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STARS OF VIBRATING STRINGS: SWITCHING BOUNDARY FEEDBACK STABILIZATION

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Abstract. We consider a star-shaped network consisting of a single node with $N \geq 3$ connected arcs. The dynamics on each arc is governed by the wave equation. The arcs are coupled at the node and each arc is controlled at the other end. Without assumptions on the lengths of the arcs, we show that if the feedback control is active at all exterior ends, the system velocity vanishes in finite time.

In order to achieve exponential decay to zero of the system velocity, it is not necessary that the system is controlled at all N exterior ends, but stabilization is still possible if, from time to time, one of the feedback controllers breaks down. We give sufficient conditions that guarantee that such a switching feedback stabilization where not all controls are necessarily active at each time is successful.

1. Introduction.

We consider a star-shaped network of $N \geq 3$ finite strings (with possibly different lengths) that are governed by the wave equation. At the boundary point zero the strings are coupled. At the other end of each string a feedback law is prescribed that requires the time derivative at this point to be proportional to the space derivative at this point. For a single string, this feedback law has been considered in [5], and it has been shown that the energy vanishes in finite time. In [7] it is shown that the result from [5] is stable in the sense that also with moving boundaries, the energy is driven to zero in finite time. In this paper we show that also on the network, the energy is driven to zero in finite time if the feedback control is active on all N boundary nodes for a sufficiently long time. Our particular interest in this paper is the question: What happens if at one of the nodes the feedback control becomes inactive? This need not be a fixed boundary node on the whole time interval but the inactivity may switch between different boundary nodes in

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time. The idea is that at each moment, it may happen that one of the N controllers is inactive, and we still want to have a stable system.

The boundary control of the wave equation has been studied by many authors (see e.g. [18], [16], [11], [12], [2], [21] and the references therein). A problem of optimal switching boundary control of a single string to rest in finite time has been considered in [8], where a string with boundary control at both ends has been considered and, at each moment, at most one of the controls is allowed to be active. The corresponding problem for the heat equation has been analyzed in [20] using an adapted adjoint calculus.

Networks of strings have been considered for example in [14, 15, 1, 13, 19] and an overview is given in [6]. In these works about networks, the nodes where a feedback control acts on the system are constant during the control process.

In contrast to this situation, in this paper we consider a system where these nodes may change as time proceeds. We are interested in the question: How many feedback controls must be switched on at each moment in time to achieve exponential decay? We show that $N-1$ controls are sufficient. It is essential that the choice of the inactive control need not be constant but can vary in a quite general way with time.

In a similar spirit in [10] it has been studied how often the control should be active in order to stabilize a string with interior damping. Analogous questions have been addressed in [3, 4] for finite dimensional systems.

This paper has the following structure: First we define the problem of switching feedback boundary stabilization of a network of strings. Then we state our main results, which are two sufficient conditions for exponential decay of the derivatives in our system. First we state a backwards in time condition and then we state a forward condition.

The proofs of the sufficiency of both conditions take advantage of the reformulation of the initial boundary value problem in terms of Riemann invariants. We show that if N feedback nodes are active for a sufficiently long time interval, after finite time the partial derivatives of the solution are equal to zero. For the backward condition, we show that if at each moment in time only one wave arrives at the coupling note of the network, the partial derivatives of the solution go to zero exponentially fast. The proof of the sufficiency of the forward condition is based upon the construction of a suitable Lyapunov function.

2. The system.

Let $N \geq 3$ and consider N strings of length $L_i > 0$ ($i \in \{1, ..., N\}$). Define $L = \max\{L_1, ..., L_N\}$. Let the corresponding wave speeds $c_i > 0$ be given. Define $c = \min\{c_1, ..., c_N\}$. For $i \in \{1, ..., N\}$ define the sets $\Omega_i = (0, \infty) \times (0,\ L_i)$. Define the set

$$B = \{(y_0^{(i)}, y_1^{(i)})_{i=1}^N :\ \partial_x y_0^{(i)} \in L^\infty(0, L_i),\ y_1^{(i)} \in L^\infty(0, L_i),\ i \in \{1, ..., N\},$$
$$y_0^{(i)}(0) = y_0^{(j)}(0),\ i, j \in \{1, ..., N\}\}.$$

For $i \in \{1, ..., N\}$, let $\sigma_i : (0, \infty) \to \{0, 1\}$ be a measurable function. The equation $\sigma_i(t) = 0$ will indicate that at time t the feedback at the end of string i is not active, whereas $\sigma_i(t) = 1$ means that the feedback is active.

Let feedback parameters $\kappa_i > 0$ be given. For $(y_0^{(i)}, y_1^{(i)})_{i=1}^N \in B$ we consider the system (**S**) given by the equations

$$v^{(i)}(0, x) = y_0^{(i)}(x),\ v_t^{(i)}(0, x) = y_1^{(i)}(x),\ x \in (0, L_i),\ i \in \{1, ..., N\} \tag{1}$$

$$v_{tt}^{(i)}(t, x) = c_i^2\, v_{xx}^{(i)}(t, x),\ (t, x) \in \Omega_i,\ i \in \{1, ..., N\} \tag{2}$$

$$v^{(i)}(t,0) = v^{(j)}(t,0),\ t \in (0,\infty), i,j \in \{1,...,N\} \tag{3}$$

$$c_1 v_x^{(1)}(t,0) + c_2 v_x^{(2)}(t,0) + \cdots + c_N v_x^{(N)}(t,0) = 0,\ t \in (0,\infty) \tag{4}$$

$$c_i v_x^{(i)}(t,L_i) = -\sigma_i(t)\ \kappa_i\ v_t^{(i)}(t,L_i),\ t \in (0,\infty),\ i \in \{1,...,N\}. \tag{5}$$

Conditions (1) describe the initial state of the system. The dynamics on the strings is given by the wave equation (2). Equations (3) and (4) describe how the strings are coupled. The feedback control (that is switched off if $\sigma_i(t) = 0$) is given by (5).

Remark 1. With homogeneous Neumann conditions (that is $\sigma_i(t) \equiv 0$) at all nodes, the coupling conditions (3) and (4) guarantee the conservation of the energy of the system (see Section 2.2 for details).

2.1. **Solution of the System.** In this section we define the d'Alembert solution in the sense of characteristics for system (**S**) analogously to Theorem 1 in [9].

