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FLOWS IN POROUS MEDIA WITH EROSION OF THE SOLID MATRIX

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Abstract. We consider the flow of an incompressible Newtonian fluid through an idealized porous medium consisting of an array of identical solid symmetric lamellae, whose profile varies in space and time due to a stress induced erosion process. The focus is on the influence of mass exchange between solid and fluid on the macroscopic flow. By means of the upscaling procedure illustrated in [6] we derive the governing system of equations for the macroscopic flow, encompassing various physical situations. We show that Darcy's law no longer applies in the classical sense. The corresponding mathematical problem turns out to be surprisingly complicated. Existence and uniqueness are proved. Numerical simulations are presented.

1. **Introduction.** Flows in porous media with mass exchange between the fluid and the solid matrix is a typical subject in the area of mixture theory (see, for instance, [13], [3] and [10]) and it has been investigated in a number of different contexts such as biofilm growth [2], manufacturing of composites materials [8], soil internal erosion [1], [7], [11], [12], [14]. A natural question which arises is whether Darcy's law is still applicable in the presence of mass exchange and in what form. This issue has been addressed in [9] by selecting a sort of symmetry principle to be satisfied by the momentum exchange associated to the mass transfer. The conclusion of [9] was that Darcy's law could be adapted by incorporating in it a slight correction accounting for the mass exchange rate. More recently the problem of incompressible saturated flows through porous media with mass exchange has been treated rather extensively in [6], both in the framework of mixture theory, and by means of an upscaling technique applied to some idealized geometry. In the latter paper several different situations have been considered, on the basis of two main elements:

(*i*). The density difference between the mass exchanging components.
(*ii*). The relative size of the time scales of the mass exchange process and of the macroscopically observable convective flow.

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The specific microscopic geometry considered in [6] consists of parallel identical symmetric lamellae confining a two-dimensional microscopic flow. Denoting by x^* the longitudinal coordinate[1] (along the one-dimensional macroscopic flow), and by y^* the transversal coordinate, if $y^* = 0$ is the middle plane of a lamella, the width of the pore space at a point x^* at time t^* was taken to be the interval $-s^* < y^* < s^*$, with $s^* = s^*(x^*, t^*)$ unknown and initially prescribed as a function $s_o^*(x^*)$. The function s^* could not vanish (medium clogging) and had to be less than some length h^* ($s^* = h^*$ corresponding to the complete disappearance of the solid matrix).

In [6] several examples have been illustrated. One of them was erosion, i.e. stress induced ablation of solid particles by the fluid, making the function s^* increase in time. In particular, in [6] a model for a stress induced erosion has been proposed, assuming that the erosion rate is proportional to the fluid stress at the pores wall. Actually, a threshold (related to the solid particles "bond energy") has been introduced, so that erosion occurs only when the fluid stress overcomes such a threshold.

In literature, a law similar to the one we have proposed can be found in [1], but also different options are available. For instance in [12] the erosion rate is proportional to the power dissipated by the flow (we will return to this question in the conclusion). The original aspect of our approach with respect to the quoted literature is that, instead of adopting directly a macroscopic description, we make use of an upscaling procedure, starting from the microscopic level and deriving the macroscopic laws through homogenization.

The most interesting case for the erosion processes, is when the erosion time scale is comparable to the time scale of the convective flow. In such a case it was shown that the problem is in general considerably complicated from the mathematical point of view (with the exception in which the initial profile of the interface is flat and there is no stress threshold for solid removal). Since it is amazing how such a conceptually simple problem has a by no means trivial mathematical description, we believe it is worthwhile to study it in some detail. This is the aim of the present paper. In practice we are dealing with a non–local free boundary problem of a very peculiar type.

After the presentation of the model, we will go through the analysis of four cases, of increasing difficulty culminating with the general problem. In each case we will proceed to a reformulation of the governing equation by means of suitable transformations. Then we will prove that the problem is well posed and we will present some numerical simulation.

2. **Microscopic modelling and upscaling.** The aim of this section is to model the dynamics a porous system at the microscopic level (i.e. the pore scale). Such a procedure is rather involved in a general 3D geometry. Therefore, with the aim of capturing some essential information, we consider the following assumptions:

A1. The space between two lamellae is saturated by a Newtonian incompressible fluid, whose viscosity and density are μ^* and $\rho^{f\,*}$, respectively. In particular, μ^* and $\rho^{f\,*}$ remain constant during the erosion process.

A2. The solid is rigid and its density is $\rho^{s\,*}$. In general $\rho^{s\,*} \neq \rho^{f\,*}$, even though we shall consider also the case $\rho^{s\,*} = \rho^{f\,*}$. A fundamental parameter in the theory we are going to develop is

$$\gamma = \frac{\rho^{s\,*}}{\rho^{f\,*}}. \tag{1}$$

[1] Throughout this paper the superscript "$*$" means that the quantity has physical dimension.

A3. The solid exchanges mass with the fluid, i.e. part of the solid mass is converted into fluid by an erosion process that takes place at the interface between the phases. Therefore the channel width (which may be nonuniform along the channel) varies in time. When $\gamma \neq 1$, our model is confined to the case in which the material extracted from the solid takes the physical properties of the liquid (as it happens in phase change processes) so that $\rho^{f\,*}$ does not change.

A4. Temperature θ^* is considered to be uniform and constant in time. In other words, mass exchange involves no latent heat and we neglect the temperature changes due to dissipation mechanisms accompanying the flow.

A5. The Helmoltz free energy[2] is uniform and constant in each constituent and is set equal to zero in the liquid. We refer to this kind of materials as inert materials.

We remark that assumption A3 is a strong limitation for our model. Indeed, when $\gamma \neq 1$ the fluid should be treated as a mixture whose components are the liquid and the dispersed solid particles. In particular, the mixture density is no longer constant, but varies according to the volume fraction occupied by the solid particles. This case will be treated in a forthcoming paper.

2.1. **Definitions and scaling.** As mentioned in the introduction, $x^* \in (0, L^*)$ is the coordinate parallel to the lamellae axis, and we denote by y^* the coordinate orthogonal to x^*. The distance between the median plane of two adjacent lamellae is $2h^*$ (see Fig.1).

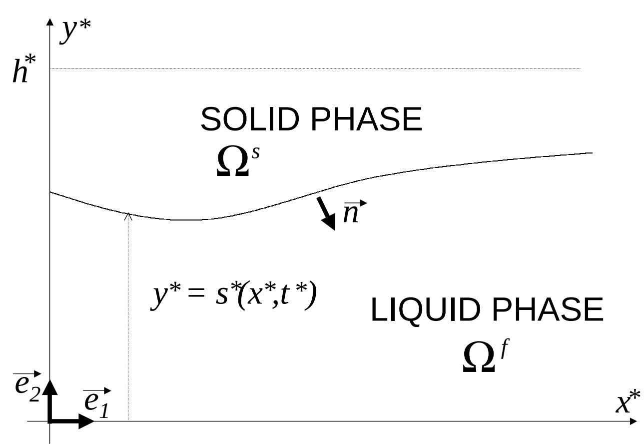

FIGURE 1. A schematic representation of the system at the pores scale.

Hereafter we give the main dependent variables considered in the model:

- $\vec{v}^{\,f\,*} = v_1^{f\,*}(x^*, y^*, t^*)\,\vec{e}_1 + v_2^{f\,*}(x^*, y^*, t^*)\,\vec{e}_2$, fluid velocity.
- $\vec{v}^{\,s\,*}$ solid velocity. Since the solid is rigid, its velocity vanishes in the chosen reference frame.
- $p^* = p^*(x^*, y^*, t^*)$, liquid pressure.

[2] We shall define the Helmoltz free energy in section 2.4.

- $s^* = s^*(x^*, t^*) \geq 0$, channel width. In particular, during erosion $\dfrac{\partial s^*}{\partial t^*} > 0$, i.e. the channel width grows.

In each channel we identify the following regions (see again Fig. 1):

- $\Omega^f = \{0 < y^* < s^*(x^*, t^*)\}$, saturated channel, i.e. liquid phase.
- $\Omega^s = \{s^*(x^*, t^*) < y^* < h^*\}$, solid phase.
- $\Omega = \Omega^f \cup \Omega^s$.

We denote by Σ the evolving interface separating the regions and by $\vec{n}$ (see Fig. 1) the normal to Σ, pointing toward the liquid. Thus Σ is described by the equation

$$\mathcal{S}(x^*, y^*, t^*) = 0, \quad \text{with} \quad \mathcal{S}(x^*, y^*, t^*) = s^*(x^*, t^*) - y^*,$$

and is a free boundary, since its evolution is not a priori known. In particular, Σ may be a discontinuity surface. Hence we denote by

$$[\![(\cdot)]\!] = \lim_{\substack{y^* \to s^* \\ y^* \in \Omega^f}} (\cdot) - \lim_{\substack{y^* \to s^* \\ y^* \in \Omega^s}} (\cdot) \,, \tag{2}$$

the jump of the quantity $(\cdot)$ across Σ.

Next, we introduce:

- $\phi^s = \dfrac{h^* - s^*}{h^*}$, volume fraction occupied by the solid matrix.
- $\phi^f = \dfrac{s^*}{h^*}$, volume fraction occupied by the liquid phase, and we recall the saturation condition
$$\phi^f + \phi^s = 1.$$

We use a double scaling for the spatial variables x^* and y^*

$$x = \frac{x^*}{L^*}, \quad y = \frac{y^*}{h^*} = \frac{y^*}{\varepsilon L^*}. \tag{3}$$

with L^* and h^* macroscopic and microscopic length scales, respectively, and

$$\varepsilon = \frac{h^*}{L^*}. \tag{4}$$

Key assumption of this theory is

$$\varepsilon \ll 1. \tag{5}$$

We thus have

$$\frac{\partial (\cdot)}{\partial x^*} = \frac{1}{L^*} \frac{\partial (\cdot)}{\partial x}, \quad \frac{\partial (\cdot)}{\partial y^*} = \frac{1}{L^* \varepsilon} \frac{\partial (\cdot)}{\partial y}. \tag{6}$$

We also introduce

$$s = \frac{s^*}{h^*}, \quad \Rightarrow \quad s^* = L^* \varepsilon \, s. \tag{7}$$

Of course, $s = \phi^f$.

Concerning the time variable t^*, we introduce t_v^*, the characteristic convective time scale

$$t_v^* = \frac{L^*}{v_c^*},$$

with v_c^* characteristic macroscopic fluid velocity, so that

$$t = \frac{t^*}{t_v^*},$$

is the dimensionless time.

The normal to the "surface" Σ pointing toward the fluid (see Fig. 1) is

$$\vec{n} = \frac{1}{\sqrt{1+\left(\frac{\partial s^*}{\partial x^*}\right)^2}}\left[\frac{\partial s^*}{\partial x^*}\vec{e}_1 - \vec{e}_2\right],$$

which, after the scaling (3) and (7), becomes

$$\vec{n} = \frac{1}{\sqrt{1+\varepsilon^2\left(\frac{\partial s}{\partial x}\right)^2}}\left[\varepsilon\left(\frac{\partial s}{\partial x}\right)\vec{e}_1 - \vec{e}_2\right]. \tag{8}$$

Similarly, we denote by $\vec{t}$ the tangential vector to Σ

$$\vec{t} = \frac{1}{\sqrt{1+\varepsilon^2\left(\frac{\partial s}{\partial x}\right)^2}}\left[\vec{e}_1 + \varepsilon\left(\frac{\partial s}{\partial x}\right)\vec{e}_2\right]. \tag{9}$$

Next, we denote by $\vec{u}^*$ the interface velocity. Referring to [4], Chapter 8, the component of $\vec{u}^*$ normal to Σ, namely $u^* = \vec{u}^* \cdot \vec{n}$, has the expression

$$u^* = -\frac{1}{\sqrt{1+\varepsilon^2\left(\frac{\partial s}{\partial x}\right)^2}}\left(\frac{h^*}{t_v^*}\right)\frac{\partial s}{\partial t}. \tag{10}$$

So, referring again to Fig. 1, we easily realize that when the channel is expanding (i.e. positive $\frac{\partial s^*}{\partial t^*}$) u^* is negative.

