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QUASISTATIC EVOLUTION FOR CAM-CLAY PLASTICITY: THE SPATIALLY HOMOGENEOUS CASE

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Abstract. We study the spatially uniform case of the quasistatic evolution in Cam-Clay plasticity, a relevant example of small strain nonassociative elastoplasticity. Introducing a viscous approximation, the problem reduces to determine the limit behavior of the solutions of a singularly perturbed system of ODE's in a finite dimensional Banach space. Depending on the sign of two explicit scalar indicators, we see that the limit dynamics presents, under quite generic assumptions, the alternation of three possible regimes: the elastic regime, when the limit equation is just the equation of linearized elasticity; the slow dynamics, when the stress evolves smoothly on the yield surface and plastic flow is produced; the fast dynamics, which may happen only in the softening regime, when viscous solutions exhibit a jump determined by the heteroclinic orbit of an auxiliary system. We give an iterative procedure to construct a viscous solution.

Contents

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1. **Introduction.** The modified Cam-Clay model has been introduced in the engineering literature on soil mechanics as a conceptual tool to understand the irreversible deformations experienced by some fine grained soils (clays); one of the interesting features of this model is that, depending on the loading conditions, the stress-strain response may exhibit a hardening or a softening behavior. Furthermore, it is an important example of nonassociative plasticity.

A general approach to the instabilities due to the softening regime has been developed in [3] using a vanishing viscosity approximation. The goal of the present paper, however, is rather different. We study the spatially homogeneous case in dimension N, with no volume forces. In this simplified setting we do not investigate the well-posedness of the problem, which is the object of [3]. Instead we carry out a qualitative study of the limit behavior of the solutions as the viscosity parameter ε goes to 0. This is done using only differential equations techniques and disregarding the variational structure of (part of) the problem. A similar study was done in [2] for a particular loading program and for a very special yield surface. Here we extend the results of that paper to a very general class of loading paths and yield surfaces, subject only to minor restrictions.

To be definite, we assume that the system is driven by a time-dependent affine boundary condition $w(t,x)$, whose symmetrized spatial gradient $Ew(t,x)$ is independent of the space variable x and is denoted by $\xi(t)$. In this situation, the displacement $u(t,x)$ coincides with $w(t,x)$ and the unknowns, independent of x, are the elastic part $e(t)$ and the plastic part $p(t)$ appearing in the additive decomposition of the strain $Eu(t,x) = e(t) + p(t)$, as well as a scalar internal variable $z(t)$, which describes the time evolving yield surface. The stress $\sigma(t)$ is determined by the elastic part of the strain through the usual relation $\sigma(t) = \mathbb{C}e(t)$, where $\mathbb{C}$ is the tensor of elastic moduli.

One ingredient of the model is a closed convex cone $K \subset \mathbb{M}^{N\times N}_{sym} \times [0,+\infty)$, where $\mathbb{M}^{N\times N}_{sym}$ is the space of symmetric $N\times N$ matrices. It is assumed that K contains the half-line $\{0\}\times[0,+\infty)$. The stress is constrained by the inclusion $\sigma(t) \in K(z(t))$, where for every $z \in [0,+\infty)$ we define $K(z) := \{\sigma \in \mathbb{M}^{N\times N}_{sym} : (\sigma, z) \in K\}$. The interior of $K(z)$ is the elastic domain corresponding to the value z of the internal variable, while its boundary $\partial K(z)$ is the yield surface. In the typical applications, $\partial K(z)$ is a suitable ellipsoid in the space $\mathbb{M}^{N\times N}_{sym}$. Due to mathematical reasons, we shall impose some restrictions on $K(z)$ (see (2.14)-(2.17)), even if most of the results can be proved without these additional assumptions.

The other ingredients of the model are the evolution laws for $p(t)$ and $z(t)$, resulting in the system

$$\begin{cases} e(t) + p(t) = \xi(t)\,, \qquad \sigma(t) = \mathbb{C}e(t) \in K(z(t))\,, \\ \dot p(t) \in N_{K(z(t))}(\sigma(t))\,, \\ \dot z(t) = \operatorname{tr}(\sigma(t))\operatorname{tr}(\dot p(t))\,, \end{cases} \tag{1.1}$$

where $N_{K(z)}(\sigma)$ denotes the normal cone to $K(z)$ at σ, in the sense of convex analysis. The nonassociative nature of the problem is due to the fact that the second equation in (1.1) does not depend on K. In view of the hypotheses on K, we have the monotonicity condition $z_1 < z_2 \Rightarrow K(z_1) \subset K(z_2)$. Therefore if $\dot z(t) > 0$ the set $K(z(t))$ expands leading to a hardening response. On the contrary, if $\dot z(t) < 0$ the set $K(z(t))$ shrinks leading to a softening response. We shall assume that $\operatorname{tr}(\sigma) \le 0$ for every $\sigma \in K(z)$, which reflects the compressive conditions typical of soil mechanics. Therefore, by the second equation in (1.1), the hardening or softening behaviour is determined only by the sign of $\mathrm{tr}(\dot p)$.

To deal with the instabilities of the softening regime, we propose a viscosity approximation to (1.1), in agreement with [3]. Denoting the minimal distance projection of σ onto $K(z)$ by $\pi_{K(z)}(\sigma)$, for every $\varepsilon > 0$ we consider the unconstrained system

$$\begin{cases} e_\varepsilon(t) + p_\varepsilon(t) = \xi(t)\,, \qquad \sigma_\varepsilon(t) = \mathbb{C}e_\varepsilon(t)\,, \\ \dot p_\varepsilon(t) = N^\varepsilon_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t))\,, \\ \dot z_\varepsilon(t) = \mathrm{tr}(\sigma_\varepsilon(t))\,\mathrm{tr}(\dot p_\varepsilon(t))\,, \end{cases} \tag{1.2}$$

where $N^\varepsilon_{K(z)}(\sigma) := \frac{1}{\varepsilon}(\sigma - \pi_{K(z)}(\sigma))$ is the usual approximation of the normal to $K(z)$. A viscosity solution $(e(t), p(t), \sigma(t), z(t))$ to (1.1) is defined as a left continuous map which, for almost every time t, is the pointwise limit of a sequence $(e_\varepsilon(t), p_\varepsilon(t), \sigma_\varepsilon(t), z_\varepsilon(t))$ of solutions of (1.2). Notice that system (1.2) is slightly different from the one considered in [2], where a particular case has been studied; here indeed, in the equation for the internal variable, the term $\mathrm{tr}(\pi_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t)))$ is replaced by simply $\mathrm{tr}(\sigma_\varepsilon(t))$, in agreement with [3].

In this paper we study in detail the limit behavior as ε goes to 0 of the solutions of (1.2). We will see that the limit dynamics presents, for a generic choice of the initial data – some degenerate cases have indeed to be excluded – the alternation of three possible regimes:

a) **Elastic regime**. This situation occurs in a time interval $[t_1, t_2]$ when the plastic part, and thus the internal variable, do not evolve, while the stress is completely determined by the prescribed boundary displacement through the relation $\sigma(t) = \mathbb{C}(\xi(t) - \xi(t_1))$, for every $t \in [t_1, t_2]$; a necessary condition for this behavior to occur is clearly $(\mathbb{C}(\xi(t) - \xi(t_1)), z(t_1)) \in K$ for every $t \in [t_1, t_2]$.
b) **Slow dynamics**. In this situation the stress evolves smoothly on the yield surface and the limit equation (3.1), called the equation of the slow dynamics, takes into account the production of plastic flow. The evolution can be studied using the standard time t; during this regime both hardening and softening behavior can occur.
c) **Fast dynamics**. In the softening regime, a singular behavior can occur, which requires the use of a *fast time* $s := \frac{1}{\varepsilon}t$. The corresponding limit equation (4.1) is called the equation of the fast dynamics. We will see that, at a jump time t, the right limit $(\sigma(t+), z(t+))$ of the solution is given by the asymptotic value for $s \to +\infty$ of the heteroclinic solution of the equation of the fast dynamics (4.1) issuing from the point $(\sigma(t-), z(t-))$ at $s = -\infty$.

As in the associative case, studied in [7] and in [1, Section 7], the alternation of these three regimes is determined by the sign of two scalar indicators; the first one, depending explicitly on time and on the state of the system, will be called the *elastic-inelastic indicator*. It is given by

$$\Phi(t, \sigma, z) := n_{K(z)}(\sigma) \cdot \mathbb{C}\dot\xi(t)$$

for every $(t, \sigma, z) \in [0, +\infty] \times \partial K$. Here $n_{K(z)}(\sigma)$ denotes the outward unit normal to $K(z)$ at σ. The second one, only depending on the state of the system, will be called the *slow-fast indicator*; its explicit expression is given by

$$\Psi(\sigma, z) := -n_{K(z)}(\sigma) \cdot \mathbb{C}n_{K(z)}(\sigma) - \tfrac{\mathrm{tr}(\sigma)\ \mathrm{tr}(n_{K(z)}(\sigma))}{z}[\sigma \cdot n_{K(z)}(\sigma)]$$

for every $(\sigma, z) \in \partial K$. For mathematical reasons, both indicators will be suitably extended to the whole space, but what only matters are their values on the yield surface.

The main difference with the model studied in [7] is in the nonassociative nature of the problem. Here indeed the inner variable does not follows to an associative flow rule depending on K. It is governed by a different equation, which destroys the variational structure of the problem. Moreover, this equation allows both for hardening and softening behavior, while in [1] and [7] only softening is considered. Therefore, the indicator Ψ has a different form and the limit equations (3.1) and (4.1) are rather different from those studied in [7]. In particular, showing the existence of a heteroclinic orbit joining two equilibrium points of (4.1) where the indicator Ψ takes different signs is a harder task than in [7] and needs further hypotheses on the yield surface (see Sections 2 and 4). Nevertheless, to show the convergence of the viscoplastic solutions to a limit satisfying either (3.1) or (4.1) we can use some methods developed in [7]. This is why the proof of some technical lemmas needs only a slight adaptation of the corresponding results in [7].

We now briefly describe how the two indicators determine the limit dynamics. We take an initial condition $(\sigma_0, z_0) \in \operatorname{int} K$; then initially the solution is following the elastic regime, till it reaches the yield surface at a time t_1 at a certain point (σ_1, z_1). Here the elastic-inelastic indicator must be nonnegative. In a generic situation it will be strictly positive, and this determines the appearance of a plastic behavior after the time t_1. The choice between the slow and the fast dynamics depends on the sign of the slow-fast indicator.

a) If $\Psi(\sigma_1, z_1) < 0$ the solution has no jump and is obtained by solving the system of the slow dynamics

$$\begin{cases} \dot\sigma_{sl}(t) &= \frac{\Phi(t,\sigma_{sl}(t),z_{sl}(t))}{\Psi(\sigma_{sl}(t),z_{sl}(t))}\, \mathbb{C}\, n_{K(z_{sl}(t))}(\sigma_{sl}(t)) + \mathbb{C}\dot\xi(t), \\ \dot z_{sl}(t) &= -\frac{\Phi(t,\sigma_{sl}(t),z_{sl}(t))}{\Psi(\sigma_{sl}(t),z_{sl}(t))}\, \operatorname{tr}(\sigma_{sl}(t))\, \operatorname{tr}(n_{K(z_{sl}(t))}(\sigma_{sl}(t))), \end{cases} \tag{1.3}$$

defined on ∂K, with Cauchy data (σ_1, z_1) at time t_1; this situation is studied in Section 3. This behavior persists as long as one of the two indicators does not vanish along the motion.

If at a time $\bar t$ we have that $\Phi(\bar t, \sigma_{sl}(\bar t), z_{sl}(\bar t)) = 0$, while Ψ remains strictly negative, elastic behavior may reappear starting from the point $(\sigma_{sl}(\bar t), z_{sl}(\bar t))$ in presence of suitable higher order conditions. This situation is discussed in Section 3.3.

If Φ remains stricly positive, the solution follows the equation of the slow dynamics for all its maximal interval of existence, that is to say as long as Ψ does not vanish.

b) If $\Psi(\sigma_1, z_1) > 0$ the solution is discontinuous at time t_1 and jumps to the limit as $s \to +\infty$ of the solution of the problem

$$\begin{cases} \dot\sigma_f(s) = \mathbb{C}(\pi_{K(z_f(s))}(\sigma_f(s)) - \sigma_f(s)), \\ \dot z_f(s) = \operatorname{tr}(\sigma_f(s))\operatorname{tr}(\sigma_f(s) - \pi_{K(z_f(s))}(\sigma_f(s))), \\ \lim\limits_{s\to-\infty} (\sigma_f(s), z_f(s)) = (\sigma_1, z_1), \end{cases} \tag{1.4}$$

which is formally obtained by rescaling time in (1.2) according to $s = \frac{t}{\varepsilon}$, and neglecting all terms of order ε. This situation is studied in Section 4. We will see that the internal variable is decreasing along the solution of (1.4), thus we are in the softening regime in this case. At the end of the jump the slow-fast indicator is nonpositive (in some cases, see for instance Example 4.4, we can prove that it is always strictly negative); excluding the degenerate case when it vanishes, this means that the evolution is continuous in a right neighborhood of t_1 and follows the elastic regime or the slow dynamics equation, depending on the sign of the elastic-inelastic indicator. Moreover in the latter case we prove that softening behavior occurs after the end of the jump.

c) If, during a continuous evolution, the indicator Ψ vanishes at a time t_2 in a point (σ_2, z_2) on the yield surface (we will see that this situation can never occur as long as we are in the hardening regime), the following higher order condition must be satisfied

$$\nabla\Psi(\sigma_2, z_2) \cdot \big(\tfrac{-\mathbb{C}\, n_{K(z_2)}(\sigma_2)}{\mathrm{tr}(\sigma_2)\,\mathrm{tr}(n_{K(z_2)}(\sigma_2))}, 1\big) \le 0; \tag{1.5}$$

if strict inequality holds, this implies a transition from the slow dynamics to the fast dynamics regime. Also this case will be discussed in Section 4. Then, the viscous solution is discontinuous at time t_2 and jumps to the limit as $s \to +\infty$ of the solution of problem (1.4), with (σ_2, z_2) in place of (σ_1, z_1). At the end of the jump, exactly as in case b), the evolution is continuous and follows the elastic regime or the slow dynamics equation, with softening behavior, depending on the sign of the elastic-inelastic indicator.

By repeating our arguments at each critical time, we can completely describe the solution, except for some degenerate cases. The precise statement is given in Theorem 5.2. It gives an iterative procedure to construct explicitly a viscous solution, upon the verification of some nondegeneracy hypotheses at each step. If these hypotheses are satisfied, the viscous solution is also unique.

2. **Formulation of the problem and preliminary results.** Let $\mathbb{M}^{N\times N}_{sym}$ be the vector space of all symmetric $N \times N$ matrices with real entries, endowed with the scalar product $\sigma \cdot \xi := \sum_{ij} \sigma_{ij}\xi_{ij}$; the norm of $\sigma \in \mathbb{M}^{N\times N}_{sym}$ will be denoted by $|\sigma|$.

Let K be a closed convex cone in $\mathbb{M}^{N\times N}_{sym} \times [0, +\infty)$. For every $z \in [0, +\infty)$ we define

$$K(z) := \{\sigma \in \mathbb{M}^{N\times N}_{sym} : (\sigma, z) \in K\}\,.$$

Each set $K(z)$ is closed and convex, and we have

$$K(z) = z\,K(1) \qquad \text{for every } z \in (0, +\infty)\,. \tag{2.1}$$

Throughout the paper, we shall assume that $K(1)$ is a bounded domain of class C^2 and that $0 \in \partial K(1)$, hence

$$0 \in \partial K(z) \qquad \text{for every } z \in [0, +\infty)\,, \tag{2.2}$$

and

$$|\sigma| \le M_K z \qquad \text{for every } (\sigma, z) \in K \tag{2.3}$$

for a suitable constant $M_K < +\infty$. For every $z > 0$, we obviuosly have

$$\sigma \in \partial K(z) \Longleftrightarrow (\sigma, z) \in \partial K. \tag{2.4}$$

For every $\sigma \in \partial K(z)$, we will denote the outward unit normal to $K(z)$ at σ by $n_{K(z)}(\sigma)$, while $n_K(\sigma, z)$ will denote the outward unit normal to K at (σ, z).

We shall also assume that

$$\mathrm{tr}(\sigma) \le 0 \text{ for every } \sigma \in K(1); \tag{2.5}$$

this reflects the compressive conditions typical of soil mechanics.

For every closed convex set $C \subset \mathbb{M}^{N\times N}_{sym}$ let $\pi_C\colon \mathbb{M}^{N\times N}_{sym} \to C$ be the minimal distance projection onto C. It follows from (2.1) that

$$\pi_{K(z)}(\sigma) = z\pi_{K(1)}(\tfrac{1}{z}\sigma) \tag{2.6}$$

for every $z>0$ and every $\sigma \in \mathbb{M}^{N\times N}_{sym}$. We also define, for every $(\sigma,z) \in \mathbb{M}^{N\times N}_{sym} \times (0,+\infty)$, the function

$$\varrho(\sigma,z) = |\sigma - \pi_{K(z)}(\sigma)|; \tag{2.7}$$

it is a Lipschitz function, moreover it is C^1 for every $(\sigma,z) \in [\mathbb{M}^{N\times N}_{sym} \times (0,+\infty)]\setminus K$. An elementary consequence of (2.6) is the following relation:

$$\varrho(\sigma,z) = z\,\varrho(\tfrac{\sigma}{z},1) \qquad \text{for every } (\sigma,z) \in \mathbb{M}^{N\times N}_{sym} \times (0,+\infty). \tag{2.8}$$

The next proposition collects some elementary properties which will be useful in what follows.

Proposition 2.1. *Let K be a closed convex cone in $\mathbb{M}^{N\times N}_{sym}\times[0,+\infty)$, and let $K(z)$ be as in (2.1). Assume that $K(1)$ is bounded and of class C^2 and that $0 \in \partial K(1)$. Then, for every $z>0$ and every $\sigma \in \mathbb{M}^{N\times N}_{sym} \setminus \operatorname{int} K(z)$, we have*

$$n_{K(z)}(\pi_{K(z)}(\sigma)) = n_{K(1)}(\pi_{K(1)}(\tfrac{1}{z}\sigma)). \tag{2.9}$$

Moreover, for every $(\sigma,z) \in \partial K$

$$n_K(\sigma,z) = \frac{1}{\sqrt{z^2+|\sigma\cdot n_{K(z)}(\sigma))|^2}}(z\,n_{K(z)}(\sigma), -\sigma\cdot n_{K(z)}(\sigma)). \tag{2.10}$$

For every $(\sigma,z) \in [\mathbb{M}^{N\times N}_{sym} \times (0,+\infty)]\setminus K$, we have

$$\nabla\varrho(\sigma,z) = \tfrac{1}{z}(z\,n_{K(z)}(\pi_{K(z)}(\sigma)),\ -\pi_{K(z)}(\sigma)\cdot n_{K(z)}(\pi_{K(z)}(\sigma))). \tag{2.11}$$

Proof. To prove (2.9) it suffices to consider the case when $\sigma \notin K(z)$, which is equivalent to say that $\frac{\sigma}{z} \notin K(1)$. We then have, applying (2.6) and (2.8), that

$$\begin{aligned} n_{K(z)}(\sigma) &= \frac{\sigma - \pi_{K(z)}(\sigma)}{\varrho(\sigma,z)} \\ &= \frac{z(\frac{\sigma}{z} - \pi_{K(1)}(\frac{\sigma}{z}))}{z\,\varrho(\frac{\sigma}{z},1)} = n_{K(1)}(\pi_{K(1)}(\tfrac{1}{z}\sigma)), \end{aligned}$$

which proves (2.9).

