1\left(\Omega^+ \cup \Omega^-\right)$ the following relation holds*

$$\lim_{\delta\to 0}\overline{\lim_{s\to\infty}}\,\Phi_\delta^{(s)}(u) = \lim_{\delta\to 0}\underline{\lim}_{s\to\infty}\,\Phi_\delta^{(s)}(u) = \int_\Gamma c\left|u^+ - u^-\right|^p d\Gamma.$$

Proof. Denote by $\mathcal{M}$ the set of functions $u(x) \in W_p^1\left(\Omega^+ \cup \Omega^-\right)$ such that

$$u(x) = \begin{cases} u^+(x), & \text{if } x \in \Omega^+, \\ u^-(x), & \text{if } x \in \Omega^-, \end{cases}$$

where $u^\pm \in W_p^1\left(\Omega^\pm\right)$ and is continuous on $\overline{\Omega^\pm}$. Since Γ is smooth, it is a well known fact that $\mathcal{M}$ is dense in $W_p^1\left(\Omega^+ \cup \Omega^-\right)$. We prove the theorem for functions in the set $\mathcal{M}$ and later by means of approximation, we recover it in the general case. Let γ_j, $j = 1, 2, ...$ be some disjoint sets of sufficiently small diameter d_j and such that $\Gamma = \cup_j \gamma_j$. Let u be an arbitrary function in $\mathcal{M}$ and $v^{(s)}$ a minimizer of the functional $\Phi_\delta^{(s)}$. Then

$$\begin{aligned}\sum_j \int_{T(\gamma_j,\delta,s)} \sum_{i=1}^n A_i\left(x, \frac{\partial v^{(s)}}{\partial x}\right)\frac{\partial v^{(s)}}{\partial x_i}dx &= \int_{T(\Gamma,\delta,s)} \sum_{i=1}^n A_i\left(x, \frac{\partial v^{(s)}}{\partial x}\right)\frac{\partial v^{(s)}}{\partial x_i}dx \\ &\leq \Phi_\delta^{(s)}(u). \qquad (17)\end{aligned}$$

Since $v^{(s)} \in \tilde{W}$, then $v^{(s)}$ takes on the surfaces Γ_δ^+ and Γ_δ^- the values of $u(x)$. Thus by Lemma 4

$$\int_{T(\gamma_j,\delta,s)} \sum_{i=1}^n A_i\left(x, \frac{\partial v^{(s)}}{\partial x}\right)\frac{\partial v^{(s)}}{\partial x_i}dx \geq a_j^p C_A\left(\gamma_j, \delta, s\right), \qquad (18)$$

where a_j is defined as in Lemma 4 with γ replaced by γ_j. u is continuous as a function in $\mathcal{M}$ thus for all $x_j \in \gamma_j$ there exists an $\varepsilon = \varepsilon\left(\gamma_j, \delta, x_j\right)$ such that

$$a_j^p = \left|u^+(x_j) - u^-(x_j)\right|^p + \varepsilon\left(\gamma_j, \delta, x_j\right) \qquad (19)$$

and $\varepsilon\left(\gamma_j, \delta, x_j\right) \to 0$ as $\delta \to 0$ and $d_j \to 0$. Condition (9) in Theorem 1 implies that there exists an $\varepsilon' = \varepsilon'\left(\gamma_j, \delta, s\right)$ such that

$$C_A\left(\gamma_j, \delta, s\right) = \int_{\gamma_j} c(x)\, d\Gamma\left(1 + \varepsilon'\left(\gamma_j, \delta, s\right)\right), \qquad (20)$$

and $\lim_{\delta\to 0}\underline{\lim}_{s\to\infty}\,\varepsilon'\left(\gamma_j, \delta, s\right) = 0$. Substituting (19) and (20) in (18) and passing to the limit we get from (17) that

$$\lim_{\delta\to 0}\underline{\lim}_{s\to\infty}\,\Phi_\delta^{(s)}(u) \geq \int_\Gamma c\left|u^+ - u^-\right|^p d\Gamma. \qquad (21)$$

Let us establish the reverse inequality, namely

$$\lim_{\delta\to 0}\underline{\lim}_{s\to\infty}\,\Phi_\delta^{(s)}(u) \leq \int_\Gamma c\left|u^+ - u^-\right|^p d\Gamma. \qquad (22)$$

The proof is technically involved and rely on some suitable test functions. We consider the sets γ_j introduced earlier. They partition Γ and are bounded by manifolds l_k of codimension 2. We set $L = \cup_k l_k$. Let δ' be such that $2\delta' < \delta/4$. We introduce in the neighbourhood of each point of l_k a coordinate system in which l_k is contained in the submanifold $\mathbf{R}^{n-2} = \{x = (x_1, x_2, ..., x_n) \in \mathbf{R}^n : x_1 = 0, x_2 = 0\}$. In this system of coordinates, we consider the function $\eta(x) \in C^\infty(\mathbf{R}^n) :\ \eta(x) = 1$ in the $\delta/4$-neighbourhood of l_k and $\eta(x) = 0$ outside the $\delta/2$-neighbourhood of l_k and $0 \le \eta(x) \le 1$. Let $\rho = \sqrt{x_1^2 + x_2^2}$. We define the function

$$\varphi_{\delta'}^k(x) = \begin{cases} 1 - \eta(x), \text{ if } \rho \le 2\delta', \\ 1 - \eta(x) \dfrac{\ln\delta - \ln 2\rho}{\ln\delta - \ln 2\delta'}, \text{ if } 2\delta' \le \rho \le \delta/2, \\ 1, \text{ if } \rho \ge \delta/2, \end{cases}$$

and let $\varphi_{\delta'}(x) = \Pi_k \varphi_{\delta'}^k(x)$. The function $\varphi_{\delta'} \in W_p^1(\Omega)$ and satisfies the following conditions: $\varphi_{\delta'}(x) = 0$ in a $2\delta'$-neighbourhood of l, $\varphi_{\delta'}(x) = 1$ in a $\delta/2$ neighbourhood of l and for fixed δ

$$\lim_{\delta' \to 0} \int_\Omega \left| \frac{\partial \varphi_{\delta'}}{\partial x} \right|^p dx = 0.$$

By means of an appropriate diffeomorphism we can extend the above construction to l_k in its original coordinates.

Let $\tilde{\Omega}^\pm = \Omega^\pm \cup T(\Gamma, \delta)$ and for $u \in \mathcal{M}$, let $u^\pm$ be the restriction of u to the domain $\Omega^\pm$. We denote by $\tilde{u}^\pm(x)$, the continuous extension of $u^\pm(x)$ to $\tilde{\Omega}^\pm$. The existence of such an extension is guaranteed by the smoothness of Γ and we have

$$\left|\left|\tilde{u}^\pm\right|\right|_{W_p^1(\tilde{\Omega}^\pm)} \le C \left|\left|u^\pm\right|\right|_{W_p^1(\Omega^\pm)}, \tag{23}$$

with the constant C independent of u.