We rescale the velocity components in the following way

$$\begin{aligned} v_1^f &= \left(\frac{L^*}{t_v^*}\right)^{-1} v_1^{f\,*} = \frac{v_1^{f\,*}}{v_c^*}, \\ v_2^f &= \left(\frac{h^*}{t_v^*}\right)^{-1} v_2^{f\,*} = \frac{1}{\varepsilon}\left(\frac{L^*}{t_v^*}\right)^{-1} v_2^{f\,*} = \frac{v_2^{f\,*}}{\varepsilon\, v_c^*}, \end{aligned}$$

so that

$$\vec{v}^{\,f} = \frac{\vec{v}^{\,f\,*}}{v_c^*} = v_1^f \vec{e}_1 + \varepsilon\, v_2^f \vec{e}_2\,. \tag{11}$$

Next, we introduce u_Σ^* as the interface characteristic velocity[3]. Hence, the time scale characterizing the free boundary dynamics is

$$t_\Sigma^* = \frac{h^*}{u_\Sigma^*}, \tag{12}$$

We set

$$\vec{u} = \frac{\vec{u}^*}{u_\Sigma^*},$$

dimensionless interface velocity. Recalling (10), we define

$$u = \frac{u^*}{u_\Sigma^*}, \tag{13}$$

[3]The definition of u_Σ^* will be better specified in section 2.2.

namely

$$u = \frac{t_\Sigma^*}{h^*}\, u^* = -\frac{1}{\sqrt{1+\varepsilon^2\left(\dfrac{\partial s}{\partial x}\right)^2}} \left(\frac{t_\Sigma^*}{t_v^*}\right)\frac{\partial s}{\partial t}. \tag{14}$$

So, up to now, we have introduced two characteristic velocities u_Σ^* and v_c^*. It is important to evaluate their ratio, i.e.

$$\frac{u_\Sigma^*}{v_c^*} = \varepsilon\,\frac{t_v^*}{t_\Sigma^*}. \tag{15}$$

Concerning the liquid pressure, we normalize it by

$$p_c^* = \frac{v_c^*\,\mu^*\,L^*}{h^{*2}},$$

which comes by considering a stationary channel flow of a Newtonian fluid whose mean velocity is v_c^*.

We finally recall that the dependent macroscopic variables have to be obtained by the microscopic ones averaging on the REV. This task will be performed in section 2.5.

2.2. **Interface evolution equation: Erosion model.** A simple model for a purely mechanical erosion process is the following[4] (see also [1])

$$u^* = -\kappa^*\left(\left|\left(\mathbf{t}^{f\,*}\vec{n}\right)\cdot\vec{t}\right| - \tau_o^*\right)_+, \tag{16}$$

where the right hand side is evaluated at the pore wall and:

- $\mathbf{t}^{f\,*}$ is the fluid stress[5].
- κ^* and τ_o^* are given non–negative parameters characterizing the solid material. For simplicity we take them constant, but our analysis can be extended to let them depend on x^* and t^*.
- $(\,\cdot\,)_+$ denotes the positive part, i.e.

$$(\,f\,)_+ = \begin{cases} f & if \quad f>0, \\ 0 & if \quad f\le 0. \end{cases}$$

Since the fluid is Newtonian and incompressible (assumption **A1**), $\mathbf{t}^{f\,*}$ is given by

$$\mathbf{t}^{f\,*} = -p^*\mathbf{I} + 2\mu^*\,\mathbf{d}^{f\,*}, \tag{17}$$

with $\mathbf{d}^{f\,*}$ strain rate tensor

$$\mathbf{d}^{f\,*} = \frac{1}{2}\left(\nabla^*\,\vec{v}^{f\,*} + \left(\nabla^*\,\vec{v}^{f\,*}\right)^{\mathrm{T}}\right),$$

whose dimensionless form is

$$\mathbf{d}^{f} = \frac{L^*}{v_c^*}\mathbf{d}^{f\,*} = \begin{pmatrix} \dfrac{\partial v_1^f}{\partial x} & \dfrac{1}{2}\left(\dfrac{1}{\varepsilon}\dfrac{\partial v_1^f}{\partial y} + \varepsilon\dfrac{\partial v_2^f}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{1}{\varepsilon}\dfrac{\partial v_1^f}{\partial y} + \varepsilon\dfrac{\partial v_2^f}{\partial x}\right) & \dfrac{\partial v_2^f}{\partial y} \end{pmatrix}.$$

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[4] We remark that in case of erosion $\dfrac{\partial s^*}{\partial t^*}$ is positive since the channel expands. Hence, by (14), the interface velocity has to be negative.

[5] Tensors at the microscopic scale will be denoted by lower case symbols.

So, from (17), we have

$$\left(\mathbf{t}^{f\,*}\vec{n}\right)\cdot\vec{t}=\frac{\mu^* v_c^*}{h^*}\left[\left(-\frac{\partial\, v_1^f}{\partial y}\right)+\mathcal{O}\left(\varepsilon\right)\right],$$

and, neglecting $\mathcal{O}\left(\varepsilon\right)$ terms, (16) rewrites as

$$u^*=-u_\Sigma^*\left(\left|-\frac{\partial\, v_1^f}{\partial y}\right|-\tau_o\right)_+. \tag{18}$$

with

$$u_\Sigma^*=\frac{\kappa^* v_c^*\mu^*}{h^*},\quad\Leftrightarrow\quad t_\Sigma^*=\frac{h^{*\,2}}{\kappa^* v_c^*\mu^*}=\varepsilon\frac{t_v^*\, h^*}{\kappa^*\,\mu^*}, \tag{19}$$

and

$$\tau_o=\frac{\tau_o^* h^*}{v_c^*\mu^*}. \tag{20}$$

Note that the interesting case corresponds to $\tau_o<\mathcal{O}\left(1\right)$. Indeed, $\tau_o\gg 1$ prevents any erosion since, by virtue of the adopted scaling, $\left|v_{1\,y}^f\right|=\mathcal{O}\left(1\right)$.

So, recalling (13), (14) and considering the initial condition $s\left(x,0\right)=s_o\left(x\right)$, the system (18), (19) and (20) gives rise to the following Cauchy problem

$$\begin{cases}\dfrac{\partial s}{\partial t}=\dfrac{t_v^*}{t_\Sigma^*}\left(\left|-v_{1\,y}^f\right|-\tau_o\right)_+, \\ \\ s\left(x,0\right)=s_o\left(x\right),\end{cases} \tag{21}$$

where $\mathcal{O}\left(\varepsilon^2\right)$ terms have been neglected.

2.3. **Fluid flow equations.** The incompressibility of the fluid (recall assumption A3) implies

$$\nabla^*\cdot\vec{v}^{\,f\,*}=0,$$

where $\nabla^*\cdot$ denotes the divergence operator with respect to x^* and y^* space coordinates. Recalling (11) and (6), we have

$$\frac{\partial v_1^f}{\partial x}+\frac{\partial v_2^f}{\partial y}=0.$$

The fluid flow, within the region $0<y^*<s^*$, is governed by the Navier–Stokes equation

$$\rho^{f\,*}\left[\frac{\partial\vec{v}^{\,f\,*}}{\partial t^*}+\left(\vec{v}^{\,f\,*}\cdot\nabla^*\right)\,\vec{v}^{\,f\,*}\right]=-\nabla^* p^*+\mu^*\,\Delta^*\vec{v}^{\,f\,*}, \tag{22}$$

where the body forces (e.g. gravity) have been neglected. The dimensionless form of (22) is

$$\begin{aligned}\mathsf{Re}\left(\frac{\partial v_1^f}{\partial t}+v_1^f\frac{\partial v_1^f}{\partial x}+v_2^f\frac{\partial v_1^f}{\partial y}\right)&=-\frac{1}{\varepsilon^2}\frac{\partial p}{\partial x}+\frac{\partial^2 v_1^f}{\partial x^2}+\frac{1}{\varepsilon^2}\frac{\partial^2 v_1^f}{\partial y^2},\\ \mathsf{Re}\left(\frac{\partial v_2^f}{\partial t}+v_1^f\frac{\partial v_2^f}{\partial x}+v_2^f\frac{\partial v_2^f}{\partial y}\right)&=-\frac{1}{\varepsilon^4}\frac{\partial p}{\partial y}+\frac{\partial^2 v_2^f}{\partial x^2}+\frac{1}{\varepsilon^2}\frac{\partial^2 v_2^f}{\partial y^2},\end{aligned}$$

where

$$\mathsf{Re}=\frac{\rho^{f\,*}\,v_c^* L^*}{\mu^*},$$

is the "macroscopic" Reynolds number.

We introduce also the so–called microscopic Reynolds number, i.e. the Reynolds number referred to the channel

$$\mathsf{Re}_{micro}=\frac{\rho^{f\,*}\,v_c^*h^*}{\mu^*},\quad\Leftrightarrow\quad\mathsf{Re}_{micro}=\varepsilon\mathsf{Re}.\tag{23}$$

Concerning the boundary conditions on the interface Σ, we consider the mass flux continuity (see [4]), namely

$$\rho^{f\,*}\,\left(\vec{v}^{\,f\,*}-\vec{u}^{\,*}\right)\cdot\vec{n}=-\rho^{s\,*}\,\vec{u}^{\,*}\cdot\vec{n},\tag{24}$$

that is, by virtue of (14) and (15),

$$\vec{v}^{\,f}\cdot\vec{n}=\frac{u_\Sigma^*}{v_c^*}\,(1-\gamma)\,u=-(1-\gamma)\frac{\varepsilon}{\sqrt{1+\varepsilon^2\left(\dfrac{\partial s}{\partial x}\right)^2}}\frac{\partial s}{\partial t},$$

with γ given by (1). In particular, recalling (11), (8) and neglecting $\mathcal{O}\left(\varepsilon^2\right)$ terms, equation (24) can be rewritten as

$$\left.v_1^f\right|_\Sigma\frac{\partial s}{\partial x}-\left.v_2^f\right|_\Sigma=-(1-\gamma)\frac{\partial s}{\partial t}.$$

Next, we assume a no–slip condition,

$$\vec{v}^{\,f\,*}\cdot\vec{t}=0,$$

namely, recalling (9) and (11),

$$v_1^f+\varepsilon^2v_2^f\frac{\partial s}{\partial x}=0,\ \Rightarrow\ \left.v_1^f\right|_\Sigma=-\varepsilon^2\ \left.v_2^f\right|_\Sigma\frac{\partial s}{\partial x}.$$

On the symmetry line $y=0$, we require

$$\begin{aligned}\left.\frac{\partial v_1^f}{\partial y}\right|_{y=0}&=0,\\ \left.v_2^f\right|_{y=0}&=0.\end{aligned}$$

At both ends of the channel, i.e. $x=0$ and $x=1$, we specify the pressure

$$p\,(0,y,t)=p_{in}\,(t)=\frac{p_{in}^*\,(t)}{p_c^*},\qquad p\,(1,y,t)=p_{out}\,(t)=\frac{p_{out}^*\,(t)}{p_c^*}.$$

It is useful to introduce the so–called mass transfer rate per unit surface[6]

$$\chi^*=\rho^{f\,*}\,\left(\vec{v}^{\,f\,*}-\vec{u}^{\,*}\right)\cdot\vec{n}=-\rho^{s\,*}\,\vec{u}^{\,*}\cdot\vec{n}=-\rho^{s\,*}\,u^*.$$

This definition shows that the natural scale for χ^* is $\rho^{s\,*}u_\Sigma^*$. Therefore, recalling (14) and neglecting, as usual, $\mathcal{O}\left(\varepsilon^2\right)$ terms, the dimensionless mass transfer rate is

$$\chi=\frac{\chi^*}{\rho^{s\,*}u_\Sigma^*}=-u=\frac{t_\Sigma^*}{t_v^*}\frac{\partial s}{\partial t}\,.$$

[6]Note that $\chi^*>0$ corresponds to erosion.

2.4. **Energy dissipation.** Referring to the entire domain Ω, we write the entropy balance globally, obtaining

$$\rho^* \left(\frac{\partial \eta^*}{\partial t^*} + \vec{v}^* \cdot \nabla^* \eta^* \right) + [\![\chi^* \eta^* + \frac{\vec{q}^*}{\theta^*} \cdot \vec{n}]\!] \delta^*_\Sigma = -\nabla^* \cdot \left(\frac{\vec{q}^*}{\theta^*} \right) + \zeta^*,$$

where θ^* is the absolute temperature (uniform and constant, assumption **A4**), and:

- η^* is the specific entropy, i.e. entropy per unit mass.
- ζ^* is the internal entropy production rate. $\zeta^* \geq 0$, by Clausius–Duhem inequality.
- $\vec{q}^{\,*}$ is the heat flux vector.
- δ^*_Σ is the Dirac delta "centered" along Σ.