For what concerns (2.11), it is well known that, for every $(\sigma,z) \in [\mathbb{M}^{N\times N}_{sym} \times (0,+\infty)]\setminus K$, $\nabla_\sigma\,\varrho(\sigma,z) = n_{K(z)}(\pi_{K(z)}(\sigma))$ so only the last component of the gradient has to be calculated. Together with (2.8) this implies that

$$\tfrac{\partial}{\partial z}\varrho(\sigma,z) = \tfrac{\partial}{\partial z}[z\,\varrho(\tfrac{\sigma}{z},1)] = \tfrac{1}{z}(\varrho(\sigma,z) - \sigma\cdot n_{K(1)}(\pi_{K(1)}(\tfrac{\sigma}{z}))),$$

hence we get (2.11) by (2.9) and the equality

$$\varrho(\sigma,z) - \sigma\cdot n_{K(z)}(\pi_{K(z)}(\sigma)) = -\pi_{K(z)}(\sigma)\cdot n_{K(z)}(\pi_{K(z)}(\sigma)).$$

This also implies (2.10); indeed, by the C^2 regularity of the boundary, for every fixed $(\bar\sigma,\bar z) \in \partial K$ we may locally define an oriented distance function r form ∂K, which is a C^1-extension of ϱ to the interior of K. Then, locally we have that $K = \{(\sigma,z)|\, r(\sigma,z) \le 0\}$. It follows that the outward unit normal to K at $(\bar\sigma,\bar z)$ must be parallel to $\nabla r(\bar\sigma,\bar z)$, which by continuity is obtained by extending the right-hand side of (2.11) to ∂K, and this proves (2.10). ∎

Another useful property, which will be used in what follows, comes directly from the characterization of the minimal distance projection and from the fact that $0 \in K(z)$ for every z; we have indeed that, for every $(\sigma, z) \in [\mathbb{M}^{N\times N}_{sym} \times (0,+\infty)] \setminus K$

$$\pi_{K(z)}(\sigma) \cdot n_{K(z)}(\pi_{K(z)}(\sigma)) \geq 0. \tag{2.12}$$

We shall often decompose $\sigma \in \mathbb{M}^{N\times N}_{sym}$ in its spherical and deviatoric part through the relation

$$\sigma = x\tfrac{I}{\sqrt{N}} + y \tag{2.13}$$

where $x \in \mathbb{R}$ and $y \in \mathbb{M}^{N\times N}_{D}$ are uniquely determined; here as usual $\mathbb{M}^{N\times N}_{D}$ denotes the space of trace-free symmetric matrices of order N. Notice that $\sqrt{N}x = \mathrm{tr}(\sigma)$; in particular, for every $\sigma \in K(1)$, we shall have $x \leq 0$. Similarly, $\eta(t)$ and $\gamma(t)$ will denote the spherical and the deviatoric part, respectively, of the function $\xi(t)$ mentioned in the introduction.

For mathematical reasons, we shall make some additional hypotheses on the set $K(1)$, even if most of the results we are going to prove do not need them. Precisely, we shall suppose that there exist a constant $a > 0$ and two not identically zero functions g and h, defined on a bounded convex domain D of class C^2, satifying $g = h = 0$ on ∂D and $g, h \in C^2(D) \cap C(\bar{D})$ such that, decomposing $\sigma \in \mathbb{M}^{N\times N}_{sym}$ as in (2.13), we have

$$K(1) = \{\sigma \in \mathbb{M}^{N\times N}_{sym} | g(y) \leq x + a \leq h(y)\} \tag{2.14}$$

Convexity of the domain $K(1)$ is then easily equivalent to the fact that g is convex and h is concave; as they do not identically vanish on D and they are zero on the boundary, this implies that

$$g(y) < 0 \quad \text{and} \quad h(y) > 0 \text{ for every } y \in D.$$

Regularity of $\partial K(1)$ implies, that, for every $\omega \in \partial D$

$$\lim_{y\to\omega,\, y\in D} |\nabla g(y)| = \lim_{y\to\omega,\, y\in D} |\nabla h(y)| = +\infty. \tag{2.15}$$

Moreover, both (2.2) and (2.5) are satisfied, provided we have

$$\max_{x\in D} h = h(0) = a. \tag{2.16}$$

We shall also suppose that

$$g^2, h^2 \text{ are concave }. \tag{2.17}$$

An example of set satisfying all these assumptions is, for instance, any ellipsoid of the form

$$(\tfrac{x}{a} + 1)^2 + \sum_{i=1}^{m} \tfrac{y_i^2}{b_i^2} = 1,$$

where $m = \frac{N(N+1)}{2} - 1$ and y_i are the components of y with respect to an orthonormal basis of $\mathbb{M}^{N\times N}_{D}$. We then have the following Proposition.

Proposition 2.2. *Assume that (2.14)-(2.17) are satisfied. Then, there exists a constant $F > 0$ such that, for every $\sigma \in \partial K(1)$*

$$|\mathrm{tr}(n_{K(1)}(\sigma))| \leq F|x + a|, \tag{2.18}$$

where x is defined as in (2.13). Moreover

$$\mathrm{tr}(n_{K(1)}(\sigma)) = 0 \Longleftrightarrow x = -a. \tag{2.19}$$

The proof of this proposition relies on the following Lemma, whose proof is left to the reader.

Lemma 2.3. *Let $m \geq 1$ and let Ω a bounded convex open subset of $\mathbb{R}^m$ with C^2 boundary. Let $f \in C^2(\Omega) \cap C(\bar{\Omega})$ a concave function satifying*

$$f(y) > 0 \text{ for every } y \in \Omega \quad \text{and} \quad f \equiv 0 \text{ on } \partial\Omega. \tag{2.20}$$

Then, for every $\omega \in \partial\Omega$,

$$\liminf_{y \to \omega,\, y \in \Omega} |\nabla f(y)| > 0.$$

We now prove Proposition 2.2.

Proof. Let $\bar\sigma \in \partial K(1)$ and let $\bar x \in \mathbb{R}$, $\bar y \in \mathbb{M}^{N\times N}_D$ as in (2.13). First, we suppose that $\bar y \in D$, which is equivalent to $\bar x \neq -a$. Then only one of the two is possible: $\bar x + a = g(\bar y)$ or $\bar x + a = h(\bar y)$. Suppose the first is true, then locally $K(1) = \{(x,y) \in \mathbb{R} \times \mathbb{M}^{N\times N}_D | g(y) - x - a \leq 0\}$; recalling that $\mathrm{tr}(n_{K(1)}(\sigma))$ is obtained multiplying by $\sqrt{N}$ the first component of the outward unit normal to $K(1)$ at $\bar\sigma$, we obtain

$$\mathrm{tr}(n_{K(1)}(\bar\sigma)) = \tfrac{-\sqrt{N}}{\sqrt{1+|\nabla g(\bar y)|^2}} = \tfrac{-(\bar x + a)\sqrt{N}}{g(\bar y)\sqrt{1+|\nabla g(\bar y)|^2}} < 0; \tag{2.21}$$

in the other case we have with similar reasonings that

$$\mathrm{tr}(n_{K(1)}(\bar\sigma)) = \tfrac{\sqrt{N}}{\sqrt{1+|\nabla h(\bar y)|^2}} = \tfrac{(\bar x + a)\sqrt{N}}{h(\bar y)\sqrt{1+|\nabla h(\bar y)|^2}} > 0. \tag{2.22}$$

in both the equations, the latter equalities are justified since g, h never vanish in D.

This in particular proves that $\mathrm{tr}(n_{K(1)}(\bar\sigma)) \neq 0$ when $\bar x \neq -a$. Conversely, suppose that $\bar x = -a$, which is equivalent to saying that $\bar y \in \partial D$. Then take a sequence $(y_n)_{n\in\mathbb{N}} \subset D$ converging to $\bar y$ and put

$$\sigma_n := [g(y_n) - a]\tfrac{I}{\sqrt{N}} + y_n;$$

easily we have that $\sigma_n \in \partial K(1)$ for every n and that σ_n converges to $\bar\sigma$. Then (2.15), and (2.21), applied to σ_n, immediately imply that $\mathrm{tr}(n_{K(1)}(\bar\sigma)) = 0$. This concludes the proof of (2.19).

By (2.19), we see that, to prove (2.18), we may suppose that, given $\bar\sigma \in \partial K(1)$ and $\bar x \in \mathbb{R}$, $\bar y \in \mathbb{M}^{N\times N}_D$ in correspondance, one has $\bar y \in D$. By this fact, (2.21), and (2.22), it clearly suffices to show that

$$\inf_{y\in D} |g(y)\sqrt{1+|\nabla g(y)|^2}| > 0 \quad \text{and} \quad \inf_{y\in D} |h(y)\sqrt{1+|\nabla h(y)|^2}| > 0.$$

We only prove the first of the two, the other being completely analogous. As g never vanishes in D it suffices to show that, for every $\omega \in \partial D$ one has

$$\liminf_{y\to\omega,\, y\in D} |g(y)\sqrt{1+|\nabla g(y)|^2}| > 0;$$

as g vanishes on the boundary,

$$\liminf_{y\to\omega,\, y\in D} |g(y)\sqrt{1+|\nabla g(y)|^2}| = \liminf_{y\to\omega,\, y\in D} \tfrac{|\nabla g^2|}{2}$$

and conclusion follows applying Lemma 2.3 to the function $f := \frac{g^2}{2}$. ☐

Remark 2.4. Let $\sigma \in \partial K(1)$ and let $x \in \mathbb{R}$, $y \in \mathbb{M}^{N\times N}_D$ as in (2.13). As g is nonpositive and h is nonnegative, $x+a>0$ is easily equivalent to $x+a=h(y)$, then (2.21) and (2.22) imply that

$$\operatorname{tr}(n_{K(1)}(\sigma)) > 0 \Longleftrightarrow x+a>0. \tag{2.23}$$

Let us fix $\xi \in C^1([0,+\infty);\mathbb{M}^{N\times N}_{sym})$. We introduce the elastic tensor $\mathbb{C}:\mathbb{M}^{N\times N}_{sym} \to \mathbb{M}^{N\times N}_{sym}$ and we suppose it is *isotropic*, that is to say

$$\mathbb{C}\xi = 2\mu\xi_D + \kappa(\operatorname{tr}\xi)I\,, \tag{2.24}$$

where the constant $\mu>0$ is the *shear modulus*, the constant $\kappa>0$ is called *modulus of compression*, and ξ_D denotes the projection of ξ onto the space of deviatoric matrices. In particular, the quadratic form associated to $\mathbb{C}$ is positive definite. For every $\varepsilon>0$ system (1.2) is equivalent to

$$\begin{cases} \varepsilon\dot e_\varepsilon(t) = \varepsilon\dot\xi(t) - \mathbb{C}e_\varepsilon(t) + \pi_{K(z_\varepsilon(t))}(\mathbb{C}e_\varepsilon(t))\,,\\ \varepsilon\dot z_\varepsilon(t) = \operatorname{tr}(\mathbb{C}e_\varepsilon(t))\operatorname{tr}(\mathbb{C}e_\varepsilon(t) - \pi_{K(z_\varepsilon(t))}(\mathbb{C}e_\varepsilon(t)))\,. \end{cases} \tag{2.25}$$

Lemma 2.5. *For every $\varepsilon>0$ and for every initial condition $e_\varepsilon(0)=e_0$ and $z_\varepsilon(0)=z_0>0$ system (2.25) has a unique solution defined for every $t\in[0,+\infty)$. Moreover the solution $(e_\varepsilon,z_\varepsilon)$ of (2.25) with initial condition $e_\varepsilon(0)=e_0$ and $z_\varepsilon(0)=z_0>0$ satisfies $z_\varepsilon(t)>0$ for every $t\in[0,+\infty)$.*

Proof. The first part of the statement can be proved as in [2, Lemma 2.2]; we also have, in particular that for every $\varepsilon>0$ and $T>0$ there exists a positive constant $M_{T,\varepsilon}$ such that $|\mathbb{C}e_\varepsilon(t)|\le M_{T,\varepsilon}$ for every $t\in[0,T]$. Let now T be the first time such that $z_\varepsilon(T)=0$ and suppose by contradiction that $T<+\infty$. Fix $\hat t<T$ such that $T-\hat t<\frac{\varepsilon}{2M_{T,\varepsilon}M_K}$, where M_K is given by (2.3) and let $a<T$ be a maximum point for $z_\varepsilon(t)$ in $[\hat t,T]$. We shall have, by (2.25) and (2.3)

$$\begin{aligned} 0 &= \varepsilon z_\varepsilon(a) + \varepsilon\int_a^T \dot z_\varepsilon(s)\,ds\\ &= \varepsilon z_\varepsilon(a) + \int_a^T [\operatorname{tr}(\mathbb{C}e_\varepsilon(s))^2 - \operatorname{tr}(\mathbb{C}e_\varepsilon(s))\operatorname{tr}(\pi_{K(z_\varepsilon(s))}(\mathbb{C}e_\varepsilon(s)))]\,ds\\ &\ge \varepsilon z_\varepsilon(a) - \int_a^T |\operatorname{tr}(\mathbb{C}e_\varepsilon(s))|\,|\operatorname{tr}(\pi_{K(z_\varepsilon(s))}(\mathbb{C}e_\varepsilon(s)))|\,ds\\ &\ge \varepsilon z_\varepsilon(a) - M_{T,\varepsilon}M_K\int_a^T z_\varepsilon(s)\,ds\\ &\ge z_\varepsilon(a)[\varepsilon-(T-a)M_{T,\varepsilon}M_K] \ge \tfrac{\varepsilon}{2}z_\varepsilon(a), \end{aligned}$$

a contradiction. ∎

Introducing the dual variable σ, the system becomes

$$\begin{cases} \varepsilon\dot\sigma_\varepsilon(t) = \varepsilon\mathbb{C}\dot\xi(t) + \mathbb{C}[\pi_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t)) - \sigma_\varepsilon(t)]\,,\\ \varepsilon\dot z_\varepsilon(t) = \operatorname{tr}(\sigma_\varepsilon(t))\operatorname{tr}(\sigma_\varepsilon(t) - \pi_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t)))\,. \end{cases} \tag{2.26}$$

Since we want to consider a system which is initially in the elastic regime, for every $\varepsilon>0$ we will consider an initial condition satisfying $(\sigma_0,z_0)\in \operatorname{int}K$; in particular, we shall have $z_0>0$. For every ε the solution of (2.26) is trivially given, by

$$(\sigma(t),z(t)) = (\sigma_0+\mathbb{C}(\xi(t)-\xi(0)),z_0) \tag{2.27}$$

for t small; actually, this formula gives the solution in the time interval $[0,t_1]$, where

$$t_1 = \inf\{t > 0 : \ (\sigma_0 + \mathbb{C}(\xi(t) - \xi(0)), z_0) \in \partial K\}. \tag{2.28}$$

In terms of the function ϱ defined by (2.7), for every t such that $\varrho(\sigma_\varepsilon(t), z_\varepsilon(t)) > 0$, equations (2.26) become

$$\begin{cases} \mathbb{C}\dot{\xi}(t) - \dot{\sigma}_\varepsilon(t) = \frac{1}{\varepsilon}\varrho(\sigma_\varepsilon(t), z_\varepsilon(t))\, \mathbb{C}\, n_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t), z_\varepsilon(t)), \\ \dot{z}_\varepsilon(t) = \frac{1}{\varepsilon}\varrho(\sigma_\varepsilon(t), z_\varepsilon(t))\ \mathrm{tr}(\sigma_\varepsilon(t))\ \mathrm{tr}(n_{K(z_\varepsilon(t))}(\pi_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t))). \end{cases} \tag{2.29}$$

Given the solution of (2.26) with the prescribed initial data we define

$$\varrho_\varepsilon(t) := \varrho(\sigma_\varepsilon(t), z_\varepsilon(t)); \tag{2.30}$$

notice that $\varrho_\varepsilon(t)$ is Lipschitz continuous, thus differentiable, for almost every t; in particular it is differentiable for every t such that $\varrho_\varepsilon(t) > 0$, and we have, by a direct computation, taking into account (2.29) and (2.11), that

$$\frac{d}{dt}\varrho_\varepsilon(t) = \Phi(t, \sigma_\varepsilon(t), z_\varepsilon(t)) + \frac{\varrho_\varepsilon(t)}{\varepsilon}\Psi(\sigma_\varepsilon(t), z_\varepsilon(t)) \quad \text{whenever } \varrho_\varepsilon(t) > 0, \tag{2.31}$$

where

$$\Phi(t,\sigma,z) := \quad n_{K(z)}(\pi_{K(z)}(\sigma)) \cdot \mathbb{C}\dot{\xi}(t), \tag{2.32}$$

$$\Psi(\sigma,z) := \quad -n_{K(z)}(\pi_{K(z)}(\sigma)) \cdot \mathbb{C}n_{K(z)}(\pi_{K(z)}(\sigma)) - \tfrac{\mathrm{tr}(\sigma)\ \mathrm{tr}(n_{K(z)}(\pi_{K(z)}(\sigma)))}{z}[\pi_{K(z)}(\sigma) \cdot n_{K(z)}(\pi_{K(z)}(\sigma))]. \tag{2.33}$$

The function Φ is defined on $[0,+\infty) \times \{[\mathbb{M}^{N\times N}_{sym} \times (0,+\infty)] \setminus \mathrm{int}\, K\}$ and is continuous, while Ψ is defined on $[\mathbb{M}^{N\times N}_{sym} \times (0,+\infty)] \setminus \mathrm{int}\, K$ and is of class C^1. In what follows, it is often convenient to consider extensions of Φ and Ψ to $[0,+\infty) \times \mathbb{M}^{N\times N}_{sym} \times (0,+\infty)$ and $\mathbb{M}^{N\times N}_{sym} \times (0,+\infty)$ of class C^0 and C^1, respectively. Notice that the partial derivatives of Ψ at each point of ∂K do not depend on the extension.

As in [7], we will see that the sign of Φ determines the transition from elastic to inelastic regime at times when the stress meets the yield surface, while in case of inelastic regime the sign of Ψ determines whether the quasistatic evolution follows the equation of the slow dynamics (continuous evolution) or jumps along the trajectory of the fast dynamics. For these reasons, Φ will be called *elastic-inelastic indicator*, while Ψ will be called *slow-fast indicator*. Even if, for mathematical reasons, the two indicators are defined on the whole space, we will also see that what only matters are the values they attain on the yield surface.

Remark 2.6. By positive definiteness of $\mathbb{C}$ and by (2.12) it is immediate to deduce that, for every (σ,z) such that $\mathrm{tr}(\sigma)\ \mathrm{tr}(n_{K(z)}(\pi_{K(z)}(\sigma))) \geq 0$, the indicator Ψ is strictly negative; as we are going to see in what follows, this reflects the fact that, as long as we are in the hardening regime, the evolution does not present discontinuities.

In general, it is easy to verify, taking into account (2.24) and (2.3), that the following bounds on Ψ hold: from above, we have, for every $(\sigma,z) \in [\mathbb{M}^{N\times N}_{sym} \times (0,+\infty)] \setminus \mathrm{int}\, K$,

$$\Psi(\sigma,z) \leq -\min\{\kappa, 2\mu\} + M_K\sqrt{N}|\mathrm{tr}(\sigma)|, \tag{2.34}$$

while from below

$$\Psi(\sigma,z) \geq -\max\{\kappa, 2\mu\} - M_K\sqrt{N}|\mathrm{tr}(\sigma)| \tag{2.35}$$

where $k, 2\mu$ are defined by (2.24) and M_K is as in (2.3); clearly we may assume that any extension of Ψ we will consider preserves these bounds in the whole space. Notice that, by (2.34) and (2.3), if z is sufficiently close to 0, and $(\sigma, z) \in K$, then the indicator Ψ is strictly negative uniformly in σ; according to what we shall see in the following sections, this means that when the internal variable is sufficiently small the evolution is continuous.