Let $T^\pm(\delta, \delta') =: \Omega^\pm \cap (T(\Gamma, \delta) \backslash T(\Gamma, \delta'))$ be the domain lying between $\Gamma_\delta^\pm$ and $\Gamma_{\delta'}^\pm$ and let $v_j^{(s)}$ be a function from $W(\gamma_j, \delta, s)$ minimizing the functional in the right-hand side of (8) in the domain $T(\gamma_j, \delta', s)$, i.e.,

$$C_A(\gamma_j, \delta', s) = \int_{T(\gamma_j, \delta', s)} \sum_{i=1}^n A_i \left(x, \frac{\partial v_j^{(s)}}{\partial x}\right) \frac{\partial v_j^{(s)}}{\partial x_i} dx.$$

We consider the function

$$w^{(s)}(x) = \begin{cases} u^+(x) \varphi_{\delta'}(x), \ \text{ if } x \in T^+(\delta, \delta'), \\ u^-(x) \varphi_{\delta'}(x), \text{ if } x \in T^-(\delta, \delta'), \\ \tilde{u}^+(x) v_j^{(s)}(x) \varphi_{\delta'}(x) - \tilde{u}^-(x) \left(1 - v_j^{(s)}(x)\right) \varphi_{\delta'}(x), \text{ if } x \in T(\gamma_j, \delta', s), \end{cases}$$

Easy verifications show that $w^{(s)} \in W_p^1(T(\Gamma, \delta, s))$ and $w^{(s)}(x) = u^\pm(x)$ on $\Gamma_\delta^\pm$. Therefore

$$\Phi_\delta^{(s)}(u) \le \int_{T(\Gamma, \delta, s)} \sum_{i=1}^n A_i \left(x, \frac{\partial w^{(s)}}{\partial x}\right) \frac{\partial w^{(s)}}{\partial x_i} dx = J_1^+ + J_1^- + J_2, \tag{24}$$

where

$$J_1^{\pm} = \int_{T^{\pm}(\delta,\delta')} \sum_{i=1}^{n} A_i\left(x, \frac{\partial w^{(s)}}{\partial x}\right) \frac{\partial w^{(s)}}{\partial x_i} dx,$$

$$J_2 = \int_{T(\gamma_j,\delta',s)} \sum_{i=1}^{n} A_i\left(x, \frac{\partial w^{(s)}}{\partial x}\right) \frac{\partial w^{(s)}}{\partial x_i} dx.$$

Let us estimate $J_1^{\pm}$. By (6), we have

$$J_1^{\pm} \leq C \int_{T^{\pm}(\delta,\delta')} \left[\varphi_{\delta'}^p \left|\frac{\partial u^{\pm}}{\partial x}\right|^p + \left(u^{\pm}\right)^p \left|\frac{\partial \varphi_{\delta'}}{\partial x}\right|^p\right] dx.$$

Since $u^{\pm}$ is continuous in $\overline{\Omega^{\pm}}$, then it is bounded there. Thus from the properties of $\varphi_{\delta'}$, we get

$$\int_{T^{\pm}(\delta,\delta')} \left(u^{\pm}\right)^p \left|\frac{\partial \varphi_{\delta'}}{\partial x}\right|^p dx \leq \tilde{\varepsilon}(\delta),$$

with $\tilde{\varepsilon}(\delta) \to 0$ as $\delta \to 0$. For s sufficiently large let δ' be so small that $F^{(s)} \subset T(\Gamma, \delta')$, then

$$\int_{T^{\pm}(\delta,\delta')} \varphi_{\delta'}^p \left|\frac{\partial u^{\pm}}{\partial x}\right|^p dx \leq C \int_{\Omega^{\pm}(\delta)} \left|\frac{\partial u^{\pm}}{\partial x}\right|^p dx,$$

where $\Omega^{\pm}(\delta') = \Omega^{\pm} \cap T(\Gamma, \delta)$ and the constant C is independent of $u^{\pm}$, δ' and s. Combining the two previous inequalities and taking into account the absolute continuity of Lebesgue's integrals we readily get that $J_1^{\pm} \to 0$ as $\delta \to 0$.

Next we estimate J_2. For $x \in T(\gamma_j, \delta', s)$

$$\frac{\partial w^{(s)}}{\partial x_i} = h_j(x) + g_j(x),$$

where

$$h_j(x) = \left(\tilde{u}^+ - \tilde{u}^-\right) \varphi_{\delta'} \frac{\partial v_j^{(s)}}{\partial x_i},$$

and

$$g_j(x) = \left(1 - v_j^{(s)}\right) \varphi_{\delta'} \frac{\partial \tilde{u}^-}{\partial x_i} + v_j^{(s)} \varphi_{\delta'} \frac{\partial \tilde{u}^+}{\partial x_i} + \left[\tilde{u}^- \left(1 - v_j^{(s)}\right) + \tilde{u}^+ v_j^{(s)}\right] \frac{\partial \varphi_{\delta'}}{\partial x_i}.$$

We write

$$J_2 = J_{21} + J_{22} + J_{23}, \tag{25}$$

where

$$J_{21} = \sum_{i=1}^{n} \int_{T(\gamma_j,\delta',s)} A_i(x, h_j) h_j dx,$$

$$J_{22} = \sum_{i=1}^{n} \int_{T(\gamma_j,\delta',s)} \left[A_i\left(x, \frac{\partial w^{(s)}}{\partial x}\right) - A_i(x, h_j)\right] h_j dx,$$

$$J_{23} = \int_{T(\gamma_j,\delta',s)} A_i\left(x, \frac{\partial w^{(s)}}{\partial x}\right) g_j dx.$$

The properties of $\varphi_{\delta'}$, $v_j^{(s)}$ and the homogeneity of A_i (see (4)) imply that

$$J_{21} \leq \int_{T(\gamma_j,\delta',s)} \left|\tilde{u}^+ - \tilde{u}^-\right|^p \sum_{i=1}^{n} A_i\left(x, \frac{\partial v_j^{(s)}}{\partial x}\right) \frac{\partial v_j^{(s)}}{\partial x_i} dx. \tag{26}$$

By (6) and Hölder's inequality, we have

$$J_{22} \leq C\left(\int_{T(\gamma_j,\delta',s)}\left|\frac{\partial w^{(s)}}{\partial x}\right|^p dx\right)^{(p-2)/p}\left(\int_{T(\gamma_j,\delta',s)}\left[\left|\frac{\partial u^+}{\partial x}\right|^p+\left|\frac{\partial u^-}{\partial x}\right|^p\right]dx\right)^{1/p} \times\left(\int_{T(\gamma_j,\delta',s)}|h_j|^p\,dx\right)^{1/p}. \tag{27}$$

Similarly we have

$$J_{23} \leq C\left(\int_{T(\gamma_j,\delta',s)}\left|\frac{\partial w^{(s)}}{\partial x}\right|^p dx\right)^{(p-1)/p}\left(\int_{T(\gamma_j,\delta',s)}\left[\left|\frac{\partial u^+}{\partial x}\right|^p+\left|\frac{\partial u^-}{\partial x}\right|^p\right]dx\right)^{1/p}. \tag{28}$$