Next, we introduce the specific Helmoltz free energy

$$\psi^* = e^* - \theta^* \, \eta^*,$$

with e^* internal energy per unit mass, and

$$\psi = \frac{\psi^*}{\psi^*_c},$$

with ψ^*_c characteristic value[7]. In particular, because of assumption **A5**, we have

$$\psi^{s\,*} = \psi^*_c, \quad \text{and} \quad \psi^{f\,*} = 0. \tag{25}$$

From the physical point of view, ψ^*_c can be interpreted as the "bonds energy" (potential energy) possessed by the particles in the solid phase. We shall return to this point later.

Then we define the rate of dissipation as

$$\xi^* = \zeta^* \theta^*,$$

and exploiting the energy equation[8]

$$\begin{aligned} \rho^* \left(\frac{\partial e^*}{\partial t^*} + \vec{v}^* \cdot \nabla^* e^* \right) \quad &+ \quad [\![\chi^* \left(e^* + \frac{v^{*\,2}}{2} \right) \\ + \vec{q}^* \cdot \vec{n} - (\mathbf{t}^* \vec{n}) \cdot \vec{v}^*]\!] \delta^*_\Sigma \quad &= \quad \mathbf{t}^* : \mathbf{d}^* - \nabla^* \cdot \vec{q}^{\,*}, \end{aligned}$$

we have

$$\begin{aligned} \xi^* \quad = \quad & \mathbf{t}^* : \mathbf{d}^* - \rho^* \left(\frac{\partial \psi^*}{\partial t^*} + \vec{v}^* \cdot \nabla^* \psi^* \right) \\ & - \chi^* [\![\psi^*]\!] \delta^*_\Sigma - \chi^* [\![\frac{v^{*2}}{2}]\!] \delta^*_\Sigma + [\![(\mathbf{t}^* \vec{n}) \cdot \vec{v}^*]\!] \delta^*_\Sigma \,. \end{aligned} \tag{26}$$

The first term of (26) represents the mechanical energy dissipated (i.e. converted into heat) in the bulk per unit time. Of course**,**

$$\mathbf{t}^* : \mathbf{d}^* = \mathbf{t}^{\,f\,*} : \mathbf{d}^{\,f\,*} = 2\mu \, \mathbf{d}^{\,f\,*} : \mathbf{d}^{\,f\,*},$$

since $\mathbf{d}^{s\,*} \equiv 0$, and $\mathbf{d}^{f\,*} : \mathbf{I} = \nabla^* \cdot \vec{v}^{\,f\,*} = 0$.

[7] $[\psi^*_c] = J/Kg$. In particular, $[\rho^* \psi^*_c] = Pa$.

[8] We recall that $\mathbf{t}^* : \mathbf{d}^* = \mathrm{tr}\left(\mathbf{t}^* \mathbf{d}^{*\mathrm{T}}\right)$, and $\mathbf{t}^*$ is the Cauchy stress tensor. In particular, the fluid Cauchy stress is given by (17), while $\mathbf{t}^{\,s\,*}$ is indeterminate because of the rigidity constraint.

The second term of (26) vanishes because of assumption **A5**. So, in this case (26) can be rewritten as

$$\xi^* = \underbrace{\mathbf{t}^* : \mathbf{d}^*}_{\xi^*_{bulk}} + \underbrace{\left([\![(\mathbf{t}^* \vec{n}) \cdot \vec{v}^*]\!] - \chi^* \left([\![\psi^*]\!] + [\![\frac{v^{*2}}{2}]\!] \right) \right)}_{\xi^*_\Sigma} \delta^*_\Sigma , \tag{27}$$

where ξ^*_{bulk} is the dissipation within the fluid while ξ^*_Σ represents the dissipation due to the phenomena that take place on the interface Σ. Clausius–Duhem postulate assumes that both ξ^*_Σ and ξ^*_{bulk} are non–negative.

The energy dissipated (in the unit time) at the interface is thus a global balance between the mechanical power associated to the motion of the interface and the kinetic and "potential" energy.

Defining the dimensionless dissipation

$$\xi = \frac{\xi^*}{\mu^* \left(\frac{v^*_c}{L^*} \right)^2},$$

and the "dimensionless" Dirac delta δ_Σ, i.e.

$$\delta^*_\Sigma = \frac{\delta_\Sigma}{h^*},$$

after some algebra, we have

$$\begin{aligned} \xi_{bulk} &= \frac{1}{\varepsilon^2} \left(\frac{\partial v^f_1}{\partial y} \right)^2 + 2 \left[\left(\frac{\partial v^f_1}{\partial x} \right)^2 + \frac{\partial v^f_1}{\partial y} \frac{\partial v^f_2}{\partial x} + \left(\frac{\partial v^f_2}{\partial y} \right)^2 \right] \\ &\quad + \varepsilon^2 \left(\frac{\partial v^f_2}{\partial x} \right)^2 , \end{aligned} \tag{28}$$

$$\begin{aligned} \xi_\Sigma &= \left\{ - \frac{\rho^{s\,*} \, \psi^*_c \, [\![\psi]\!] L^*}{\mu^* \, v^*_c} - \varepsilon^2 \, \mathsf{Re} \, \gamma (1-\gamma)^2 \left(\frac{\partial s}{\partial t} + \mathcal{O}\left(\varepsilon^2\right) \right)^2 \right. \\ &\quad + \varepsilon^2 (1-\gamma) \left[\varepsilon^2 p + 2 \left(\frac{\partial v^f_2}{\partial y} - \frac{\partial s}{\partial x} \frac{\partial v^f_1}{\partial y} \right. \right. \\ &\quad \left. \left. \left. + \, \varepsilon^2 \frac{\partial s}{\partial x} \left(\frac{\partial v^f_1}{\partial x} - \frac{\partial v^f_2}{\partial y} \right) \right) \right] \right\} \left(- \frac{t^*_v}{t^*_\Sigma} u \right) , \end{aligned} \tag{29}$$

clearly indicating that the first term of (28) is the leading–one. Note that, because of (25),

$$[\![\psi]\!] = -1.$$

Concerning (29), if $\mathsf{Re} \le \mathcal{O}\left(\varepsilon^{-1}\right)$ and $\mathcal{O}\left(\varepsilon\right)$ terms are neglected, the leading term of ξ_Σ is the first–one, i.e.

$$\xi_\Sigma = \gamma \, \frac{\mathsf{Re}}{\mathsf{Ec}} [\![\psi]\!] \left(\frac{t^*_v}{t^*_\Sigma} \right) u, \quad \text{with} \quad \xi_\Sigma \ge 0,$$

(recall $u \le 0$, and $[\![\psi]\!] = -1$) where

$$\mathsf{Ec} = \frac{v^{*\,2}_c}{\psi^*_c},$$

is the so–called Eckert's number.

So, during erosion the energy dissipated in the unit time on the interface is the free energy stored within the solid phase. Thus the physical interpretation of ψ_c^* as solid particles "bonds" energy is consistent with this framework. Indeed, when the solid material is converted into fluid (whose free energy vanishes), the "bonds" between solid particles are broken and the corresponding potential energy is lost. Actually, according to (2), (21) and (25), we have

$$[\![\,\psi\,]\!]\, u = \left(\left|-v^f_{1\,y}\right| - \tau_o\right)_+ \geq 0,$$

as expected. The model proposed is thus consistent with Clausius–Duhem postulate.

Now, since

$$\xi_\Sigma = \frac{1}{\varepsilon^2}\frac{\kappa^* \rho^{s\,*} \psi_c^*}{v_c^*}\left(\left|-v^f_{1\,y}\right| - \tau_o\right)_+,$$

namely

$$-\gamma\,\frac{\mathsf{Re}}{\mathsf{Ec}}[\![\,\psi\,]\!]\left(\frac{t_v^*}{t_\Sigma^*}\right) = \frac{1}{\varepsilon^2}\frac{\kappa^* \rho^{s\,*} \psi_c^*}{v_c^*}.$$

we stipulate

$$\tau_o^* = \rho^{s\,*}\psi_c^*. \tag{30}$$

Such an identification is physically consistent. Indeed, ξ_Σ vanishes when $\tau_o = 0$, i.e. when the "bonds" energy vanishes. In other words, according to (30), a vanishing τ_o^* characterizes a solid material so "soft" that erosion occurs practically without dissipation. In the opposite case, high values of τ_o^* characterize solids whose "bonds" energy between particles is so large that strong stress is required for particles detachment, accompanied by high dissipation.

We also note that τ_o^* is a quantity that can be estimated experimentally.

Finally, recalling (20), we have

$$-\gamma\,\frac{\mathsf{Re}}{\mathsf{Ec}}[\![\,\psi\,]\!]\left(\frac{t_v^*}{t_\Sigma^*}\right) = \frac{\tau_o}{\varepsilon^2}\frac{u_\Sigma^*}{v_c^*},$$

where, as remarked before, $\tau_o < \mathcal{O}\,(1)$.

2.5. **Model upscaling.** The condition of separation of scales (5) enable us to use the homogenization method of double scale expansion. We have seen that both characteristic lengths introduce the dimensionless space variables x and y. The variable x is the macroscopic (or slow) variable where y is the microscopic (or fast) variable.

The unknown fields (i.e. v_1^f, v_2^f, s, etc.) appear at the microscopic scale as functions of these two dimensionless space variables and are looked for in the following form

$$f\,(x,y,t) = f^{\,(0)}\,(x,y,t) + \varepsilon f^{\,(1)}\,(x,y,t) + \varepsilon^2 f^{\,(2)}\,(x,y,t) + \ldots\,.$$

Next, introducing the asymptotic expansions in the dimensionless equations, we solve the boundary value problems arising at successive order of ε. We then attach to each dependent variable f its macroscopic value

$$F = \langle\, f\,\rangle = \frac{1}{|\Omega|}\int_\Omega f\,dy = \frac{1}{|\Omega|}\sum_{\alpha=s,f}\int_{\Omega^\alpha} f^{\,\alpha}\,dy,$$

which, neglecting $\mathcal{O}(\varepsilon)$ corrections, reduces to

$$F = \left\langle f^{(0)} \right\rangle = \frac{1}{|\Omega|} \int_{\Omega} f^{(0)} dy.$$

We start specifying two further assumptions:

A6. We assume

$$\mathsf{Re} \leq \mathcal{O}\left(\varepsilon^{-1}\right). \tag{31}$$

Recalling (23), we have

$$\mathsf{Re}_{micro} \leq \mathcal{O}(1),$$

i.e. laminar flow within the channel. From the physical point of view, (31) implies also that inertia terms in the momentum balance equation are negligible.

A7. $\dfrac{t_v^*}{t_\Sigma^*} = \mathcal{O}(1) \Leftrightarrow \dfrac{u_\Sigma^*}{v_c^*} = \mathcal{O}(\varepsilon)$, i.e. the convective time scale is comparable with the interface evolution time scale. Recalling (19), we have

$$\frac{t_\Sigma^*}{t_v^*} = \varepsilon \frac{h^*}{\kappa^* \mu^*}.$$

and so

$$\frac{h^*}{\kappa^* \mu^*} = \mathcal{O}\left(\varepsilon^{-1}\right)$$

When $1-\gamma = \mathcal{O}(1)$, we have the following mathematical model[9]

$$\begin{cases} \dfrac{\partial v_1}{\partial x} + \dfrac{\partial v_2}{\partial y} = 0, \\ \dfrac{1}{\varepsilon}\left(\dfrac{\partial v_1}{\partial t} + v_1 \dfrac{\partial v_1}{\partial x} + v_2 \dfrac{\partial v_1}{\partial y}\right) = -\dfrac{1}{\varepsilon^2}\dfrac{\partial p}{\partial x} + \dfrac{\partial^2 v_1}{\partial x^2} + \dfrac{1}{\varepsilon^2}\dfrac{\partial^2 v_1}{\partial y^2}, \\ \dfrac{1}{\varepsilon}\left(\dfrac{\partial v_2}{\partial t} + v_1 \dfrac{\partial v_2}{\partial x} + v_2 \dfrac{\partial v_2}{\partial y}\right) = -\dfrac{1}{\varepsilon^4}\dfrac{\partial p}{\partial y} + \dfrac{\partial^2 v_2}{\partial x^2} + \dfrac{1}{\varepsilon^2}\dfrac{\partial^2 v_2}{\partial y^2}, \\ \left[v_1 \dfrac{\partial s}{\partial x} - v_2 + (1-\gamma)\dfrac{\partial s}{\partial t}\right]_{y=s} = 0, \\ \left[v_1 + \varepsilon^2 v_2 \dfrac{\partial s}{\partial x}\right]_{y=s} = 0, \\ \left.\dfrac{\partial v_1}{\partial y}\right|_{y=0} = 0, \\ v_2\big|_{y=0} = 0, \\ p(0,y,t) = p_{in}, \quad p(1,y,t) = p_{out}, \\ \dfrac{\partial s}{\partial t} = \left(\left|-\dfrac{\partial v_1}{\partial y}\right| - \tau_o\right)_+, \\ + \text{ I. C.}, \end{cases} \tag{32}$$

and we proceed to analyze the various orders in ε.