In what follows we shall define, for every $\sigma \in \mathbb{M}^{N\times N}_{sym}$,

$$\lambda(\sigma) := \max\{\kappa, 2\mu\} + M_K\sqrt{N}|\mathrm{tr}(\sigma)|. \tag{2.36}$$

3. **Continuous evolution.**

3.1. **The equation of the slow dynamics.** In this section we study in detail the equation

$$\begin{cases} \dot\sigma_{sl}(t) &= \frac{\Phi(t,\sigma_{sl}(t),z_{sl}(t))}{\Psi(\sigma_{sl}(t),z_{sl}(t))}\,\mathbb{C}\, n_{K(z_{sl}(t))}(\sigma_{sl}(t)) + \mathbb{C}\dot\xi(t), \\ \dot z_{sl}(t) &= -\frac{\Phi(t,\sigma_{sl}(t),z_{sl}(t))}{\Psi(\sigma_{sl}(t),z_{sl}(t))}\,\mathrm{tr}(\sigma_{sl}(t))\;\mathrm{tr}(n_{K(z_{sl}(t))}(\sigma_{sl}(t))), \end{cases} \tag{3.1}$$

defined on the open submanifold $\partial K \cap \{\Psi(\sigma,z)\neq 0\}\setminus\{(0,0)\}$. This will be called the equation of the slow dynamics: observe that this is a well-defined equation, since, for every $t \in [0,+\infty)$, the vector field

$$\chi_t(\sigma,z) = (\mathbb{C}\dot\xi(t) + \tfrac{\Phi(t,\sigma,z)}{\Psi(\sigma,z)}\,\mathbb{C}\, n_{K(z)}(\sigma),\, \tfrac{-\Phi(t,\sigma,z)}{\Psi(\sigma,z)}\mathrm{tr}(\sigma)\,\mathrm{tr}(n_{K(z)}(\sigma)))$$

is a tangent vector field to $\partial K \cap \{\Psi(\sigma,z)\neq 0\}\setminus\{(0,0)\}$; indeed, by (2.10), it suffices to show that $\chi_t(\sigma,z)\cdot(z\, n_{K(z)}(\sigma), -\sigma\cdot n_{K(z)}(\sigma)) = 0$, which follows by a direct computation, recalling (2.32), and (2.33).

Remark 3.1. Let $(\sigma(t), z(t))$ a solution of (3.1) and define $e(t)$, $p(t)$ through the constitutive relations in (1.1); we have that $\dot p(t) = -\frac{\Phi(t,\sigma(t),z(t))}{\Psi(\sigma(t),z(t))} n_{K(z(t))}(\sigma(t))$, thus the flow rule in (1.1) is satisfied as long as $-\frac{\Phi(t,\sigma(t),z(t))}{\Psi(\sigma(t),z(t))} \geq 0$; that is, in our case, as long as Φ does not become negative along the trajectory. We will see indeed that equation (3.1) appears in the limit of (2.26) when the slow-fast indicator Ψ is negative.

Viceversa, let $(\sigma(t), z(t))$ a C^1 function with values on ∂K satisfying (1.1) in a certain interval of time; if we suppose $\Psi(\sigma(t),z(t)) \neq 0$, the flow rule and the condition

$$0 = n_K((\sigma(t), z(t)))\cdot(\dot\sigma(t), \dot z(t)),$$

with the help of (2.10), easily imply that $(\sigma(t), z(t))$ satisfies (3.1) and that

$$-\tfrac{\Phi(t,\sigma(t),z(t))}{\Psi(\sigma(t),z(t))} \geq 0.$$

We endow equation (3.1) with initial data $(\sigma_1, z_1) \in \partial K$ at a time $t_1 > 0$, with $z_1 > 0$ and $\Psi(\sigma_1, z_1) \neq 0$. We may thus apply all standard results about local existence and uniqueness and the existence of a maximal interval where solutions to (3.1) are defined. So, let (t_1, t_2) be the maximal interval of existence for the Cauchy problem associated to (3.1) with datum (σ_1, z_1). As said in (2.13), we denote the spherical and the deviatoric part of $\sigma_{sl}(t)$ with $x_{sl}(t)$ and $y_{sl}(t)$, and the spherical and the deviatoric part of $\xi(t)$ with $\eta(t)$ and $\gamma(t)$. Using the identity $\mathrm{tr}(\mathbb{C}\sigma) = \kappa N \mathrm{tr}(\sigma)$, from (3.1) we obtain

$$\kappa \dot z_{sl}(t) = x_{sl}(t)(\kappa\, N\dot\eta(t) - \dot x_{sl}(t)). \tag{3.2}$$

The next Proposition shows an useful consequence of this equation.

Proposition 3.2. *Assume (2.1)-(2.5), and (2.24); let Φ, Ψ as in (2.32), and (2.33), respectively. Let $(\sigma_{sl}(t), z_{sl}(t))$ the unique solution to the Cauchy problem associated to (3.1) with Cauchy data $(\sigma_1, z_1) \in \partial K$ at a time $t_1 > 0$, with $z_1 > 0$ and $\Psi(\sigma_1, z_1) \neq 0$, and let $[t_1, t_2)$ be its maximal interval of existence. If $t_2 < +\infty$, there exists a positive constant M such that*

$$|(\sigma_{sl}(t), z_{sl}(t))| < M \text{ for every } t \in [t_1, t_2) \tag{3.3}$$

Proof. By (2.3), it suffices to show that $z_{sl}(t)$ is bounded. Let $L > 0$ such that $|\dot\eta(t)| < L$ for every $t \in [t_1, t_2]$: by (3.2), and (2.3) we have, for every $t \in [t_1, t_2)$

$$\begin{aligned}
\kappa(z_{sl}(t) - z_{sl}(t_1)) &= \kappa \int_{t_1}^{t} \dot z_{sl}(s)\,ds \\
&= -\int_{t_1}^{t} x_{sl}(s)\dot x_{sl}(s)\,ds + \kappa N \int_{t_1}^{t} \dot\eta(s) x_{sl}(s)\,ds \\
&\leq \tfrac12 [x_{sl}^2(t_1) - x_{sl}^2(t)] + \kappa N \int_{t_1}^{t} |\dot\eta(s)||x_{sl}(s)|\,ds \\
&\leq \tfrac12 x_{sl}^2(t_1) + \kappa L N M_K \int_{t_1}^{t} z_{sl}(s)\,ds
\end{aligned}$$

and conclusion follows by Gronwall's inequality. □

By the use of (3.2) we are also able to show that $z_{sl}(t)$ cannot vanish at $t = t_2$.

Proposition 3.3. *Assume (2.1)-(2.5), and (2.24); let Φ, Ψ as in (2.32), and (2.33), respectively. Let $(\sigma_{sl}(t), z_{sl}(t))$ the unique solution to the Cauchy problem associated to (3.1) with Cauchy data $(\sigma_1, z_1) \in \partial K$ at a time $t_1 > 0$, with $z_1 > 0$ and $\Psi(\sigma_1, z_1) \neq 0$, and let $[t_1, t_2)$ be its maximal interval of existence. If $t_2 < +\infty$, then*

$$\liminf_{t \to t_2} z_{sl}(t) > 0. \tag{3.4}$$

Proof. Suppose by contradiction that $\liminf_{t \to t_2} z_{sl}(t) = 0$; we first show that this liminf is a limit. Let $L > 0$ such that $|\dot\eta(t)| < L$ for every $t \in (t_1, t_2)$, and M_K as in (2.3), and let $c := \limsup_{t \to t_2} z_{sl}(t)$; if we suppose $c > 0$, we may fix $\hat t < t_2$ such that

1) $LNM_K(t_2 - \hat t) < \frac18$;
2) $z_{sl}(t) < 2c$ for every $t > \hat t$;
3) $z_{sl}(\hat t) > \frac{c}{2}$.

We shall then have, by (3.2), (2.3), and the previous assumptions, that, for every $t > \hat t$

$$\begin{aligned}
\kappa z_{sl}(t) &= \kappa z_{sl}(\hat t) + \int_{\hat t}^{t} \dot z_{sl}(s)\,ds \\
&= \kappa z_{sl}(\hat t) - \int_{\hat t}^{t} x_{sl}(s)\dot x_{sl}(s)\,ds + \kappa N \int_{\hat t}^{t} \dot\eta(s) x_{sl}(s)\,ds \\
&\geq \kappa\tfrac{c}{2} + \tfrac12 [x_{sl}^2(\hat t) - x_{sl}^2(t)] - \kappa N \int_{\hat t}^{t} |\dot\eta(s)||x_{sl}(s)|\,ds \\
&\geq \kappa\tfrac{c}{2} - \tfrac12 x_{sl}^2(t) - \kappa N L M_K \int_{\hat t}^{t} z_{sl}(s)\,ds \\
&\geq \kappa\tfrac{c}{2} - \tfrac12 x_{sl}^2(t) - \kappa\tfrac{c}{4}.
\end{aligned}$$

So, let t_n a sequence converging to t_2 realizing the liminf; by (2.3) we shall get that $\lim\limits_{n\to+\infty} x_{sl}(t_n) = 0$. As $t_n > \hat{t}$ for n sufficiently large, we shall have

$$\kappa z_{sl}(t_n) \geq \kappa \tfrac{c}{4} - \tfrac{1}{2} x_{sl}^2(t_n),$$

which in the limit yields $\frac{c}{4} \leq 0$, a contradiction. We thus have that $\lim_{t\to t_2} z_{sl}(t) = 0$, which immediately implies, by (2.3), that $\lim\limits_{t\to t_2} x_{sl}(t) = 0$. We now fix $\bar{t} < t_2$ such that $LNM_K(t_2 - \bar{t}) < \frac{1}{2}$; as $z_{sl}(t) > 0$ in (t_1, t_2) and $\lim\limits_{t\to t_2} z_{sl}(t) = 0$, there exists a maximum point t_3 for $z_{sl}(t)$ in $[\bar{t}, t_2)$. Repeating the previous estimates, we shall have, for every $t > t_3$, that

$$\kappa z_{sl}(t) \geq \kappa z_{sl}(t_3) - \tfrac{1}{2} x_{sl}^2(t) - \kappa NLM_K z_{sl}(t_3)(t_2 - \bar{t}) \geq \kappa \tfrac{z_{sl}(t_3)}{2} - \tfrac{1}{2} x_{sl}^2(t),$$

which in the limit as $t \to t_2$ gives $z_{sl}(t_3) \leq 0$, a contradiction. □

Proposition 3.4. *Assume (2.1)-(2.5), and (2.24); let Φ, Ψ as in (2.32), and (2.33), respectively. Let $(\sigma_{sl}(t), z_{sl}(t))$ the unique solution to the Cauchy problem associated to (3.1) with Cauchy data $(\sigma_1, z_1) \in \partial K$ at a time $t_1 > 0$, with $z_1 > 0$ and such that $\Psi(\sigma_1, z_1) \neq 0$, and let $[t_1, t_2)$ be its maximal interval of existence. If $t_2 < +\infty$, then*

$$\lim_{t\to t_2^-} \Psi(\sigma_{sl}(t), z_{sl}(t)) = 0 \tag{3.5}$$

Proof. Suppose by contradiction that there exists a sequence $t_k \to t_2$ such that

$$\lim_{k\to+\infty} \Psi(\sigma_{sl}(t_k), z_{sl}(t_k)) \neq 0. \tag{3.6}$$

By Proposition 3.2, we may assume that $(\sigma_{sl}(t_k), z_{sl}(t_k))$ tends to a finite limit (σ_2, z_2) as $k \to +\infty$; by Proposition 3.3 we have that $z_2 > 0$. By continuity of Ψ, (3.6) implies that $\Psi(\sigma_2, z_2) \neq 0$; it follows now from Lemma 3.5 below that

$$\lim_{t\to t_2} (\sigma_{sl}(t), z_{sl}(t)) = (\sigma_2, z_2);$$

we may then solve the Cauchy problem associated to (3.1) with data (σ_2, z_2) at time t_2, contradicting the maximality of $[t_1, t_2)$. □

In the previous Proposition we have used the following elementary Lemma about differential equations, whose proof can be found in [4], Chapter 1, Lemma 3.1; we state it for the reader's convenience.

Lemma 3.5. *Let E be a subset of $\mathbb{R} \times \mathbb{R}^n$, let $f\colon E \to \mathbb{R}^n$ a continuous function, and let $u(t)$ a solution of the ODE $\dot{v}(t) = f(t, v(t))$ on an interval $[a, \delta)$ or $(\delta, a]$ where $|\delta| < +\infty$. If there exists a sequence t_k converging to δ such that $u(t_k) \to \bar{u} \in \mathbb{R}^n$ and $f(t, v)$ is bounded on the intersection of E with an open neighborhood of the point $(\delta, \bar{u})$, then*

$$\lim_{t\to\delta} u(t) = \bar{u}.$$

In the next Proposition, we use Lemma 3.5 to prove that, if Ψ vanishes at time $t_2 < +\infty$, then $(\sigma_{sl}(t), z_{sl}(t))$ have a limit at $t = t_2$; the proof is obtained by $z_{sl}(t)$ must be monotone in a neighborhood of t_2. We also need the additional hypothesis that the elastic-inelastic indicator is not vanishing at t_2, that is to say

$$\liminf_{t\to t_2^-} |\Phi(t, \sigma_{sl}(t), z_{sl}(t))| > 0. \tag{3.7}$$

Proposition 3.6. *Assume* (2.1)-(2.5), *and* (2.24)*; let* Φ, Ψ *as in* (2.32), *and* (2.33), *respectively. Let* $(\sigma_{sl}(t), z_{sl}(t))$ *the unique solution to the Cauchy problem associated to* (3.1) *with Cauchy data* (σ_1, z_1) *at a time* $t_1 > 0$, *with* $z_1 > 0$ *and such that* $\Psi(\sigma_1, z_1) \neq 0$, *and let* $[t_1, t_2)$ *be its maximal interval of existence. If* $t_2 < +\infty$, *and* (3.7) *holds, then there exists*

$$\lim_{t \to t_2^-} (\sigma_{sl}(t), z_{sl}(t)) := (\sigma_2, z_2) \in \partial K. \tag{3.8}$$

Proof. By Proposition 3.4 we have $\lim_{t \to t_2^-} \Psi(\sigma_{sl}(t), z_{sl}(t)) = 0$; as seen in Remark 2.6, this implies that

$$\liminf_{t \to t_2^-} x_{sl}(t) < 0 \quad \text{and} \quad \liminf_{s \to t_2^-} \operatorname{tr}(n_{K(z_{sl}(t))}(\sigma_{sl}(t))) > 0;$$

if not, in both cases we may find a sequence t_n converging to t_2 along which

$$\limsup_{n \to +\infty} \Psi(\sigma_{sl}(t_n), z_{sl}(t_n)) \leq -\min\{\kappa, 2\mu\} < 0,$$

a contradiction. By (3.1), (3.5), and (3.7) we easily get that there exists a left neighborhood of t_2, denoted with $(\hat{t}, t_2)$, where $\dot{z}_{sl}(t) \neq 0$; thus $z_{sl}(t)$ is invertible in this interval, with inverse $t(z)$, and converges to a limit z_2, which is finite by Proposition 3.2. We now suppose, for instance, that $z_{sl}(t)$ is strictly decreasing, the proof in the other case being completely analogous. We put $\hat{z} := z_{sl}(\hat{t})$ and we express σ in function of z; by (3.1), we then get that

$$-\sigma'_{sl}(z) = \tfrac{1}{\operatorname{tr}(\sigma_{sl}(z)) \operatorname{tr}(n_{K(z)}(\sigma_{sl}(z)))} [\mathbb{C}\, n_{K(z)}(\sigma_{sl}(z)) - \mathbb{C}\, \chi(z) \tfrac{\Psi(\sigma_{sl}(z), z)}{\Phi(t(z), \sigma_{sl}(z), z)}] \tag{3.9}$$

for every $z \in (z_2, \hat{z})$; here we have put: $\chi(z) := \dot{\xi}(t(z))$. So, as

$$\liminf_{z \to z_2} |\operatorname{tr}(\sigma_{sl}(z)) \operatorname{tr}(n_{K(z)}(\sigma_{sl}(z)))| > 0$$

by the previous discussion, and taking into account (2.3) and (3.7), $|\sigma'_{sl}(z)|$ remains uniformly bounded in this interval. The conclusion follows. ☐

Remark 3.7. We will see in the next subsection that the solutions of (2.26) uniformly converge to the solution of (3.1) in a right neighborhood of t_1 if we suppose that

$$\Phi(t_1, \sigma_1, z_1) > 0 \tag{3.10}$$

and

$$\Psi(\sigma_1, z_1) < 0. \tag{3.11}$$

In general, $[t_1, t_2)$ may not be the maximal interval of convergence, as positivity of Φ may fail before of t_2. We will show that this convergence holds on $[t_1, t_2)$ whenever

$$\Phi(t, \sigma_{sl}(t), z_{sl}(t)) > 0 \qquad \text{for every } t < t_2. \tag{3.12}$$

Assume this inequality, as well as (3.7), suppose that $t_2 < +\infty$, and let (σ_2, z_2) be as in (3.8); then

$$\Psi(\sigma_2, z_2) = 0. \tag{3.13}$$

Let us prove that

$$\nabla\Psi(\sigma_2, z_2) \cdot (\tfrac{-\mathbb{C}\, n_{K(z_2)}(\sigma_2)}{\operatorname{tr}(\sigma_2) \operatorname{tr}(n_{K(z_2)}(\sigma_2))}, 1) \leq 0. \tag{3.14}$$

Indeed, as seen in Proposition 3.6 $z_{sl}(t)$ is strictly decreasing in a left neighborhood of t_2, with inverse $t(z)$. If we define $\sigma_{sl}(z) := \sigma_{sl}(t(z))$, we shall then have that $\Psi(\sigma_{sl}(z), z) < 0$ in a right neighborhood of z_2, which yields

$$\lim_{z \to z_2} \frac{d}{dz} \Psi(\sigma_{sl}(z), z) \leq 0;$$

a direct computation involving (3.9) and (3.13) gives us condition (3.14).

We claim that the vector $(\frac{-\mathbb{C}\, n_{K(z_2)}(\sigma_2)}{\operatorname{tr}(\sigma_2)\operatorname{tr}(n_{K(z_2)}(\sigma_2))}, 1)$ is tangent to ∂K at (σ_2, z_2). To prove that, by (2.10), it suffices to show that

$$(\tfrac{-\mathbb{C}\, n_{K(z_2)}(\sigma_2)}{\operatorname{tr}(\sigma_2)\operatorname{tr}(n_{K(z_2)}(\sigma_2))}, 1) \cdot (z_2\, n_{K(z_2)}(\sigma_2), -\sigma_2 \cdot n_{K(z_2)}(\sigma_2)) = 0.$$

Recalling (2.33), the left-hand side is equal to $\frac{z_2 \Psi(\sigma_2, z_2)}{\operatorname{tr}(\sigma_2)\operatorname{tr}(n_{K(z_2)}(\sigma_2))}$, and the conclusion follows by (3.13). Thus the left-hand side of (3.14) is a tangential derivative and depends only on the values Ψ attains on ∂K.

Due to the presence of the forcing term $\mathbb{C}\dot{\xi}(t)$, the sign of $\dot{z}_{sl}(t)$ may change, causing the alternance of hardening and softening regime; we end this subsection by presenting a simple condition that prevents this phenomenon. To be definite, we consider the case where the spherical part of $\xi(t)$ is constant, as in [2]. Observe that here we are assuming (2.14)-(2.17), in order to apply Proposition 2.2.

Proposition 3.8. *Assume that (2.24), and (2.14)-(2.17) are satisfied; let Φ, Ψ as in (2.32), and (2.33), respectively. Let $(\sigma_{sl}(t), z_{sl}(t))$ the unique solution to (3.1) with Cauchy data (σ_1, z_1) at a time $t_1 > 0$, with $z_1 > 0$ and $\Psi(\sigma_1, z_1) < 0$, and let $[t_1, t_2)$ be its maximal interval of existence. Let $\hat{t} \in [t_1, t_2)$ such that*

$$\Phi(t, \sigma_{sl}(t), z_{sl}(t)) > 0 \qquad \text{for every } t \in [t_1, \hat{t}] \tag{3.15}$$

and suppose that $\dot{\eta}(t) = 0$ for every $t \in [t_1, \hat{t}]$. If there exists $\bar{t} \in (t_1, \hat{t})$ such that $\dot{z}_{sl}(\bar{t}) = 0$, then $\dot{z}_{sl}(t) = 0$ for every $t \in [t_1, \hat{t}]$.