Owing to the absolute continuity of Lebesgue's integrals, the expressions in the right-hand side of inequalities (27)-(28) vanish as $\delta\to 0(\delta'\to 0)$. Hence by (25) and the definition of J_2, we get

$$\sum_j\int_{T(\gamma_j,\delta',s)}\sum_{i=1}^n A_i\left(x,\frac{\partial w^{(s)}}{\partial x}\right)\frac{\partial w^{(s)}}{\partial x_i}dx \leq \sum_j\int_{T(\gamma_j,\delta',s)}\left|\tilde u^+-\tilde u^-\right|^p\sum_{i=1}^n A_i\left(x,\frac{\partial v_j^{(s)}}{\partial x}\right)\frac{\partial v_j^{(s)}}{\partial x_i}dx+\varepsilon(\Gamma,\delta), \tag{29}$$

where $\varepsilon(\Gamma,\delta)\to 0$ as $\delta\to 0$. The functions $\tilde u^+$, $\tilde u^-$ are continuous as continuous extensions of the restriction of u to Ω_+ and Ω_- respectively. Thus by the definition of P_A, we get

$$\sum_j\int_{T(\gamma_j,\delta',s)}\left|\tilde u^+-\tilde u^-\right|^p\sum_{i=1}^n A_i\left(x,\frac{\partial v_j^{(s)}}{\partial x}\right)\frac{\partial v_j^{(s)}}{\partial x_i} \leq \sum_j\left(\left|\tilde u^+\left(x^j\right)-\tilde u^-\left(x^j\right)\right|^p+\varepsilon\left(\delta',\gamma_j,u,x^j\right)\right)C_A\left(\gamma_j,\delta',s\right), \tag{30}$$

where $x^j\in\gamma_j$ and $\varepsilon\left(\delta',\gamma_j,u,x^j\right)\to 0$ as $\delta'\to 0$ and $d_j\to 0$. Appealing to condition (9) of Theorem 1, the relations (29)-(30), we deduce that

$$\sum_j\int_{T(\gamma_j,\delta',s)}\sum_{i=1}^n A_i\left(x,\frac{\partial w^{(s)}}{\partial x}\right)\frac{\partial w^{(s)}}{\partial x_i}dx\leq\int_\Gamma c(x)\left|u^+-u^-\right|^p d\Gamma+\varepsilon\left(\delta',s,d\right), \tag{31}$$

where $\lim_{d\to 0}\overline{\lim}_{\delta\to 0}\underline{\lim}_{s\to\infty}\varepsilon\left(\delta',s,d\right)=0$ and $d=\max\{d_j\}$.

Since the first two terms in the right-hand side of (24) vanish as $\delta\to 0$, we deduce from (31) and (24) that

$$\lim_{\delta\to 0}\overline{\lim}_{s\to\infty}\Phi_\delta^{(s)}(u)\leq\int_\Gamma c(x)\left|u^+-u^-\right|^p dx. \tag{32}$$

(21) and (32) prove the theorem for $u\in\mathcal{M}$.

Let us show that the theorem holds for $u\in W_p^1\left(\Omega^+\cup\Omega^-\right)$. For such u, there exists a sequence of functions $\{u_l\}$ from $\mathcal{M}$ such that

$$\lim_{l\to\infty}||u_l-u||_{W_p^1(\Omega^+\cup\Omega^-)}=0;$$

if $u_l^{\pm}$ is the restriction of u_l on $\Omega^{\pm}$, we have

$$\lim_{l\to\infty} ||u_l^{\pm} - u||_{W_p^1(\Omega^{\pm})} = 0.$$

By the definition of $\Phi_\delta^{(s)}$, the inequality (6) and Hölder's inequality, we have

$$\begin{aligned} &\left|\Phi_\delta^{(s)}(u) - \Phi_\delta^{(s)}(u_l)\right| \\ \le\ & C\left(\int_{T(\Gamma,\delta,s)} \left|\frac{\partial u}{\partial x}\right|^p dx\right)^{(p-1)/p} \left(\int_{T(\Gamma,\delta,s)} \left|\frac{\partial (u_l - u)}{\partial x}\right|^p dx\right)^{1/p} \\ & C\left(\int_{T(\Gamma,\delta,s)} \left[\left|\frac{\partial u}{\partial x}\right|^p + \left|\frac{\partial u_l}{\partial x}\right|^p\right] dx\right)^{(p-2)/p} \left(\int_{T(\Gamma,\delta,s)} \left|\frac{\partial u_l}{\partial x}\right| dx\right)^{1/p} \\ & \times \left(\int_{T(\Gamma,\delta,s)} \left|\frac{\partial (u_l - u)}{\partial x}\right|^p dx\right)^{1/p}. \end{aligned}$$

As $l \to \infty$ the right-hand side vanishes. Thus

$$\Phi_\delta^{(s)}(u) = \lim_{l\to\infty} \Phi_\delta^{(s)}(u_l).$$

This shows that the theorem holds for $u \in W_p^1(\Omega^+ \cup \Omega^-)$. □

4. **Proof of Theorem 1.** Straightforward calculations give

$$\left|\left|u^{(s)}\right|\right|_{W_p^1(\Omega^{(s)})} \le C$$

with a constant C independent of δ. Let $\Omega_\delta^{\pm} = \Omega^{\pm} \backslash T(\Gamma, \delta)$. We assume that for sufficiently large s, $F^{(s)}$ lies inside $T(\Gamma, \delta)$, therefore the sets $\Omega_\delta^{\pm}$ are independent of s. From this inequality it follows that

$$\left|\left|u^{(s)}\right|\right|_{W_p^1(\Omega_\delta^- \cup \Omega_\delta^+)} \le C,$$

We have that $u^{(s)}$ is independent of δ. Thus passing to the limit as $\delta \to 0$ in this inequality we get

$$\left|\left|u^{(s)}\right|\right|_{W_p^1(\Omega^- \cup \Omega^+)} \le C. \tag{33}$$

There is therefore a function $u \in W_p^1(\Omega^- \cup \Omega^+)$ such that

$$u^{(s)} \rightharpoonup u, \text{ weakly in } W_p^1\left(\Omega^- \cup \Omega^+\right). \tag{34}$$

Furthermore we have the following result which follows from a straightforward adaptation of the arguments of Boccardo and Murat [12].

Lemma 6. *Under the conditions on problem (1)-(3), we have*

$$u^{(s)} \to u \text{ strongly in } W_q^1\left(\Omega^- \cup \Omega^+\right), \tag{35}$$

for $1 < q < p$.

For any $\delta > 0$ and sufficiently large s, $u^{(s)}$ is defined in $\Omega_\delta^{\pm}$ and we have from (33)

$$\left|\left|u^{(s)}\right|\right|_{W_p^1(\Omega_\delta^{\pm})} \le C. \tag{36}$$

Thus letting $u^{\pm}$ be the restriction of u to $\Omega_{\delta}^{\pm}$, we have from this inequality and Lemma 6, that

$$u^{(s)} \to u^{\pm} \text{ in } W_p^1\left(\Omega_{\delta}^{\pm}\right) \text{ weakly,} \tag{37}$$

$$u^{(s)} \to u^{\pm} \text{ in } W_q^1\left(\Omega_{\delta}^{\pm}\right) \text{ strongly.} \tag{38}$$

Lemma 7. *The function $u^{\pm}$ satisfies the integral identity*

$$\int_{\Omega_{\delta}^{\pm}} Au^{\pm}\varphi dx = \int_{\Omega_{\delta}^{\pm}} f\varphi dx, \tag{39}$$

for all $\varphi \in \overset{o}{W}{}_p^1\left(\Omega_{\delta}^{\pm}\right)$.

