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EXISTENCE AND STABILITY OF A MULTIDIMENSIONAL SCALAR CONSERVATION LAW WITH DISCONTINUOUS FLUX

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Abstract. We prove the existence and stability of an entropy solution to a multidimensional scalar conservation law with discontinuous flux with no genuine nonlinearity assumptions. The proof is based on the corresponding kinetic formulation of the equation under consideration and a "smart" change of an unknown function.

1. **Introduction.** In the current contribution, we consider the following Cauchy problem:

$$\partial_t u + \mathrm{div}_x\, f(x,u) = 0, \quad u = u(t,x), \quad t \geq 0,\ x \in \mathbb{R}^d. \tag{1}$$

$$u|_{t=0} = u_0(x) \in L^1(\mathbb{R}^d), \quad a \leq u_0 \leq b. \tag{2}$$

Here, the flux vector $f(x,\lambda) = (f_1(x,\lambda),\dots,f_d(x,\lambda))$, $\lambda \in \mathbb{R}$, is assumed to be continuously differentiable with respect to $u \in \mathbb{R}$ and discontinuous with respect to $x \in \mathbb{R}^d$ so that, for every $\lambda \in \mathbb{R}$, the discontinuity is placed on the manifold $\Gamma \subset \mathbb{R}^d$ of co-dimension one which divides the space $\mathbb{R}^d$ into two domains.

More precisely, we assume that there exist two domains Ω_L and Ω_R such that:

$$\mathbb{R}^d = \Omega_L \cup \Gamma \cup \Omega_R, \quad \overline{\Omega_L} \cap \overline{\Omega_R} = \Gamma, \tag{3}$$

and that, by denoting

$$\kappa_L(x) = \begin{cases} 1, & x \in \Omega_L \\ 0, & x \notin \Omega_L \end{cases}, \quad \kappa_R(x) = \begin{cases} 1, & x \in \Omega_R \\ 0, & x \notin \Omega_R \end{cases},$$

we can rewrite (1) in the form:

$$\partial_t u + \mathrm{div}_x \left(g_L(x,u)\kappa_L(x) + g_R(x,u)\kappa_R(x)\right) = 0. \tag{4}$$

Furthermore, we assume that the functions $g_L, g_R \in C^1(\mathbb{R}^{d+1};\mathbb{R}^d)$ are of the form:

$$g_L(x,u) = (g_{1L}(\hat{x_1},u),\dots,g_{dL}(\hat{x_d},u)),$$
$$g_R(x,u) = (g_{1R}(\hat{x_1},u),\dots,g_{dR}(\hat{x_d},u)).$$

Remark 1. Here and in the sequel, by $A(\hat{x_i})$ we imply that the quantity A does not depend on x_i but only on $x_1,\dots,x_{i-1},x_{i+1},\dots,x_d$.

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The scalar conservation law with discontinuous flux has attracted a great deal of attention in recent years. It models different physical phenomena, for instance flow in porous media, sedimentation processes, traffic flow, radar shape-from-shading problems and blood flow. Nonetheless, almost all results have been restricted to the one dimensional case. The following incomplete list ranges over different admissibility concepts and methods for proving the existence and/or uniqueness of a weak solution to the one dimensional scalar conservation law [8, 17, 10, 11, 20, 1, 18, 19, 7, 9, 3, 2, 12, 15, 29]. Besides confinement to one dimension, in all the listed papers, some structural demands have been imposed on the flux (such as genuine nonlinearity, convexity, crossing condition) or on the form of a solution (such as piecewise smoothness).

On the other hand, there are incomparably fewer results concerning questions of existence and uniqueness for a multidimensional scalar conservation law with a discontinuous flux. In the two-dimensional case, the existence of a weak solution to the corresponding Cauchy problem is obtained in [22] by using the compensated compactness [32] under the genuine nonlinearity assumptions on the flux (see also [6]). Under the same assumptions, in [29] the existence is proved in the d-dimensional case, for arbitrary $d \in I\!\!N$. The basic tool was a modification of the H-measures [33, 16, 28]. Here, in a way, we generalize this result since we demand no genuine nonlinearity conditions.

Probably only results on the uniqueness of a certain class of solutions to the Cauchy problem (1), (2) can be found in [30] and a recent preprint [4].