Proof. As $\hat{t} < +\infty$, by the same arguments as in Proposition 3.2 and Proposition 3.3, we may assume that $Z := \inf_{t \in [t_1, \hat{t}]} z_{sl}(t) > 0$ and that $|x_{sl}(t)|$ is bounded by a finite constant M. By (3.1) we have that

$$\dot{x}_{sl}(t) = \sqrt{N} \tfrac{\Phi(t, \sigma_{sl}(t), z_{sl}(t))}{\Psi(\sigma_{sl}(t), z_{sl}(t))}\, \kappa \operatorname{tr}(n_{K(z_{sl}(t))}(\sigma_{sl}(t))), \tag{3.16}$$

while (3.2) reduces to

$$\kappa \dot{z}_{sl}(t) = -x_{sl}(t)\dot{x}_{sl}(t). \tag{3.17}$$

By (3.16), (3.15), (2.9), and (2.19), we have that

$$\dot{x}_{sl}(t) = 0 \Longleftrightarrow x_{sl}(t) + a\, z_{sl}(t) = 0, \tag{3.18}$$

where $a > 0$ is as in (2.14). Let us prove that $x_{sl}(t) \neq 0$ for every $t \in (t_1, \hat{t}]$; indeed, by (2.5), which is equivalent to (2.16), if the value 0 is assumed, it is a maximum value for $x_{sl}(t)$, thus, if for some $t \in (t_1, \hat{t}]$ we have $x_{sl}(t) = 0$, it must be also $\dot{x}_{sl}(t) = 0$, but this is excluded by (3.18), as $z_{sl}(t) > 0$.

Suppose that there exists $\bar{t} \in (t_1, \hat{t})$ such that $\dot{z}_{sl}(\bar{t}) = 0$; as $x_{sl}(\bar{t}) \neq 0$, by (3.17) we must have $\dot{x}_{sl}(\bar{t}) = 0$, that is to say $x_{sl}(\bar{t}) + a\, z_{sl}(\bar{t}) = 0$. Let $f(t) := x_{sl}(t) + a\, z_{sl}(t)$; under our hypotheses, by (3.16) and (3.17) there exists a positive constant W such that

$$|\dot{f}(t)| \leq W |\operatorname{tr}(n_{K(z_{sl}(t))}(\sigma_{sl}(t))| \qquad \text{for every } t \in [t_1, \hat{t}];$$

(2.9) and (2.19) imply that

$$|\mathrm{tr}(n_{K(z_{sl}(t))}(\sigma_{sl}(t))| \leq \tfrac{F}{Z}|x_{sl}(t) + a\, z_{sl}(t)|,$$

where $F > 0$ is as in (2.18). We conclude that

$$|\dot{f}(t)| \leq W\tfrac{F}{Z}|f(t)| \qquad \text{for every } t \in [t_1, \hat{t}];$$

as $f(\bar{t}) = 0$, Gronwall's inequality implies that $f(t) = 0$ for every $t \in [t_1, \hat{t}]$, which in its turn entails that $\dot{x}_{sl}(t) = 0$ for every $t \in [t_1, \hat{t}]$, and conclusion follows by (3.17). □

3.2. **Convergence to the slow dynamics.** In this subsection we examine how to recover equation (3.1) from (2.26) in the limit as ε goes to 0, under suitable hypotheses on the sign of the indicators Φ and Ψ: as the arguments are essentially the same as in [7, Section 3], some of the proofs will be only sketched.

Throughout this part of the paper, $\hat{t}$ denotes a time such that there exist a left continuous function $t \mapsto (\sigma(t), z(t))$ defined on $[0, \hat{t})$ with values in $\mathbb{M}^{N\times N}_{sym} \times [0, +\infty)$ and an element $(\hat{\sigma}, \hat{z})$ of $\mathbb{M}^{N\times N}_{sym} \times [0, +\infty)$ satisfying the following properties:

$$(\sigma_\varepsilon(t), z_\varepsilon(t)) \to (\sigma(t), z(t)) \quad \text{for a.e. } t \in [0, \hat{t}), \tag{3.19}$$

$$\text{there exists } \hat{t}_\varepsilon \to \hat{t} \text{ such that } (\sigma_\varepsilon(\hat{t}_\varepsilon), z_\varepsilon(\hat{t}_\varepsilon)) \to (\hat{\sigma}, \hat{z}), \tag{3.20}$$

$$(\hat{\sigma}, \hat{z}) \in \partial K \quad \text{and} \quad \hat{z} > 0, \tag{3.21}$$

$$\Phi(\hat{t}, \hat{\sigma}, \hat{z}) > 0. \tag{3.22}$$

For instance, we can take $\hat{t} = t_1$ defined by (2.28), if $t_1 < +\infty$ and, setting

$$(\sigma_1, z_1) := (\sigma_0 + \mathbb{C}(\xi(t_1) - \xi(0)), z_0). \tag{3.23}$$

we have

$$\Phi(t_1, \sigma_1, z_1) > 0; \tag{3.24}$$

notice that in general we have $\Phi(t_1, \sigma_1, z_1) \geq 0$, as the solution was in K at all previous times, thus we are only excluding the degenerate case when equality holds. The case $\Phi(t_1, \sigma_1, z_1) = 0$ will be discussed in the next subsection.

Lemma 3.9. *Assume (2.1)-(2.5), and (2.24), and let Φ as in (2.32). Let $\hat{t} > 0$ satisfy (3.19)-(3.22), and let $\hat{t}_\varepsilon$ be as in (3.20); then, for every $t^* > \hat{t}$, the set $\{\varrho_\varepsilon(t) > 0\} \cap [\hat{t}_\varepsilon, t^*]$ is nonempty, when ε is sufficiently small.*

Proof. Assume on the contrary that along a suitable subsequence, that we shall not relabel, one has $\varrho_\varepsilon(t) = 0$ for every $t \in [\hat{t}_\varepsilon, t^*]$; we then get

$$(\sigma_\varepsilon(t), z_\varepsilon(t)) = (\sigma_\varepsilon(\hat{t}_\varepsilon) + \mathbb{C}(\xi(t) - \xi(\hat{t}_\varepsilon)), z_\varepsilon(t_\varepsilon)) \in K \tag{3.25}$$

for every $t \in [\hat{t}_\varepsilon, t^*]$. In the limit we obtain that $(\hat{\sigma} + \mathbb{C}(\xi(t) - \xi(\hat{t})), \hat{z}) \in K$ for every $t \in [\hat{t}, t^*]$; by (3.21) we easily deduce that it must be $\Phi(\hat{t}, \hat{\sigma}, \hat{z}) \leq 0$, contradicting (3.22). □

Remark 3.10. Notice that if $\hat{t} = t_1$, the statement of the Lemma holds with $\hat{t}_\varepsilon = t_1$.

We fix an open neighborhood $U_\delta := (\hat{t} - \delta, \hat{t} + \delta) \times B_\delta(\hat{\sigma}, \hat{z})$, where $B_\delta(\hat{\sigma}, \hat{z})$ denotes the open ball of radius $\delta > 0$ centered at $(\hat{\sigma}, \hat{z})$, in a way that there exists a positive constant $\gamma_2 > 0$ such that

$$\Phi(t, \sigma, z) \geq \gamma_2 > 0 \qquad \text{for every } (t, \sigma, z) \in U_\delta. \tag{3.26}$$

We may clearly assume that $\delta < \frac{\max\{\kappa, 2\mu\}}{2M_K\sqrt{N}}$, where $k, 2\mu$ are defined by (2.24) and M_K is as in (2.3), in a way that for every $(\sigma, z) \in B_\delta(\hat{\sigma}, \hat{z})$, the following holds:

$$\frac{\lambda(\sigma)}{\lambda(\hat{\sigma})} < \frac{3}{2}, \tag{3.27}$$

where $\lambda(\sigma)$ is defined as in (2.36). We define

$$a_\varepsilon := \inf\{t \in (\hat{t}_\varepsilon, \hat{t}_\varepsilon + \delta) : \ (\sigma_\varepsilon(t), z_\varepsilon(t)) \in \partial B_\delta(\hat{\sigma}, \hat{z})\}, \tag{3.28}$$

where $\hat{t}_\varepsilon$ is given by (3.20). The following lemma shows that, thanks to (3.26), the function $\frac{1}{\varepsilon}\varrho_\varepsilon(t)$ becomes greater than a fixed positive constant after a time t_ε converging to $\hat{t}$ as $\varepsilon \to 0$, while the motion is still in $B_\delta(\hat{\sigma}, \hat{z})$; we shall see that this implies a transition to the inelastic regime.

Lemma 3.11. *Assume (2.1)-(2.5), and (2.24), and let Φ as in (2.32). Let $\hat{t} > 0$ satisfy (3.19)-(3.22), let $\hat{t}_\varepsilon$ be as in (3.20), and let δ, a_ε, and γ_2, be as in (3.26) and (3.28). Let $\varepsilon > 0$ and $\varrho_\varepsilon(t)$ be as in (2.30). Define*

$$t_\varepsilon := \inf\{t \in (\hat{t}_\varepsilon, \hat{t}_\varepsilon + \delta) : \ \tfrac{1}{\varepsilon}\varrho_\varepsilon(t) \geq \tfrac{\gamma_2}{3\lambda(\hat{\sigma})}\}. \tag{3.29}$$

Then:

a) $t_\varepsilon - \hat{t} \to 0$ *as* $\varepsilon \to 0^+$*;*
b) $t_\varepsilon < a_\varepsilon$ *for* ε *sufficiently small;*
c) $\frac{1}{\varepsilon}\varrho_\varepsilon(t) \geq \frac{\gamma_2}{3\lambda(\hat{\sigma})}$ *for every* $t \in [t_\varepsilon, a_\varepsilon]$.

Proof. Concerning part a) and part b) of the statement, we may clearly suppose that $t_\varepsilon > \hat{t}_\varepsilon$. Let $s_\varepsilon := t_\varepsilon \wedge a_\varepsilon$. We first claim that, for small ε, in $(\hat{t}_\varepsilon, s_\varepsilon)$ one has $\varrho_\varepsilon(t) > 0$.

Indeed, we first observe that if the set $\{\varrho_\varepsilon(t) > 0\} \cap [\hat{t}_\varepsilon, s_\varepsilon]$ is empty along a suitable subsequence (unrelabelled), then clearly $s_\varepsilon = a_\varepsilon$, and (3.25) holds for every $t \in [\hat{t}_\varepsilon, t^*]$; we then easily get that $\liminf a_\varepsilon > \hat{t}$, and this contradicts Lemma 3.9. Then, for ε sufficiently small, the set $\{\varrho_\varepsilon(t) > 0\} \cap [\hat{t}_\varepsilon, s_\varepsilon]$ has positive measure. Now, observe that $\dot{\varrho}_\varepsilon(t) = 0$ a.e. in $\{\varrho_\varepsilon(t) = 0\} \cap [\hat{t}_\varepsilon, s_\varepsilon]$, while in the set $\{\varrho_\varepsilon(t) > 0\} \cap [\hat{t}_\varepsilon, s_\varepsilon]$ one has

$$\dot{\varrho}_\varepsilon(t) \geq \tfrac{\gamma_2}{2} \tag{3.30}$$

by (2.31), (3.26), (2.35), and (3.27). Then, by the fundamental theorem of calculus and by Lemma 3.9, we get

$$\varrho_\varepsilon(\tau) = \int_{\{\varrho_\varepsilon(t)>0\}\cap[\hat{t}_\varepsilon,\tau]} \dot{\varrho}_\varepsilon(t)\,dt \geq \tfrac{\gamma_2}{2}\mathcal{L}^1(\{\varrho_\varepsilon(t) > 0\} \cap [\hat{t}_\varepsilon, \tau]) > 0$$

for every $\tau \in [\hat{t}_\varepsilon, s_\varepsilon]$, which proves our claim. Therefore $\{\varrho_\varepsilon(t) > 0\} \cap (\hat{t}_\varepsilon, s_\varepsilon] = (\hat{t}_\varepsilon, s_\varepsilon]$ so that the previous estimate and the definition of s_ε yield

$$\varepsilon\tfrac{\gamma_2}{3\lambda(\hat{\sigma})} \geq \varrho_\varepsilon(s_\varepsilon) \geq \tfrac{\gamma_2}{2}(s_\varepsilon - \hat{t}_\varepsilon),$$

which implies, by (3.20), that

$$s_\varepsilon - \hat{t} \to 0 \quad \text{as } \varepsilon \to 0^+. \tag{3.31}$$

Now suppose, by contradiction, that $s_\varepsilon = a_\varepsilon$ as $\varepsilon \to 0$ along a suitable sequence. Then $a_\varepsilon - \hat{t}_\varepsilon \to 0$ as $\varepsilon \to 0^+$ and

$$\sup_{t\in[\hat{t}_\varepsilon, a_\varepsilon]} \tfrac{1}{\varepsilon}\varrho_\varepsilon(t) \leq \tfrac{\gamma_2}{3\lambda(\hat{\sigma})};$$

by the definition of a_ε, (2.29), and (3.20), this implies

$$\begin{aligned}
\delta + o(1) &= |(\sigma_\varepsilon(a_\varepsilon), z_\varepsilon(a_\varepsilon)) - (\sigma_\varepsilon(\hat{t}_\varepsilon), z_\varepsilon(\hat{t}_\varepsilon))| \\
&\leq |(\sigma_\varepsilon(a_\varepsilon) - \sigma_\varepsilon(\hat{t}_\varepsilon), 0)| + |(0, z_\varepsilon(a_\varepsilon) - z_\varepsilon(\hat{t}_\varepsilon))| \\
&\leq \int_{\hat{t}_\varepsilon}^{a_\varepsilon} |\dot{\sigma}_\varepsilon(t)| + |\dot{z}_\varepsilon(t)|\, dt \qquad (3.32)\\
&\leq (|\mathbb{C}| + |\mathrm{tr}(\hat{\sigma})| + \delta + o(1)) \int_{\hat{t}_\varepsilon}^{a_\varepsilon} \tfrac{\varrho_\varepsilon(t)}{\varepsilon}\, dt + |\mathbb{C}| \int_{\hat{t}_\varepsilon}^{a_\varepsilon} |\dot{\xi}(t)|\, dt \\
&\leq [(|\mathbb{C}| + |\mathrm{tr}(\hat{\sigma})| + \delta + o(1)) \tfrac{\gamma_2}{3\lambda(\hat{\sigma})}](a_\varepsilon - t_1) + |\mathbb{C}| \int_{\hat{t}_\varepsilon}^{a_\varepsilon} |\dot{\xi}(t)|\, dt,
\end{aligned}$$

a contradiction, since the right-hand side tends to 0 as $\varepsilon \to 0$. This proves part a) and part b) of the statement.

Observe now that (3.30) yields $\dot{\varrho}_\varepsilon(t_\varepsilon) \geq \frac{\gamma_2}{2}$. Thus, if c) is false, let t_ε^1 be the first time in $(t_\varepsilon, a_\varepsilon)$ such that $\varrho_\varepsilon(t_\varepsilon^1) = \frac{\gamma_2}{3\lambda(\sigma_1)}$; then $\dot{\varrho}_\varepsilon(t_\varepsilon^1) \leq 0$. Repeating the proof of (3.30) we find $\dot{\varrho}_\varepsilon(t_\varepsilon^1) \geq \frac{\gamma_2}{2} > 0$, a contradiction. □

Remark 3.12. Notice that if $\hat{t} = t_1$, the statement of the Lemma holds with $\hat{t}_\varepsilon = t_1$.

We now focus on the case where the slow-fast indicator is negative at $(\hat{\sigma}, \hat{z})$. As in [7], this allows to show that, in a neighborhood of $\hat{t}$, the function $\frac{1}{\varepsilon}\varrho_\varepsilon(t)$ remains uniformly bounded. This is the key ingredient to prove that the limit evolution is continuous.

For a suitable choice of δ in the definition of the neighborhood U_δ satisfying (3.26), we may assume that there exists a positive constant γ_1 such that

$$\Psi(\sigma, z) \leq -\gamma_1 \qquad \text{for every } (\sigma, z) \in B_\delta(\hat{\sigma}, \hat{z}). \tag{3.33}$$

We now state an auxiliary lemma, analogous to [7, Lemma 3.6], which will be used also in Section 4. Notice that in the statement of the lemma we make no assumption on the sign of the indicator Φ.

Lemma 3.13. *Assume (2.1)-(2.5), and (2.24); let Ψ be as in (2.33). Let $\tilde{t} > 0$, $(\tilde{\sigma}, \tilde{z}) \in \partial K$, and $\tilde{t}_\varepsilon$ a sequence such that*

$$\begin{aligned}
&\tilde{t}_\varepsilon \to \tilde{t} \text{ as } \varepsilon \to 0^+, \\
&(\sigma_\varepsilon(\tilde{t}_\varepsilon), z_\varepsilon(\tilde{t}_\varepsilon)) \to (\tilde{\sigma}, \tilde{z}) \text{ as } \varepsilon \to 0^+.
\end{aligned}$$

Suppose that there exist two constants $\eta > 0$, $\gamma > 0$ such that, for every (σ, z) satisfying $|(\sigma, z) - (\tilde{\sigma}, \tilde{z})| < \eta$, one has

$$\Psi(\sigma, z) < -\gamma.$$

Let

$$b_\varepsilon^\eta := \inf\{t \in (\tilde{t}_\varepsilon, \tilde{t} + \eta) :\ (\sigma_\varepsilon(t), z_\varepsilon(t)) \in \partial B_\eta(\tilde{\sigma}, \tilde{z})\}.$$

Then there exist $L > 0$ and a sequence $\tilde{s}_\varepsilon$, which may be taken equal to $\tilde{t}_\varepsilon$ whenever $\limsup\limits_{\varepsilon\to 0} \frac{\varrho_\varepsilon(\tilde{t}_\varepsilon)}{\varepsilon} < +\infty$, such that

a) $\tilde{s}_\varepsilon \to \tilde{t}$ *as* $\varepsilon \to 0^+$,

b) $(\sigma_\varepsilon(\tilde{s}_\varepsilon), z_\varepsilon(\tilde{s}_\varepsilon)) \to (\tilde{\sigma}, \tilde{z})$ *as* $\varepsilon \to 0^+$,

c) $\frac{\varrho_\varepsilon(t)}{\varepsilon} \le \frac{L}{\gamma}$ *for every* $t \in [\tilde{s}_\varepsilon, b^\eta_\varepsilon]$,

d) $\liminf_{\varepsilon\to 0} b^\eta_\varepsilon \ge \tilde{t} + C(\tilde{\sigma}, \eta, \gamma)$,

where $C(\tilde{\sigma}, \eta, \gamma) := \min\{\eta, \frac{\eta\gamma}{L[(1+\gamma)|\mathbb{C}|+|\mathrm{tr}(\tilde{\sigma})|+\eta]}\}$.

Proof. To prove a), b), c) it suffices to adapt the arguments of [7, Lemma 3.6]; to prove d) one can proceed as in (3.32), using the above bound on $\frac{\varrho_\varepsilon(t)}{\varepsilon}$ given by c); this explains why, differently from [7, Lemma 3.6], here the constant C may also depend on $\tilde{\sigma}$. □

The proof of the main theorem of this section involves of the following general result on continuous dependence on a parameter, whose proof can be found in [6] (see also [5]).

Theorem 3.14. *Let* f_ε *and* f_0 *be Carathéodory functions defined on* $[a, b]\times\mathbb{R}^m$ *with values in* $\mathbb{R}^m$, *let* t_ε, $t_0 \in [a, b]$, *and let* x_ε, $x_0 \in \mathbb{R}^m$. *Assume that there exist two constants* $L > 0$ *and* $M > 0$ *such that*

$$|f_\varepsilon(t, x_2) - f_\varepsilon(t, x_1)| \le L\, |x_2 - x_1|\,, \tag{3.34}$$

$$|f_\varepsilon(t, x)| \le M\,, \tag{3.35}$$

for every $\varepsilon > 0$, *every* $t \in [a, b]$, *and every* x, x_1, $x_2 \in \mathbb{R}^m$. *Let* $y_\varepsilon(t)$ *and* $y_0(t)$ *be the solutions of the Cauchy problems*

$$\begin{cases} \dot{y}_\varepsilon(t) = f_\varepsilon(t, y(t))\,, \\ y_\varepsilon(t_\varepsilon) = x_\varepsilon\,, \end{cases} \qquad\qquad \begin{cases} \dot{y}_0(t) = f_0(t, y(t))\,, \\ y_\varepsilon(t_0) = x_0\,. \end{cases} \tag{3.36}$$

If $t_\varepsilon \to t_0$, $x_\varepsilon \to x_0$, *and for every* $x \in \mathbb{R}^m$

$$\int_a^t f_\varepsilon(s, x)\, ds \to \int_a^t f(s, x)\, ds \qquad \text{uniformly for } t \in [a, b]\,,$$

then $y_\varepsilon(t) \to y_0(t)$ *uniformly for* $t \in [a, b]$.