Proof. We have

$$Au^{(s)} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i\left(x, \frac{\partial u^{(s)}}{\partial x}\right) = f, \text{ in } \Omega_{\delta}^{\pm}.$$

An integration by parts and Hölder's inequality yield

$$\begin{aligned} \left|\left|Au^{(s)}\right|\right|_{W_p^{-1}\left(\Omega_{\delta}^{\pm}\right)} &= \sup_{v\in \overset{o}{W}{}_p^1\left(\Omega_{\delta}^{\pm}\right)} \int_{\Omega_{\delta}^{\pm}} \frac{\partial}{\partial x_i} A_i\left(x, \frac{\partial u^{(s)}}{\partial x}\right) v dx \\ &= \sup_{v\in \overset{o}{W}{}_p^1\left(\Omega_{\delta}^{\pm}\right)} \int_{\Omega_{\delta}^{\pm}} f v dx \le C. \end{aligned}$$

Thus

$$Au^{(s)} \to \chi, \text{ weakly in } W_{p'}^{-1}\left(\Omega_{\delta}^{\pm}\right).$$

Hence since

$$\int_{\Omega_{\delta}^{\pm}} Au^{(s)}\varphi dx = \int_{\Omega_{\delta}^{\pm}} f\varphi dx,$$

for all $\varphi \in \overset{o}{W}{}_p^1\left(\Omega_{\delta}^{\pm}\right)$, passing to the limit we get

$$\int_{\Omega_{\delta}^{\pm}} \chi\varphi dx = \int_{\Omega_{\delta}^{\pm}} f\varphi dx.$$

Thus $\chi = f$ almost everywhere in $\Omega_{\delta}^{\pm}$. Since the operator A is monotone and $u^{(s)}$ converges weakly to $u^{\pm}$ in $W_p^1\left(\Omega_{\delta}^{\pm}\right)$, arguing as in [31] (Chap. 2, page 170), we get $Au^{\pm} = f$, for $x \in \Omega_{\delta}^{\pm}$. □

Lemma 8. *Let $u^{\pm}$ be as in the previous lemma. Then*

$$\lim_{s\to\infty} \left|\left|u^{(s)} - u^{\pm}\right|\right|_{W_p^1\left(\Omega_{\delta}^{\pm}\right)} = 0. \tag{40}$$

Proof. Let

$$\chi_{\delta\varepsilon}^{\pm}(x) = \begin{cases} 1 \text{ in } \Omega_{\delta}^{\pm} \\ 0 \text{ outside } \Omega_{\delta+\varepsilon}^{\pm} \end{cases},$$

and $\chi_{\delta\varepsilon}^{\pm}(x) \in [0,1]$; $\varepsilon > 0$ is chosen so that $\Omega_{\delta+\varepsilon}^{\pm} \cap F^{(s)} = \emptyset$, $\Omega_{\delta+\varepsilon}^{\pm} \subset \Omega$, $\chi_{\delta\varepsilon}^{\pm} \in C_o^{\infty}\left(\Omega_{\delta+\varepsilon}^{\pm}\right)$ and $\chi_{\delta\varepsilon}^{\pm} \to 1$ strongly in $W_p^1\left(\Omega_{\delta}^{\pm}\right)$ as $\varepsilon \to 0$. Let $u^{\pm}$ be an extension of $u^{\pm}$ to $\Omega_{\delta+\varepsilon}^{\pm}$ such that

$$u^{(s)} \rightharpoonup u^{\pm} \text{ weakly in } W_p^1\left(\Omega_{\delta+\varepsilon}^{\pm}\right), \tag{41}$$

$$u^{(s)} \to u \text{ strongly in } W_q^1\left(\Omega_{\delta+\varepsilon}^{\pm}\right),\ 1 < q < p$$

Multiply (1) by $\left(u^{(s)}-u^{\pm}\right)\chi^{\pm}_{\delta\varepsilon}$ and substitute in (39) φ by $\left(u^{(s)}-u^{\pm}\right)\chi^{\pm}_{\delta\varepsilon}$ and subtract the resulting equations to get

$$\begin{aligned}&\sum_{i=1}^{n}\int_{\Omega^{\pm}_{\delta+\varepsilon}}\left[A_i\left(x,\frac{\partial u^{(s)}}{\partial x}\right)-A_i\left(x,\frac{\partial u^{\pm}}{\partial x}\right)\right]\frac{\partial\left(u^{(s)}-u^{\pm}\right)\chi^{\pm}_{\delta\varepsilon}}{\partial x_i}\\ =\;&\int_{\Omega^{\pm}_{\delta+\varepsilon}} f\left(u^{(s)}-u^{\pm}\right)\chi^{\pm}_{\delta\varepsilon}dx.\end{aligned}$$

Applying the inequalities (5)-(6) and Hölder s inequality, we have

$$\begin{aligned}&\int_{\Omega^{\pm}_{\delta+\varepsilon}}\left|\frac{\partial\left(u^{(s)}-u^{\pm}\right)}{\partial x}\right|^{p}\chi^{\pm}_{\delta\varepsilon}dx\\ \leq\;& C(\varepsilon)\left(\int_{\Omega}\left(\left|\frac{\partial u^{(s)}}{\partial x}\right|^{p-2}+\left|\frac{\partial u^{\pm}}{\partial x}\right|^{p-2}\right)dx\right)^{(p-2)/p}\left(\int_{\Omega}\left|\frac{\partial u^{(s)}-u^{\pm}}{\partial x}\right|^{p}dx\right)^{1/p}\\ &\times\left(\int_{\Omega}\left|u^{(s)}-u^{\pm}\right|^{p}dx\right)^{1/p}+\left(\int_{\Omega}|f|^{p'}dx\right)^{1/p'}\left(\int_{\Omega}\left|u^{(s)}-u^{\pm}\right|^{p}dx\right)^{1/p}.\end{aligned}$$

(41) implies that $u^{(s)}$ strongly converges to $u^{\pm}$ in $L_p\left(\Omega^{\pm}_{\delta+\varepsilon}\right)$. Thus the right-hand side of the above inequality vanishes when $s\to\infty$. Hence

$$\int_{\Omega^{\pm}_{\delta+\varepsilon}}\left|\frac{\partial\left(u^{(s)}-u^{\pm}\right)}{\partial x}\right|^{p}\chi^{\pm}_{\delta\varepsilon}dx\to 0,\text{ as } s\to\infty.$$

Passing to the limit as $\varepsilon\to 0$, we get (40). □

Let $\overset{o}{W}{}^1_p\left(\Omega^+\cup\Omega^-\right)$ denote the class of functions in $W^1_p\left(\Omega^+\cup\Omega^-\right)$ which vanish on $\partial\Omega$. We consider the functional J defined as

$$J(w)=\int_{\Omega^+\cup\Omega^-}\left[\sum_{i=1}^{n}A_i\left(x,\frac{\partial w}{\partial x}\right)\frac{\partial w}{\partial x_i}-fw\right]dx+\int_{\Gamma}c(x)\left|w^+-w^-\right|^{p}d\Gamma, \quad (42)$$

We have with A_i satisfying the conditions formulated earlier with regard to the functional I.