Define the number

$$\lambda_{\min} = \min\{L_1/c_1, ..., L_n/c_n\}.$$

For $i \in \{1,...,N\}$ let $\Omega_i = [0,\infty) \times [0, L_i]$. Define the orthogonal symmetric $N \times N$ reverberation matrix

$$A = \frac{N-2}{N} \begin{pmatrix} 1 & \frac{2}{2-N} & \frac{2}{2-N} & \cdots & \frac{2}{2-N} \\ \frac{2}{2-N} & 1 & \frac{2}{2-N} & \cdots & \frac{2}{2-N} \\ \vdots & & \ddots & & \vdots \\ \frac{2}{2-N} & \cdots & \frac{2}{2-N} & 1 & \frac{2}{2-N} \\ \frac{2}{2-N} & \frac{2}{2-N} & \cdots & \frac{2}{2-N} & 1 \end{pmatrix}. \tag{6}$$

Theorem 2.1. [Well-posedness of (**S**)] *Let the initial state $(y_0^{(i)},\ y_1^{(i)})_{i=1}^N \in B$ be given. For $x \in [0, L_i]$ define the functions α_i, β_i by*

$$\alpha_i(x) \;=\; \frac{1}{2} y_0^{(i)}(x) + \frac{1}{2c_i}\int_0^x y_1^{(i)}(s)\,ds,\ x \in (0, L_i), \tag{7}$$

$$\beta_i(x) \;=\; \frac{1}{2} y_0^{(i)}(x) - \frac{1}{2c_i}\int_0^x y_1^{(i)}(s)\,ds,\ x \in (0, L_i). \tag{8}$$

For $t \in [L_i/c_i, 2L_i/c_i]$ let

$$\alpha_i'(c_i t) = \left[\frac{\kappa_i\,\sigma_i\left(t - \frac{L_i}{c_i}\right) - 1}{\kappa_i\,\sigma_i\left(t - \frac{L_i}{c_i}\right) + 1} \right] \beta_i'\left(2L_i - c_i t\right). \tag{9}$$

Together with (7) this yields the values of $\alpha_i'(c_i t)$ for $t \in [0,\ 2\lambda_{\min}]$.

For $t \in [0, 2\lambda_{\min}]$ let the equation

$$\begin{pmatrix} c_1\beta_1'(-c_1 t) \\ c_2\beta_2'(-c_2 t) \\ \vdots \\ c_N\beta_N'(-c_N t) \end{pmatrix} = A \begin{pmatrix} c_1\alpha_1'(c_1 t) \\ c_2\alpha_2'(c_2 t) \\ \vdots \\ c_N\alpha_N'(c_N t) \end{pmatrix} \tag{10}$$

define the values of $\beta_i'(c_i t)$ for $t \in [-2\lambda_{\min}, 0)$.

For $t \in [2L_i/c_i,\ 2\lambda_{\min} + 2L_i/c_i]$ equation (9) yields the values of $\alpha_i'(c_i t)$. Hence for $t \in [0,\ 4\lambda_{\min}]$ the values of $\alpha_i'(c_i t)$ are well-defined. Now (10) defines the values of $\beta_i'(c_i t)$ for $t \in [-4\lambda_{\min}, 0)$.

For $t \in [2L_i/c_i + 2\lambda_{\min},\, 2L_i/c_i + 4\lambda_{\min}]$ equation (9) yields the values of $\alpha_i'(c_i t)$. Hence for $t \in [0,\, 6\lambda_{\min}]$ the values of $\alpha_i'(c_i t)$ are well-defined. Now (10) defines the values of $\beta_i'(c_i t)$ for $t \in [-6\lambda_{\min}, 0)$.

By repeating the process we define the functions $\alpha_i'(c_i t)$, $\beta_i'(-c_i t)$ inductively and obtain functions $\alpha_i' \in L^\infty_{\rm loc}(0,\infty)$ and $\beta_i' \in L^\infty_{\rm loc}(-\infty, L_i)$. For $x > L_i$ define $\alpha_i(x) = \alpha_i(L_i) + \int_{L_i}^x \alpha_i'(s)\,ds$. For $x < 0$, let $\beta_i(x) = \beta_i(0) + \int_0^x \beta_i'(s)\,ds$. Then $(v^{(i)})_{i=1}^N$ given by

$$v^{(i)}(t,x) = \alpha_i(x + c_i t) + \beta_i(x - c_i t) \tag{11}$$

are solutions of $(\mathbf{S})$ *in the sense described below.*

The function $v^{(i)}$ is continuous on Ω_i and $v_t^{(i)}$, $v_x^{(i)} \in L^\infty_{\rm loc}(\Omega_i)$. Given the family of test functions

$$\begin{aligned} T_i \;=\; &\{\varphi \in C^2(\Omega_i) : \text{There exists a set } Q = [t_1,t_2] \times [x_1,x_2] \subset \Omega_i \\ &\text{such that the support of } \varphi \text{ is contained in the interior of } Q\}, \end{aligned}$$

the function $v^{(i)}$ satisfies the wave equation (2) in the following weak sense:

$$\int_{\Omega_i} v_t^{(i)}(t,x)\varphi_t(t,x)\,d(t,x) = c_i^2 \int_{\Omega_i} v_x^{(i)}(t,x)\varphi_x(t,x)\,d(t,x) \text{ for all } \varphi \in T_i.$$

The functions $v^{(i)}$ satisfy (1) for almost every x, (3) for every t, and (4), (5) for almost every t.