In the following corollary inequalities (3.34) and (3.35) are satisfied only in the intervals $[t_\varepsilon, b]$, and the conclusion is slightly weaker.

Corollary 3.15. *Let* f_ε *and* f_0 *be Carathéodory functions defined on* $[a, b]\times\mathbb{R}^m$ *with values in* $\mathbb{R}^m$, *let* $t_\varepsilon \to a$, *and let* x_ε, $x_0 \in \mathbb{R}^m$. *Assume that there exist two constants* $L > 0$ *and* $M > 0$ *such that* (3.34) *and* (3.35) *hold for every* $\varepsilon > 0$, *every* $t \in [t_\varepsilon, b]$, *and every* x, x_1, $x_2 \in \mathbb{R}^m$. *Let* $y_\varepsilon(t)$ *and* $y_0(t)$ *be the solutions of the Cauchy problems* (3.36). *If* $x_\varepsilon \to x_0$, *and for every* $x \in \mathbb{R}^m$ *and every* $\eta > 0$

$$\int_{a+\eta}^t f_\varepsilon(s, x)\, ds \to \int_{a+\eta}^t f(s, x)\, ds \qquad \text{uniformly for } t \in [a+\eta, b]\,,$$

then

$$\sup_{t_\varepsilon \le t \le b} |y_\varepsilon(t) - y_0(t)| \to 0$$

Proof. See [7, Corollary 3.5]. □

We are now ready to prove the main result of this section.

Theorem 3.16. *Assume* (2.1)-(2.5), (2.24), *and let* Φ, Ψ *be as in* (2.32), *and* (2.33), *respectively. Let* $\hat{t} > 0$ *satisfy* (3.19)-(3.22), *let* $\hat{t}_\varepsilon$ *be as in* (3.20), *and suppose that* (3.33) *holds. Let* $(\sigma_{sl}(s), z_{sl}(s))$ *be the unique solution to the equation* *of the slow dynamics* (3.1) *with Cauchy datum* $(\hat\sigma, \hat z)$ *at* $\hat t$, *and let* $t_2 > \hat t$ *be as in* (3.5). *Let* $\bar t < t_2$ *and suppose that there exists a constant* $\gamma_3 > 0$ *such that*

$$\Phi(s, \sigma_{sl}(s), z_{sl}(s)) \geq \gamma_3 \qquad \text{for every } s \in [\hat t, \bar t]. \tag{3.37}$$

Then $(\sigma_\varepsilon, z_\varepsilon)$ *converges uniformly to* (σ_{sl}, z_{sl}) *as* $\varepsilon \to 0^+$ *on compact subsets of* $(\hat t, \bar t]$.

Proof. Let δ, γ_2, γ_1, $\hat t_\varepsilon$, and a_ε be given by (3.26), (3.33), (3.20), and (3.28), respectively. We put $t^* = \liminf_{\varepsilon \to 0^+} a_\varepsilon$, and we apply Lemma 3.13 with $\tilde t = \hat t$, $\tilde t_\varepsilon = \hat t_\varepsilon$, and $b_\varepsilon^\eta = a_\varepsilon$; we have that $t^* > \hat t$, and, by part c) of the Lemma, we may assume that there exists a nonnegative function $\omega(t)$ such that, for every $\eta > 0$, $\frac{\varrho_\varepsilon(t)}{\varepsilon}$ w^*-converges in $L^\infty((\hat t + \eta, t^*))$ to $\omega(t)$.

We write equation (2.26) in the form

$$\begin{cases} \mathbb{C}\dot\xi(t) - \dot\sigma(t) = \omega_1^\varepsilon(t, \sigma(t), z(t)) \\ \dot z(t) = \omega_2^\varepsilon(t, \sigma(t), z(t)), \end{cases}$$

where

$$\omega_1^\varepsilon(t, \sigma(t), z(t)) := \tfrac{\varrho_\varepsilon(t)}{\varepsilon} h_1(\sigma(t), z(t)) \tag{3.38}$$

$$\omega_2^\varepsilon(t, \sigma(t), z(t)) := \tfrac{\varrho_\varepsilon(t)}{\varepsilon} h_2(\sigma(t), z(t)); \tag{3.39}$$

here $h_1(\sigma, z)$ and $h_2(\sigma, z)$ denote two C^1 globally Lipschitz continuous functions, which coincide with $\mathbb{C} n_{K(z)}(\pi_{K(z)}(\sigma))$, and $\mathrm{tr}(\sigma)\mathrm{tr}(n_{K(z)}(\pi_{K(z)}(\sigma)))$, respectively, in $B_\delta(\hat\sigma, \hat z) \setminus \mathrm{int}\, K$. Corollary 3.15 now provides the uniform convergence of the solutions of (2.26) to the solution of the problem

$$\begin{cases} \mathbb{C}\dot\xi(t) - \dot\sigma(t) = \omega(t) h_1(\sigma(t), z(t)) \\ \dot z(t) = \omega(t) h_2(\sigma(t), z(t)), \end{cases} \tag{3.40}$$

with the required Cauchy data, on the compact subintervals of $(\hat t, t^*]$.

Now, Lemma 3.13, part c), implies that $(\sigma(t), z(t)) \in K$ for every $t \in (\hat t, t^*]$, while Lemma 3.11 entails that, for every $t \in (\hat t, t^*]$, the points $(\sigma_\varepsilon(t), z_\varepsilon(t))$ do not belong to K when ε is sufficiently small; this proves that $(\sigma(t), z(t)) \in \partial K$ for every $t \in (\hat t, t^*]$. Thus, for every $t \in (\hat t, t^*]$, the functions $h_1(\sigma(t), z(t))$ and $h_2(\sigma(t), z(t))$ coincide with $\mathbb{C} n_{K(z)}(\sigma)$ and $\mathrm{tr}(\sigma)\mathrm{tr}(n_{K(z)}(\sigma))$, respectively. Since $(\sigma(t), z(t)) \in \partial K$, we must have, for every $t \in (\hat t, t^*]$

$$0 = n_K((\sigma(t), z(t))) \cdot (\dot\sigma(t), \dot z(t));$$

this in turn, recalling (2.10), is equivalent to

$$0 = (z\, n_{K(z)}(\sigma), -\sigma \cdot n_{K(z)}(\sigma)) \cdot (\dot\sigma(t), \dot z(t)).$$

Then (3.40), (2.32), and (2.33) imply that

$$0 = \omega(t)\Psi(\sigma(t), z(t)) + \Phi(t, \sigma(t), z(t)). \tag{3.41}$$

Therefore (3.40) coincides with (3.1). We conclude that the solutions of (2.26) converge uniformly on compact subintervals of $(\hat t, t^*]$ to the solution of the equation (3.1) with Cauchy data $(\hat\sigma, \hat z)$ at $\hat t$, and by uniqueness, the limit is exactly $(\sigma_{sl}(t), z_{sl}(t))$.

Now, let $t^\dagger$ the maximal time such that $(\sigma_\varepsilon, z_\varepsilon)$ converges uniformly to (σ_{sl}, z_{sl}) as $\varepsilon \to 0^+$ on compact subintervals of $(\hat t, t^\dagger)$; to conclude the proof, we have to show that $t^\dagger > \bar t$. Let us argue by contradiction, supposing $t^\dagger \leq \bar t$. Define $(\sigma^\dagger, z^\dagger) :=$

$(\sigma_{sl}(t^\dagger), z_{sl}(t^\dagger))$ and observe that, by the hypotheses, there exist two constants $\eta > 0$ and $\gamma > 0$ such that, for every $(t, \sigma, z) \in [t^\dagger - \eta, t^\dagger + \eta] \times B_\eta(\sigma^\dagger, z^\dagger)$, one has $\Psi(\sigma, z) < -\gamma$ and $\Phi(t, \sigma, z) > \gamma$. We define $c(\frac{\eta}{2}, \gamma)$ as the infimum in $B_{\frac{\eta}{2}}(\sigma^\dagger, z^\dagger)$ of $C(\sigma, \frac{\eta}{2}, \gamma)$, where the latter is the constant defined in Lemma 3.13. Now we may fix $t^\dagger - \frac{\eta}{2} < t_1^\dagger < t_2^\dagger < t^\dagger < t_3^\dagger < t_1^\dagger + c(\frac{\eta}{2}, \gamma)$ in a way that $(\sigma_{sl}(t_1^\dagger), z_{sl}(t_1^\dagger)) \in B_{\frac{\eta}{2}}(\sigma^\dagger, z^\dagger)$ and we shall have that for every $(t, \sigma, z) \in [t_1^\dagger - \frac{\eta}{2}, t_1^\dagger + \frac{\eta}{2}] \times B_{\frac{\eta}{2}}(\sigma_{sl}(t_1^\dagger), z_{sl}(t_1^\dagger))$,

$$\Psi(\sigma, z) < -\gamma \quad \text{and} \quad \Phi(t, \sigma, z) > \gamma. \tag{3.42}$$

By Lemma 3.13, applied with $\tilde{t} = \tilde{t}_\varepsilon = t_1^\dagger$, we have that there exists $L > 0$ such that for ε sufficiently small $\frac{\varrho_\varepsilon(t)}{\varepsilon} \leq \frac{L}{\gamma}$ for every $t \in [t_2^\dagger, t_3^\dagger]$. By Lemma 3.11, applied with $\hat{t} = \hat{t}_\varepsilon = t_1^\dagger$, and $a_\varepsilon = b_\varepsilon^{\frac{\eta}{2}}$ we get that

$$\frac{\varrho_\varepsilon(t)}{\varepsilon} \geq \frac{\gamma}{3\lambda(\sigma_{sl}(t_1^\dagger))} \qquad \text{for every } t \in [t_2^\dagger, t_3^\dagger], \tag{3.43}$$

when ε is sufficiently small; here $\lambda(\sigma)$ is defined by (2.36). We repeat the arguments of the previous step of the proof, and we also notice that we are in position to apply Theorem 3.14 in place of Corollary 3.15, to get that the solutions of (2.26) converges uniformly in the interval $[t_2^\dagger, t_3^\dagger]$ to the solution of the problem (3.1) with Cauchy data $(\sigma(t_2^\dagger), z(t_2^\dagger)) = (\sigma_{sl}(t_2^\dagger), z_{sl}(t_2^\dagger))$, that is, by uniqueness, to $(\sigma_{sl}(t), z_{sl}(t))$. This contradicts the maximality of $t^\dagger$. □

Remark 3.17. A slight adaptation of the proof, taking into account Remark 3.12, easily shows that in the particular case $\hat{t} = t_1$ the conclusion of the Theorem holds on the whole closed interval $[t_1, \bar{t}]$.

The previous theorem shows that, if one has

$$\Phi(t, \sigma_{sl}(t), z_{sl}(t)) > 0 \qquad \text{for every } \hat{t} \leq t < t_2, \tag{3.44}$$

then $(\sigma_\varepsilon, z_\varepsilon)$ converges uniformly to (σ_{sl}, z_{sl}) as $\varepsilon \to 0^+$ on compact subintervals of $(\hat{t}, t_2)$. On the contrary, if

$$\Phi(\bar{t}, \sigma_{sl}(\bar{t}), z_{sl}(\bar{t})) = 0 \tag{3.45}$$

for some $\hat{t} < \bar{t} < t_2$, then the elastic behavior may re-appear starting from the point $(\bar{\sigma}, \bar{z}) := (\sigma_{sl}(\bar{t}), z_{sl}(\bar{t})) \in \partial K$, as we are going to discuss in the next subsection.

In the last section of the paper we will consider the case when (3.44) holds, and $t_2 < +\infty$; we will show that a transition from the slow to the fast dynamics occurs at time t_2 when(3.7) and (3.14) hold with strict inequality.

3.3. **Return to the elastic regime.** In this subsection we take $\hat{t}$ and t_2 as in Theorem 3.16, and we assume that there exists $\hat{t} < \bar{t} < t_2$ satisfying (3.45) and such that $\Phi(t, \sigma_{sl}(t), z_{sl}(t)) > 0$ for every $\hat{t} \leq t < \bar{t}$. Our purpose is to give some conditions which imply the return of the system to the elastic behavior after the time $\bar{t}$. The discussion will be completely analogous to that in [7, Section 3.3], hence the proofs will be only sketched.

Assume that there exists a sequence $t_n \to \bar{t}$ such that

$$\Phi(t_n, \sigma_{sl}(t_n), z_{sl}(t_n)) < 0 \tag{3.46}$$

and that there exists $\eta > 0$ such that, for every $(t, s, \sigma, z) \in (\bar{t}, \bar{t} + \eta) \times (0, \eta) \times (B_\eta(\bar{\sigma}, \bar{z})) \cap \partial K$ satisfying $\Phi(t, \sigma, z) \leq 0$,

$$(\sigma + \mathbb{C}(\xi(t+s) - \xi(t)), z) \in \operatorname{int} K. \tag{3.47}$$

We then have the following theorem.

Theorem 3.18. *Assume (2.1)-(2.5), (2.24), and let Φ, Ψ be as in (2.32), and (2.33), respectively. Let $\hat{t} > 0$, $(\sigma_{sl}(s), z_{sl}(s))$, and $t_2 > \hat{t}$ be as in Theorem 3.16. Let $\bar{t} < t_2$ satisfy (3.45) and suppose that $\Phi(t, \sigma_{sl}(t), z_{sl}(t)) > 0$ for every $\hat{t} \le t < \bar{t}$. Let $(\bar{\sigma}, \bar{z}) := (\sigma_{sl}(\bar{t}), z_{sl}(\bar{t}))$, and assume that (3.46) and (3.47) hold. Let $(\sigma_{el}(t), z_{el}(t)) := (\bar{\sigma} + \mathbb{C}(\xi(t) - \xi(\bar{t})), \bar{z})$ and*

$$\tau := \sup\{t > \bar{t} \,|\, (\sigma_{el}(s), z_{el}(s)) \in \operatorname{int} K \text{ for every } s \in (\bar{t}, t)\}.$$

Then $(\sigma_\varepsilon, z_\varepsilon)$ converges uniformly on compact subsets of $(\hat{t}, \tau)$ to the function (σ, z) defined by

$$(\sigma(t), z(t)) := \begin{cases} (\sigma_{sl}(t), z_{sl}(t)) & \text{for } \hat{t} < t \le \bar{t}, \\ (\sigma_{el}(t), z_{el}(t)) & \text{for } \bar{t} \le t < \tau. \end{cases} \tag{3.48}$$

Proof. Let $\hat{\tau}$ be the maximal time such that $(\sigma_\varepsilon, z_\varepsilon)$ converges uniformly to (σ, z) on compact subintervals of $(\hat{t}, \hat{\tau})$; we have to show that $\hat{\tau} = \tau$. By Theorem 3.16, it follows that $\hat{\tau} \ge \bar{t}$. As in [7, Theorem 3.11], it is easy to see that $\hat{\tau} = \tau$ when $\hat{\tau} > \bar{t}$, therefore we have only to exclude $\hat{\tau} = \bar{t}$.

In this case, there exist two constants $\eta > 0$ and $\gamma > 0$ such that, for every $(t, \sigma, z) \in [\bar{t} - \eta, \bar{t} + \eta] \times B_\eta(\bar{\sigma}, \bar{z})$, one has $\Psi(\sigma, z) < -\gamma$. We define $c(\frac{\eta}{2}, \gamma)$ as the infimum in $B_{\frac{\eta}{2}}(\sigma^\dagger, z^\dagger)$ of $C(\sigma, \frac{\eta}{2}, \gamma)$, where the latter is the constant defined in Lemma 3.13. Now we may fix $t^\dagger - \frac{\eta}{2} < t_1^\dagger < t_2^\dagger < t^\dagger < t_3^\dagger < t_1^\dagger + c(\frac{\eta}{2}, \gamma)$ in a way that $(\sigma_{sl}(t_1^\dagger), z_{sl}(t_1^\dagger)) \in B_{\frac{\eta}{2}}(\sigma^\dagger, z^\dagger)$ and we shall have that for every $(t, \sigma, z) \in [t_1^\dagger - \frac{\eta}{2}, t_1^\dagger + \frac{\eta}{2}] \times B_{\frac{\eta}{2}}(\sigma_{sl}(t_1^\dagger), z_{sl}(t_1^\dagger))$,

$$\Psi(\sigma, z) < -\gamma.$$

By Lemma 3.13, applied with $\tilde{t} = \tilde{t}_\varepsilon = t_1^\dagger$, we have that there exists $L > 0$ such that for ε sufficiently small $\frac{\varrho_\varepsilon(t)}{\varepsilon} \le \frac{L}{\gamma}$ for every $t \in [t_2^\dagger, t_3^\dagger]$, thus we may assume $\frac{\varrho_\varepsilon(t)}{\varepsilon}$ w^*-converges in $L^\infty((t_2^\dagger, t_3^\dagger))$ to some nonnegative function $\omega(t)$. By (3.38), (3.39), and Theorem 3.14 the sequence $(\sigma_\varepsilon, z_\varepsilon)$ converges uniformly in $[t_2^\dagger, t_3^\dagger]$ to a continuous function $(\tilde{\sigma}, \tilde{z})$. Theorem 3.16 gives $(\tilde{\sigma}, \tilde{z}) = (\sigma_{sl}, z_{sl})$ in $[t_2^\dagger, \bar{t})$, while [7, Theorem 3.11] gives $(\tilde{\sigma}, \tilde{z}) = (\sigma_{el}, z_{el})$ in $[\bar{t}, t_3^\dagger]$, thus $(\tilde{\sigma}, \tilde{z}) = (\sigma, z)$ in $[t_2^\dagger, t_3^\dagger]$. This contradicts the maximality of $\hat{\tau}$, when $\hat{\tau} = \bar{t}$. □

Remark 3.19. When ξ is at least C^2 regular, by adapting the argument of [7, Remark 3.12] we obtain that the inequality

$$\mathbb{C}\ddot{\xi}(\bar{t}) \cdot n_{K(\bar{z})}(\bar{\sigma}) + \mathbb{C}\dot{\xi}(\bar{t}) \cdot [\nabla_\sigma n_{K(\bar{z})}(\bar{\sigma})]\, \mathbb{C}\dot{\xi}(\bar{t}) < 0. \tag{3.49}$$

implies both (3.46) and (3.47). Notice that, since $\bar{t}$ is the first time such that (3.45) is satisfied, we always have

$$\mathbb{C}\ddot{\xi}(\bar{t}) \cdot n_{K(\bar{z})}(\bar{\sigma}) + \mathbb{C}\dot{\xi}(\bar{t}) \cdot [\nabla_\sigma n_{K(\bar{z})}(\bar{\sigma})]\, \mathbb{C}\dot{\xi}(\bar{t}) \le 0.$$

It follows from the definition of Φ, from (3.45), and from (2.10), that the vector $\mathbb{C}\dot{\xi}(\bar{t})$ is tangent to $\partial K(\bar{z})$ at $\bar{\sigma}$, hence $\mathbb{C}\dot{\xi}(\bar{t}) \cdot [\nabla_\sigma n_{K(\bar{z})}(\bar{\sigma})]\, \mathbb{C}\dot{\xi}(\bar{t})$ is exactly the second fundamental form of $\partial K(\bar{z})$ at $\bar{\sigma}$, applied to the tangent vector $\mathbb{C}\dot{\xi}(\bar{t})$.

4. **Softening with discontinuities.**

4.1. **The equation of the fast dynamics.**

The goal of this section is a qualitative study of the equation

$$\begin{cases}\dot\sigma_f(s) = \mathbb{C}(\pi_{K(z_f(s))}(\sigma_f(s)) - \sigma_f(s))\\ \dot z_f(s) = \operatorname{tr}(\sigma_f(s))\operatorname{tr}\big(\sigma_f(s) - \pi_{K(z_f(s))}(\sigma_f(s))\big);\end{cases} \tag{4.1}$$

this is called the fast dynamics equation and appears, as we shall see, as limit of a rescaled version of (2.26) near a discontinuity point of a viscosity solution.

Under suitable conditions, we shall see the viscosity solution will jump between the two endpoints of a heteroclinic orbit of (4.1), whose existence, together with other properties, is the object of this subsection.