<u>Order ε^{-4} and ε^{-3}</u>

[9] In the sequel, when possible, we shall omit the index " f " to keep notation simple.

$$\begin{aligned}\frac{\partial p^{(0)}}{\partial y} &= 0, \;\Rightarrow\; p^{(0)} = p^{(0)}(x,t),\\ \frac{\partial p^{(1)}}{\partial y} &= 0, \;\Rightarrow\; p^{(1)} = p^{(1)}(x,t),\end{aligned}$$

Order ε^{-2}

$$\frac{\partial p^{(2)}}{\partial y} = \frac{\partial^2 v_2^{(0)}}{\partial y^2},$$

$$\begin{cases} -\dfrac{\partial p^{(0)}}{\partial x} + \dfrac{\partial^2 v_1^{(0)}}{\partial y^2} = 0 \\ v_1^{(0)}\Big|_{y=s} = 0 \\ \dfrac{\partial v_1^{(0)}}{\partial y}\Big|_{y=0} = 0, \end{cases} \Rightarrow\; v_1^{(0)}(x,y,t) = \frac{1}{2}\frac{\partial p^{(0)}}{\partial x}\left(y^2 - s^2\right), \tag{33}$$

and, from $(32)_1$ and $(32)_7$,

$$\begin{cases} \dfrac{\partial v_2^{(0)}}{\partial y} = -\dfrac{\partial v_1^{(0)}}{\partial x}, \\ v_2^{(0)}\Big|_{y=0} = 0 \end{cases} \Rightarrow\; v_2^{(0)}(x,y,t) = -\int_0^y \frac{\partial v_1^{(0)}}{\partial x}(x,\eta,t)\,d\eta,$$

that is

$$v_2^{(0)}(x,y,t) = -\frac{1}{2}\frac{\partial^2 p^{(0)}}{\partial x^2}\left(\frac{y^3}{3} - ys^2\right) + \frac{\partial p^{(0)}}{\partial x}\frac{\partial s}{\partial x}sy.$$

Finally, the pressure field $p^{(0)}$ and s will be obtained by equations $(32)_4$ and $(32)_9$, namely

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{s^3}{3}\dfrac{\partial p^{(0)}}{\partial x}\right) = (1-\gamma)\left(\left|-\dfrac{\partial p^{(0)}}{\partial x}\right| s - \tau_o\right)_+, \\ \dfrac{\partial s}{\partial t} = \left(\left|-\dfrac{\partial p^{(0)}}{\partial x}\right| s - \tau_o\right)_+, \\ s(x,0) = s_o(x), \\ p(0,y,t) = p_{in}, \;\; p(1,y,t) = p_{in} - \Delta p, \end{cases} \tag{34}$$

with $s_o(x)$ continuously differentiable in $[0,1]$, with values in$(0,1)$, and

$$\Delta p = p_{in} - p_{out}, \quad \text{with} \quad \Delta p \geq 0. \tag{35}$$

Both, p_{in} and p_{out} will be supposed to be continuous functions of t.

In case $\tau_o = 0$, the free boundary evolution equation $(32)_9$ becomes

$$\frac{\partial s}{\partial t} = \left|-\frac{\partial p^{(0)}}{\partial x}\right| s, \tag{36}$$

so that (34) acquires the form

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{s^3}{3}\dfrac{\partial p^{(0)}}{\partial x}\right) = (1-\gamma)\left|-\dfrac{\partial p^{(0)}}{\partial x}\right| s\,, \\[2ex] \dfrac{\partial s}{\partial t} = \left|-\dfrac{\partial p^{(0)}}{\partial x}\right| s\,, \\[2ex] s\,(x,0) = s_o\,(x)\,, \\[2ex] p\,(0,y,t) = p_{in}, \quad p\,(1,y,t) = p_{in} - \Delta p. \end{cases} \tag{37}$$

The latter simplifies further if $\gamma = 1 + \mathcal{O}\,(\varepsilon)$, which may occur in biological systems where the solid matrix and the fluid have essentially the same density. In this case we have

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{s^3}{3}\dfrac{\partial p^{(0)}}{\partial x}\right) = 0, \\[2ex] \dfrac{\partial s}{\partial t} = s\left|-\dfrac{\partial p^{(0)}}{\partial x}\right|, \\[2ex] s\,(x,0) = s_o\,(x)\,, \\[2ex] p|_{x=0} = p_{in}, \quad p|_{x=1} = p_{in} - \Delta p. \end{cases} \tag{38}$$

Finally, when $\tau_o \neq 0$ but $\gamma = 1 + \mathcal{O}\,(\varepsilon)$, problem (34) reduces to the following

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{s^3}{3}\dfrac{\partial p^{(0)}}{\partial x}\right) = 0\,, \\[2ex] \dfrac{\partial s}{\partial t} = \left(\left|-\dfrac{\partial p^{(0)}}{\partial x}\right| s - \tau_o\right)_+\,, \\[2ex] s\,(x,0) = s_o\,(x)\,, \\[2ex] p\,(0,y,t) = p_{in}, \quad p\,(1,y,t) = p_{in} - \Delta p. \end{cases} \tag{39}$$

Remark 1. Introducing the macroscopic variables

$$P^* = p_c^*\left\langle p^{(0)}\right\rangle, \quad V_1^{f\,*} = v_c^*\left\langle v_1^{(0)}\right\rangle, \quad V_2^{f\,*} = v_c^*\left\langle v_2^{(0)}\right\rangle,$$

we have

$$P^* = p_c^*\,p^{(0)}\,(x,t)\,, \quad i.e. \quad \left\langle p^{(0)}\right\rangle = p^{(0)},$$

since $p^{(0)}$ does not depends on y. Concerning $V_1^{f\,*}$, from (33), we obtain

$$\left\langle v_1^{(0)}\right\rangle = -\frac{s^2}{3}\frac{\partial p^{(0)}}{\partial x},$$

and, consequently

$$V_1^{f\,*} = -\frac{K^*\left(\phi^f\right)}{\phi^f\,\mu^*}\frac{\partial P^*}{\partial x^*}, \quad \text{with} \quad K^*\left(\phi^f\right) = \frac{h^{*\,2}\left(\phi^f\right)^3}{3}, \tag{40}$$

and with ϕ^f evolving in time according to some specific law involving the pressure gradient. However we remark that, although (40) formally resembles to Darcy's law, actually we may not speak of "Darcyan" flow in the classical sense, since the flow does not depend linearly on the pressure gradient. As a matter of fact, the pressure gradient influences the dynamics that governs ϕ^f in a complicate way, as we shall see explicitly in example 1.

Remark 2. It is interesting to write explicitly the upscaled version of model (39). Adopting the standard notation we have

$$\begin{cases} \dfrac{\partial \phi^f}{\partial t^*} = \mathfrak{n}^*, \\ \\ \dfrac{\partial Q_1^*}{\partial x^*} = 0, \\ \\ Q_1^* = -\dfrac{K^*\left(\phi^f\right)}{\mu^*}\dfrac{\partial P^*}{\partial x^*}, \end{cases}$$

where Q_1^* is the longitudinal component of the discharge, i.e. $Q_1^* = \phi^f V_1^{f\,*}$, and $\mathfrak{n}^*$ is the erosion rate, namely

$$\mathfrak{n}^* = \kappa^* \left(\left| -\frac{\partial P^*}{\partial x^*} \right| \phi^f - \frac{\tau_o^*}{h^*} \right)_+ .$$

3. **Analysis of the mathematical problem.** The aim of this section is to develop a qualitative analysis for the mathematical problems presented in the previous section. We start with the general case, i.e. with problem (34), assuming for the time being that the stress at the interface always exceeds the threshold τ_o (a circumstance to be checked a posteriori, which implies a constraint on the initial conditions, as we shall see in example 2). We set

$$\Pi\left(x,t\right) = \frac{\partial p^{(0)}}{\partial x} < 0,$$

and we rewrite the differential system in the form

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{s^3}{3}\Pi\right) = -\left(1-\gamma\right)\left(s\Pi + \tau_o\right), \\ \\ \dfrac{\partial s}{\partial t} = -\left(s\Pi + \tau_o\right), \end{cases} \tag{41}$$

with the assumption $(s\Pi + \tau_o) < 0$.

By integrating $(41)_2$ we deduce the relationship

$$s\left(x,t\right) = e^{-Z(x,t)}\left(s_o\left(x\right) - \tau_o \int_0^t e^{Z\left(x,t'\right)}dt'\right), \tag{42}$$

where we have defined

$$Z\left(x,t\right) = \int_0^t \Pi\left(x,t'\right)dt'.$$

With the help of (42) we rewrite $(41)_1$ in the form

$$\begin{aligned}&\frac{1}{3}\frac{\partial}{\partial x}\left\{\Pi\left(x,t\right)e^{-3Z(x,t)}\left[s_o\left(x\right)-\tau_o\int_0^t e^{Z\left(x,t'\right)}dt'\right]^3\right\}\\&=-\left(1-\gamma\right)\left\{\tau_o+\Pi\left(x,t\right)e^{-Z(x,t)}\left[s_o\left(x\right)-\tau_o\int_0^t e^{Z\left(x,t'\right)}dt'\right]\right\}.\end{aligned}\tag{43}$$

At this point we introduce the mapping $\Pi\to\Gamma$

$$\Gamma=\Pi\,e^{-3Z},\tag{44}$$

and compute its inverse

$$\Pi=\mathcal{F}\left(\Gamma\right),\tag{45}$$

for Γ in a suitable class of negative continuous functions.

Next, if we set

$$W=\frac{\Pi}{\Gamma},$$

then we can rewrite (44) as

$$W\left(x,t\right)=\exp\left\{3\int_0^t W\left(x,t'\right)\,\Gamma\left(x,t'\right)\,dt'\right\},$$

which can be put in a differential form

$$\frac{\partial W}{\partial t}=3W^2\Gamma,\quad W\left(x,0\right)=1,$$

having the solution

$$W\left(x,t\right)=\left[1-3\int_0^t\Gamma\left(x,t'\right)dt'\right]^{-1}.$$

We thus obtain the explicit expression for the mapping (45), i.e.

$$\mathcal{F}\left(\Gamma\right)=\frac{\Gamma\left(x,t\right)}{1-3\displaystyle\int_0^t\Gamma\left(x,t'\right)dt'},\tag{46}$$

Now, using mappings (44) and (45), we can recast equation (43) in the form

$$\begin{aligned}&\frac{1}{3}\frac{\partial}{\partial x}\left\{\Gamma\left(x,t\right)\left[s_o\left(x\right)-\tau_o\int_0^t\exp\left(\int_0^{t'}\mathcal{F}\left(\Gamma\right)\left(x,t''\right)dt''\right)dt'\right]^3\right\}\\&=-\left(1-\gamma\right)\left\{\tau_o+\left(\mathcal{F}\left(\Gamma\right)\right)^{2/3}\Gamma^{1/3}\left[s_o\left(x\right)\right.\right.\\&\left.\left.-\tau_o\int_0^t\exp\left(\int_0^{t'}\mathcal{F}\left(\Gamma\right)\left(x,t''\right)dt''\right)dt'\right]\right\},\end{aligned}\tag{47}$$

containing the only unknown $\Gamma\left(x,t\right)$. Recall that $\Delta p\left(t\right)$ is a continuous function of t.