Proof. Due to (7), (8), it is easy to check that $(v^{(i)})_{i=1}^N$ satisfies the initial conditions (1). Similar as in the proof of Theorem 1 in [9], integration by parts shows that $v^{(i)}$ satisfies the wave equation in the weak sense given in Theorem 2.1. Writing the boundary conditions (5) in terms of α_i' and β_i' yields equation (9). By definition of $\beta_i'(\cdot)$, for almost every $t > 0$ we have (10). Hence we have

$$\begin{aligned} \begin{pmatrix} v_t^{(1)}(t,0) \\ v_t^{(2)}(t,0) \\ \vdots \\ v_t^{(N)}(t,0) \end{pmatrix} &= \begin{pmatrix} c_1(\alpha_1'(c_1 t) - \beta_1'(-c_1 t)) \\ c_2(\alpha_2'(c_2 t) - \beta_2'(-c_2 t)) \\ \vdots \\ c_N(\alpha_N'(c_N t) - \beta_N'(-c_N t)) \end{pmatrix} = (I - A) \begin{pmatrix} c_1\alpha_1'(c_1 t) \\ c_2\alpha_2'(c_2 t) \\ \vdots \\ c_N\alpha_N'(c_N t) \end{pmatrix} \\ &= \frac{N-2}{N} \begin{pmatrix} \frac{2}{N-2} & \frac{2}{N-2} & \cdots & \frac{2}{N-2} \\ \vdots & \vdots & & \vdots \\ \frac{2}{N-2} & \frac{2}{N-2} & \cdots & \frac{2}{N-2} \end{pmatrix} \begin{pmatrix} c_1\alpha_1'(c_1 t) \\ c_2\alpha_2'(c_2 t) \\ \vdots \\ c_N\alpha_N'(c_N t) \end{pmatrix}, \end{aligned}$$

which implies the equation $v_t^{(i)}(t,0) = v_t^{(j)}(t,0)$ for $t \in (0,\infty)$ and $i, j \in \{1, ..., N\}$. Due to the definition of the set B, this implies that (3) is valid. Moreover, we have

$$\begin{pmatrix} c_1 v_x^{(1)}(t,0) \\ c_2 v_x^{(2)}(t,0) \\ \vdots \\ c_N v_x^{(N)}(t,0) \end{pmatrix} = \begin{pmatrix} c_1(\alpha_1'(c_1 t) + \beta_1'(-c_1 t)) \\ c_2(\alpha_2'(c_2 t) + \beta_2'(-c_2 t)) \\ \vdots \\ c_N(\alpha_N'(c_N t) + \beta_N'(-c_N t)) \end{pmatrix} = (I + A) \begin{pmatrix} c_1\alpha_1'(c_1 t) \\ c_2\alpha_2'(c_2 t) \\ \vdots \\ c_N\alpha_N'(c_N t) \end{pmatrix}$$

$$= \frac{N-2}{N} \begin{pmatrix} \frac{2N-2}{N-2} & \frac{2}{2-N} & \cdots & \frac{2}{2-N} \\ \frac{2}{2-N} & \ddots & & \vdots \\ \vdots & & \ddots & \frac{2}{2-N} \\ \frac{2}{2-N} & \cdots & \frac{2}{2-N} & \frac{2N-2}{N-2} \end{pmatrix} \begin{pmatrix} c_1\alpha_1'(c_1 t) \\ c_2\alpha_2'(c_2 t) \\ \vdots \\ c_N\alpha_N'(c_N t) \end{pmatrix},$$

which implies the equation $c_1 v_x^{(1)}(t,0) + c_2 v_x^{(2)}(t,0) + \cdots + c_N v_x^{(N)}(t,0) = 0$ for almost $t \in (0,\infty)$, hence (4) holds. □

2.2. **The Energy of the System.** Define the energy of the solution of system (**S**) as given in Theorem 2.1 by

$$E(t) = \frac{1}{2}\sum_{i=1}^{N} c_i \int_0^{L_i} \left(\frac{v_t^{(i)}(t,x)^2}{c_i^2} + v_x^{(i)}(t,x)^2 \right) dx.$$

Equation (11) implies

$$v_x^{(i)}(t,\,x) = \alpha_i'(x + c_i t) + \beta_i'(x - c_i t),\; v_t^{(i)}(t,\,x) = c_i[\alpha_i'(x + c_i t) - \beta_i'(x - c_i t)]. \quad (12)$$

Hence we have

$$\frac{v_t^{(i)}(t,x)^2}{c_i^2} + v_x^{(i)}(t,x)^2 = (\alpha_i'(x + c_i t) + \beta_i'(x - c_i t))^2 + (\alpha_i'(x + c_i t) - \beta_i'(x - c_i t))^2$$
$$= 2[\alpha_i'(x + c_i t)^2 + \beta_i'(x - c_i t)^2].$$

Thus we have

$$E(t) = \sum_{i=1}^{N} c_i \int_0^{L_i} \left(\alpha_i'(c_i t + x)^2 + \beta_i'(-c_i t + x)^2\right) dx$$
$$= \sum_{i=1}^{N} c_i \left[\int_{c_i t}^{L_i + c_i t} \alpha_i'(s)^2\, ds + \int_{-c_i t}^{L_i - c_i t} \beta_i'(s)^2\, ds \right]$$

Hence, for almost every $t \geq 0$ the time-derivative of the energy $E(t)$ exists and is given by

$$\dot{E}(t) = \sum_{i=0}^{N} c_i^2 \left[\alpha_i'(L_i + c_i t)^2 - \beta_i'(L_i - c_i t)^2\right] - c_i^2 \left[\alpha_i'(c_i t)^2 - \beta_i'(-c_i t)^2\right]$$
$$= -\sum_{i=0}^{N} \frac{4\kappa_i \sigma_i(t)}{(\kappa_i \sigma_i(t) + 1)^2} c_i^2 \beta_i'(L_i - c_i t)^2 - \|(c_1\alpha_1'(c_1 t), \ldots, c_N \alpha_N'(c_N t))\|^2$$
$$+ \|(c_1\beta_1'(-c_1 t), \ldots, c_N \beta_N'(-c_N t))\|^2$$
$$= -\sum_{i=0}^{N} \frac{4\kappa_i \sigma_i(t)}{(\kappa_i \sigma_i(t) + 1)^2} c_i^2 \beta_i'(L_i - c_i t)^2, \quad (13)$$

where the last equality follows from (10) and the orthogonality of the matrix A. In particular, E is non-increasing.

Notice that, in terms of $v^{(i)}$,

$$\dot{E}(t) = -\sum_{i=0}^{N} \frac{\kappa_i \sigma_i(t)}{(\kappa_i \sigma_i(t) + 1)^2} \left(c_i v_x^{(i)}(t, L_i) - v_t^{(i)}(t, L_i) \right)^2$$

for almost every t.

3. **Main results.** In this section we state the main results of the paper, which provide conditions on the switching functions σ_i that guarantee exponential decay of the first order derivatives of the solutions of system (**S**).