In [30], the flux vector $f = (f_1, \dots, f_d)$ has a rather special form. Namely, it is assumed that $f = f(\beta(x,u)) = (f_1(\beta(x,u), \dots, f_d(\beta(x,u))$, where the function $\beta \in C^1(I\!\!R_u; L^1(I\!\!R^d))$ is increasing with respect to $u \in I\!\!R$ and is discontinuous with respect to $x \in I\!\!R^d$. Since the function β increases with respect to u, there exists a function $\alpha(x,v)$ such that:

$$\beta(x,u) = v \Rightarrow v = \alpha(x,u).$$

Thus, equation (1) can be rewritten as:

$$\begin{aligned} \partial_t \alpha(x,v) + \operatorname{div} f(v) &= 0, \\ v|_{t=0} &= \alpha(x, u_0(x)). \end{aligned}$$

Since the discontinuity in $x \in I\!\!R^d$ is removed from the derivative in x, we can apply the standard Kruzhkov theory to prove the uniqueness.

Roughly speaking, in [4], the authors consider solutions that can be obtained in the strong L^1_{loc} limit of the standard vanishing viscosity-smoothed flux approximation. Now, having two such solutions, u and v say, we can adjoint them the families (u_ε) and (v_ε), respectively, solving

$$\begin{aligned} \partial_t u_\varepsilon + \operatorname{div}_x f(x, u_\varepsilon) &= \varepsilon \Delta u_\varepsilon, \\ \partial_t v_\varepsilon + \operatorname{div}_x f(x, v_\varepsilon) &= \varepsilon \Delta v_\varepsilon, \\ u_\varepsilon|_{t=0} = u_0(x) \in L^1(I\!\!R^d), &\quad v_\varepsilon|_{t=0} = v_0(x) \in L^1(I\!\!R^d). \end{aligned}$$

Subtracting the latter equations and multiplying them by $\operatorname{sgn}(u_\varepsilon - v_\varepsilon)$ we get (avoiding standard technical moments):

$$\partial_t |u_\varepsilon - v_\varepsilon| + \operatorname{div}_x (f(x,u_\varepsilon) - f(x,v_\varepsilon)) \leq \varepsilon \Delta(|u_\varepsilon - v_\varepsilon|),$$

and, letting $\varepsilon \to 0$, we obtain:

$$\partial_t |u - v| + \operatorname{div}_x (f(x,u) - f(x,v)) \leq 0.$$

From here, stability follows according to standard arguments (e.g. [24]). We reiterate that this is only a rather rough presentation of [4]. The admissibility conditions introduced there are well justified and compared with previous works on the subject.

In the first part of the paper, we shall consider the case which we call the "special case". More precisely, we shall assume that, for every $i = 1, \dots, d$, there exist the functions $\alpha_i = \alpha(\hat{x_i}) \in C^1(I\!\!R^{d-1})$, $i = 1, \dots, d$, such that:

a) The flux $f = f(x, u)$, $x \in I\!\!R^d$, $u \in I\!\!R$, from (1) has a compact support with respect to $u \in I\!\!R$ which is a common and rather natural assumption and provides the maximum principle for the considered problem (see Remark 2; actually, this means that the form of the flux out of the range of initial data plays no role).

b) The discontinuity manifold is such that it holds for every $i = 1, \dots, d$:

$$\begin{aligned} &\Gamma = \{x \in I\!\!R^d :\ x_i = \alpha_i(\hat{x_i})\}, \quad \text{and} \\ &x \in \Omega_L \text{ if } x_i \leq \alpha_i(\hat{x_i}) \quad \text{and} \quad x \in \Omega_R \text{ if } x_i > \alpha_i(\hat{x_i}). \end{aligned} \tag{5}$$

Finally, we assume that we can rewrite equation (4) in the form:

$$\partial_t u + \sum_{i=1}^{d} \partial_{x_i} \left(g_{iL}(\hat{x_i}, u) H(\alpha_i(\hat{x_i}) - x_i) + g_{iR}(\hat{x_i}, u) H(x_i - \alpha_i(\hat{x_i})) \right) = 0, \tag{6}$$

where H is the Heaviside function and the functions $g_{iL}(\hat{x_i}, \lambda)$ and $g_{iR}(\hat{x_i}, \lambda)$ depend on $\lambda \in I\!\!R$ and the coordinates $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d$. Furthermore, we assume that they are non-negative, continuously differentiable with respect to all variables and that for every $i = 1, \dots, d$,

$$\text{supp} g_{iL}(\hat{x_i}, \cdot), \text{supp} g_{iR}(\hat{x_i}, \cdot) \subset (a, b) \subset I\!\!R, \tag{7}$$

independently on $x \in I\!\!R^d$.