In order to prove the main theorem of this subsection, we need a preliminary lemma, showing that the internal variable is constant along the unique solution of (4.1), with an initial condition $(\bar\sigma, \bar z)$ satisfying

$$(\bar\sigma, \bar z) \notin K \quad \text{and} \quad \operatorname{tr}\big(n_{K(\bar z)}(\pi_{K(\bar z)}(\bar\sigma))\big) = 0. \tag{4.2}$$

We preliminarly observe that taking an initial condition outside K easily implies that we can never reach K in finite time, as the set K is made of critical points of the autonomous equation (4.1). Through the decomposition (2.13) we identify $\mathbb{M}^{N\times N}_{sym}$ with $\mathbb{R}\times\mathbb{M}^{N\times N}_D$; in particular $\sigma_f(s)$ is identified with the pair $(x_f(s), y_f(s))$ of its spherical and deviatoric parts. Introducing the function ϱ defined by (2.7), which is positive by the previous remark, we may rewrite equation (4.1) in the form

$$\begin{cases}\dot x_f(s) = -\kappa\sqrt{N}\,\varrho(x_f(s), y_f(s), z_f(s))\operatorname{tr}\Big(n_{K(z_f(s))}\big(\pi_{K(z_f(s))}(x_f(s), y_f(s))\big)\Big),\\ \dot y_f(s) = -2\mu\,\varrho(x_f(s), y_f(s), z_f(s))\, n^D_{K(z_f(s))}\big(\pi_{K(z_f(s))}(x_f(s), y_f(s))\big),\\ \dot z_f(s) = \sqrt{N}x_f(s)\,\varrho(x_f(s), y_f(s), z_f(s))\operatorname{tr}\Big(n_{K(z_f(s))}\big(\pi_{K(z_f(s))}(x_f(s), y_f(s))\big)\Big).\end{cases} \tag{4.3}$$

Here κ and μ are defined in (2.24) and $n^D_{K(z_f(s))}\big(\pi_{K(z_f(s))}(x_f(s), y_f(s))\big)$ is the deviatoric part of $n_{K(z_f(s))}\big(\pi_{K(z_f(s))}(x_f(s), y_f(s))\big)$.

Lemma 4.1. *Let $(\bar\sigma, \bar z) \in [\mathbb{M}^{N\times N}_{sym} \times (0, +\infty)] \setminus K$ satisfying (4.2), and let $\bar x$ and $\bar y$ the spherical and the deviatoric part of $\bar\sigma$, respectively. Then, for every $t \in \mathbb{R}$, the unique solution to equation (4.3) with Cauchy data $(x_f(0), y_f(0), z_f(0)) = (\bar x, \bar y, \bar z)$ is given by*

$$(x_f(s), y_f(s), z_f(s)) = (\bar x, y(s), \bar z)$$

where $y(s)$ solves the equation

$$\dot y(s) = -2\mu\,\varrho(\bar x, y(s), \bar z)N^D_{K(\bar z)}(\pi_{K(\bar z)}(\bar x, \bar y)) \tag{4.4}$$

with Cauchy condition $y(0) = \bar y$.

Proof. Let $y(s)$ be the unique solution to (4.4) with Cauchy condition $y(0) = \bar{y}$. Then, for every $s' > 0$

$$\begin{aligned}(\bar{x}, y(s')) &= \Big(\bar{x}, \bar{y} - 2\mu \int_0^{s'} \varrho(\bar{x}, y(s), \bar{z}) n^D_{K(\bar{z})}(\pi_{K(\bar{z})}(\bar{x}, \bar{y}))\, ds\Big) \\ &= \Big(\bar{x}, \bar{y} - 2\mu\, n^D_{K(\bar{z})}(\pi_{K(\bar{z})}(\bar{x}, \bar{y})) \int_0^{s'} \varrho(\bar{x}, y(s), \bar{z})\, ds\Big) \\ &= \Big((\bar{x}, \bar{y}) - 2\mu\, n_{K(\bar{z})}(\pi_{K(\bar{z})}(\bar{x}, \bar{y})) \int_0^{s'} \varrho(\bar{x}, y(s), \bar{z})\, ds\Big),\end{aligned}$$

Therefore $\pi_{K(\bar{z})}(\bar{x}, y(s')) = \pi_{K(\bar{z})}(\bar{x}, \bar{y})$, provided $(\bar{x}, y(s'), \bar{z}) \notin K$; this allows to check that $(\bar{x}, y(s), \bar{z})$ solves (4.3), at least for small $|s|$. The conclusion for every s follows, as solutions to (4.3) can never reach K in finite time. ☐

Now we are able to prove the existence of an heteroclinic orbit of (4.1) starting from a point $(\hat{\sigma}, \hat{z}) \in \partial K$ under suitable hypotheses on the slow-fast indicator Ψ.

Theorem 4.2. *Assume that (2.24) and (2.14)-(2.17) are satisfied; let Φ, Ψ as in (2.32) and (2.33), respectively. Let $(\hat{\sigma}, \hat{z}) \in \partial K$ and suppose that*

$$\Psi(\hat{\sigma}, \hat{z}) > 0 \tag{4.5}$$

or

$$\Psi(\hat{\sigma}, \hat{z}) = 0 \quad \text{and} \quad \nabla\Psi(\hat{\sigma}, \hat{z}) \cdot \big(\tfrac{-\mathbb{C}\, n_{K(\hat{z})}(\hat{\sigma})}{\operatorname{tr}(\hat{\sigma}) \operatorname{tr}(n_{K(\hat{z})}(\hat{\sigma}))}, 1\big) < 0. \tag{4.6}$$

Then equation (4.1) has a unique solution $(\hat{\sigma}_f(s), \hat{z}_f(s))$ (up to time-translations) satisfying

$$\lim_{s \to -\infty} (\hat{\sigma}_f(s), \hat{z}_f(s)) = (\hat{\sigma}, \hat{z}). \tag{4.7}$$

Moreover, the limit

$$(\sigma_\infty, z_\infty) := \lim_{s \to +\infty} (\hat{\sigma}_f(s), \hat{z}_f(s)) \tag{4.8}$$

exists and satisfies the following conditions

$$(\sigma_\infty, z_\infty) \in \partial K,\ z_\infty > 0, \tag{4.9}$$

$$\Psi(\sigma_\infty, z_\infty) \le 0, \tag{4.10}$$

$$\operatorname{tr}(\sigma_\infty) < 0,\ \operatorname{tr}(n_{K(z_\infty)}(\sigma_\infty)) > 0. \tag{4.11}$$

Proof. We first observe that, by (2.5), (2.12), and by (2.33), both (4.5) and (4.6) imply that

$$\operatorname{tr}(\hat{\sigma}) < 0,\ \operatorname{tr}(n_{K(\hat{z})}(\hat{\sigma})) > 0. \tag{4.12}$$

Moreover, due to our regularity assumptions on K we may assume that in a suitably small neighborhood of $(\hat{\sigma}, \hat{z})$ an oriented distance function r from ∂K is well-defined; this is a C^1-extension of the function ϱ, defined by (2.7), to the interior of K. In view of the same assumptions, we may also locally define a minimal distance projection onto $\partial K(z)$, denoted by $\pi_{\partial K(z)}$, which obviously coincides with $\pi_{K(z)}$ outside of $K(z)$. For all these reasons, the Cauchy problem

$$\begin{cases} \sigma'(z) = \dfrac{-\mathbb{C}\, n_{K(z)}(\sigma(z))}{\operatorname{tr}(\sigma(z)) \operatorname{tr}(n_{K(z)}(\pi_{\partial K(z)}(\sigma(z))))} \\ \sigma(\hat{z}) = \hat{\sigma} \end{cases} \tag{4.13}$$

is well defined and admits a unique solution, which shall be denoted by $\hat{\sigma}(z)$. For z sufficiently close to $\hat{z}$ we then have that $\operatorname{tr}(\hat{\sigma}(z)) < 0$ and $\operatorname{tr}(n_{K(z)}(\pi_{\partial K(z)}(\sigma(z))) >$

0; moreover for $z < \hat{z}$, sufficiently close to $\hat{z}$ we can prove that $(\hat{\sigma}(z), z) \notin K$. Indeed, as $r(\hat{\sigma}, \hat{z}) = 0$, it suffices to show that in a left open neighborhood of $\hat{z}$ one has

$$\tfrac{d}{dz} r(\hat{\sigma}(z), z) < 0. \tag{4.14}$$

By a direct computation, similar to that in (2.11), exploiting (4.13) and (2.33) we get:

$$\frac{d}{dz} r(\hat{\sigma}(z), z) = \frac{\Psi(\hat{\sigma}(z), z)}{\operatorname{tr}(\hat{\sigma}(z)) \operatorname{tr}(n_{K(z)}(\pi_{\partial K(z)}(\hat{\sigma}(z))))}. \tag{4.15}$$

Then (4.5) implies that $\frac{d}{dz} r(\hat{\sigma}(z), z) < 0$ for $z = \hat{z}$, thus (4.14) follows; if (4.6) holds, deriving $\Psi(\hat{\sigma}(z), z)$, we get that

$$\tfrac{d}{dz} r(\hat{\sigma}(\hat{z}), \hat{z}) = 0 \text{ and } \tfrac{d^2}{dz^2} r(\hat{\sigma}(\hat{z}), \hat{z}) > 0,$$

which in its turn implies (4.14). We thus may fix $\bar{z} < \hat{z}$ such that, for every $z \in [\bar{z}, \hat{z})$, the following three hold

$$\varrho(\hat{\sigma}(z), z) > 0, \tag{4.16}$$

$$\operatorname{tr}(\hat{\sigma}(z)) < 0, \tag{4.17}$$

$$\operatorname{tr}(n_{K(z)}(\pi_{K(z)}(\hat{\sigma}(z)))) > 0; \tag{4.18}$$

we may indeed replace $\pi_{\partial K}$ with π_K as $(\hat{\sigma}(z), z) \notin K$. Now, let $\hat{z}_f(s)$ the unique solution to the autonomous Cauchy problem

$$\begin{cases} \dot{z}_f(s) = \operatorname{tr}(\hat{\sigma}(z_f(s))) \operatorname{tr}(\hat{\sigma}(z_f(s)) - \pi_{K(z_f(s))}(\hat{\sigma}(z_f(s)))) \\ z_f(0) = \hat{z}; \end{cases}$$

by (4.16)-(4.18), we have that $\operatorname{tr}(\hat{\sigma}(z)) \operatorname{tr}(\hat{\sigma}(z) - \pi_{K(z)}(\hat{\sigma}(z)) < 0$, for every $z \in [\bar{z}, \hat{z})$, with equality in $z = \hat{z}$; the theory of autonomous equations implies that $\hat{z}_f(s)$ is defined for every $s \leq 0$ and satisfies

$$\lim_{t \to -\infty} \hat{z}_f(s) = \hat{z}, \qquad \dot{\hat{z}}_f(s) < 0 \text{ for every } t \leq 0;$$

it now suffices to put $\hat{\sigma}_f(s) := \hat{\sigma}(\hat{z}_f(s))$, to get a solution to (4.1) satifying (4.7).

To prove uniqueness, let $(\sigma(s), z(s))$ a solution to (4.1) satisfying (4.7); (4.12) implies that there exists $\bar{s} \in \mathbb{R}$ such that, for every $s \leq \bar{s}$, one has $\dot{z}(s) < 0$. Then $z(s)$ is invertible in $(-\infty, \bar{t})$ with inverse $s(z)$. If we put $\sigma(z) := \sigma(s(z))$, it is easy to see that $\sigma(z)$ solves (4.13), thus coincides with $\hat{\sigma}(z)$; the theory of autonomous equation now implies that $(\sigma(s), z(s))$ and $(\hat{\sigma}_f(s), \hat{z}_f(s))$ may only differ by a time translation, thus the first part of the statement is proven.

Now, let $(-\infty, S)$ the maximal interval of definition for $(\hat{\sigma}_f(s), \hat{z}_f(s))$; observe that, as orbits can never reach K in finite time, $(\hat{\sigma}_f(s), \hat{z}_f(s))$ also solves (4.3). We split $\hat{\sigma}_f(s)$ in its spherical part $\hat{x}_f(s)$ and in its deviatoric part $\hat{y}_f(s)$ as in (2.13), and we observe that, by (4.3), the following equality holds:

$$\kappa \dot{\hat{z}}_f(s) = -\hat{x}_f(s) \dot{\hat{x}}_f(s). \tag{4.19}$$

Moreover, (4.12) implies that there exist $\bar{s} < S$ such that $\dot{\hat{x}}_f(s) < 0$ for every $s \leq \bar{s}$. Let us prove that $\dot{\hat{x}}_f(s) < 0$ for every $s < S$. Indeed, if there exists $s_1 < S$ such that $\dot{\hat{x}}_f(s_1) = 0$, by (4.3), as $\varrho(\hat{x}_f(s_1), \hat{y}_f(s_1), z_f(s_1) > 0$, it must be

$$\operatorname{tr}(n_{K(\hat{z}_f(s_1))}(\hat{\sigma}_f(s_1))) = 0;$$

by Lemma 4.1, this implies $\hat{x}_f(s) = \hat{x}_f(s_1)$ for all s, a contradiction. In particular there exists

$$x_S := \lim_{s\to S} \hat{x}_f(s) < \hat{x} < 0, \tag{4.20}$$

where $\hat{x}$ is the spherical part of $\hat{\sigma}$. Now (4.19) implies that $\dot{\hat{z}}_f(s) < 0$ for every $s < S$. In particular there exists $z_S := \lim_{s\to S} \hat{z}_f(s) < \hat{z}$.

We now show that z_S is greater than zero. Indeed, by (4.3), the fact that $\dot{\hat{x}}_f(s) < 0$ for every $s < S$ is equivalent to the inequality

$$\operatorname{tr}(n_{K(\hat{z}_f(s))}(\pi_{K(\hat{z}_f(s))}(\hat{x}_f(s), \hat{y}_f(s)))) > 0 \text{ for every } s < S, \tag{4.21}$$

and also, as $\varrho(\hat{x}_f(s), \hat{y}_f(s), \hat{z}_f(s)) > 0$, to the inequality

$$\operatorname{tr}(\pi_{K(\hat{z}_f(s))}(\hat{\sigma}_f(s)) < \operatorname{tr}(\hat{\sigma}_f(s)) = \sqrt{N}\hat{x}_f(s) \text{ for every } s < S. \tag{4.22}$$

By (2.9) and (2.23), (4.21) is equivalent to

$$\operatorname{tr}(\pi_{K(\hat{z}_f(s))}(\hat{\sigma}_f(s))) + a\sqrt{N}\hat{z}_f(s) > 0,$$

where a is the positive constant defined by (2.14); thus, by (4.22) we conclude that

$$\hat{x}_f(s) + a\hat{z}_f(s) > 0 \text{ for every } s < S \tag{4.23}$$

which in the limit gives $z_S > \frac{|x_S|}{a} > 0$, as claimed.

We now show that $(\hat{\sigma}_f(s), \hat{z}_f(s))$ is bounded, which in particular implies that $S = +\infty$. Clearly, it suffices to prove that $\hat{y}_f(s)$ is bounded. We have, by (4.1), the negativeness of $\hat{x}_f(s)$ and (4.22), that

$$\begin{aligned}
\frac{d}{ds}\frac{|\hat{y}_f(s)|^2}{2} &= \hat{y}_f(s)\cdot\dot{\hat{y}}_f(s)\\
&= 2\mu\,\hat{y}_f(s)\cdot\big(\pi_{K(\hat{z}_f(s))}(\hat{\sigma}_f(s)) - \hat{\sigma}_f(s)\big)\\
&= 2\mu\,\hat{\sigma}_f(s)\cdot\big(\pi_{K(\hat{z}_f(s))}(\hat{\sigma}_f(s)) - \hat{\sigma}_f(s)\big)\\
&\quad - 2\tfrac{\mu}{\sqrt{N}}\,\hat{x}_f(s)\operatorname{tr}\big(\pi_{K(\hat{z}_f(s))}(\hat{\sigma}_f(s)) - \hat{\sigma}_f(s)\big)\\
&\le 2\mu\,\hat{\sigma}_f(s)\cdot\big(\pi_{K(\hat{z}_f(s))}(\hat{\sigma}_f(s)) - \hat{\sigma}_f(s)\big) \le 0,
\end{aligned}$$

as a consequence of (2.12); this proves that $|\hat{y}_f(s)|^2$ is decreasing, thus $\hat{y}_f(s)$ is bounded.

Thus $S = +\infty$ and z_S is the limit of $\hat{z}_f(s)$ at $+\infty$, which shall be denoted with z_∞ from now on; by the previous discussion, we also have that $z_\infty > 0$, as required by (4.9). Now we prove that $\hat{\sigma}_f(s)$ has a limit at $+\infty$. To do that, we observe that $\hat{z}_f(s)$ is strictly decreasing, thus globally invertible; we thus express $\hat{\sigma}$ in function of z and we have to show that there exists $\lim_{z\to z_\infty} \hat{\sigma}(z)$. We already know that $\hat{\sigma}(z)$ is bounded and that its derivative satisfies

$$\hat{\sigma}'(z) = \frac{-\mathbb{C}\, n_{K(z)}(\hat{\sigma}(z))}{\operatorname{tr}(\hat{\sigma}(z))\operatorname{tr}(n_{K(z)}(\pi_{K(z)}(\hat{\sigma}(z))))} \tag{4.24}$$

thus the claim will follow once we get that

$$\liminf_{z\to z_\infty} \operatorname{tr}(n_{K(z)}(\pi_{K(z)}(\hat{\sigma}(z)))) > 0. \tag{4.25}$$

Suppose that (4.25) is false; first, observe that in this case the liminf must be a limit, as a consequence of the boundedness of $\hat{\sigma}(z)$ and of Lemma 3.5. Therefore we will have, exploiting (2.33),

$$\lim_{z\to z_\infty} \Psi(\hat{\sigma}(z), z) = -2\mu. \tag{4.26}$$

Moreover, observe that by (2.9) and (2.19),

$$\lim_{z\to z_\infty} \operatorname{tr}(n_{K(z)}(\pi_{K(z)}(\hat\sigma(z)))) = 0 \Leftrightarrow \lim_{z\to z_\infty} \tfrac{1}{z}[\tfrac{\operatorname{tr}(\pi_{K(z)}(\hat\sigma(z)))}{\sqrt{N}} + az] = 0; \tag{4.27}$$

on the other hand, clearly $\lim_{z\to z_\infty} \operatorname{tr}(n_{K(z)}(\pi_{K(z)}(\hat\sigma(z)))) = 0$ implies that

$$\lim_{z\to z_\infty} [\operatorname{tr}(\pi_{K(z)}(\hat\sigma(z))) - \sqrt{N}\hat x(z)] = 0, \tag{4.28}$$

thus combining (4.27) and (4.28), we get that

$$\lim_{z\to z_\infty} \hat x(z) = -az_\infty. \tag{4.29}$$

Now, by (2.9), (2.19), (2.23), and (4.22), we have that

$$\begin{aligned} |\operatorname{tr}(n_{K(z)}(\pi_{K(z)}(\hat\sigma(z))))| &\le |\tfrac{1}{z}[\tfrac{\operatorname{tr}(\pi_{K(z)}(\hat\sigma(z)))}{\sqrt{N}} + az]| \\ &\le \tfrac{1}{z_\infty}[\tfrac{\operatorname{tr}(\pi_{K(z)}(\hat\sigma(z)))}{\sqrt{N}} + az] \\ &\le \tfrac{1}{z_\infty}[\hat x(z) + az]. \end{aligned} \tag{4.30}$$

By (4.24), $\hat x'(z) = \frac{-\kappa}{\hat x(z)}$; this fact, together with (4.29) and (4.30), yields that

$$\limsup_{z\to z_\infty} \frac{|\operatorname{tr}(n_{K(z)}(\pi_{K(z)}(\hat\sigma(z))))|}{z - z_\infty} \le \frac{1}{z_\infty}(\frac{\kappa}{az_\infty} + a). \tag{4.31}$$

Since (4.15) gives

$$\tfrac{d}{dz}\varrho(\hat\sigma(z), z) = \tfrac{\Psi(\hat\sigma(z),z)}{\operatorname{tr}(\hat\sigma(z))\operatorname{tr}(n_{K(z)}(\pi_{K(z)}(\hat\sigma(z))))}, \tag{4.32}$$

recalling that $\operatorname{tr}(n_{K(z)}(\pi_{K(z)}(\hat\sigma(z)))) > 0$ for all $z > z_\infty$, we conclude by (4.26), (4.29), and (4.31), that

$$\liminf_{z\to z_\infty} (z - z_\infty)\tfrac{d}{dz}\varrho(\hat\sigma(z), z) \ge \tfrac{2\mu z_\infty}{\sqrt{N}(\kappa + az_\infty)} > 0.$$

This finally implies that

$$\lim_{z\to z_\infty} \varrho(\hat\sigma(z), z) = -\infty,$$

contradicting the nonnegativeness of ϱ.