Theorem 3.1. [Switching feedback stabilization of (**S**): Backward Condition] *Consider system* (**S**) *defined in (1)–(5). Let*

$$\lambda = \max\{L_1/c_1, \ldots, L_N/c_N\}.$$

If $\kappa_i = 1$ for all $i \in \{1, ..., N\}$ and there exists a time $T > 0$ such that

$$\sum_{i=1}^{N} \sigma_i \left(t - \frac{L_i}{c_i} \right) = N \text{ almost everywhere on } (T, T + 2\lambda) \tag{14}$$

then the system reaches a constant state after finite time, in the sense that for all $t > T + 2\lambda$ we have $v_x^{(i)}(t, x) = 0$ and $v_t^{(i)}(t, x) = 0$, $x \in (0, L_i)$, $i \in \{1, ..., N\}$.

Define $f = \max\{\frac{2}{N}, \frac{N-2}{N}\}$. If for all $i \in \{1, ..., N\}$ we have

$$\left| \frac{\kappa_i - 1}{\kappa_i + 1} \right| < \frac{1 - f}{\sqrt{N}}$$

and

$$\sum_{i=1}^{N} \sigma_i \left(t - \frac{L_i}{c_i} \right) \geq N - 1 \text{ almost everywhere on } (\lambda, \infty) \tag{15}$$

then the system state converges exponentially fast to a constant state, in the sense that for almost every t in $(0, \infty)$ we have the inequality

$$\operatorname{ess\,sup}\{|v_x^{(i)}(t, x)|,\ |v_t^{(i)}(t, x)| : x \in (0, L_i),\ i \in \{1, ..., N\}\} \leq C \exp\left(\frac{\ln(F)}{2\lambda} t \right) \tag{16}$$

where $F = \sqrt{N} \max_{i \in \{1,\ldots,N\}} \left| \frac{\kappa_i - 1}{\kappa_i + 1} \right| + f < 1$. The decay is uniform with respect to σ, that is the constant C in (16) is independent of the choice of σ verifying (15).

Proof. The proof of the first assertion is given in Section 4.1. The proof of the second assertion is given in Section 4.2. □

Theorem 3.2. [Switching feedback stabilization of (**S**): Forward Condition] *Consider system* (**S**) *defined in (1)–(5). If $\kappa_i = 1$ for all $i \in \{1, ..., N\}$ and there exists a time $T > 0$ such that*

$$\sum_{i=1}^{N} \sigma_i \left(t + \frac{L_i}{c_i} \right) = N \text{ almost everywhere on } (T, T + 2\lambda),$$

then the system reaches a constant state after finite time.

If $\kappa_i > 0$ for all $i \in \{1, ..., N\}$ and

$$\sum_{i=1}^{N} \sigma_i \left(t + \frac{L_i}{c_i}\right) \geq N - 1 \text{ almost everywhere on } (0, \infty) \tag{17}$$

then the energy of the state converges exponentially fast to zero, in the sense that

$$E(t) \leq C_1 \exp\left(-C_2 t\right) E(0), \tag{18}$$

for some $C_1, C_2 > 0$. The decay is uniform with respect to σ and the initial condition, that is, the constants C_1 and C_2 in (18) are independent of $(y_0^{(i)}, y_1^{(i)})_{i=1}^N$ and of the choice of σ verifying (17).

Proof. The proof of the first assertion can be easily obtained by adapting the arguments of Section 4.1. The proof of the second assertion is given in Section 5. □

Remark 2. In the case $N = 2$ the star reduces to a single string. (The optimal switching control problem for the interval case $N = 2$ is studied in [8].) Neither condition (15) nor condition (17) guarantee the exponential stabilization in this case. Indeed, it is possible to construct nonzero periodic solutions of (**S**) satisfying both (15) and (17): Take for simplicity $c_1 = c_2 = c$ and $L_1 = L_2 = L$, in such a way that both (15) and (17) provide the same condition $\sigma_1(t) + \sigma_2(t) \geq 1$. Notice, as in [17], that there exist σ_1 and σ_2 satisfying $\sigma_1 + \sigma_2 \equiv 1$, piecewise constant, $2L/c$-periodic, and an optic ray never touching the boundary points when the damping is active (see Figure 1). This allows to construct a periodic nonzero solution of (**S**)

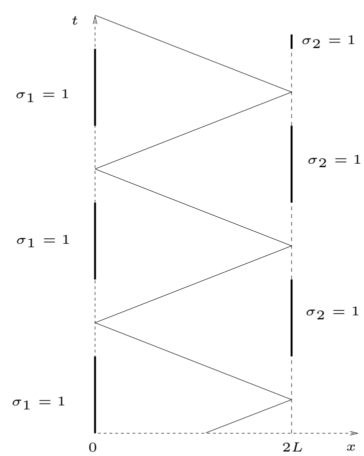

FIGURE 1. Optic ray never hitting the boundary points when the damping is active.

for such choice of σ_1 and σ_2 (see [17] for details).

Remark 3. Conditions (15) and (17) in Theorems 3.1 and 3.2 cannot, in general, be relaxed by taking $N-2$ instead of $N-1$. Indeed, consider the case in which $c_1 = c_2$ and $L_1/L_2 \in \mathbf{Q}$. It is known that there exist nonzero periodic solutions to the uncontrolled wave equation on the network (with Neumann boundary conditions) which are supported on the union of the first two strings. In order to construct any such solution it suffices to consider a string of length $L_1 + L_2$, parameterized on the interval $[0, L_1 + L_2]$, and a periodic solution having L_1 as nodal point. Then, identifying the intervals $[0, L_1]$ and $[L_1, L_1 + L_2]$ with the first and the second string of the network respectively, the extension by zero of the solution on the whole network satisfies $(\mathbf{S})$ with $\sigma_1, \sigma_2 \equiv 0$ (see [6, Section 4.7] for the explicit expression of the solution). This proves the existence of solutions of $(\mathbf{S})$ with $\sum_{i=1}^N \sigma_i(t \pm L_i/c_i) \equiv N - 2$ that do not converge to any constant function.

4. Proof of Theorem 3.1.

4.1. Velocity Decay to Zero in Finite Time. In this section we prove the first part of Theorem 3.1. So assume that for all $i \in \{1, ..., N\}$ we have $\kappa_i = 1$.