Example 1. *We give two examples of equation* (6).

a) *Assume that we have a two-dimensional scalar conservation law of form* (6) *where the corresponding discontinuity manifold is the hyperplane* $x_2 = 0$. *More precisely, we assume that* $\Gamma = \{(x_1, x_2) \in I\!\!R^2 :\ x_2 = 0\}$.

Conservation law (6) *can be rewritten as:*

$$\partial_t u + \partial_{x_1} g_1(x_2, u) + \partial_{x_2} \left(g_{2L}(x_1, u) H(-x_2) + g_{1L}(x_1, u) H(x_2) \right) = 0,$$

i.e. here, $g_{1L}(x_2, u) = g_{1R}(x_2, u) = g_1(x_2, u)$, *since there is no discontinuity with respect to* x_1 *and we do not need the function* α_1, *while* $\alpha_2(\hat{x_2}) = \alpha_2(x_1) = 0$.

b) *We consider again the two-dimensional scalar conservation law of form* (6), *this time assuming that* $\Gamma = \{(x_1, x_2) \in I\!\!R^2 :\ x_2 + x_1 = 1\}$.

Equation (6) *can be rewritten as:*

$$\begin{aligned} \partial_t u &+ \partial_{x_1} \left(g_{1L}(x_2, u) H(x_2 + x_1) + g_{1R}(x_2, u) H(x_1 + x_2) \right) \\ &+ \partial_{x_2} \left(g_{2L}(x_1, u) H(x_1 + x_2) + g_{1L}(x_1, u) H(x_2 + x_1) \right) = 0, \end{aligned}$$

i.e. $\alpha_1(\hat{x_1}) = \alpha_1(x_2) = -x_2$ *and* $\alpha_2(\hat{x_2}) = \alpha_2(x_1) = -x_1$.

In the last part of the paper, we consider equation (4) in the "general case". It represents a generalization of the "special case" (see Example 1).

More precisely, without loss of generality, we assume that there exists a finite partition of the set $I\!\!R^d$ such that

$$Cl(\dot{\bigcup}_{j=1}^{n} \Omega_j) = I\!\!R^d,$$

where $\Omega_j \subset \mathbb{R}^d$ are domains in $\mathbb{R}^d$, Cl is the closure of a set, and $\dot{\bigcup}$ denotes the disjoint union. Furthermore, we assume that for every $j = 1, \dots, n$, there exist functions $\alpha_i^j = \alpha_i^j(\hat{x}_i) \in C^1(\mathbb{R}^{d-1})$, $i = 1, \dots, d$, such that:

$$\Gamma \cap \Omega_j = \{x \in \Omega_j : \ x_i = \alpha_i^j(\hat{x}_i)\}.$$

Also, we assume that:

$$codim(\overline{\Omega_p} \cap \overline{\Omega_q} \cap \Gamma) \geq 2, \quad p, q = 1, \dots, n.$$

Denote

$$\kappa_j(x) = \begin{cases} 1, & x \in \Omega_j \\ 0, & x \notin \Omega_j \end{cases},$$

i.e. κ_j is the characteristic function of the set Ω_j, $j = 1, \dots, n$.

According to the latter assumptions, we can rewrite equation (4) in the form:

$$\begin{aligned} \partial_t u + \sum_{i=1}^d \partial_{x_i} \Big(\sum_{j=1}^n \kappa_j(x) \Big(g_{iL}^j(\hat{x}_i, u) H(\alpha_i^j(\hat{x}_i) - x_i) \\ + g_{iR}^j(\hat{x}_i, u) H(x_i - \alpha_i^j(\hat{x}_i)) \Big) \Big) = 0, \end{aligned} \tag{8}$$

where g_{iL}^j and g_{iR}^j are nonnegative functions such that for every $j = 1, \dots, n$

$$\begin{aligned} & g_{iL}^j(\hat{x}_i, \cdot) = g_{iL}(\hat{x}_i, \cdot) \text{ and } g_{iR}^j(\hat{x}_i, \cdot) = g_{iR}(\hat{x}_i, \cdot), \quad i = 1, \dots, d, \ \textit{or} \\ & g_{iL}^j(\hat{x}_i, \cdot) = g_{iR}(\hat{x}_i, \cdot) \text{ and } g_{iR}^j(\hat{x}_i, \cdot) = g_{iL}(\hat{x}_i, \cdot), \quad i = 1, \dots, d. \end{aligned} \tag{9}$$

The discontinuity manifold in the first part of the following example is not admissible in the sense that it does not satisfy conditions (9). Still, as we shall see in the second part of the same example, there exist many admissible manifolds; in particular, any inadmissible manifold can be approximated by an admissible one. Furthermore, in the one-dimensional case, conditions (9) are always fulfilled making the current work a step forward with respect to previous contributions (since we do not have any constraint on the flux or initial data).