The peculiar aspect of the problem for $\Gamma\left(x,t\right)$ is that we have no direct information on $\Gamma\left(0,t\right)$. Therefore we have to go through a kind of shooting technique: supposing we are able to integrate (47) with the condition

$$\Gamma\left(0,t\right)=\Gamma_o\left(t\right),$$

for each t we must impose

$$\int_0^1 \mathcal{F}(\Gamma)(x,t)\,dx = -\Delta p(t)\,, \tag{48}$$

which is supposed to determine $\Gamma_o(t)$.

The last step is to calculate the interface $s(x,t)$. To this end we elaborate a bit further the expression e^Z appearing in (42). Introducing

$$\Xi(x,t) = \int_0^t \Gamma(x,t')\,dt' < 0, \quad \Xi(x,0) = 0,$$

and, remembering (46), we can rewrite (45) as

$$\mathcal{F}(\Gamma) = \frac{1}{1-3\Xi}\frac{\partial \Xi}{\partial t},$$

and compute

$$\begin{aligned}\exp\left\{\int_0^t \mathcal{F}(\Gamma)(x,t')\,dt'\right\} &= (1-3\Xi(x,t))^{-1/3} \\ &= \left(1-3\int_0^t \Gamma(x,t')\,dt'\right)^{-1/3}. \end{aligned}\tag{49}$$

Hence

$$Z = -\frac{1}{3}\ln(1-3\Xi)\,, \tag{50}$$

and

$$e^{Z(x,t)} = \exp\left\{\int_0^t \mathcal{F}(\Gamma)(x,t')\,dt'\right\} = (1-3\Xi(x,t))^{-1/3}\,. \tag{51}$$

With the help of (42), (50), (51) we obtain the final expression of the interface

$$\begin{aligned} s(x,t) &= \left[1-3\int_0^t \Gamma(x,t')\,dt'\right]^{1/3} \\ &\cdot \left\{ s_o(x) - \tau_o \int_0^t \left[1-3\int_0^{t'} \Gamma(x,t'')\,dt''\right]^{-1/3} dt' \right\}. \end{aligned}\tag{52}$$

We will now consider separately the following problems since they require different techniques:

- Case I: $\gamma = 1,\ \tau_o = 0$, problem (38).
- Case II: $\gamma = 1,\ \tau_o > 0$, problem (39).
- Case III: $\gamma \neq 1,\ \tau_o = 0$, problem (37).
- Case IV: $\gamma \neq 1,\ \tau_o \neq 0$, problem (34).

In all cases the data are the initial interface profile $s_o(x)$ and the pressure difference $\Delta p(t)$. The goal is to find the functions $\Gamma(x,t)$, $\Gamma_o(t)$ satisfying (47) and (48). We emphasize that (48) is a non–local condition. Once Γ is known, (52) provides the free boundary $s(x,t)$.

3.1. **Case I: $\gamma = 1$, $\tau_o = 0$, problem (38).** In this case equation (47) becomes trivial. We have the integral

$$\Gamma(x,t) = \Gamma_o(t) \frac{s_o^3(0)}{s_o^3(x)}, \tag{53}$$

which allows to write (48) as follows

$$\int_0^1 \Gamma_o(t) \frac{s_o^3(0)}{s_o^3(x)} \left[1 - 3\frac{s_o^3(0)}{s_o^3(x)} \int_0^t \Gamma_o(t')\, dt'\right]^{-1} dx = -\Delta p(t). \tag{54}$$

Introducing

$$g(t) = \int_0^t \Gamma_o(t')\, dt', \quad \Leftrightarrow \quad \Gamma_o(t) = \dot{g}(t), \tag{55}$$

with $g(0) = 0$, we find in place of (54)

$$\dot{g}(t) \int_0^1 \frac{1}{\frac{s_o^3(x)}{s_o^3(0)} - 3g(t)} dx = -\Delta p(t).$$

Looking for $t(g)$ instead of $g(t)$, we have

$$\begin{cases} -\Delta p(t(g))\ t'(g) = \displaystyle\int_0^1 \frac{dx'}{\frac{s_o^3(x')}{s_o^3(0)} - 3g}, \\ t(0) = 0, \end{cases}$$

whose solution is (separate the variables and interchange the integration order on the r.h.s.)

$$\int_0^t \Delta p(t')\ dt' = \frac{1}{3} \int_0^1 \ln\left(1 - 3\frac{s_o^3(0)}{s_o^3(x')} g\right) dx'.$$

The r.h.s. is a positive monotone decreasing function of $g \in (-\infty, 0)$, while the l.h.s. is positive, monotonically increasing in t. Thus the function $g(t)$ is defined uniquely for all t as a C^1 decreasing function. Hence, by (55) and (53) we obtain first $\Gamma_o(t)$ and then $\Gamma(x,t)$. Next, from (46) we compute the pressure gradient Π and the interface $s(x,t)$ is finally provided by (42), namely

$$s(x,t) = s_o(x) \exp\left(-\int_0^t \Pi(x,t')\, dt'\right). \tag{56}$$

Example 1. Let us consider the particular case $s_o(x) = s_o$, $0 \le x \le 1$, and Δp constant in time. We have

$$g(t) = \frac{1}{3}\left(1 - e^{3\Delta p\, t}\right), \quad \Rightarrow \quad \dot{g} = \Gamma_o = -\Delta p\, e^{3\Delta p\, t}$$

Hence, by (53) we obtain

$$\Gamma(t) = -\Delta p\, e^{3\Delta p\, t},$$

and, by (46)

$$\Pi = \frac{-\Delta p e^{3\Delta p\, t}}{1 + 3\displaystyle\int_0^t \Delta p\, e^{3\Delta p\, t'} dt'} = -\Delta p,$$

so that (56) yields the following interface profile

$$s(t) = s_o e^{\Delta p\, t} \tag{57}$$

and $p(x) = p_{in} - \Delta p\, x$, up to the time $\hat{t} = (\Delta p)^{-1} |\ln s_o|$ (total erosion time).

Next, considering the macroscopic quantities introduced in remark 1, we have

$$\begin{aligned} \phi^f(t^*) &= \phi_o^f \exp\left\{ \kappa^* \left(\frac{\Delta P^*}{L^*} \right) t^* \right\}, \\ V_1^{f\,*} &= \frac{K^*\left(\phi^f\right)}{\phi^f \mu^*} \frac{\Delta P^*}{L^*}, \end{aligned}$$

emphasizing the nonlinear dependence of $V_1^{f\,*}$ on $\Delta P^*/L^*$.

Remark 3. We may obtain the same results by means of an alternative technique. Indeed, combining $(38)_1$ and $(38)_2$, we get

$$\frac{\partial}{\partial x}\left(\frac{\partial s^3}{\partial t}\right) = 0, \quad \Rightarrow \quad s^3 = f(x) + \varphi(t),$$

with $\varphi(0) = 0$. In particular, imposing $(38)_3$ we get $f(x) = s_o^3(x)$. Thus, from $(38)_2$ we obtain

$$\Pi = -\frac{\dot{\varphi}(t)}{3\left(s_o^3 + \varphi(t)\right)}, \quad \Rightarrow \quad p^{(0)}(x,t) - p_{in}(t) = -\frac{\dot{\varphi}(t)}{3} \int_0^x \frac{dx'}{s_o^3(x') + \varphi(t)}.$$

Imposing now $(38)_4$, i.e. $p|_{x=1} = p_{in} - \Delta p$, we have

$$\begin{cases} \dot{\varphi}(t) \displaystyle\int_0^1 \frac{dx'}{s_o^3(x') + \varphi(t)} = 3\Delta p(t), \\ \varphi(0) = 0. \end{cases}$$

which, if we look for $t(\varphi)$ instead of $\varphi(t)$, can be rewritten as

$$\begin{cases} 3\Delta p(t(\varphi))\; t'(\varphi) = \displaystyle\int_0^1 \frac{dx'}{s_o^3(x') + \varphi}, \\ t(0) = 0, \end{cases}$$

whose solution is

$$\int_0^t \Delta p(r)\, dr = \frac{1}{3} \int_0^1 \ln\left(\frac{1}{s_o^3(x') + \varphi(t)}\right) dx'.$$

In particular, considering the data of example 1 yields

$$\varphi(t) = s_o^3\left(e^{3\Delta p\, t} - 1\right), \quad \Rightarrow \quad s(t) = s_o\, e^{\Delta p\, t},$$

namely (57).

3.2. **Case II:** $\gamma = 1,\ \tau_o > 0$**, problem (39).** This time we need the assumptions:
(i). Δp continuously differentiable.
(ii). $s_o(x)$ is Lipschitz continuous.
Recalling (49), from (47) we derive the following nonlinear integral equation for Γ

$$\begin{aligned} &\Gamma(x,t)\left[s_o(x) - \tau_o\int_0^t\left(1-3\int_0^{t'}\Gamma(x,t'')\,dt''\right)^{-1/3}dt'\right]^3 \\ &= \Gamma_o(t)\left[s_o(0) - \tau_o\int_0^t\left(1-3\int_0^{t'}\Gamma_o(t'')\,dt''\right)^{-1/3}dt'\right]^3, \end{aligned} \tag{58}$$

where, as usual, $\Gamma_o(t) = \Gamma(0,t)$ is also unknown and $x \in [0,1]$ is a parameter. It is worth recalling that we are looking for Γ_o, Γ both negative and the quantities in brackets strictly positive. The latter requirement is guaranteed if we take

$$0 \le t \le \frac{\inf s_o(x) - \alpha}{\tau_o},$$

for some $0 < \alpha < \inf s_o(x)$. Then for given $\Gamma_o(t)$ such that

$$-\Gamma^* \le \Gamma_o(t) \le 0, \tag{59}$$

we have the a priori estimate $\Gamma \le 0$, and

$$|\Gamma_o(t)|\frac{\alpha^3}{[\sup s_o(x)]^3} \le |\Gamma(x,t)| \le |\Gamma_o(t)|\frac{s_o^3(0)}{\alpha^3}.$$

By means of a standard fixed point argument it is easy to show that (58) has one unique solution $\Gamma(x,t)$, such that

$$-\Gamma^*\frac{s_o^3(0)}{\alpha^3} \le \Gamma(x,t) \le 0, \quad \text{for} \quad 0 \le t \le t_\alpha,$$

for any $\Gamma_o(t)$ satisfying (59) and belonging to a Hölder class.

The next step is to look for $\Gamma_o(t)$ such that

$$-\int_0^1 \frac{\Gamma(x,t)}{1-3\int_0^t\Gamma(x,t')\,dt'}dx = \Delta p(t), \tag{60}$$

for all t. We proceed as follows. First of all we remark that at time $t = 0$ equation (58) defines

$$\Gamma(x,0) = \Gamma_o(0)\left(\frac{s_o(0)}{s_o(x)}\right)^3, \tag{61}$$

which of course coincides with formula (53) we found for case I. Putting (61) in (60), for $t = 0$, we find $\Gamma_o(0)$

$$\Gamma_o(0) = -\Delta p(0)\left[\int_0^1 \frac{s_o^3(0)}{s_o^3(x)}dx\right]^{-1},$$

so that $\Gamma(x,0)$ turns out to be

$$\Gamma(x,0) = -\Delta p(0)\left(\frac{s_o(0)}{s_o(x)}\right)^3\left[\int_0^1 \frac{s_o^3(0)}{s_o^3(x)}dx\right]^{-1}. \tag{62}$$

Now we take a sequence $t_k = k\epsilon$ (now ϵ is a "small" parameter which has absolutely no relation with ε given by the ratio (4)) and we approximate $\Gamma_o(t)$ by

$$\Gamma_o(t_k) = \Gamma_o(0) + \epsilon \sum_{j=1}^{k} \Gamma_j \,.$$

For $t_1 = \epsilon$, in (60) we take for $\Gamma(x,t)$ in the numerator the approximation ensuing from (58)

$$\begin{aligned}\Gamma(x,\epsilon) &\approx [\Gamma_o(0) + \epsilon\,\Gamma_1] \frac{[s_o(0) - \tau_o\epsilon]^3}{[s_o(x) - \tau_o\epsilon]^3} \\ &\approx \left[\frac{s_o(0)}{s_o(x)}\right]^3 \left\{\Gamma_o(0) + \epsilon\left[\Gamma_1 - 3\tau_o\Gamma_o(0)\left(\frac{1}{s_o(0)} - \frac{1}{s_o(x)}\right)\right]\right\}, \quad (63)\end{aligned}$$

and in the denominator we just take the leading term of (63), i.e. (62). In this way we obtain

$$\begin{aligned}-\Gamma_1 \int_0^1 \left(\frac{s_o(0)}{s_o(x)}\right)^3 dx &= \frac{\Delta p(\epsilon) - \Delta p(0)}{\epsilon} + 3\Delta p(0) \\ &\quad + 3\tau_o \int_0^1 \Gamma(x,0)\left(\frac{1}{s_o(0)} - \frac{1}{s_o(x)}\right) dx.\end{aligned}$$