We thus have that there exists

$$\sigma_\infty := \lim_{z\to z_\infty} \hat\sigma(z),$$

thus the proof of (4.8) is concluded. It is obvious that $(\sigma_\infty, z_\infty) \in \partial K$ as it must be a critical point of (4.1), thus (4.9) is proved. Concerning (4.11), it immediately follows from (4.25) and (4.20). Finally, as $\varrho(\hat\sigma(z), z) \ge 0$ for $z > z_\infty$, we must have $\frac{d}{dz}\varrho(\hat\sigma(z), z) \ge 0$ for $z = z_\infty$; observing that $\operatorname{tr}(\sigma_\infty)\operatorname{tr}(n_{K(z_\infty)}(\sigma_\infty)) < 0$ by (4.11), from (4.32) we immediately get (4.10). □

Remark 4.3. It is easy to show that, if an orbit of the system (4.1) has $(\hat\sigma, \hat z)$ as an α-limit point, then $(\hat\sigma, \hat z)$ is indeed its unique α-limit point; indeed, by the same arguments used in the proof of the previous theorem we can show that in this case $z(s)$ is strictly decreasing in a neighborhood of $-\infty$, thus it has $\hat z$ as a limit; the rest of the proof follows from (4.24), and Lemma 3.5.

We end up this analysis of equation (4.1) by showing an example where we can improve (4.10), that is a case where $\Psi(\sigma_\infty, z_\infty) < 0$.

Example 4.4. We suppose that for every $z \in (0, +\infty)$, $K(z)$ is an ellipsoid of the form

$$K(z) := \{\sigma \in \mathbb{M}^{N\times N}_{sym} | (x+z)^2 + \tfrac{|y|^2}{b^2} = z^2\}, \tag{4.33}$$

where x and y are as in (2.13). Notice that $K(1)$ satisfies (2.14)-(2.17) with $a = 1$. Suppose that, if κ and μ are as in (2.24) and b as in (4.33) the following condition holds:

$$\kappa N \geq \tfrac{2\mu}{b^2}. \tag{4.34}$$

We first compute the expression of Ψ on the yield surface in this case. Let $(\sigma, z) \in \partial K$, with $z > 0$. We define

$$F(x, y, z) = \sqrt{(x+z)^2 + \tfrac{|y|^2}{b^4}}; \tag{4.35}$$

to compute the expression of Ψ, it suffices to take into account the following facts:

a) $n_{K(z)}(\sigma) = \frac{1}{F(x,y,z)}[(x+z)\frac{I}{\sqrt{N}} + \frac{y}{b^2}]$;

b) $\mathrm{tr}(n_{K(z)}(\sigma)) = \frac{\sqrt{N}(x+z)}{F(x,y,z)}$ and $\mathrm{tr}(\sigma) = \sqrt{N}x$;

c) $\mathbb{C}n_{K(z)}(\sigma) = \frac{1}{F(x,y,z)}[\kappa N(x+z)\frac{I}{\sqrt{N}} + \frac{2\mu y}{b^2}]$.

It follows that

$$\begin{aligned} -n_{K(z)}(\sigma)\cdot \mathbb{C}n_{K(z)}(\sigma) &= -\tfrac{1}{F(x,y,z)^2}[\kappa N(x+z)^2 + \tfrac{2\mu|y|^2}{b^4}] \\ &= -\tfrac{1}{F(x,y,z)^2}[(\kappa N - \tfrac{2\mu}{b^2})(x+z)^2 + \tfrac{2\mu z^2}{b^2}], \end{aligned} \tag{4.36}$$

exploting (4.33). On the other hand, again by the use of (4.33),

$$\begin{aligned} \tfrac{\mathrm{tr}(\sigma)\ \mathrm{tr}(n_{K(z)}(\sigma))}{z}[\sigma\cdot n_{K(z)}(\sigma)] &= \tfrac{N(x+z)x}{zF(x,y,z)^2}(x(x+z) + \tfrac{|y|^2}{b^2}) \\ &= \tfrac{N(x+z)x}{zF(x,y,z)^2}(x(x+z) + z^2 - (x+z)^2) \\ &= \tfrac{-N(x+z)x^2}{F(x,y,z)^2}. \end{aligned} \tag{4.37}$$

Recalling (2.33), by the use of (4.36) and (4.37), we have that

$$\Psi(\sigma, z) = -\tfrac{1}{F(x,y,z)^2}[(\kappa N - \tfrac{2\mu}{b^2})(x+z)^2 + \tfrac{2\mu z^2}{b^2} - Nx^2(x+z)] \tag{4.38}$$

for every $(\sigma, z) \in \partial K$, $z > 0$. We put

$$G(x,z) := (\kappa N - \tfrac{2\mu}{b^2})(x+z)^2 + \tfrac{2\mu z^2}{b^2} - Nx^2(x+z) \tag{4.39}$$

and

$$H(\sigma, z) = -\frac{G(x,z)}{F(x,y,z)}. \tag{4.40}$$

Now, let $(\hat{\sigma}(z), z)$ be the heteroclinic trajectory joining the points $(\hat{\sigma}, \hat{z})$ and $(\sigma_\infty, z_\infty)$ whose existence is guaranteed by the previous theorem; we shall denote the spherical and the deviatoric part of $\hat{\sigma}(z)$ by $\hat{x}(z)$ and $\hat{y}(z)$, respectively. Let x_∞ and y_∞ be the spherical and the deviatoric part of σ_∞. Recall that, by (4.24), $\hat{x}(z)$ satisfies

$$\hat{x}'(z) = -\tfrac{\kappa}{\hat{x}(z)}. \tag{4.41}$$

We claim that if (4.34) holds, one has

$$\Psi(\sigma_\infty, z_\infty) < 0. \tag{4.42}$$

Suppose, by contradiction, that $\Psi(\sigma_\infty, z_\infty) = 0$; this means, according to (4.38), that

$$G(x_\infty, z_\infty) = 0. \tag{4.43}$$

Observe now that by (4.32), we have $\frac{d}{dz}\varrho(\hat\sigma(z_\infty),z_\infty)=0$; as $\varrho(\hat\sigma(z),z)$ is strictly positive for $z>z_\infty$ while it is 0 for $z=z_\infty$, we must have

$$\tfrac{d^2}{dz^2}\varrho(\hat\sigma(z_\infty),z_\infty)\geq 0,$$

and by explictly calculating this derivative with the help of (4.24), and recalling (4.11), we find that it must be

$$\nabla\Psi(\sigma_\infty,z_\infty)\cdot\big(\tfrac{-\mathbb{C}\,n_{K(z_\infty)}(\sigma_\infty)}{\operatorname{tr}(\sigma_\infty)\operatorname{tr}(n_{K(z_\infty)}(\sigma_\infty))},1\big)\leq 0. \tag{4.44}$$

As we have already discussed in Remark 3.7, the directional derivative in (4.44) is calculated in a tangential direction with respect to ∂K, whenever we suppose $\Psi(\sigma_\infty,z_\infty)=0$; as Ψ and H coincide on ∂K, we conclude that (4.44) is equivalent to

$$\nabla H(\sigma_\infty,z_\infty)\cdot\big(\tfrac{-\mathbb{C}\,n_{K(z_\infty)}(\sigma_\infty)}{\operatorname{tr}(\sigma_\infty)\operatorname{tr}(n_{K(z_\infty)}(\sigma_\infty))},1\big)\leq 0, \tag{4.45}$$

where the left-hand side is, taking again into account (4.24), nothing more that $\frac{d}{dz}H(\hat\sigma(z),z)$ calculated for $z=z_\infty$. By (4.43) and (4.40), we conclude that we have

$$\tfrac{d}{dz}G(\hat x(z),z)\geq 0 \text{ for } z=z_\infty. \tag{4.46}$$

Now, by (4.41), recalling that $\hat x(z)<0$ for every $z\in[z_\infty,\hat z]$, we have that $\hat x'(z)>0$ for all $z\in[z_\infty,\hat z]$; moreover, by (4.41) it follows that, for every z,

$$\tfrac{d}{dz}\hat x^2(z)=-2\kappa,\qquad \tfrac{d^2}{dz^2}\hat x^2(z)=0,\qquad \hat x''(z)=\tfrac{\kappa}{\hat x^2(z)}\hat x'(z)>0. \tag{4.47}$$

As in our case the constant a defined in (2.14) is equal to 1, (4.23) gives us that $\hat x(z)+z>0$ for every z; as $\hat x''(z)>0$ by (4.47), we easily conclude that if (4.34) holds

$$\tfrac{d^2}{dz^2}[(\kappa N-\tfrac{2\mu}{b^2})(\hat x(z)+z)^2]\geq 0\qquad\text{for every } z\in[z_\infty,\hat z]. \tag{4.48}$$

Therefore, recalling (4.39), by the use of (4.47), we get

$$\begin{aligned}\tfrac{d^2}{dz^2}G(\hat x(z),z) &\geq \tfrac{d^2}{dz^2}[\tfrac{2\mu z^2}{b^2}-N\hat x^2(z)(\hat x(z)+z)]\\ &= \tfrac{4\mu}{b^2}-N\tfrac{d^2}{dz^2}[\hat x^2(z)(\hat x(z)+z)]\\ &= \tfrac{4\mu}{b^2}-N\,[2(\tfrac{d}{dz}\hat x^2(z))(\tfrac{d}{dz}(\hat x(z)+z))+\hat x^2(z)\tfrac{d^2}{dz^2}(\hat x(z)+z)]\\ &= \tfrac{4\mu}{b^2}-N\,[-4\kappa(\hat x'(z)+1)+\kappa\hat x'(z)]\\ &= \tfrac{4\mu}{b^2}+4\kappa N+3\kappa N\hat x'(z)>0;\end{aligned}$$

thus, by (4.43) and (4.46) we have that

$$G(\hat x(z),z)>0\qquad\text{for every } z\in(z_\infty,\hat z];$$

in particular, for $z=\hat z$ we get $G(\hat x,\hat z)>0$, and then, by (4.38), (4.39), and (4.40), we conclude that $\Psi(\hat\sigma,\hat z)<0$, which contradicts both (4.5) and (4.6).

4.2. **Convergence to the fast dynamics.** We want now to investigate how equation (4.1) governs the jump of our viscosity solution when it reaches a point on the yield surface where the elastic-inelastic indicator is strictly positive (which means that we are in the inelastic regime), while the slow-fast indicator satisfies (4.5), or (4.6); we will see how a rescaled version of the solution converges to a heteroclinic solution of the auxiliary system (4.1), whose asymptotic values at $s=\pm\infty$ give the asymptotic values of the viscosity solution before and after the jump time. Both the cases where (4.5) and (4.6) hold will be treated simultaneously; the discussion will closely follow Section 4 and Section 5 of [7].

Throughout this part of the paper, $\hat{t}$ denotes a time such that there exist a left continuous function $t \mapsto (\sigma(t), z(t))$ defined on $[0, \hat{t})$ with values in $\mathbb{M}^{N\times N}_{sym} \times [0, +\infty)$ and an element $(\hat{\sigma}, \hat{z})$ of $\mathbb{M}^{N\times N}_{sym} \times [0, +\infty)$ satisfying the following properties:

$$(\sigma_\varepsilon(t), z_\varepsilon(t)) \to (\sigma(s), z(t)) \quad \text{for a.e. } t \in [0, \hat{t}), \tag{4.49}$$

$$(\sigma(t), z(t)) \to (\hat{\sigma}, \hat{z}) \quad \text{as } t \to \hat{t}^-, \tag{4.50}$$

$$(\hat{\sigma}, \hat{z}) \in \partial K \quad \text{and} \quad \hat{z} > 0, \tag{4.51}$$

$$\Psi(\hat{\sigma}, \hat{z}) \text{ satisfies (4.5) or (4.6)}, \tag{4.52}$$

$$\Phi(\hat{t}, \hat{\sigma}, \hat{z}) > 0. \tag{4.53}$$

For instance, we can take $\hat{t} = t_1$ defined by (2.28), if (3.24) holds and $\Psi(\sigma_1, z_1) > 0$, or $\hat{t} = t_2$ defined by (3.5), provided that (4.6) holds for $(\hat{\sigma}, \hat{z}) = (\sigma_2, z_2)$ defined in Proposition 3.6. In the latter case we have $\Psi(\sigma_2, z_2) = 0$ and in general, by Remark 3.7, we have the weak inequality

$$\nabla\Psi(\sigma_2, z_2) \cdot \big(\tfrac{-\mathbb{C}\, n_{K(z_2)}(\sigma_2)}{\operatorname{tr}(\sigma_2)\operatorname{tr}(n_{K(z_2)}(\sigma_2))}, 1\big) \le 0;$$

thus, assuming (4.6), we are excluding the degenerate case when equality holds.

By (4.49) and (4.50) we also may fix a sequence $\hat{t}_\varepsilon \to \hat{t}$ such that

$$(\sigma_\varepsilon(\hat{t}_\varepsilon), z_\varepsilon(\hat{t}_\varepsilon)) \to (\hat{\sigma}, \hat{z}); \tag{4.54}$$

Indeed, by (4.53), and Lemma 3.11 we can find another sequence, still denoted by $\hat{t}_\varepsilon$, which preserves (4.54), and satisfies in addition, for every $\varepsilon > 0$,

$$\varrho(\sigma_\varepsilon(\hat{t}_\varepsilon), z_\varepsilon(\hat{t}_\varepsilon)) > c\varepsilon, \tag{4.55}$$

where c is a positive constant independent of ε.

We finally recall, as we have already dicussed in Remark 2.6 and in Proposition 3.6, that in the case $\hat{t} = t_2$ the internal variable z is strictly decreasing in a left neighborhood of t_2, thus discontinuities can appear only in the softening regime.

We start by fixing an open neighborhood $U_{\delta_1} := (\hat{t} - \delta_1, \hat{t} + \delta_1) \times B_{\delta_1}(\hat{\sigma}, \hat{z})$ of $(\hat{t}, \hat{\sigma}, \hat{z})$, in a way that (3.26) holds. If (4.5) holds, we may assume for a suitable choice of δ_1 there exists a positive constant γ_1 such that

$$\Psi(\sigma, z) \ge \gamma_1 \qquad \text{for every } (\sigma, z) \in B_{\delta_1}(\hat{\sigma}, \hat{z}); \tag{4.56}$$

if instead (4.6) holds, we may assume that there exists a positive constant γ_4 such that

$$\nabla\Psi(\sigma, z) \cdot \big(\tfrac{-\mathbb{C}\, n_{K(z)}(\pi_{K(z)}(\sigma))}{\operatorname{tr}(\sigma)\operatorname{tr}(n_{K(z)}(\pi_{K(z)}(\sigma)))}, 1\big) \le -\gamma_4 \tag{4.57}$$

for every $(\sigma, z) \in B_{\delta_1}(\hat{\sigma}, \hat{z}) \setminus \operatorname{int} K$.

We now define the exit time from $B_{\delta_1}(\hat{\sigma}, \hat{z})$

$$b^1_\varepsilon := \inf\{t \in (\hat{t}_\varepsilon, \hat{t}_\varepsilon + \delta_1) : \ (\sigma_\varepsilon(t), z_\varepsilon(t)) \in \partial B_{\delta_1}(\hat{\sigma}, \hat{z})\}; \tag{4.58}$$

by the previous assumptions for small ε we will trivially have $\hat{t}_\varepsilon < b^1_\varepsilon$. We then fix a positive decreasing sequence $\delta_k \searrow 0^+$, starting from δ_1, and consequently we define, for every $k \in \mathbb{N}$,

$$b^k_\varepsilon := \sup\{t \in (\hat{t}_\varepsilon, b^1_\varepsilon) : \ (\sigma_\varepsilon(t), z_\varepsilon(t)) \in \partial B_{\delta_k}(\hat{\sigma}, \hat{z})\}. \tag{4.59}$$

Next lemma, which will be crucial in the remainder of the section, shows that the exit times b^k_ε tend to $\hat{t}$ when ε goes to 0 and that the difference $b^1_\varepsilon - b^k_\varepsilon$ is of order ε for fixed k.

Lemma 4.5. *Assume* (2.1)-(2.5)*, and* (2.24)*; let* Φ*,* Ψ *as in* (2.32)*, and* (2.33)*, respectively. Let* $\hat{t} > 0$ *satisfying* (4.49)-(4.53)*. Let* b^1_ε *be given by* (4.58) *and* b^k_ε *be given for every* $k \in \mathbb{N}$*,* $k > 1$ *by (4.59). Then, for every* $k \in \mathbb{N}$*:*

a) $b^k_\varepsilon \to \hat{t}$ *as* $\varepsilon \to 0^+$*;*

b) $\sup_{\varepsilon>0} \frac{b^1_\varepsilon - b^k_\varepsilon}{\varepsilon} \leq c_k < +\infty$,

where c_k *is a constant depending on* k*. Moreover, for every* $k \in \mathbb{N}$*, there exists a constant* m_k *such that*

$$\varrho(\sigma_\varepsilon(b^k_\varepsilon), z_\varepsilon(b^k_\varepsilon)) > m_k. \tag{4.60}$$

Proof. We limit ourselves to giving a brief outline of the proof, as the argument is similar to that of [7, Lemmas 4.3 and 5.1]. Concerning part a) of the statement, it clearly suffices to show this is true for b^1_ε. As $\hat{t}_\varepsilon \to \hat{t}$ this will be proved once we get:

$$\limsup_{\varepsilon\to 0^+} (b^1_\varepsilon - \hat{t}_\varepsilon) = 0. \tag{4.61}$$

By Lemma 3.11 we have that $\varrho_\varepsilon(t) > 0$ for every $t \in (\hat{t}_\varepsilon, b^1_\varepsilon)$, hence (2.31) holds.

Now, assume that (4.5) holds, which implies on his turn (4.56). With this condition, with the help of (3.26) and (2.31), we get that $\dot{\varrho}_\varepsilon(t) \geq \gamma_1 \frac{1}{\varepsilon} \varrho_\varepsilon(t)$; dividing by $\varrho_\varepsilon(t)$, we get

$$\frac{\dot{\varrho}_\varepsilon(t)}{\varrho_\varepsilon(t)} \geq \frac{\gamma_1}{\varepsilon} \qquad \text{for every } t \in (\hat{t}_\varepsilon, b^1_\varepsilon), \tag{4.62}$$

which is the analogue of [7, formula (4.17)]; now the proof of (4.61), of part b) of the statement and of (4.60) can be easily achieved by simply adapting the arguments of [7, Lemma 4.3].

Assume instead (4.6), which implies (4.57). We have already observed that this implies $\mathrm{tr}(\hat{\sigma}) < 0$; by (2.34) this means that

$$\mathrm{tr}(\hat{\sigma}) < -\tfrac{\min\{\kappa, 2\mu\}}{M_K \sqrt{N}},$$

where M_K is as in (2.3) and $\kappa, 2\mu$ as in (2.24). Provided we have chosen δ_1 suitably small, we may clearly assume that

$$\mathrm{tr}(\sigma_\varepsilon(t)) < -\tfrac{\min\{\kappa, 2\mu\}}{2M_K \sqrt{N}} \qquad \text{for every } t \in (\hat{t}_\varepsilon, b^1_\varepsilon);$$

analogously, as (4.6) implies $\mathrm{tr}(n_{K(z)}(\hat{\sigma})) > 0$ we may assume

$$\mathrm{tr}(n_{K(z_\varepsilon(t))}(\pi_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t))) > 0 \qquad \text{for every } t \in (\hat{t}_\varepsilon, b^1_\varepsilon).$$

By these facts and (2.29) we then easily get the existence of a positive constant C such that

$$\dot{z}_\varepsilon(t) \leq -C \frac{\varrho_\varepsilon(t)}{\varepsilon} \qquad \text{for every } t \in (\hat{t}_\varepsilon, b^1_\varepsilon). \tag{4.63}$$

In particular, for fixed $\varepsilon > 0$, the function $\dot{z}_\varepsilon(t)$ never vanishes in the prescribed interval. We also immediately get, as $z_\varepsilon(t) < \hat{z} + \delta_1$ for every $t \in (\hat{t}_\varepsilon, b^1_\varepsilon)$ that there exists a positive constant $\tilde{R}$ independent of ε such that:

$$\int_{\hat{t}_\varepsilon}^{b^1_\varepsilon} \frac{\varrho_\varepsilon(t)}{\varepsilon}\, dt \leq \tilde{R},$$

as in [7, formula (5.10)].