Lemma 9. *Under the conditions imposed on the functions $A_i(x,p)$, $(x,p)\in\mathbf{R}^{2n}$, any minimizer of the functional J in $\overset{o}{W}{}^1_p\left(\Omega^+\cup\Omega^-\right)$ is also a weak solution of problem (10)-(12).*

Proof. Let ϕ be any function from $\overset{o}{W}{}^1_p\left(\Omega^+\cup\Omega^-\right)$ and let $t\in\mathbf{R}$. We readily have

$$\begin{aligned}\left.\frac{dJ(w+t\phi)}{dt}\right|_{t=0}=\;&\int_{\Omega^+\cup\Omega^-}\left[\sum_{i=1}^{n}a_i\left(x,\frac{\partial w}{\partial x}\right)\frac{\partial\phi}{\partial x_i}-f\phi\right]dx\\ &+p\int_{\Gamma}c(x)\left|w^+-w^-\right|^{p-2}\left(w^+-w^-\right)\left[\phi^+-\phi^-\right]d\Gamma.\end{aligned}$$

Integrating by parts in the integral over $\Omega^+ \cup \Omega^-$ and setting $dJ/dt|_{t=0} = 0$, we get

$$\begin{aligned} & -\int_{\Omega^+\cup\Omega^-} \phi \left[\sum_{i=1}^n \frac{\partial}{\partial x_i} A_i \left(x, \frac{\partial w}{\partial x} \right) + f \right] dx \\ = & \int_{\Gamma^+} \phi \frac{\partial w}{\partial \nu_A^+} d\Gamma + \int_{\Gamma^-} \phi \frac{\partial w}{\partial \nu_A^-} d\Gamma \\ & -p \int_{\Gamma} c(x) \left| w^+ - w^- \right|^{p-2} \left(w^+ - w^- \right) \left[\phi^+ - \phi^- \right] d\Gamma, \end{aligned}$$

where $\partial w / \partial \nu_A^{\pm}$ are the normals directed toward the interior of $\Omega^{\pm}$. This integral identity is the weak formulation of problem (10)-(12). □

Now we prove that the function u minimizes J in $\overset{o}{W}{}_p^1 (\Omega^+ \cup \Omega^-)$. Let w be an arbitrary function in $\overset{o}{W}{}_p^1 (\Omega^+ \cup \Omega^-)$, we introduce the function

$$w_\delta^{(s)}(x) = \begin{cases} w(x), \text{ if } x \in \Omega_\delta^{\pm} \\ w^{(s)}(x) \text{ if } x \in T(\Gamma, \delta) \end{cases},$$

where $w^{(s)}$ is a function from $W_p^1 (T(\Gamma, \delta, s))$ which minimizes the functional $\Phi_\delta^{(s)}$ in the class of functions equal to $w(x)$ on $\Gamma_\delta^{\pm}$. Clearly $w_\delta^{(s)} \in W_p^1 \left(\Omega^{(s)} \right)$. As discussed at the beginning of the paper, minimizers in $\overset{o}{W}{}_p^1 \left(\Omega^{(s)} \right)$ of the functional

$$I(v) = \int_{\Omega^{(s)}} \left[\sum_{i=1}^n A_i \left(x, \frac{\partial v}{\partial x} \right) \frac{\partial v}{\partial x_i} + fv \right] dx$$

under the assumptions made on A_i are weak solutions of the problem (1)-(3). We assumed that the support of f does not intersect Γ. Thus for sufficiently small δ, f is located outside $T(\Gamma, \delta)$, and from the definitions of the functionals I and $\Phi_\delta^{(s)}$, we have

$$I\left(w_\delta^{(s)} \right) = \int_{\Omega_\delta^+ \cup \Omega_\delta^-} \left[\sum_{i=1}^n A_i \left(x, \frac{\partial w}{\partial x} \right) \frac{\partial w}{\partial x_i} + fw \right] dx + \Phi_\delta^{(s)}(w). \tag{43}$$

By Theorem 5 and the definition of J, we have

$$\lim_{\delta \to 0} \overline{\lim}_{s \to \infty} I\left(w_\delta^{(s)} \right) = J(w). \tag{44}$$

Let $u \in \overset{o}{W}{}_p^1 (\Omega^+ \cup \Omega^-)$ be the limit of $u^{(s)}$ along the subsequence $s = s_k \to \infty$. By Lemma 8, we have shown that

$$\lim_{s \to \infty} \left| \left| u^{(s)} - u^{\pm} \right| \right|_{W_p^1 \left(\Omega_\delta^{\pm} \right)} = 0,$$

where $u^{\pm}$ is a restriction to $\Omega_\delta^{\pm}$. Let $u_\delta^{(s)} \in \overset{o}{W}{}_p^1 (\Omega^+ \cup \Omega^-)$ be an extension of $u^{(s)}$ from $\Omega_\delta^+ \cup \Omega_\delta^-$ to $\Omega^+ \cup \Omega^-$ such that

$$\lim_{s \to \infty} \left| \left| u_\delta^{(s)} - u \right| \right|_{\overset{o}{W}{}_p^1 (\Omega^+ \cup \Omega^-)} = 0. \tag{45}$$

We have

$$I\left(u^{(s)} \right) = \int_{\Omega_\delta^{=} \cup \Omega_\delta^-} \left[\sum_{i=1}^n A_i \left(x, \frac{\partial u_\delta^{(s)}}{\partial x} \right) \frac{\partial u_\delta^{(s)}}{\partial x_i} + fu_\delta^{(s)} \right] dx + \Phi_\delta^{(s)} \left(u_\delta^{(s)} \right). \tag{46}$$

Let us estimate the last term. We have

$$\Phi_\delta^{(s)}\left(u_\delta^{(s)}\right)=\Phi_\delta^{(s)}(u)+\int_{T(\Gamma,\delta,s)}\sum_{i=1}^{n}\left[A_i\left(x,\frac{\partial u_\delta^{(s)}}{\partial x}\right)\frac{\partial u_\delta^{(s)}}{\partial x_i}-A_i\left(x,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x_i}\right]dx. \tag{47}$$

Denoting the last term in the right-hand side of this relation by $\Delta_s\Phi$, we have from (6)

$$\begin{aligned}\Delta_s\Phi &= \int_{T(\Gamma,\delta,s)}\sum_{i=1}^{n}\left[A_i\left(x,\frac{\partial u_\delta^{(s)}}{\partial x}\right)-A_i\left(x,\frac{\partial u}{\partial x}\right)\right]\frac{\partial u_\delta^{(s)}}{\partial x_i}dx+\\ &\quad \int_{T(\Gamma,\delta,s)}\sum_{i=1}^{n}A_i\left(x,\frac{\partial u}{\partial x}\right)\frac{\partial\left(u_\delta^{(s)}-u\right)}{\partial x_i}dx\\ &\le C\int_{T(\Gamma,\delta,s)}\left[\left|\frac{\partial u_\delta^{(s)}}{\partial x_i}\right|^{p-2}-\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\right]\left|\frac{\partial\left(u_\delta^{(s)}-u\right)}{\partial x_i}\right|dx\\ &\quad +C\int_{T(\Gamma,\delta,s)}\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\left|\frac{\partial\left(u_\delta^{(s)}-u\right)}{\partial x_i}\right|dx.\end{aligned}$$

(45) and Hölder's inequality imply that

$$\lim_{s\to\infty}\Delta_s\Phi=0.$$

Analogously we have

$$\begin{aligned}&\int_{\Omega_\delta^{=}\cup\Omega_\delta^{-}}\left[\sum_{i=1}^{n}A_i\left(x,\frac{\partial u_\delta^{(s)}}{\partial x}\right)\frac{\partial u_\delta^{(s)}}{\partial x_i}+fu_\delta^{(s)}\right]dx\\ \to\ &\int_{\Omega_\delta^{=}\cup\Omega_\delta^{-}}\left[\sum_{i=1}^{n}A_i\left(x,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x_i}+fu\right]dx\end{aligned}$$

as $s\to\infty$. Hence passing to the limit in (46), (47), we get

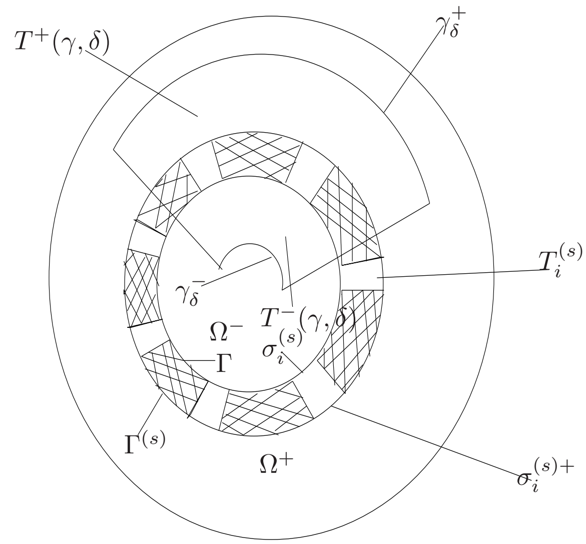
$$\underline{\lim}_{s\to\infty}I\left(u^{(s)}\right)\ge J(u). \tag{48}$$