Back to (63) we have now the approximation sought for $\Gamma(x,\epsilon)$

$$\begin{aligned}\Gamma(x,\epsilon) &\approx \Gamma(x,0)\left[1 - 3\epsilon\tau_o\left(\frac{1}{s_o(0)} - \frac{1}{s_o(x)}\right)\right] \\ &\quad -\epsilon\frac{\left(\frac{s_o(0)}{s_o(x)}\right)^3}{\int_0^1 \left(\frac{s_o(0)}{s_o(x)}\right)^3 dx}\left\{\frac{\Delta p(\epsilon) - \Delta p(0)}{\epsilon} + 3\Delta p(0)\right. \\ &\quad \left. + 3\tau_o \int_0^1 \Gamma(x,0)\left(\frac{1}{s_o(0)} - \frac{1}{s_o(x)}\right) dx\right\}.\end{aligned}$$

The procedure can be iterated: suppose we have calculated the approximations $\Gamma_o(t_k)$, $\Gamma(x,t_k)$. Then from (58) we get an expression for $\Gamma(x,t_{k+1})$

$$\Gamma(x,t_{k+1}) \approx [\Gamma_o(t_k) + \epsilon\,\Gamma_{k+1}] \frac{\left\{s_o(0) - \tau_o\epsilon \sum_{j=0}^{k}\left[1 - 3\epsilon\sum_{i=0}^{j}\Gamma_o(t_i)\right]^{-1/3}\right\}^3}{\left\{s_o(x) - \tau_o\epsilon \sum_{j=0}^{k}\left[1 - 3\epsilon\sum_{i=0}^{j}\Gamma(x,t_i)\right]^{-1/3}\right\}^3}. \quad (64)$$

Next, we approximate (60) as follows

$$-\int_0^1 \frac{\Gamma(x,t_{k+1})}{1 - 3\epsilon \sum\limits_{j=0}^{t_k} \Gamma(x,t_j)} dx = \Delta p(t_{k+1}),$$

which, introducing (64), yields Γ_{k+1}. The latter in turn provides $\Gamma(x,t_{k+1})$ using again (64).

It is easy to realize that, thanks to the differentiability of Δp, the ratios

$$\frac{\Gamma_{k+1}-\Gamma_k}{\epsilon}, \quad \text{and} \quad \frac{\Gamma\left(x,t_{k+1}\right)-\Gamma\left(x,t_k\right)}{\epsilon},$$

are uniformly bounded. In this way we have constructed a uniformly Lipschitz continuous sequence of approximations $\Gamma_o^{(\epsilon)}(t)$, $\Gamma_o^{(\epsilon)}(x,t)$ (it is enough to complete the functions by linear time interpolations). Any convergent subsequence defines a solution to (60), also Lipschitz continuous. As before, knowing Γ, we can compute the interface.

Uniqueness can be demonstrated by means of the following argument. Suppose there are two solutions $\Gamma_o^{(1)}(t)$ and $\Gamma_o^{(2)}(t)$, to which there correspond the respective functions $\Gamma^{(1)}(x,t)$ and $\Gamma^{(2)}(x,t)$. From (58) we can deduce that $\Gamma^{(1)}-\Gamma^{(2)}$ can be expressed as $\Lambda(x,t)\left(\Gamma_o^{(1)}-\Gamma_o^{(2)}\right)$, where Λ depends on all the Γ's, but is bounded. Using (46), we may conclude that the difference $\Gamma_o^{(1)}-\Gamma_o^{(2)}$ satisfies a homogeneous Volterra integral equation of the second kind (with the coefficients depending on all Γ's), and therefore is vanishing.

Remark 4. Of course the inequality $s<1$ and $s\Pi+\tau_o<0$, must be checked at all times. If the second one is violated, say at $(\bar{x},\bar{t})$, then the problem becomes much more complicated because, for $t>\bar{t}$, one or more unknowns appears, namely the moving point bounding a non–erosion zone (there can be many of such free boundaries). We will not examine this case.

Remark 5. Case II can be studied also using a technique similar to the one introduced in remark 3. Such a technique has the advantage to be slightly simpler when used for cases I and II. However, if applied to cases III and IV, it gives rise to extremely involved functional equations. So, considering $\Pi<0$, we proceed as in remark 3 combining together $(39)_1$ and $(39)_2$ so to get

$$\frac{\partial}{\partial x}\left[s^2\left(\frac{\partial s}{\partial t}+\tau_o\right)\right]=0,$$

which entails

$$s^2\left(\frac{\partial s}{\partial t}+\tau_o\right)=\varphi(t), \quad \Rightarrow \quad \left(\frac{\partial s}{\partial t}+\tau_o\right)=\frac{\varphi(t)}{s^2}. \tag{65}$$

Using now $(39)_2$ we have

$$-\Pi=\frac{\varphi(t)}{s^3},$$

which, once integrated in x between 0 and 1, gives

$$\Delta p(t)=\varphi(t)\int_0^1\frac{dx}{s^3(x,t)}, \quad \Rightarrow \quad \varphi(t)=\Delta p(t)\left(\int_0^1\frac{dx}{s^3(x,t)}\right)^{-1}.$$

Thus, returning to equation (65), we obtain the following problem

$$\begin{cases} s^2\left(\dfrac{\partial s}{\partial t}+\tau_o\right)=\Delta p(t)\left(\displaystyle\int_0^1\frac{dx}{s^3(x,t)}\right)^{-1}, \\ \\ s(x,0)=s_o(x). \end{cases} \tag{66}$$

which shows clearly the "non–local" structure of the model. Existence and uniqueness can be proved using fixed–point arguments.

Example 2. We consider the data of example 1, namely $s_o(x) = s_o$, $0 \le x \le 1$, and Δp constant in time. We look for a solution of (66) by means of the iterative procedure based on the following mapping

$$\begin{cases} \dfrac{\partial s^{(k+1)}}{\partial t} + \tau_o = \dfrac{\Delta p}{\left(s^{(k)}\right)^2}\left(\displaystyle\int_0^1 \dfrac{dx}{\left(s^{(k)}\right)^3}\right)^{-1}, \\ \\ s^{(k+1)}(x,0) = s_o. \end{cases} \tag{67}$$

It is easy to realize that $s^{(k+1)}$ does not depend on x if $s^{(k)}$ does not. Therefore, since $s_o(x) = s_o$, $\forall\, x \in [0,1]$, $s^{(k)} = s^{(k)}(t)$. Thus solving (67) we have

$$s^{(k+1)}(t) = s_o - \tau_o t + \Delta p \int_0^t s^{(k)}(t')\, dt'.$$

Hence

$$\begin{aligned} k=0 \qquad & s^{(1)}(t) = s_o + (\Delta p s_o - \tau_o)\, t, \\ k=1 \qquad & s^{(2)}(t) = s_o + (\Delta p s_o - \tau_o)\, t + \Delta p\, (\Delta p s_o - \tau_o)\, \frac{t^2}{2}, \\ \cdots\cdots\cdots \qquad & \cdots\cdots\cdots\cdots\cdots \\ k=n, \qquad & s^{(n+1)}(t) = s_o + \sum_{k=1}^{n} \Delta p^{k-1} \frac{(s_o \Delta p - \tau_o)}{k!} t^k \\ & \qquad = s_o + \left(s_o - \frac{\tau_o}{\Delta p}\right) \sum_{k=1}^{n} \frac{(\Delta p\, t)^k}{k!}. \end{aligned}$$

In the limit $n \to \infty$, we obtain

$$s(t) = \left(s_o - \frac{\tau_o}{\Delta p}\right) e^{\Delta p\, t} + \frac{\tau_o}{\Delta p},$$

i.e. the expected solution. In this case the total erosion time is

$$\hat{t} = \frac{1}{\Delta p} \ln \frac{1 - \tau_o/\Delta p}{s_o - \tau_o/\Delta p}.$$

Concerning the pressure gradient, from $(39)_2$ we have

$$\Pi = -\Delta p, \quad \Rightarrow \quad p(x) = p_{in} - \Delta p x.$$

In particular, condition $\Pi s + \tau_o < 0$ (necessary for erosion occurrence) is fulfilled when

$$\Delta p > \frac{\tau_o}{s_o}, \tag{68}$$

which, using dimensional quantities, can be rewritten as

$$\frac{\Delta p^*}{L^*} > \frac{\tau_o^*}{s_o^*}, \quad or \quad \varepsilon \Delta p^* > \frac{\tau_o^*}{s_o}.$$

Amazingly, condition (68) is analogous to the so–called "flow condition" for a pressure driven flow in a channel of a Bingham fluid characterized by an yield stress τ_o (see [5] for instance).

3.3. **Case III: $\gamma \neq 1$, $\tau_o = 0$, problem (37).** We start recalling that, by (46),

$$\frac{\mathcal{F}(\Gamma)}{\Gamma} = \frac{1}{1 - 3\displaystyle\int_0^t \Gamma(x, t')\, dt'}, \tag{69}$$

and we rewrite (47) in the form

$$\frac{\partial}{\partial x}\left(\Gamma(x,t)\ s_o(x)\right) = -3(1-\gamma)\frac{\Gamma(x,t)\ s_o(x)}{\left[1 - 3\displaystyle\int_0^t \Gamma(x,t')\,dt'\right]^{2/3}},$$

which is equivalent to

$$\Gamma(x,t)\, s_o(x) = \Gamma_o(t)\, s_o(0) \exp\left\{ -\frac{3(1-\gamma)}{\left[1 - 3\displaystyle\int_0^t \Gamma(x,t')\,dt'\right]^{2/3}} \right\}, \tag{70}$$

$\Gamma_o(t)$ representing, as before, the unknown value of $\Gamma(0,t)$.

Setting once more

$$\Xi(x,t) = \int_0^t \Gamma(x,t')\,dt', \quad \Leftrightarrow \quad \Gamma(x,t) = \frac{\partial \Xi(x,t)}{\partial t},$$

we rewrite (70) as

$$\frac{\partial \Xi}{\partial t} \exp\left\{ \frac{3(1-\gamma)}{(1-3\Xi)^{2/3}} \right\} = \Gamma_o(t)\frac{s_o(0)}{s_o(x)},$$

so that Ξ is found implicitly as

$$G(\Xi) = -\frac{s_o(0)}{s_o(x)} \int_0^t \Gamma_o(t')\,dt', \tag{71}$$

where

$$G(\Xi) = \int_\Xi^0 e^{\frac{3(1-\gamma)}{(1-3\eta)^{2/3}}} d\eta. \tag{72}$$

Therefore, in this case too we can say that $\Gamma(x,t)$ is uniquely determined by $\Gamma_o(t)$

$$\Gamma(x,t) = \Gamma_o(t)\frac{s_o(0)}{s_o(x)} \exp\left\{ -\frac{3(1-\gamma)}{(1-3\Xi)^{2/3}} \right\}, \tag{73}$$

where Ξ is the functional of Γ_o specified by (71), (72). With the help of (73) (which replaces (58) of section 3.2) we can use the same time discretization procedure illustrated in the previous case, which through (46) provides an approximation of $\Gamma_o(t)$. We will not repeat this analysis now, since the former one can be clearly adapted to the present case. An argument parallel to the one used in section 3.2 to obtain uniqueness, leads to the same result here too.