Differentiating the function Ψ along the trajectories, we get

$$\begin{aligned}\frac{d}{dt}\Psi(\sigma_\varepsilon(t),z_\varepsilon(t)) &= \nabla\Psi(\sigma_\varepsilon(t),z_\varepsilon(t))\cdot(\dot\sigma_\varepsilon(t),\dot z_\varepsilon(t))\\ &= \nabla\Psi(\sigma_\varepsilon(t),z_\varepsilon(t))\cdot(\mathbb{C}\dot\xi(t),0)\\ &\quad + \dot z_\varepsilon(t)\nabla\Psi(\sigma_\varepsilon(t),z_\varepsilon(t))\cdot(-\tfrac{\mathbb{C}(\dot\xi(t)-\dot\sigma_\varepsilon(t))}{\dot z_\varepsilon(t)},1)\\ &= \nabla\Psi(\sigma_\varepsilon(t),z_\varepsilon(t))\cdot(\mathbb{C}\dot\xi(t),0)\\ &\quad +\dot z_\varepsilon(t)\nabla\Psi(\sigma_\varepsilon(t),z_\varepsilon(t))\cdot(\tfrac{-\mathbb{C}\, n_{K(z_\varepsilon(t))}(\pi_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t)))}{\mathrm{tr}(\sigma_\varepsilon(t))\mathrm{tr}(n_{K(z_\varepsilon(t))}(\pi_{K(z_\varepsilon(t))}(\sigma_\varepsilon(t)))},1);\end{aligned}$$

this equality, together with (4.63) and (4.57), implies that there exist two positive constants L and R such that

$$\frac{d}{dt}\Psi(\sigma_\varepsilon(t),z_\varepsilon(t))\geq R\frac{\varrho_\varepsilon(t)}{\varepsilon}-L|\mathbb{C}||\dot\xi(t)| \qquad \text{for every } t\in(\hat t_\varepsilon,b^1_\varepsilon), \tag{4.64}$$

as in [7, formula (5.11)]. Now the proof of (4.61), of part b) of the statement and of (4.60) can be achieved by repeating the arguments of [7, Lemma 5.1]. □

We are now ready to prove the main result of this section.

Theorem 4.6. *Assume* (2.24), *and* (2.14)-(2.17)*; let* Φ, Ψ *as in* (2.32), *and* (2.33), *respectively. Let* $\hat t>0$, $(\hat\sigma,\hat z)\in\partial K$, *such that* (4.5) *or* (4.6) *hold. Assume that* $\Phi(\hat t,\hat\sigma,\hat z)>0$, *and let* $\delta_1>0$ *as in* (3.26) *and let* b^1_ε *be given by* (4.58). *For every* $s\in\mathbb{R}$, *let* $(\sigma^1_\varepsilon(s),z^1_\varepsilon(s)):=(\sigma_\varepsilon(b^1_\varepsilon+\varepsilon s),z_\varepsilon(b^1_\varepsilon+\varepsilon s))$. *Then* $(\sigma^1_\varepsilon(s),z^1_\varepsilon(s))$ *converges uniformly on compact subsets of* $\mathbb{R}$ *to a solution of the problem:*

$$\begin{cases}\dot\sigma_f(s)=\mathbb{C}(\pi_{K(z_f(s))}(\sigma_f(s))-\sigma_f(s))\\ \dot z_f(s)=\mathrm{tr}(\sigma_f(s))\,\mathrm{tr}(\sigma_f(s)-\pi_{K(z_f(s))}(\sigma_f(s)))\\ \lim\limits_{s\to-\infty}(\sigma_f(s),z_f(s))=(\hat\sigma,\hat z)\end{cases} \tag{4.65}$$

whose existence and uniqueness up to time translations is guaranteed by Theorem 4.2.

Proof. This proof is reminiscent of [8, Lemma 4.3]. First of all, we claim that it suffices to prove the statement along a subsequence ε_k tending to 0. Indeed, the only difficulty is that the solutions of (4.65) may differ by a time translation, thus the limit could depend on the chosen subsequence. We are able to exclude this fact applying [8, Lemma 4.4], with the same arguments as in the proof of Theorem 3.5 of the same paper. In view of that, we shall extract from now on subsequences without relabelling. We also define $\chi_\varepsilon(s):=\dot\xi(a^1_\varepsilon+\varepsilon s)$.

We start by observing that the function $(\sigma^1_\varepsilon(s),z^1_\varepsilon(s))$ solves the problem

$$\begin{cases}\dot\sigma^1_\varepsilon(s)=\mathbb{C}(\pi_{K(z^1_\varepsilon(s))}(\sigma^1_\varepsilon(s))-\sigma^1_\varepsilon(s))+\varepsilon\mathbb{C}\,\chi_\varepsilon(s),\\ \dot z^1_\varepsilon(s)=\mathrm{tr}(\sigma^1_\varepsilon(s))\,\mathrm{tr}(\sigma^1_\varepsilon(s)-\pi_{K(z^1_\varepsilon(s))}(\sigma^1_\varepsilon(s))),\\ (\sigma^1_\varepsilon(0),z^1_\varepsilon(0))=(\sigma_\varepsilon(b^1_\varepsilon),z_\varepsilon(b^1_\varepsilon)),\end{cases} \tag{4.66}$$

in the interval $[-\frac{b^1_\varepsilon}{\varepsilon},\frac{\hat t+\delta_1-b^1_\varepsilon}{\varepsilon}]$. As $(\sigma_\varepsilon(b^1_\varepsilon),z_\varepsilon(b^1_\varepsilon))$ belongs to the compact set $\partial B_{\delta_1}(\hat\sigma,\hat z)$ we may assume, possibly passing to a subsequence that $(\sigma_\varepsilon(a^1_\varepsilon),z_\varepsilon(a^1_\varepsilon))$ converges to $(\hat\sigma_1,\hat z_1)\in\partial B_{\delta_1}(\hat\sigma,\hat z)$ as $\varepsilon\to0$. Notice that $(\hat\sigma_1,\hat z_1)$ has a strictly positive distance from K as a consequence of (4.60). Therefore, Lemma 4.5 and the Continuous Dependence Theorem imply that $(\sigma^1_\varepsilon(s), z^1_\varepsilon(s))$ converges uniformly on compact subsets of $\mathbb{R}$, as $\varepsilon \to 0$, to the solution $(\sigma^1(s), z^1(s))$ of the problem

$$\begin{cases} \dot\sigma^1(s) = \mathbb{C}(\pi_{K(z^1(s))}(\sigma^1(s)) - \sigma^1(s)), \\ \dot z^1(s) = \operatorname{tr}(\sigma^1(s)) \operatorname{tr}(\sigma^1(s) - \pi_{K(z^1(s))}(\sigma^1(s))), \\ (\sigma^1(0), z^1(0)) = (\hat\sigma_1, \hat z_1). \end{cases} \tag{4.67}$$

To conclude the proof we have to show that

$$\lim_{s\to-\infty} (\sigma^1(s), z^1(s)) = (\hat\sigma, \hat z). \tag{4.68}$$

Actually, recalling Remark 4.3, it suffices to show that there exist $s_k \to +\infty$ such that

$$\lim_{k\to+\infty} (\sigma^1(-s_k), z^1(-s_k)) = (\hat\sigma, \hat z). \tag{4.69}$$

To do that, we take δ_k and b^k_ε as in Lemma 4.5, and we define $S^{1,k}_\varepsilon := \frac{b^1_\varepsilon - b^k_\varepsilon}{\varepsilon}$; by Lemma 4.5 and a diagunal argument, we may suppose, passing to a subsequence, that for every $k \in \mathbb{N}$ there exists

$$s_k := \lim_{\varepsilon\to 0} S^{1,k}_\varepsilon \in \mathbb{R}_+.$$

We define $(\sigma^k_\varepsilon(s), z^k_\varepsilon(s)) := (\sigma_\varepsilon(b^k_\varepsilon + \varepsilon s), z_\varepsilon(b^k_\varepsilon + \varepsilon s))$; by repeating the above arguments we may suppose that for every $k \in \mathbb{N}$ there exists $(\hat\sigma_k, \hat z_k) \in \partial B_{\delta_k}(\hat\sigma, \hat z) \setminus K$ such that $(\sigma^k_\varepsilon(s), z^k_\varepsilon(s))$ converges, as $\varepsilon \to 0$, uniformly on compact subsets of $\mathbb{R}$, to the solution $(\sigma^k(s), z^k(s))$ of the problem

$$\begin{cases} \dot\sigma^k(s) = \mathbb{C}(\pi_{K(z^k(s))}(\sigma^k(s)) - \sigma^k(s)), \\ \dot z^k(s) = \operatorname{tr}(\sigma^k(s)) \operatorname{tr}(\sigma^k(s) - \pi_{K(z^k(s))}(\sigma^k(s))), \\ (\sigma^k(0), z^k(0)) = (\hat\sigma_k, \hat z_k). \end{cases} \tag{4.70}$$

Moreover, equality $(\sigma^k_\varepsilon(S^{1,k}_\varepsilon), z^k_\varepsilon(S^{1,k}_\varepsilon)) = (\sigma_\varepsilon(b^1_\varepsilon), z_\varepsilon(b^1_\varepsilon))$ implies that

$$(\sigma^k(s_k), z^k(s_k)) = (\hat\sigma_1, \hat z_1)$$

for every k, hence by the uniqueness of solutions for Cauchy problems we get

$$(\sigma^k(s), z^k(s)) = (\sigma^1(s - s_k), z^1(s - s_k)). \tag{4.71}$$

It follows that

$$(\sigma^1(-s_k), z^1(-s_k)) = (\hat\sigma_k, \hat z_k). \tag{4.72}$$

As $\delta_k \to 0$, we have that $(\hat\sigma_k, \hat z_k) \to (\hat\sigma, \hat z)$ as k goes to $+\infty$, hence

$$\lim_{k\to+\infty} (\sigma^1(-s_k), z^1(-s_k)) = (\hat\sigma, \hat z); \tag{4.73}$$

since $(\hat\sigma, \hat z)$ is an equilibrium point for equation (4.1), necessarily $s_k \to +\infty$ as $k \to +\infty$; so, (4.69) is proven and conclusion follows. ☐

Remark 4.7. (**Return to the continuous evolution).**
Let $(\sigma_\infty, z_\infty)$ the unique ω-limit point of the solution of (4.65); by Theorem 4.2, we have that $\Psi(\sigma_\infty, z_\infty) \le 0$; assume now that strict inequality holds (this is certainly true, for instance, if we are in the situation described by Example 4.4). By the previous theorem we may fix a sequence $\tilde t_\varepsilon$ converging to $\hat t$ as $\varepsilon \to 0^+$ such that $(\sigma_\varepsilon(\tilde t_\varepsilon), z_\varepsilon(\tilde t_\varepsilon)) \to (\sigma_\infty, z_\infty)$.

Now, we have three possibilities:

a) **Return to the continuous evolution in the softening regime**. This situation occurs if $\Phi(\hat{t}, \sigma_\infty, z_\infty) > 0$; by Theorem 3.16, in a right neighborhood of $\hat{t}$ the solutions of (2.26) uniformly converge, on compact subintervals, to the solution of the slow dynamics equation given by (3.1) with Cauchy datum $(\sigma_\infty, z_\infty)$ at time $\hat{t}$; notice that (4.11) implies that, when the continuous evolution restarts, we are still in the softening regime, thus no istantaneous transition between the softening and the hardening regime occurs during the jump.

b) **Return to the elastic regime** . This situation occurs if $\Phi(\hat{t}, \sigma_\infty, z_\infty) < 0$. To prove that, take $\eta > 0$, $\gamma > 0$ such that, for every $t \in [\hat{t}, \hat{t} + \eta]$ and every (σ, z) satisfying $|(\sigma, z) - (\sigma_\infty, z_\infty)| < \eta$, one has

$$\Psi(\sigma, z) < -\gamma \quad \text{and} \quad \Phi(t, \sigma, z) < -\gamma \tag{4.74}$$

We observe that (4.74) obviously implies both (3.46) and (3.47), hence repeating the arguments of [7, Theorem 3.11], we get that $(\sigma_\varepsilon(t), z_\varepsilon(t))$ uniformly converges to the solution of the equation of linearized elasticity

$$(\sigma_{el}(t), z_{el}(t)) := (\sigma_\infty + \mathbb{C}(\xi(t) - \xi(\hat{t})), z_\infty)$$

on compact subintervals of $(\hat{t}, \tau)$, where

$$\tau := \sup\{t > \hat{t}\,|(\sigma_{el}(s), z_{el}(s)) \in \operatorname{int} K \text{ for every } s \in (\bar{t}, t)\}.$$

c) If $\Phi(\hat{t}, \sigma_\infty, z_\infty) = 0$, we need some higher order conditions on the indicator Φ to establish whether the system will follow the first or the second alternative; however, by the negativeness of the indicator Ψ, applying Lemma 3.13, and Corollary 3.15, we are able to conlude that the evolution must be continuous in a right open neighborhood of $\hat{t}$.

5. **Statement of the main result.** We collect the results of the previous sections in the next theorem, which gives a procedure to construct a viscosity solution to our evolution problem under quite general assumptions; in fact, if these assumptions are satisfied at every step of the construction, the viscosity solution is also unique. The theorem will determine a possibly infinite sequence of times $t_0 < t_1 < \cdots < t_i < \dots$ such that in each interval $(t_{i-1}, t_i]$ the solution, denoted here by (σ_{i-1}, z_{i-1}) is continuous and satisfies either the slow dynamics, or the elastic regime, or a combination of the two. A jump may occur at time t_i if the value $(\sigma_{i-1}(t_i), z_{i-1}(t_i))$ satisfies (4.5) or (4.6). In this case the new starting point (σ_i^+, z_i^+) for the solution in the interval $(t_i, t_{i+1}]$ is determined by taking the limit as $s \to +\infty$ of the solution of the fast dynamics originating from $(\sigma_{i-1}(t_i), z_{i-1}(t_i))$ at $s = -\infty$. To prepare the technical statement of the theorem it is convenient to introduce some notation.

Definition 5.1. For every $(\hat{\sigma}, \hat{z}) \in \partial K$ satisfying $\Psi(\hat{\sigma}, \hat{z}) \neq 0$, and every $T > 0$ we define $(\sigma_{sl}, z_{sl})(t; \hat{\sigma}, \hat{z}, T)$ as the unique solution to (3.1) starting from the point $(\hat{\sigma}, \hat{z})$ at time T. For every $(\hat{\sigma}, \hat{z}) \in \partial K$ we define $(\sigma_{el}, z_{el})(t; \hat{\sigma}, \hat{z}, T) = (\hat{\sigma} + \mathbb{C}(\xi(t) - \xi(T)), \hat{z})$. For every $(\hat{\sigma}, \hat{z}) \in \partial K$ satisfying (4.5) or (4.6) we define $(\sigma_f, z_f)(s; \hat{\sigma}, \hat{z})$ as the unique solution to (4.1) having $(\hat{\sigma}, \hat{z})$ as an α-limit point.

To simplify our notation, in the statement of the theorem we also put

$$\partial K_f := \{(\sigma, z) \in \partial K : (\sigma, z) \text{ satisfy } (4.5) \text{ or } (4.6)\ \}.$$

Theorem 5.2. *Let* $(\sigma_0, z_0) \in \operatorname{int} K$, *let* $t_0 = 0$, t_1 *as in* (2.28), *and* $(\sigma_0(t), z_0(t)) = (\sigma_0 + \mathbb{C}(\xi(t) - \xi(0)), z_0)$. *For every* $i \geq 1$ *with* $t_i < +\infty$ *define*

$$(\sigma_i^+, z_i^+) = \begin{cases} (\sigma_{i-1}, z_{i-1})(t_i) & \text{if } \Psi(\sigma_{i-1}(t_i), z_{i-1}(t_i)) < 0, \\ \lim\limits_{s \to +\infty} (\sigma_f, z_f)(s; \sigma_{i-1}(t_i), z_{i-1}(t_i)) & \text{if } (\sigma_{i-1}(t_i), z_{i-1}(t_i)) \in \partial K_f. \end{cases}$$

If $\Psi(\sigma_i^+, z_i^+) < 0$, *let* $\hat{t}_i$ *be the maximal time of existence for* $(\sigma_{sl}, z_{sl})(t; \sigma_i^+, z_i^+, t_i)$, *and*

$$\bar{t}_i := \inf\{t \geq t_i : \Phi(t, (\sigma_{sl}, z_{sl})(t; \sigma_i^+, z_i^+, t_i)) \leq 0\}.$$

If $\hat{t}_i = \bar{t}_i$, *put* $t_{i+1} := \hat{t}_i$, *and*

$$(\sigma_i(t), z_i(t)) = (\sigma_{sl}, z_{sl})(t; \sigma_i^+, z_i^+, t_i)$$

for every $t_i \leq t \leq t_{i+1}$; *if instead* $\hat{t}_i > \bar{t}_i$, *put* $(\bar{\sigma}_i, \bar{z}_i) := (\sigma_{sl}, z_{sl})(\bar{t}_i, \sigma_i^+, z_i^+, t_i)$,

$$t_{i+1} := \inf\{t > \bar{t}_i \,|\, (\sigma_{el}, z_{el})(t; \bar{\sigma}_i, \bar{z}_i, \bar{t}_i) \in \operatorname{int} K \text{ for every } s \in (\bar{t}_i, t)\},$$

and

$$(\sigma_i(t), z_i(t)) = \begin{cases} (\sigma_{sl}, z_{sl})(t; \sigma_i^+, z_i^+, t_i) & \text{for } t_i < t \leq \bar{t}_i, \\ (\sigma_{el}, z_{el})(t; \bar{\sigma}_i, \bar{z}_i, \bar{t}_i) & \text{for } \bar{t}_i \leq t \leq t_{i+1}. \end{cases}$$

Define $(\sigma(t), z(t)) := \sum_{i \geq 1} 1_{(t_{i-1}, t_i]}(\sigma_{i-1}(t), z_{i-1}(t))$. *Assume that*

$$\Phi(\sigma(t_i), z(t_i)) > 0 \quad \text{for every } i \geq 1, \tag{5.1}$$

$$\text{(3.46) and (3.47) hold} \quad \text{for every } i \text{ with } \bar{t}_i < \hat{t}_i \tag{5.2}$$

$$\liminf_{t \to t_{i+1}^-} \Phi(t, \sigma(t), z(t)) > 0 \quad \text{for every } i \text{ with } t_{i+1} = \hat{t}_i < +\infty. \tag{5.3}$$

Define $e(t)$ *and* $p(t)$ *through the consitutive relations in* (1.1), *and put* $T := \sup_i t_i$. *Then* $(e(t), p(t), \sigma(t), z(t))$ *is the unique viscosity solution of* (1.1) *in* $[0, T)$.

Proof. The result follows from Theorem 3.16, Theorem 3.18, Theorem 4.6, and Remark 4.7. ☐

Remark 5.3. Notice that assumption (5.3) ensures that whenever $t_{i+1} = \hat{t}_i < +\infty$ we can extend by continuity (σ_{sl}, z_{sl}) in t_{i+1} thanks to Proposition 3.6, hence at every step $(\sigma(t_i), z(t_i))$ is well-defined. Concerning the other assumptions in the theorem, observe that by construction and Theorem 4.2, we always have at least the weak inequality $\Psi(\sigma_i^+, z_i^+) \leq 0$; by construction we also have $\Phi(\sigma(t_i), z(t_i)) \geq 0$ for every i. Similarly, the weak inequality in (5.3) is always true whenever $t_{i+1} = \hat{t}_i$. Thus our construction works at least for the nondegenerate cases where equality is excluded while a higher-order analysis is needed in the remaining situations to get insight of the limit behavior of the viscous approximations.

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