Since $u^{(s)}$ minimizes I in $\overset{o}{W}{}_p^1\left(\Omega^{(s)}\right)$, thus for $w_\delta^{(s)}\in\overset{o}{W}{}_p^1\left(\Omega^{(s)}\right)$, $I\left(u^{(s)}\right)\le I\left(w_\delta^{(s)}\right)$. Therefore from (44) and (48), we get

$$J(u)\le J(w),$$

for all $w\in\overset{o}{W}{}_p^1\left(\Omega^+\cup\Omega^-\right)$. This implies that u minimizes J in $\overset{o}{W}{}_p^1\left(\Omega^+\cup\Omega^-\right)$. This completes the proof of Theorem 1.

5. **Example of a geometry for $F^{(s)}$.** We consider for each s a layer $T^{(s)}$ of thickness $h^{(s)}$ bounded from one side by a fixed surface Γ and from the other side by a surface $\Gamma^{(s)}$ parallel to Γ and at a distance $h^{(s)}$ from it. We remove from Γ s disjoint connected open sets $\sigma_i=\sigma_i^{(s)}$ of diameter $d_i^{(s)}$(radius $r_i^{(s)}$). The normals through the points $x\in\sigma_i$, cut some channels $T_i^{(s)}$ through $T^{(s)}$. Set $F^{(s)}=\overline{T^{(s)}\setminus\cup_{i=1}^{s}T_i^{(s)}}$, thus $F^{(s)}$ is a set with channels for each s. Let Ω be a large



ball containing $T^{(s)}$. In the region $\Omega^{(s)} = \Omega \backslash F^{(s)}$, we consider the boundary value problem

$$\Delta_p u^{(s)} = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\left| \frac{\partial u^{(s)}}{\partial x} \right|^{p-2} \frac{\partial u^{(s)}}{\partial x_j} \right) = f \text{ in } \Omega^{(s)}, \tag{49}$$

$$\frac{\partial u^{(s)}}{\partial \nu_{\Delta_p}} = 0, \text{ on } \partial F^{(s)}; \ u = 0 \text{ on } \partial\Omega. \tag{50}$$

Let γ be a portion of the surface Γ and $T(\gamma, \delta)$ be the layer with thickness 2δ ($\delta > h^{(s)}$ for sufficiently large s) centered around γ with bases $\gamma_\delta^{\pm}$. We define the quantity

$$C_{\Delta_p}(\gamma, \delta, s) = \inf_{w^{(s)}} \int_{T(\gamma,\delta,s)} \left| \frac{\partial w^{(s)}}{\partial x} \right|^p dx, \tag{51}$$

where $T(\gamma, \delta, s) = T(\gamma, \delta) \backslash \overline{T^{(s)} \backslash \cup_{i=1}^{s} T_i^{(s)}}$, and the infimum is taken over the functions $w^{(s)}$ belonging to the set $W(\gamma, \delta, s)$ which consists of functions from $W_p^1(T(\gamma, \delta, s))$, taking on the surfaces γ_δ^+ and γ_δ^- the values one and zero respectively. Let us denote by $T^-(\gamma, \delta)$ (resp. $T^+(\gamma, \delta)$) the part of $T(\gamma, \delta)$ whose boundary intersects with γ_δ^- (resp. γ_δ^+) and Γ (resp. $\Gamma^{(s)}$). We denote $\overline{T_i^{(s)}} \cap \Gamma$ and $\overline{T_i^{(s)}} \cap \Gamma^{(s)}$ by $\sigma_i^{(s)-}$ and $\sigma_i^{(s)+}$, respectively.

We define the $p-$capacity of a set $E \subset B(x_0, R)$ (ball of center x_0 and radius R) as the number

$$C_p(E) = \inf_{\varphi \in \mathcal{M}(E)} \int_{B(x_0,2R)} \left| \frac{\partial \varphi}{\partial x} \right|^p dx,$$

where $\mathcal{M}(E) = \{\varphi \in C_o^\infty(B(x_0, 2R)) : \varphi(x) = 1 \text{ on } E\}$.

Let $R_i^{(s)}$ be the distance between $\sigma_i^{(s)-}$ and $\cup_{i \neq j} \sigma_j^{(s)-}$; we assume that

$\max\left\{R_i^{(s)}, d_i^{(s)}\right\} < \delta$. Let $B\left(x_i^{(s)}, d_i^{(s)}\right)$ be the smaller $(n-1)-$dimensional ball containing $\sigma_i^{(s)}$. Let $\tilde{B}\left(x_i^{(s)}, R\right)$ be a $(n-1)$-dimensional ball centered at

$x_i^{(s)}$ with radius R. We denote by $Q^{\pm}\left(\sigma_i^{(s)}, R, d\right)$ the cylinder in $T^{\pm}(\gamma,\delta)$ with base $\tilde{B}\left(x_i^{(s)}, R\right)$ and height d; if $R = d$, we write $Q^{\pm}\left(\sigma_i^{(s)}, R\right)$. We define the function $\varphi_i^{(s)\pm}$ sufficiently smooth in Ω and such that $\varphi_i^{(s)\pm}(x) = 1$ in $Q^{\pm}\left(\sigma_i^{(s)}, R_i^{(s)}/4, \lambda_1^{\pm} d_i^{(s)}\right)$, $\varphi_i^{(s)\pm}(x) = 0$ outside a $Q^{\pm}\left(\sigma_i^{(s)}, R_i^{(s)}/3, \lambda_2^{\pm} d_i^{(s)}\right)$, $\partial\varphi_i^{(s)\pm}/\partial\nu_{\Delta_p} = 0$ on Γ, $0 \le \varphi_i^{(s)\pm} \le 1$, $\left|\partial\varphi_i^{(s)\pm}/\partial x\right| \le C\left[R_i^{(s)}\right]^{-1}$; $\lambda_1^{\pm} < \lambda_2^{\pm}$. Further conditions on $d_i^{(s)}$ and $R_i^{(s)}$ will ensure that that the functions $\varphi_i^{(s)\pm}$ have disjoint supports.