If $\sigma_i(t - \frac{L_i}{c_i}) = 1$, equation (9) implies that

$$\alpha_i'(c_i t) = 0.$$

Therefore (14) implies that $\alpha_i'(c_i t) = 0$ for all $i \in \{1, ..., N\}$ and for almost every t in $(T, T + 2\lambda)$. Due to (10) this yields $\beta_i'(-c_i t) = 0$ for all $i \in \{1, ..., N\}$ and for almost every t in $(T, T + 2\lambda)$. Thanks to (12), for almost every x in $(0, L_i)$ we have

$$v_x^{(i)}(T + \lambda,\, x) = 0,\ v_t^{(i)}(T + \lambda,\, x) = 0.$$

So for all $t > T + 2\lambda$, the solution v is constant.

Note in particular that if the feedback is active at all boundary nodes, that is

$$\sum_{i=1}^N \sigma_i(t) = N$$

for a time interval of length greater than or equal to $3\lambda - \lambda_{\min}$, then the state becomes constant in finite time.

4.2. Exponential Decay. In this section we prove the second part of Theorem 3.1.

Lemma 4.1. *Assume that condition (15) is satisfied. Define* $f = \max\{\frac{2}{N},\ \frac{N-2}{N}\}$. *Assume that for all* $i \in \{1, ..., N\}$ *we have*

$$\left|\frac{\kappa_i - 1}{\kappa_i + 1}\right| < \frac{1 - f}{\sqrt{N}}.$$

Define the number

$$F = \sqrt{N} \max_{i \in \{1,\dots,N\}} \left|\frac{\kappa_i - 1}{\kappa_i + 1}\right| + f < 1.$$

Then the following inequality holds for all natural numbers k:

$$\operatorname*{ess\,sup}_{s > 2k\lambda} \max_{i \in \{1,\dots,N\}} \{|c_i \beta_i'(-c_i s)|\} \le F^k \operatorname*{ess\,sup}_{s \in (0, 2\lambda)} \max_{i \in \{1,\dots,N\}} |c_i \beta_i'(-c_i\, s)|. \tag{19}$$

Moreover, for all $t \ge (2k+1)\lambda$ the following inequality holds:

$$\begin{aligned} &\operatorname{ess\,sup}\{c_i |v_x^{(i)}(t, x)|,\ |v_t^{(i)}(t, x)| : x \in (0, L_i),\ i \in \{1, ..., N\}\} \\ &\le \left(1 + \sqrt{N}\right)\ F^k \operatorname*{ess\,sup}_{s \in (0, 2\lambda)} \max_{j \in \{1,\dots,N\}} |c_j \beta_j'(-c_j\, s)|. \end{aligned} \tag{20}$$

Proof. The idea of the proof is that for all $s > 2k\lambda$ we can go backwards in $(0, s)$ until a point in the interval $(0, 2\lambda)$ is reached in at least k steps of length less than or equal to 2λ. In each of these steps, the essential supremum is reduced at least by a factor F.

Define $\gamma = \max_{i\in\{1,...,N\}} \left|\frac{\kappa_i - 1}{\kappa_i + 1}\right|$. Condition (15) implies that for almost every $t > \lambda$ there exists at most one number $k \in \{1, ..., N\}$ with $\sigma_k(t - L_k/c_k) = 0$. Due to (9), we have

$$|\alpha_k'(c_k\, t)| \leq |\beta_k'(-c_k(t - 2L_k/c_k))|.$$

Moreover, for the other $N-1$ derivatives we have the inequality

$$|\alpha_j'(c_j t)| \leq \gamma |\beta_j'(-c_j(t - 2L_j/c_j))|, \quad j \neq k.$$

Define the vectors

$$a(t) = \begin{pmatrix} c_1\alpha_1'(c_1 t) \\ c_2\alpha_2'(c_2 t) \\ \vdots \\ c_N\alpha_N'(c_N t) \end{pmatrix}, \qquad b(t) = \begin{pmatrix} c_1\beta_1'(-c_1 t) \\ c_2\beta_2'(-c_2 t) \\ \vdots \\ c_N\beta_N'(-c_N t) \end{pmatrix}. \tag{21}$$

Due to (10) we have, for almost every $t > 0$,

$$b(t) = Aa(t).$$

Define by $\Pi_k : \mathbf{R}^N \to \mathbf{R}^N$ the orthogonal projection on the k-th coordinate axis. For a vector $z \in \mathbf{R}^n$ let $\|z\|_\infty = \max_{j\in\{1,...,N\}} |z_j|$ denote the corresponding maximum norm and let $\|z\|$ denote the Euclidean norm. On account of the definition (6) of the matrix A we have the inequality

$$\|A\Pi_k a(t)\|_\infty \leq f\, |a_k(t)| \leq f|b_k(t - 2L_k/c_k)|,$$

where a_k and b_k denote the k-th component of a and b respectively.

Since the matrix A is orthogonal we have, for almost every $t > 0$,

$$\begin{aligned}
\|b(t)\|_\infty &= \|A(a(t) - \Pi_k a(t)) + A\Pi_k a(t)\|_\infty \\
&\leq \|A(a(t) - \Pi_k a(t))\|_\infty + \|A\Pi_k a(t)\|_\infty \\
&\leq \|A(a(t) - \Pi_k a(t))\| + \|A\Pi_k a(t)\|_\infty \\
&= \|a(t) - \Pi_k a(t)\| + \|A\Pi_k a(t)\|_\infty \\
&\leq \sqrt{N}\|a(t) - \Pi_k a(t)\|_\infty + \|A\Pi_k a(t)\|_\infty \\
&\leq \sqrt{N}\gamma \max_{j\neq k} |b_j(t - 2L_j/c_j)| + f|b_k(t - 2L_k/c_k)| \\
&\leq \left(\sqrt{N}\gamma + f\right) \max_{j\in\{1,...,N\}} |b_j(t - 2L_j/c_j)| \\
&= F \max_{j\in\{1,...,N\}} |b_j(t - 2L_j/c_j)|.
\end{aligned}$$

Then for all $\tau_2 \geq \tau_1 \geq 2\lambda$ we have the inequality

$$\operatorname{ess}\sup_{t\in[\tau_1,\tau_2]} \|b(t)\|_\infty \leq F \operatorname{ess}\sup_{t\in[\tau_1 - 2\lambda, \tau_2 - 2\lambda_{\min}]} \|b(t)\|_\infty, \tag{22}$$

where we recall that $\lambda_{\min} = \min\{L_1/c_1, \ldots, L_N/c_N\}$.