Example 3. As in example 2 we consider $s_o(x) = s_o$, $0 \leq x \leq 1$, and Δp constant in time. Next, we assume, as usual, $\Pi = \frac{\partial p^{(0)}}{\partial x} < 0$ and $|1-\gamma|$ "small", for instance

$$|1-\gamma| \approx 10^{-1} \div 10^{-2}.$$

Problem (41) is now

$$\begin{cases} \dfrac{\partial}{\partial x}\left(\dfrac{s^3}{3}\Pi\right) = -\left(1-\gamma\right)\Pi s\,, \\ \\ \Pi = -\dfrac{1}{s}\dfrac{\partial s}{\partial t}\,. \end{cases} \tag{74}$$

From the above system we derive

$$\begin{cases} s^2\dfrac{\partial s}{\partial x}\Pi + \dfrac{s^3}{3}\dfrac{\partial \Pi}{\partial x} = \left(1-\gamma\right)\dfrac{\partial s}{\partial t}\,, \\ \\ \dfrac{\partial \Pi}{\partial x} = -\dfrac{1}{s^2}\left(\dfrac{\partial^2 s}{\partial x \partial t}s - \dfrac{\partial s}{\partial t}\dfrac{\partial s}{\partial x}\right)\,, \end{cases}$$

yielding

$$\frac{\partial}{\partial x}\left(s^2\frac{\partial s}{\partial t}\right) = -3\left(1-\gamma\right)\frac{\partial s}{\partial t}. \tag{75}$$

Now we look for a "perturbative" solution of (75) considering:

- $s\left(x,t\right) = s_o e^{\Delta p t} + \left(1-\gamma\right)\,\sigma\left(x,t\right) + \mathcal{O}\left(\left(1-\gamma\right)^2\right)$, with $\sigma\left(x,0\right)=0$.
- $p\left(x,t\right) = p_{in} - \Delta p x + \left(1-\gamma\right)\,q\left(x,t\right) + \mathcal{O}\left(\left(1-\gamma\right)^2\right)$, with $q\left(0,t\right) = q\left(1,t\right) = 0$.

So, neglecting $\mathcal{O}\left(\left(1-\gamma\right)^2\right)$ terms, we have

$$\begin{cases} \dfrac{\partial s}{\partial t} = s_o \Delta p\, e^{\Delta p t} + \left(1-\gamma\right)\dfrac{\partial \sigma}{\partial t}, \\ \\ s^2 = s_o^2\, e^{2\Delta p t} + 2\left(1-\gamma\right)\, s_o \sigma\, e^{\Delta p t}, \end{cases}$$

and equation (75) becomes

$$\frac{\partial^2}{\partial x \partial t}\left(\sigma\, e^{2\Delta p t}\right) = -\frac{3}{s_o}\Delta p\, e^{\Delta p t},$$

which we read as

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}\left(\sigma e^{2\Delta p t}\right) + 3\frac{x}{s_o}\Delta p\, e^{\Delta p t}\right] = 0.$$

We thus get

$$\frac{\partial}{\partial t}\left(\sigma e^{2\Delta p t}\right) + 3\frac{x}{s_o}\Delta p\, e^{\Delta p t} = B\left(t\right), \tag{76}$$

where $B\left(t\right)$ is a function that will be determined later.
Equation (76) can be integrated with the initial condition $\sigma\left(x,0\right)=0$. We obtain

$$\sigma\left(x,t\right) = e^{-2\Delta p t}\left\{\frac{3x}{s_o}\left(1-e^{\Delta p t}\right) + \int_0^t B\left(t'\right)dt'\right\}. \tag{77}$$

Let us now go back to the pressure equation $(74)_2$. Neglecting again $\mathcal{O}\left(\left(1-\gamma\right)^2\right)$ terms, we have

$$\frac{\partial q}{\partial x} = \frac{e^{-\Delta p t}}{s_o}\left(\sigma \Delta p - \frac{\partial \sigma}{\partial t}\right), \tag{78}$$

that integrated w.r.t. x between 0 and 1, with the boundary conditions $q(0,t) = q(1,t) = 0$, yields

$$\int_0^1 \left(\sigma \Delta p - \frac{\partial \sigma}{\partial t}\right) dx = 0,$$

namely, using (77),

$$\frac{3\Delta p}{2s_o}\left(3 - 2e^{\Delta p t}\right) + 3\Delta p \int_0^t B(t')\, dt' - B(t) = 0, \tag{79}$$

and

$$B(0) = \frac{3\Delta p}{2s_o}.$$

Differentiating w.r.t. t equation (79) we obtain the Cauchy problem

$$\begin{cases} \dfrac{dB}{dt} - 3\Delta p B = -\dfrac{3\Delta p^2}{s_o} e^{\Delta p t} \\[2ex] B(0) = \dfrac{3\Delta p}{2s_o} \end{cases}$$

whose solution is

$$B(t) = \frac{3\Delta p}{2s_o} e^{\Delta p\, t}.$$

Hence, substituting the above expression into (77), we have

$$\sigma(x,t) = \frac{3}{s_o}\left(x - \frac{1}{2}\right)\left(e^{-2\Delta p t} - e^{-\Delta p t}\right),$$

and

$$s(x,t) = s_o e^{\Delta p\, t} + \frac{3(1-\gamma)}{s_o}\left(x - \frac{1}{2}\right)\left(e^{-2\Delta p t} - e^{-\Delta p t}\right). \tag{80}$$

Next, exploiting (78) we evaluate $q(x,t)$, obtaining

$$\begin{aligned} q(x,t) - \underbrace{q(0,t)}_{=0} &= \frac{e^{-\Delta p t}}{s_o}\int_0^x (\sigma \Delta p - \sigma_t)\, dx' \\ &= \frac{3\Delta p}{2s_o^2}\left(x^2 - x\right)\left(3e^{-3\Delta p t} - 2e^{-2\Delta p t}\right) \end{aligned}$$

and therefore

$$p(x,t) = p_{in} - \Delta p x + \frac{3(1-\gamma)\Delta p}{2s_o^2}\left(x^2 - x\right)\left(3e^{-3\Delta p t} - 2e^{-2\Delta p t}\right) \tag{81}$$

At this point the following comments are in order:

- The $\mathcal{O}(|1-\gamma|)$ corrections to the interface and pressure, i.e. $\sigma(x,t)$ and $q(x,t)$, decay exponentially in time. The maximum correction to s occurs for $t = \frac{\ln 2}{\Delta p}$, and equals $\frac{3(1-\gamma)}{4s_o}\left(x - \frac{1}{2}\right)$, independent of Δp.

- Recalling remark 1, we can use formula (40) for evaluating the macroscopic fluid velocity. We get

$$
\begin{aligned}
V_1^{f\,*} &= -\frac{K^*\left(\phi^f\right)}{\phi^f\,\mu^*}\frac{\partial P^*}{\partial x^*} \\
&= -\frac{K^*\left(\phi^f\right)}{\phi^f\,\mu^*}\Big[\frac{\Delta P^*}{L^*}\Big(-1+\frac{3\,(1-\gamma)}{\phi_o^f\,L^*}\left(x^*-\frac{L^*}{2}\right) \\
&\quad \cdot\left(3e^{-\frac{3\Delta P^*}{\mu^*}\,\varepsilon^2 t^*}-2e^{-\frac{2\Delta P^*}{\mu^*}\,\varepsilon^2 t^*}\right)\Big)\Big].
\end{aligned}
$$

It is worth to remark that the correction to $V_1^{f\,*}$ vanishes at $x^* = L^*/2$, and originates a flow due to the density difference. According to the sign of $(1-\gamma)$, the fluid tends to be pushed out from the channel (or sucked into the channel) because the fluid phase specific volume is different from than the one of the solid.

3.4. **Case IV: $\gamma \neq 1,\ \tau_o > 0$, problem (34).** Now we have the full equation (47). We start defining

$$
M\,(x,t) = \Gamma\,(x,t)\left\{s_o\,(x) - \tau_o\int_0^t\left[1-3\int_0^{t'}\Gamma\,(x,t'')\,dt''\right]^{-1/3}dt'\right\}^3, \tag{82}
$$

and we interpret (47) as

$$
\frac{\partial M}{\partial x} = -3\,(1-\gamma)\left[\tau_o + \mathcal{G}\,(\Gamma,t)\,M\right], \tag{83}
$$

where

$$
\mathcal{G}\,(\Gamma,t) = \left(\frac{\mathcal{F}\,(\Gamma)}{M}\right)^{2/3} = \frac{\omega^{2/3}}{\left(s_o - \tau_o\int_0^t \omega dt'\right)^2},
$$

and

$$
\omega\,(x,t) = \frac{1}{s_o - \tau_o\int_0^t \Gamma\,(x,t')\,dt'}.
$$

Therefore, we can integrate (83) with the condition

$$
M\,(0,t) = \Gamma_o\,(t)\left\{s_o\,(0) - \tau_o\int_0^t\left[1-3\int_0^{t'}\Gamma_o\,(t'')\,dt''\right]^{-1/3}dt'\right\}^3, \tag{84}
$$

obtaining

$$
\begin{aligned}
M\,(x,t) &= M\,(0,t)\exp\left\{-3\,(1-\gamma)\int_0^x \mathcal{G}\,(\Gamma,t)\,dx'\right\} \\
&\quad -3\,(1-\gamma)\,\tau_o\int_0^x \exp\left\{-3\,(1-\gamma)\int_{x'}^x \mathcal{G}\,(\Gamma,t)\,dx''\right\}dx'.
\end{aligned} \tag{85}
$$

Setting

$$
M_o\,(x) = M\,(x,0)\,,
$$

we observe that $M_o(x) = \Gamma(x,0)\, s_o^3(x)$ and we proceed to computing the function $M_o(x)$, noting that $\omega(x,0) = 1$, implying $\mathcal{G}(\Gamma,0) = \frac{1}{s_o^2(x)}$. Hence

$$\begin{aligned} M_o(x) &= \Gamma_o(0)\, s_o^3(0) \exp\left\{-3(1-\gamma)\int_0^x \frac{1}{s_o^2(x')}dx'\right\} \\ &\quad -3(1-\gamma)\tau_o \int_0^x \exp\left\{-3(1-\gamma)\int_{x'}^x \frac{1}{s_o^2(x'')}dx''\right\} dx'. \end{aligned}$$

The unknown constant $\Gamma_o(0)$ must be found imposing (48) and remembering that for $t=0$ we have $\mathcal{F}(\Gamma) = \Gamma(x,0) = M(x,0)/s_o^3(x)$, i.e.

$$\int_0^1 \frac{M_o(x)}{s_o^3(x)}dx = -\Delta p(0).$$

As we did for the previous cases, we look for an approximate solution at time $t=\epsilon$, introducing the functions $\mathfrak{m}_1(x)$ and $\mathfrak{n}_1(x)$, via

$$\begin{aligned} M_1(x) &= M(x,\epsilon) = M_o(x) + \epsilon\mathfrak{m}_1(x), \\ \Gamma(x,\epsilon) &= \Gamma(x,0) + \epsilon\mathfrak{n}_1(x). \end{aligned}$$

The two functions are related through (82), which yields (to the first order in ϵ)

$$\mathfrak{m}_1(x) = s_o^3(x)\,\mathfrak{n}_1(x) - 3\tau_o\Gamma(x,0)\, s_o^2(x). \tag{86}$$

To the same order, the approximation of $\mathcal{G}(\Gamma,\epsilon)$ is

$$\mathcal{G}(\Gamma,\epsilon) = \frac{1}{s_o^2}\left[1 + 2\epsilon\Gamma(x,0)\left(1+\frac{\tau_o}{s_o}\right) + \mathcal{O}\left(\epsilon^2\right)\right].$$

Thus (85) gives the approximation

$$\begin{aligned} \mathfrak{m}_1(x) &= -2M_o(0)\exp\left\{\int_0^x \frac{3(1-\gamma)}{s_o^2(x')}\Gamma(x',0)\left(1+\frac{\tau_o}{s_o(x')}\right)dx'\right\} \\ &\quad +\mathfrak{m}_1(0)\exp\left\{-3(1-\gamma)\int_0^x \frac{1}{s_o^2(x')}dx'\right\} + 18(1-\gamma)^2\tau_o \\ &\quad \cdot \int_0^x \exp\left\{\int_{x'}^x \frac{1}{s_o^2(x'')}\Gamma(x'',0)\left(1+\frac{\tau_o}{s_o(x'')}\right)dx''\right\}dx'. \end{aligned} \tag{87}$$

We remark that the only unknown left is $\mathfrak{n}_1(0)$ and that it is determined applying condition (48). Indeed, at time $t=\epsilon$, we have

$$\mathcal{F}(\Gamma) = \Gamma(x,0)\left[1 + \epsilon\left(\frac{\mathfrak{n}_1(x)}{\Gamma(x,0)} + 3\Gamma(x,0)\right) + \mathcal{O}\left(\epsilon^2\right)\right],$$

and, writing $\Delta p(\epsilon) = \Delta p(0) + \epsilon p_1 + \mathcal{O}\left(\epsilon^2\right)$, we obtain

$$\int_0^1 \left(\mathfrak{n}_1(x) + 3\Gamma^2(x,0)\right)dx = -p_1.$$

Thus, using (86) and (87) also for $t=0$, i.e.

$$\mathfrak{m}_1(0) = s_o^2(0)\left[s_o(0)\,\mathfrak{n}_1(0) - 3\tau_o\Gamma_o(0)\right],$$

we easily recover $\mathfrak{n}_1(0)$.