We introduce the function $v_i^{(s)-}$ solution of the boundary value problem

$$\Delta_p v_i^{(s)-}(x) = 0, \text{ in } T^{-}(\gamma,\delta) \tag{52}$$

$$v_i^{(s)-}(x) = 1 \text{ in } \sigma_i^{(s)-}, \tag{53}$$

$$\frac{\partial v_i^{(s)-}}{\partial \nu_{\Delta_p}} = 0 \ \text{ on } \ \partial T^{-}(\gamma,\delta)\backslash\sigma_i^{(s)-}, \tag{54}$$

By a solution of problem (52)-(54), we mean a function $v_i^{(s)-} \in W_p^1(T^{-}(\gamma,\delta))$ such that $v_i^{(s)-} - 1 \in W_{0p}^1\left(T^{-}(\gamma,\delta)\cup\sigma_i^{(s)-}\right)$ and

$$\sum_{j=1}^{n}\int_{T^{-}(\gamma,\delta)}\left|\frac{\partial v_i^{(s)-}}{\partial x}\right|^{p-2}\frac{\partial v_i^{(s)-}}{\partial x_j}\frac{\partial\varphi}{\partial x_j}dx = 0, \tag{55}$$

for all $\varphi \in W_{0p}^1\left(T^{-}(\gamma,\delta)\cup\sigma_i^{(s)-}\right)$. Here $W_{0p}^1\left(T^{-}(\gamma,\delta)\cup\sigma_i^{(s)-}\right)$ denotes the closure of $C_0^1\left(T^{-}(\gamma,\delta)\cup\sigma_i^{(s)-}\right)$ in $W_p^1(T^{-}(\gamma,\delta))$.

Next we derive some a priori estimates for the functions $v_i^{(s)\pm}$. Substituting $\varphi = \min\left\{v_i^{(s)-}, 0\right\}$ and $\varphi = \max\left\{v_i^{(s)-} - 1, 0\right\}$ in (55) and performing some simple calculations we get

$$0 \le v_i^{(s)-} \le 1. \tag{56}$$

Let $\psi_{1i}^{(s)} \in C^1\left(\mathbf{R}_{-}^n\right)$, $\psi_{1i}^{(s)}(x) = 1$ on $\sigma_i^{(s)}$ and $\psi_{1i}^{(s)}(x) = 0$ outside $Q^{-}\left(\sigma_i^{(s)}, R_i^{(s)}/4\right)$, $\psi_{2i}^{(s)} \in C^1\left(\mathbf{R}_{-}^n\right)$, $\psi_{2i}^{(s)}(x) = 1$ on $B\left(\sigma_i^{(s)-}, \lambda d_i^{(s)}\right)$ and $\psi_{2i}^{(s)}(x) = 0$ outside $Q^{-}\left(\sigma_i^{(s)}, R_i^{(s)}/3\right)$, $0 \le \psi_{ki}^{(s)} \le 1$, $k = 1, 2$. We assume that $d_i^{(s)} < \lambda d_i^{(s)} < r_i^{(s)} + R_i^{(s)}/3$ then the supports of the $\psi_{ki}^{(s)}$'s do not intersect for different i's. Substitute $\varphi_i^{(s)-} = \left(v_i^{(s)-} - \psi_{1i}^{(s)}\right)\psi_{2i}^{(s)}$ in (55) Then by Young's inequality and the estimate (56), we get

$$\begin{aligned}
&\int_{Q^{-}\left(\sigma_i^{(s)}, R_i^{(s)}/3\right)}\left|\frac{\partial v_i^{(s)-}}{\partial x}\right|^p dx \\
\le\ & C\left\{\int_{Q^{-}\left(\sigma_i^{(s)}, R_i^{(s)}/4\right)}\left|\frac{\partial\psi_{1i}^{(s)}}{\partial x}\right|^p dx + \int_{Q^{-}\left(\sigma_i^{(s)}, R_i^{(s)}/3\right)}\left|\frac{\partial\psi_{2i}^{(s)}}{\partial x}\right|^p dx\right\}.
\end{aligned}$$

Passing to infimum in the right-hand side and recalling the definition of capacity we have

$$\int_{B\left(\sigma_i^{(s)-},R_i^{(s)}/2\right)}\left|\frac{\partial v_i^{(s)-}}{\partial x}\right|^p dx \leq C\left\{C_p\left(\sigma_i^{(s)-}\right)+C_p\left(B\left(\sigma_i^{(s)-},\lambda d_i^{(s)}\right)\right)\right\}. \tag{57}$$

We define the functions $\varphi_i^{(s)+}$ and $v_i^{(s)+}$ analogously with respect to $T^+(\gamma,\delta)$ and $\sigma_i^{(s)+}$ and obtain analogous corresponding estimates as (57), (56) with the obvious changes.

Let $w_i^{(s)\pm}(x)=\varphi_i^{(s)\pm}\left(1-v_i^{(s)\pm}\right)$ and consider the function

$$\hat{w}^{(s)}(x)=\begin{cases}\sum_{\gamma(s)} w_i^{(s)-}(x)/h^{(s)} \text{ if } x\in T^-(\gamma,\delta),\\ \dfrac{t}{h^{(s)}} \text{ if } x\in T_i^{(s)},\\ 1-\sum_{\gamma(s)} w_i^{(s)+}(x)/h^{(s)} \text{ if } x\in T^+(\gamma,\delta),\end{cases}$$

where t is the distance between x and Γ for $x\in T_i^{(s)}$. A verification shows that $\hat{w}^{(s)}\in W(\gamma,\delta,s)$.

Let us evaluate

$$\int_{T(\gamma,\delta,s)}\left|\frac{\partial \hat{w}^{(s)}}{\partial x}\right|^p dx.$$

We have

$$\begin{aligned}\int_{T^-(\gamma,\delta)}\left|\frac{\partial \hat{w}^{(s)}}{\partial x}\right|^p dx &\leq \frac{1}{\left[h^{(s)}\right]^p}\int_{T^-(\gamma,\delta)}\left|\frac{\partial \varphi_i^{(s)-}}{\partial x}\right|^p\left(1-v_i^{(s)-}\right)^p dx\\ &+\frac{1}{\left[h^{(s)}\right]^p}\int_{T^-(\gamma,\delta)}\left|\varphi_i^{(s)-}\right|^p\left|\frac{\partial\left(1-v_i^{(s)-}\right)}{\partial x}\right|^p dx.\end{aligned}$$

From the definition of φ_i^- and the estimates (57), (56), we get

$$\begin{aligned}\int_{T^-(\gamma,\delta)}\left|\frac{\partial \hat{w}^{(s)}}{\partial x}\right|^p dx &\leq C\sum_{\gamma(s)}\frac{d_i^{(s)}\left[d_i^{(s)}+R_i^{(s)}/3\right]^{n-1}}{\left[h^{(s)}R_i^{(s)}\right]^p}\\ &+C\sum_{\gamma(s)}\frac{1}{\left[h^{(s)}\right]^p}\left[C_p\left(\sigma_i^{(s)-}\right)+C_p\left(B\left(\sigma_i^{(s)-},\lambda d_i^{(s)}\right)\right)\right].\end{aligned} \tag{58}$$

Analogously we have

$$\begin{aligned}\int_{T^+(\gamma,\delta)}\left|\frac{\partial \hat{w}^{(s)}}{\partial x}\right|^p dx &\leq C\sum_{\gamma(s)}\frac{d_i^{(s)}\left[d_i^{(s)}+R_i^{(s)}/3\right]^{n-1}}{\left[h^{(s)}R_i^{(s)}\right]^p}\\ &+C\sum_{\gamma(s)}\frac{1}{\left[h^{(s)}\right]^p}\left[C_p\left(\sigma_i^{(s)+}\right)+C_p\left(B\left(\sigma_i^{(s)+},\lambda d_i^{(s)}\right)\right)\right].\end{aligned} \tag{59}$$

We have

$$\int_{T_i^{(s)}}\left|\frac{\partial \hat{w}^{(s)}}{\partial x}\right|^p dx=\sum_{\gamma(s)}\frac{mes\left(\sigma_i^{(s)}\right)}{\left[h^{(s)}\right]^{p-1}}. \tag{60}$$