By induction, for a natural number j and $\tau_2 \geq \tau_1 \geq 2j\lambda$ we get the inequality

$$\operatorname{ess}\sup_{t\in[\tau_1,\tau_2]} \|b(t)\|_\infty \leq F^j \operatorname{ess}\sup_{t\in[\tau_1 - 2j\lambda, \tau_2 - 2j\lambda_{\min}]} \|b(t)\|_\infty.$$

For $\tau_1 = 2j\lambda \leq \tau_2$, this yields

$$\operatorname{ess}\sup_{t\in[2j\lambda,\tau_2]} \|b(t)\|_\infty \leq F^j \operatorname{ess}\sup_{t\in[0, \tau_2 - 2j\lambda_{\min}]} \|b(t)\|_\infty. \tag{23}$$

Using (22) with $\tau_1 = 2\lambda$ and $\tau_2 = T$ in a similar way, for all $T > 2\lambda$ we obtain the inequality

$$\operatorname{ess}\sup_{t\in[0,T]} \|b(t)\|_\infty \leq \operatorname{ess}\sup_{t\in[0,\,T-2\lambda_{\min}]} \|b(t)\|_\infty.$$

By induction, for all $T \geq 2\lambda$ this implies

$$\operatorname{ess}\sup_{t\in[0,T]} \|b(t)\|_\infty \leq \operatorname{ess}\sup_{t\in[0,\,2\lambda]} \|b(t)\|_\infty. \tag{24}$$

With (24), inequality (23) yields

$$\operatorname{ess}\sup_{t\in[2j\lambda,\tau_2]} \|b(t)\|_\infty \leq F^j \operatorname{ess}\sup_{t\in[0,\tau_2-2j\lambda_{\min}]} \|b(t)\|_\infty \leq F^j \operatorname{ess}\sup_{t\in[0,\,2\lambda]} \|b(t)\|_\infty.$$

Since the number τ_2 can be chosen arbitrarily large, this yields (19).

Due to (10) and the fact that the matrix A is orthogonal we have

$$\|a(s)\|_\infty \leq \|a(s)\| = \|b(s)\| \leq \sqrt{N}\|b(s)\|_\infty$$

for almost every $s > 0$.

If $t + x/c_i \geq 2k\lambda$ and $-t + x/c_i \leq -2k\lambda$ for all $i \in \{1, \ldots, N\}$ we have

$$\begin{aligned}\max\{c_i|v_x^{(i)}(t,x)|,\ |v_t^{(i)}(t,x)|\} &\leq \max_{i\in\{1,\ldots,N\}} c_i\left[|\alpha_i'(x+c_it)| + |\beta_i'(x-c_it)|\right]\\ &\leq \left(1+\sqrt{N}\right)\ F^k \operatorname{ess}\sup_{s\in(0,2\lambda)} \|b(s)\|_\infty.\end{aligned}$$

This implies inequality (20). ☐

In order to complete the proof of Theorem 3.1 let us introduce the function

$$\rho(t) = \operatorname{ess\,sup}\{c_i|v_x^{(i)}(t,x)|,\ |v_t^{(i)}(t,x)| : x \in (0,L_i),\ i \in \{1,...,N\}\}.$$

Due to (12) we have

$$\rho(t) \leq \max_{i\in\{1,\ldots,N\}} \operatorname{ess\,sup}\{c_i(|\alpha_i'(x+c_it)| + |\beta_i'(x-c_it)|) : x \in (0,L_i)\}.$$

We claim that ρ satisfies

$$\rho(t) \leq C_0, \qquad t \in (0,3\lambda), \tag{25}$$

$$\rho(t) \leq C_1\,F^k, \quad t > (2k+1)\lambda, \quad k \in \mathbf{N}, \tag{26}$$

with C_0 and C_1 only depending on the initial condition $(y_0^{(i)}, y_1^{(i)})_{i=1}^N$ and not on the choice of σ verifying (15).

For $x \in (0,L_i)$, the values of $\alpha_i(x)$ are given by (7) and the values of $\beta_i(x)$ are given by (8). Hence the constant

$$C_* = 2\max_{i\in\{1,\ldots,N\}} \operatorname{ess\,sup}\{c_i|\alpha_i'(x)|,\ c_i|\beta_i'(x)| : x \in (0,L_i)\}$$

only depends on the initial condition and satisfies $\rho(0) \leq C_*$. Equation (9) yields the inequality

$$c_i|\alpha_i'(c_it)| \leq c_i|\beta_i'(2L_i - c_it)| \leq C_*/2, \quad t \in (L_i/c_i, 2L_i/c_i).$$

Hence for almost every $t \in (0, 2L_i/c_i)$ we have $c_i|\alpha_i'(c_it)| \leq C_*/2$. Since A has the matrix norm $\|A\|_\infty = 3 - \frac{4}{N}$, the node condition (10) implies that

$$c_i|\beta_i'(-c_it)| \leq \left(3 - \frac{4}{N}\right)\max\{c_j|\alpha_j'(c_jt)| : j = 1,...,N\} \tag{27}$$

for almost every $t > 0$.

Moreover, due again to (9), we have

$$c_i|\alpha_i'(c_it)| \le \left(3 - \frac{4}{N}\right) \max\{c_j|\alpha_j'(c_j(t-2L_i/c_i))| : j = 1, ..., N\}, \quad t > 2L_i/c_i.$$

Hence, by recurrence, for all integer k and almost all $t < 2k\lambda_{\min}$,

$$c_i|\alpha_i'(c_it)| \le \frac{C_*}{2}\left(3 - \frac{4}{N}\right)^{k-1}.$$

As a consequence, if $t < 2k\lambda_{\min}$, then, for almost every $x \in (0, L_i)$,

$$\begin{aligned} c_i|\alpha_i'(c_it + x)| &\le \frac{C_*}{2}\left(3 - \frac{4}{N}\right)^{k}, \\ c_i|\beta_i'(x - c_it)| &\le \frac{C_*}{2}\left(3 - \frac{4}{N}\right)^{k}, \end{aligned}$$

where the second inequality uses (27). It follows that

$$\rho(t) \le C_*\left(3 - \frac{4}{N}\right)^{k}, \quad t < 2k\lambda_{\min}.$$

Now we choose $\bar{k}$ such that $3\lambda < 2\bar{k}\lambda_{\min}$ and set

$$C_0 = C_*\left(3 - \frac{4}{N}\right)^{\bar{k}}.$$

If $t > (2k+1)\lambda$ for some integer k inequality (20) implies

$$\begin{aligned} \rho(t) &\le \left(1 + \sqrt{N}\right) F^k \operatorname{ess} \sup_{s\in(0,2\lambda)} \max_{j\in\{1,\ldots,N\}} |c_j\beta_j'(-c_js)| \\ &\le \left(1 + \sqrt{N}\right) C_0 F^k. \end{aligned}$$