Example 2. a) *We shall give an example of a two-dimensional variant of equation* (8) *when the discontinuity manifold* Γ *is a unit circle.*

More precisely, we assume that we deal with the equation:

$$\begin{aligned} \partial_t u + \partial_{x_1} \big(g_{1L}(x_2, u) \kappa_{D(0,1)} + g_{1R}(x_2, u) \kappa_{D^C(0,1)} \big) \\ + \partial_{x_2} \big(g_{2L}(x_1, u) \kappa_{D(0,1)} + g_{2R}(x_1, u) \kappa_{D^C(0,1)} \big) = 0, \end{aligned}$$

where $D(0,1) \subset \mathbb{R}^2$ *is the unit disc centered at* $0 \in \mathbb{R}^2$ *and* $D^C(0,1)$ *is its complement.*

With the previous notation, we partition the space $\mathbb{R}^2$ *on four domains* Ω_i, $i = 1, 2, 3, 4$*:*

$$\begin{aligned} \Omega_1 &= \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 > 0, \ x_2 > 0\}, \\ \Omega_2 &= \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 < 0, \ x_2 > 0\}, \\ \Omega_3 &= \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 < 0, \ x_2 < 0\}, \\ \Omega_4 &= \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 > 0, \ x_2 < 0\}. \end{aligned}$$

The functions g^j_{iL} *and* g^j_{iR} *as well as* α^j_i, $i=1,2$, $j=1,2,3,4$, *are given by:*

$$j=1: \quad \begin{cases} g^1_{1L}(x_2,u)=g_{1L}(x_2,u),\ g^1_{1R}(x_2,u)=g_{1R}(x_2,u),\\ g^1_{2L}(x_1,u)=g_{2L}(x_1,u),\ g^1_{2R}(x_1,u)=g_{2R}(x_1,u),\\ \alpha^1_1(x_2)=\sqrt{1-x_1^2},\quad \alpha^1_2(x_1)=\sqrt{1-x_2^2}, \end{cases}$$

$$j=2: \quad \begin{cases} g^2_{1L}(x_2,u)=g_{1R}(x_2,u),\ g^2_{1R}(x_2,u)=g_{1L}(x_2,u),\\ g^2_{2L}(x_1,u)=g_{2L}(x_1,u),\ g^2_{2R}(x_1,u)=g_{2R}(x_1,u),\\ \alpha^2_1(x_2)=\sqrt{1-x_1^2},\quad \alpha^2_2(x_1)=-\sqrt{1-x_2^2}, \end{cases}$$

$$j=3: \quad \begin{cases} g^3_{1L}(x_2,u)=g_{1R}(x_2,u),\ g^3_{1R}(x_2,u)=g_{1L}(x_2,u),\\ g^3_{2L}(x_1,u)=g_{2R}(x_1,u),\ g^3_{2R}(x_1,u)=g_{2L}(x_1,u),\\ \alpha^1_1(x_2)=-\sqrt{1-x_1^2},\quad \alpha^1_2(x_1)=-\sqrt{1-x_2^2}, \end{cases}$$

$$j=4: \quad \begin{cases} g^4_{1L}(x_2,u)=g_{1L}(x_2,u),\ g^4_{1R}(x_2,u)=g_{1R}(x_2,u),\\ g^4_{2L}(x_1,u)=g_{2R}(x_1,u),\ g^4_{2R}(x_1,u)=g_{2L}(x_1,u),\\ \alpha^4_1(x_2)=-\sqrt{1-x_1^2},\quad \alpha^4_2(x_1)=\sqrt{1-x_2^2}. \end{cases}$$

It is clear that in the domains Ω_2 *and* Ω_4 *conditions* (9) *are not fulfilled.*

b) *Assume that the discontinuity manifold* Γ *has the form plotted in Figure 1. Clearly, this manifold satisfies conditions* (8), (9), *and it approximates the sphere from part* **a)**.