By (58)-(60), we have

$$\int_{T(\gamma,\delta,s)} \left|\frac{\partial \hat{w}^{(s)}}{\partial x}\right|^p dx \geq \int_{T_i^{(s)}} \left|\frac{\partial \hat{w}^{(s)}}{\partial x}\right|^p dx - \int_{T^+(\gamma,\delta)} \left|\frac{\partial \hat{w}^{(s)}}{\partial x}\right|^p dx - \int_{T^-(\gamma,\delta)} \left|\frac{\partial \hat{w}^{(s)}}{\partial x}\right|^p dx. \tag{61}$$

We make the following assumptions :

$$d_i^{(s)} = o\left(R_i^{(s)}\right) \tag{62}$$

$$\sum_{\gamma(s)} \frac{\left[d_i^{(s)}\right]^{n-1}}{\left[h^{(s)}R_i^{(s)}\right]^p} \leq C, \tag{63}$$

$$\overline{\lim_{s\to\infty}} \sum_{\gamma(s)} \frac{mes\left(\sigma_i^{(s)}\right)}{\left[h^{(s)}\right]^{p-1}} = \int_\gamma c(x)\, d\Gamma, \tag{64}$$

where $c(x)$ is a nonnegative function on Γ, $\sum_{\gamma(s)}$ is the sum over all i for which $\sigma_i^{(s)}$ belong to $\gamma \subset \Gamma$. Assumptions (62) and (64) mean that the holes $\sigma_i^{(s)}$ are sufficiently big and the heigths of the channels $T_i^{(s)}$ are relatively larger compared to the diameters of the holes since $\left[d_i^{(s)}\right]^{n-1} / \left[h^{(s)}\right]^{p-1}$ converge to zero on account of the convergence of the series in (64).

By (62) and (63), we have

$$\begin{aligned} \lim_{\delta\to 0}\lim_{s\to\infty} C \sum_{\gamma(s)} \frac{d_i^{(s)}\left[d_i^{(s)} + R_i^{(s)}/3\right]^{n-1}}{\left[h^{(s)}R_i^{(s)}\right]^p} &= \\ \lim_{\delta\to 0}\lim_{s\to\infty} d_i^{(s)} \sum_{\gamma(s)} \frac{\left[d_i^{(s)}\right]^{n-1}}{\left[h^{(s)}R_i^{(s)}\right]^p} &= 0. \end{aligned}$$

Next we have

$$\begin{aligned} \sum_{\gamma(s)} C_p\left(\sigma_i^{(s)+}\right) / \left[h^{(s)}\right]^p &\leq C \sum_{\gamma(s)} \left[d_i^{(s)}\right]^{n-p-1} / \left[h^{(s)}\right]^p \\ &\leq C \max_s \left[\frac{R_i^{(s)}}{d_i^{(s)}}\right] \sum_{\gamma(s)} \frac{\left[d_i^{(s)}\right]^{n-1}}{\left[h^{(s)}R_i^{(s)}\right]^p}. \end{aligned}$$

The first inequality follows from [36](Chap. 9). (62)-(63) imply

$$\lim_{\delta\to 0}\lim_{s\to\infty} \sum_{\gamma(s)} C_p\left(\sigma_i^{(s)+}\right) = 0.$$

Similarly since

$$\sum_{\gamma(s)} C_p\left(B\left(\sigma_i^{(s)+}, \lambda d_i^{(s)}\right)\right) \leq C\sum_{\gamma(s)}\left(d_i^{(s)}\right)^{n-p} \leq C\max\left\{d_i^{(s)}\right\}\sum_{\gamma(s)}\left[d_i^{(s)}\right]^{n-p-1},$$

we get

$$\lim_{\delta\to 0}\overline{\lim_{s\to\infty}}\sum_{\gamma(s)} C_p\left(B\left(\sigma_i^{(s)+}, \lambda d_i^{(s)}\right)\right) = 0.$$

Thus we have shown that

$$\lim_{\delta\to 0}\lim_{s\to\infty}\int_{T^+(\gamma,\delta)}\left|\frac{\partial \hat{w}^{(s)}}{\partial x}\right|^p dx = 0. \tag{65}$$

Analogously

$$\lim_{\delta\to 0}\lim_{s\to\infty}\int_{T^-(\gamma,\delta)}\left|\frac{\partial \hat{w}^{(s)}}{\partial x}\right|^p dx = 0. \tag{66}$$

Combining (65), (66) with (61), we get from (51)

$$\lim_{\delta\to 0}\underline{\lim_{s\to\infty}} C_{\Delta_p}(\gamma,\delta,s) \geq \lim_{s\to\infty}\sum_{\gamma(s)}\frac{mes\left(\sigma_i^{(s)}\right)}{\left[h^{(s)}\right]^{p-1}} \tag{67}$$

Next we consider in the domain $T(\gamma,\delta,s)$ the function

$$w^{(s)}(x) = \begin{cases} 0 \text{ if } x\in T^-(\gamma,\delta), \\ \dfrac{t}{h^{(s)}} \text{ if } x\in T_i^{(s)}, \\ 1 \text{ if } x\in T^+(\gamma,\delta), \end{cases}$$

where $t = t(x)$ is defined above. $w^{(s)} \in W(\gamma,\delta,s)$ and

$$\int_{T(\gamma,\delta,s)}\left|\frac{\partial w^{(s)}}{\partial x}\right|^p dx = \sum_{\gamma(s)}\frac{mes\left(\sigma_i^{(s)}\right)}{\left[h^{(s)}\right]^{p-1}}.$$

Using the formula (51), we have

$$\overline{\lim_{s\to\infty}} C_{\Delta_p}(\gamma,\delta,s) \leq \lim_{s\to\infty}\sum_{\gamma(s)}\frac{mes\left(\sigma_i^{(s)}\right)}{\left[h^{(s)}\right]^{p-1}}. \tag{68}$$

The inequalities (67) and (68) imply

$$\lim_{\delta\to 0}\overline{\lim_{s\to\infty}} C_{\Delta_p}(\gamma,\delta,s) = \lim_{\delta\to 0}\underline{\lim_{s\to\infty}} C_{\Delta_p}(\gamma,\delta,s) = \lim_{s\to\infty}\sum_{\gamma(s)}\frac{mes\left(\sigma_i^{(s)}\right)}{\left[h^{(s)}\right]^{p-1}}.$$

We have thus proved the following

Theorem 10. *Let the conditions (62)-(64) be satisfied and $n > p+1$, then the sequence of solutions of problem (49)-(50) converges to a function $u(x)$ which is a solution of the problem*

$$\Delta_p u = f, \quad in\ \Omega, \tag{69}$$

$$\left(\frac{\partial u}{\partial \Delta_p}\right)_+ + \left(\frac{\partial u}{\partial \Delta_p}\right)_- = pc(x)\left|u_+ - u_-\right|^{p-2}(u_+ - u_-) \quad on\ \Gamma, \tag{70}$$

$$u = 0 \text{ on } \partial\Omega, \tag{71}$$

where c is the function defined in (64).

Remark 11. The case of Neumann Sieve where the layer $T^{(s)}$ has zero thickness can be considered as well and explicit formula for the function $c(x)$ can be found if the perforations are periodically distributed balls in convenient way. In this case the Δ_p-conductivity of $T(\gamma, \delta, s)$ reduces to its capacity, cf. [35].

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