Hence we choose

$$C_1 = \left(1 + \sqrt{N}\right) C_0.$$

Since $\rho(t)$ satisfies (25) and (26), it can be bounded from above by $F^{-2}\max\{C_0, C_1\}\exp\left(\frac{\ln(F)}{2\lambda}t\right)$.

5. **Proof of Theorem 3.2.** Let ε be a positive real number to be fixed later and define, for $i = 1, ..., N$,

$$\Xi_i = \left\{ x \in \mathbf{R}^N : \sum_{j\ne i} x_j^2 < \varepsilon x_i^2 \right\}.$$

Hence, Ξ_i is a cone with axial symmetry with respect to the axis spanned by the i-th vector of the canonical basis of $\mathbf{R}^N$. Let $\Xi = \cup_{i=1}^N \Xi_i$.

Lemma 5.1. *There exists $k_1 > 0$ depending only on ε such that, if x belongs to $\mathbf{R}^N \setminus \Xi$, then*

$$\min_{i\in\{1,\ldots,N\}} \sum_{j\ne i} x_j^2 \ge k_1\|x\|^2. \tag{28}$$

Proof. Let i be an index achieving the minimization in (28). Since $x \notin \Xi_i$, then

$$\sum_{j\neq i} x_j^2 \geq \frac{1}{2}\sum_{j\neq i} x_j^2 + \frac{\varepsilon x_i^2}{2} \geq \min\left(\frac{1}{2},\frac{\varepsilon}{2}\right)\|x\|^2$$

and the lemma is proved. □

Let a and b be defined as in (21). We can consider a and b as measurable vector-valued functions defined on the entire half-line $[0,+\infty)$. Define the set

$$\Theta = \{t \geq 0 : b(t) \in \Xi\}.$$

Lemma 5.2. *Let $k_2 \in (0, \min\{2, N-2\}/N)$. Then, for every $\varepsilon > 0$ small enough, almost every $t \in \Theta$ and every $i = 1, ..., N$,*

$$|a_i(t)| \geq k_2\|a(t)\|. \tag{29}$$

Proof. Let

$$\Upsilon = \{x \in \mathbf{R}^N : |x_i| \geq k_2\|x\| \text{ for every } i = 1, ..., N\}.$$

In order to prove the lemma, we have to show that for every $\varepsilon > 0$ small enough and almost every $t \in \Theta$, $a(t) \in \Upsilon$.

Since A is idempotent and because of (10), $a(t) \in A\Xi$ for almost every $t \in \Theta$. Notice that $A\Xi$ is the union of the N cones with axial symmetry with respect to the columns of A and with the same aperture as the Ξ_i's.

We have to show that for every $\varepsilon > 0$ small enough $A\Xi$ is contained in Υ. It suffices to notice that the boundary of Υ is invariant by multiplication by a scalar and that each vector corresponding to a column of A is in the interior of Υ. (Indeed, if x is a column of A, then $\|x\| = 1$ and $|x_i| = (N-2)/N$ or $|x_i| = 2/N$.) Then for ε small enough every vector of $A\Xi \setminus \{0\}$ belongs to the interior of Υ. □

In the following ε will be fixed fulfilling the smallness requirement of Lemma 5.2.

Lemma 5.3. *Let*

$$\mathcal{T} = \{\tau : \tau \geq \lambda,\ \sigma_i(\tau - L_i/c_i) = 0 \text{ for every } i = 1, \ldots, N\}.$$

If σ satisfies (17), then, for almost every $t \in \mathcal{T} \cap (4\lambda, +\infty)$, $t - 2L_i/c_i \in \mathcal{T}$ for at most one $i \in \{1, ..., N\}$.

Proof. Assume by contradiction that for all t in a subset of positive measure of $\mathcal{T} \cap (4\lambda, +\infty)$ we have $t - 2L_i/c_i, t - 2L_j/c_j \in \mathcal{T}$ with $i \neq j$. In particular, for all t is such set,

$$\sigma_i\left(t - 2\frac{L_j}{c_j} - \frac{L_i}{c_i}\right) = \sigma_j\left(t - 2\frac{L_i}{c_i} - \frac{L_j}{c_j}\right) = 0.$$

This implies that condition (17) is not satisfied when we take as t the time $t - 2(L_i/c_i + L_j/c_j)$ (see Figure 2). Thus (17) is not satisfied on a set of positive measure and the contradiction is reached. □

Let us complete the proof of Theorem 3.2.

The time-derivative of the energy $E(t)$ is given by (13). Notice that

$$\dot{E}(t) = -4\sum_{i=1}^{N} \frac{\kappa_i\sigma_i(t)}{(\kappa_i\,\sigma_i\,(t)+1)^2} c_i^2\beta_i'(L_i - c_i t)^2 \leq -\nu\sum_{i=1}^{N}\sigma_i(t)c_i^2\beta_i'(L_i - c_i t)^2,$$

where

$$\nu = 4\min\left\{\frac{\kappa_1}{(\kappa_1+1)^2}, \ldots, \frac{\kappa_N}{(\kappa_N+1)^2}\right\} > 0.$$

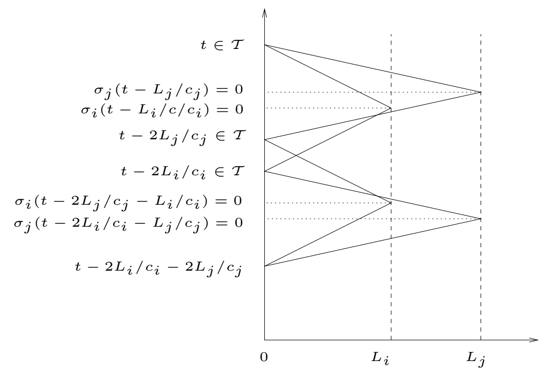

FIGURE 2. Contradiction argument for $t, t-2L_i/c_i, t-2L_j/c_j \in \mathcal{T}$.