Finally, we can update the interface profile with the help of (52),

$$s(x,\epsilon) = s_1(x) = s_o(x)\left\{1 - \epsilon\left[\Gamma(x,0) + \frac{\tau_o}{s_o(x)}\right] + \mathcal{O}\left(\epsilon^2\right)\right\}.$$

The procedure can be iterated. We may use the same formulas in the second time step, just replacing $\Gamma(x,0)$, $s_o(x)$, etc., with the corresponding quantities.

The convergence argument goes like in the previous cases.

Uniqueness can be studied directly form the system (82), (83), (48), considering the differences ΔM, $\Delta\Gamma$, $\Delta\Gamma_o$ between two possible solutions. From (84), (85) we see that

$$\Delta M = \mathfrak{A}(x,t)\,\Delta\Gamma_o + \mathfrak{B}\left(\int_0^t \Delta\Gamma\,dt'\right), \tag{88}$$

where $\mathfrak{A}$ is a bounded function and $\mathfrak{B}$ a Lipschitz functional. Moreover, we are considering solutions such that $\mathfrak{A}$ is strictly positive. Now, coupling (88) with the similar relationship

$$\Delta M = \mathfrak{C}(x,t)\,\Delta\Gamma + \mathfrak{D}\left(\int_0^t \Delta\Gamma\,dt'\right),$$

obtained directly from (82), we can derive a Gronwall type inequality for $|\Delta\Gamma|$, concluding that it can be estimated as

$$\sup_{0<x<1,\ 0<t'<t} |\Delta\Gamma(x,t')| \leq C \sup_{0<t'<t} |\Delta\Gamma_o(t')|,$$

with C depending on the data only. Using the latter estimate in (48) we realize that $|\Delta\Gamma_o|$ satisfies a homogeneous Gronwall inequality, implying that it vanishes. Hence uniqueness is proved.

We finally observe that, in this case, remark 4 must be kept into account.

4. **Numerical simulations.** In this section we report the results of some numerical simulations concerning the case III, $\gamma \neq 1$, $\tau_o = 0$.

At first we have analyzed the accuracy of the "perturbative" approach illustrated in example 3. We have indeed considered $\gamma = 1.05$, i.e. $|1-\gamma| = 0.05$, $s_o(x)$ uniform, that is $s_o = 0.5$, and constant Δp in time, namely $\Delta p = 1$. In Fig. 2 we have reported the results of the simulations, namely $s(x)$ and $p(x)$ at various times. We note that at time $t = 0.6$, the interface has reached the value 1, that is the solid phase, close to the channel end has been totally eroded. According to (80), s acquires a dependence on x and p slightly deviates from the linear behavior in x.

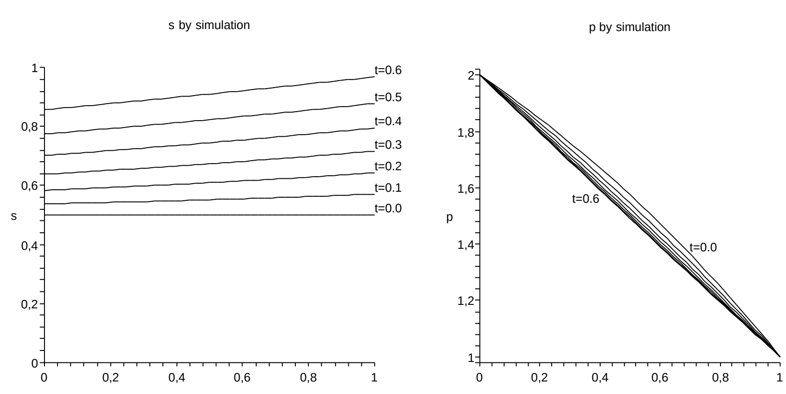

FIGURE 2. Simulations of $s(x,t)$ and $p(x,t)$ corresponding to $\gamma = 1.05$, $s_o(x) = 0.5$ and $\Delta p = 1$.

Fig. 3 shows the difference between the quantities plotted in Fig. 2 and the corresponding quantities evaluated according to the approximations (80) and (81), respectively. As we see, the difference between the simulated pressure and the one computed according to (81) does not exceed $0,3\%$. The error on $s(x,t)$ is also very small, not exceeding 2%. This fact proves what already observed concerning Fig. 2: the pressure linear behavior in x is scarcely affected by the perturbation. The latter has more influence on s, where the dependence on x is evident. We may conclude that the formulas (80) and (81) of example 3 provide rather good approximations.

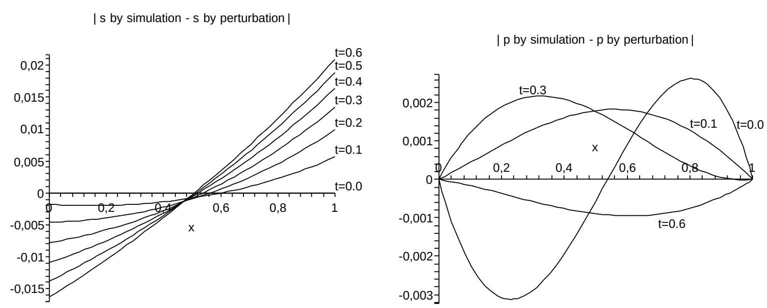

FIGURE 3. Difference between the simulated interface and pressure and the interface and pressure computed according to (80) and (81), respectively.

Next, we have considered the dynamics of the system when the initial interface is not uniform and Δp is constant in time, $\Delta p = 1$. More precisely we have considered oscillatory initial profiles

$$s_o(x) = 0.5\left(1 + 0.1\sin(30x)\right), \tag{89}$$

$$s_o(x) = 0.5\left(1 - 0.1\sin(30x)\right), \tag{90}$$

and $\gamma = 1.1$. The space and time behavior of the interface and pressure field are reported in Fig.s 4, 5. Note the inversion of the peaks of pressure in the two cases.

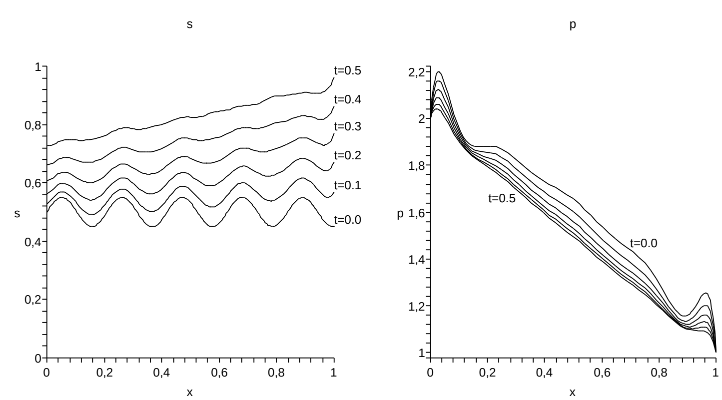

FIGURE 4. Interface and pressure when $s_o(x)$ is given by (89). $\gamma = 1.1$ and $\Delta p = 1$.

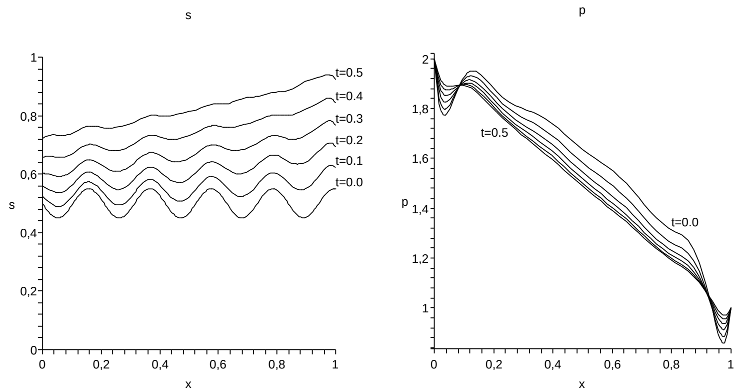

FIGURE 5. Interface and pressure when $s_o(x)$ is given by (90). $\gamma = 1.1$ and $\Delta p = 1$.

Finally we have analyzed the case corresponding to a non–uniform s_o and to a non–stationary pressure difference, considering s_o given by (89) and $\Delta p = 1 + 0.5\sin(12t)$. The results of the simulations are shown in Fig. 6.

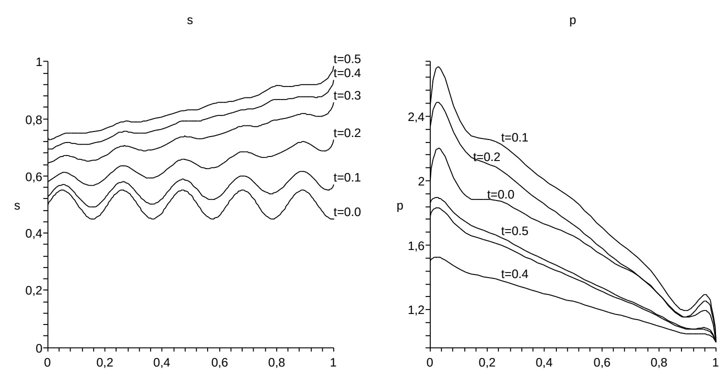

FIGURE 6. Interface and pressure when $s_o(x)$ is given by (90) and $\Delta p = 1 + 0.5\sin(12t)$. $\gamma = 1.1$.

5. **Conclusions.** We have examined the flow of a Newtonian incompressible fluid through a porous medium producing erosion. The solid matrix has been idealized as an array of identical lamellae, possessing a symmetry plane. Applying the upscaling procedure illustrated in [6], we have formulated a model for the macroscopic flow, emphasizing that the classical Darcy's law has not a simple extension.

Different macroscopic models can be found in literature. For instance, considering just the case $\gamma = 1$, according to [12] the macroscopic erosion rate, $\mathfrak{n}^*$, is proportional to the square of the discharge, i.e.

$$\mathfrak{n}^* = \alpha^* {Q_1^*}^2 = \alpha^* \left(\frac{K^*(\phi^f)}{\mu^*} \frac{\partial P^*}{\partial x^*} \right)^2,$$

while in our model (as e.g. in [1]) we arrive at the formula

$$\mathfrak{n}^* = \kappa^* \left(\left| -\frac{\partial P^*}{\partial x^*} \right| \phi^f - \frac{\tau_o^*}{h^*} \right)_+ . \tag{91}$$

We stress that, differently from other authors, we derive (91) as a macroscopic law by means of a rigorous upscaling procedure.

A limitation of our model, expressed in assumption A3, consists in the fact that, when $\gamma \neq 1$, the material removed from the solid matrix takes the physical properties of the liquid so that $\rho^{f\,*}$ and $\mu^{f\,*}$ remain unchanged (as it happens in phase change processes). More generally, one should consider the removed solid as a dispersed phase and write down a constitutive law for its flux and the corresponding mass balance. This issue will be the subject of a forthcoming paper.

From the mathematical point of view the model here studied gives rise to a very peculiar free boundary problem of non-local type. The model is characterized by the presence of two parameters: γ (ratio between the densities of the solid and of the liquid) and the stress threshold τ_o that has to be overcome by the interface stress for erosion to take place. We have distinguished four cases: (i) $\gamma = 1$, $\tau_o = 0$, (ii) $\gamma = 1$, $\tau_o > 0$, (iii) $\gamma \neq 1$, $\tau_o = 0$, (iv) $\gamma \neq 1$, $\tau_o > 0$ (the general case), presenting increasing difficulties. In each case we have adopted a suitable transformation reducing the integro-differential system to a more treatable form. Existence and uniqueness have been proved and particular examples have been discussed. In section 4 we have presented some numerical results.

Among the several problems still open, we mention the case in which erosion is switched on and off by fluctuations of the stress at the fluid–solid interface around the threshold value τ_o.

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