Let

$$F(t) = \sum_{i=1}^{N} E\left(t + \frac{L_i}{c_i}\right) + \sum_{i=1}^{N} E\left(t - \frac{L_i}{c_i}\right).$$

Then

$$\dot{F}(t) \leq -\nu \sum_{i=1}^{N} \sigma_i\left(t + \frac{L_i}{c_i}\right) b_i(t)^2 - \nu \sum_{i=1}^{N} \sigma_i\left(t - \frac{L_i}{c_i}\right) c_i^2 \beta_i'(2L_i - c_i t)^2.$$

Lemma 5.1 and condition (17) guarantee that if $t \notin \Theta$, then

$$\dot{F}(t) \leq -\nu k_1 \|b(t)\|^2 = -\nu k_1 \|a(t)\|^2.$$

On the other hand, for almost every $t \in \Theta$, (29) holds. Hence, if $t \in \Theta \cap (\lambda, +\infty)$ and $\sigma_i(t - L_i/c_i) = 1$ for some $i \in \{1, ..., N\}$, then, thanks to (9), either $a(t) = 0$ or $\kappa_i \neq 1$ and

$$|\beta_i'(2L_i - c_i t)| = \frac{\kappa_i + 1}{|\kappa_i - 1|} |\alpha_i'(c_i t)|,$$

leading to

$$\dot{F}(t) \leq -\nu \frac{\kappa_i + 1}{|\kappa_i - 1|} k_2 \|a(t)\|^2.$$

We proved that for almost every $t \in (\lambda, +\infty) \setminus \mathcal{T}$,

$$\dot{F}(t) \leq -k_3 \|a(t)\|^2,$$

with $k_3 > 0$ not depending on the initial condition nor on the choice of $(\sigma_1, \ldots, \sigma_N)$ satisfying (17).

According to Lemma 5.3, moreover, for almost every $t \in \mathcal{T} \cap \Theta \cap (4\lambda, \infty)$, for all but possibly one $i \in \{1, ..., N\}$ we have

$$\begin{aligned}\dot{F}(t - 2L_i/c_i) &\leq -k_3 \|a(t - 2L_i/c_i)\|^2 = -k_3 \|b(2L_i/c_i - t)\|^2 \\ &\leq -k_3 |b_i(2L_i/c_i - t)|^2 = -k_3 |a_i(t)|^2 \leq -k_2^2 k_3 \|a(t)\|^2.\end{aligned}$$

Let $G(t) = F(t) + \sum_{i=1}^{N} F(t - 2L_i/c_i)$. Then, for almost every $t > 4\lambda$,

$$\dot{G}(t) \leq -(N-1) k_2^2 k_3 \|a(t)\|^2, \tag{30}$$

where we used the inequality $(N-1)k_2^2 \leq 1$.

Notice that

$$\begin{aligned} E(t) &= \sum_{i=1}^{N} \frac{1}{c_i} \int_0^{L_i} \left(|a_i(t + x/c_i)|^2 + |b_i(t - x/c_i)|^2\right) dx \\ &= \sum_{i=1}^{N} \frac{1}{c_i} \int_0^{L_i} \left(|a_i(t + x/c_i)|^2 + |a_i(t - x/c_i)|^2\right) dx \\ &= \sum_{i=1}^{N} \int_{t-L_i/c_i}^{t+L_i/c_i} \left(|a_i(s)|^2\right) dx \leq \int_{t-\lambda}^{t+\lambda} \|a(s)\|^2 ds. \end{aligned}$$

This and (30) imply that for almost every $t > 5\lambda$ the inequality

$$G(t+\lambda) - G(t-\lambda) \leq -(N-1)k_2^2 k_3 E(t)$$

holds true.

By monotonicity of E and definition of G,

$$G(t) \leq 2N(N+1)E(t-3\lambda)$$

so that

$$G(t+\lambda) - G(t-\lambda) \leq -\frac{N-1}{2N(N+1)} k_2^2 k_3 G(t+3\lambda).$$

Hence $G(t)$ decays exponentially to zero as t goes to infinity. Moreover, $G(5\lambda) \leq 2N(N+1)E(0)$. Since c, k_2, k_3, N and λ do not depend on the initial conditions nor on σ, we have that $G(t) \leq C_1 \exp(-C_2 t) E(0)$ with C_1 and C_2 independent of $(y_0^{(i)}, y_1^{(i)})_{i=1}^N$ and of the choice of σ verifying (17).

Inequality (18) follows and this concludes the proof of Theorem 3.2. □

6. **Conclusion.** For a single string it is well known that a velocity feedback at one end with a special feedback parameter steers the solution to a constant state in finite time; the semigroup describing the corresponding solution is nilpotent. For a larger set of feedback parameters the energy decays exponentially.

In this paper we prove that a similar situation occurs for star-shaped networks with boundary feedback at all boundary nodes: For the special feedback parameter the partial derivatives of the solution vanish after finite time and the system state becomes constant.

If the feedback parameter is chosen in a neighborhood of the special parameter and the feedback is switched off at the boundary nodes in such a way that the l^1-norm of the switching vector $(\sigma_i(\cdot - L_i/c_i))_{i=1}^N$ is larger than or equal to $N-1$ almost always, then the partial derivatives of the solution decay exponentially fast in L^∞. Note that the node where the control is switched off need not be constant but can vary with time.

An even stronger result is obtained considering the switching vector $(\sigma_i(\cdot + L_i/c_i))_{i=1}^N$, since in this case the exponential decay of the energy holds for all strictly positive feedback parameters.

These results may be interpreted as the following robustness property of the system: The exponential stability for the boundary feedback stabilization of a star-shaped network is not destroyed if at each moment (up to suitable time-shifts) one of the feedback controllers is not active.

It is a natural question (still open, up to our knowledge) whether an analogous result holds true without considering time-shifts, that is, under the hypothesis that the l^1-norm of the switching vector $(\sigma_i(\cdot))_{i=1}^N$ is larger than or equal to $N-1$ almost always.

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