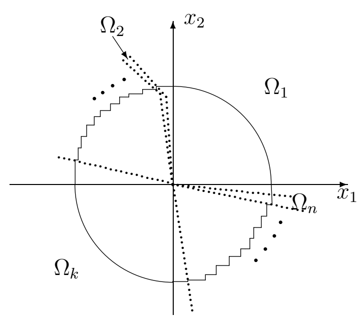

FIGURE 1. Discontinuity manifold which is admissible and approximates the sphere from Example 2, **a)**

The paper contains the following sections.

Section 1 is the Introduction where we formulate and explain the problems that are addressed.

In Section 2, we introduce several admissibility concepts. They are based on a combination of concepts used in [26] and [8]. We formulate the main theorem about existence and uniqueness of certain classes of our entropy solutions to the Cauchy problem (6), (2).

Section 3 is a collection of concepts and auxiliary results that are used in the rest of the paper.

Section 4 is the proof of the main theorem stating the existence and uniqueness of certain classes of entropy solutions to (6), (2).

Section 5 deals with the Cauchy problem (8), (2). We prove the existence and uniqueness of appropriate entropy admissible solutions to the latter Cauchy problem.

2. **Admissibility conditions.** First, we shall introduce admissibility conditions similar to the ones that we used in [26].

We need the following step function:

$$k(x)=\begin{cases} k_L, & x_i\leq \alpha_i(\hat{x_i})\\ k_R, & x_i>\alpha_i(\hat{x_i})\end{cases}, \quad k_L,k_R\in I\!R, \quad i=1,\dots,d. \tag{10}$$

Notice that, according to the assumptions on the discontinuity manifold Γ, the function k is well defined.

In the sequel, we denote as usual $I\!R^+=(0,\infty)$ and:

$$|z|^+=\begin{cases} z, & z>0\\ 0, & z\leq 0\end{cases}, \quad |z|^-=\begin{cases} 0, & z>0\\ -z, & z\leq 0\end{cases}$$

$$\operatorname{sgn}_\pm(z)=(|z|^\pm)'.$$

Definition 2.1. We say that the weak solution $u\in L^\infty([0,\infty)\times I\!R^d)$ to Cauchy problem (6), (2), is the k-entropy weak super(sub) solution if function $v(t,x)=u(t,x)-k(x)$ satisfies for every $\xi\in I\!R$:

$$\begin{aligned}\partial_t|v-\xi|^\pm+\sum_{i=1}^d\partial_{x_i}\operatorname{sgn}_\pm(v-\xi)\Big(\big(g_{iL}(\hat{x_i},v+k_L)-g_{iL}(\hat{x_i},\xi+k_L)\big)H(\alpha_i(\hat{x_i})-x_i)\\ +\left(g_{iR}(\hat{x_i},v+k_R)-g_{iR}(\hat{x_i},\xi+k_R)\right)H(x_i-\alpha_i(\hat{x_i}))\Big)\\ -\sum_{i=1}^d|g_{iL}(\hat{x_i},\xi+k_R)-g_{iR}(\hat{x_i},\xi+k_L)|^\pm\delta(x_i-\alpha_i(\hat{x_i}))\leq 0,\end{aligned}$$

where $\delta(x_i-\alpha_i(\hat{x_i}))$ is the Dirac δ distribution supported at $x_i=\alpha_i(\hat{x_i})$.

If the function u is the k-entropy weak super and sub solution at the same time then we call it the k-entropy weak solution.

Remark 2. Before we continue to analyze the existence and uniqueness of the admissible solution that has just been defined, we find it convenient to compare the k-entropy solution with its one-dimensional predecessor-entropy solutions given in [9, 20]. In [20] one of the first admissibility concepts analogous to Kruzhkov's entropy admissibility concept was given. The following definition was utilized there:

Definition 2.2. [9] Let u be a weak solution to the (one-dimensional variant) problem (1), (4) with discontinuity manifold $x=0$.

We say that u is an entropy-admissible weak solution to (1), (4) if the following entropy condition is satisfied for every fixed $\xi\in\mathbf{R}$:

$$\begin{aligned}\partial_t|u-\xi|+\partial_x\Big\{\operatorname{sgn}(u-\xi)\Big[H(x)(g_{1R}(u)-g_{1R}(\xi))+(1-H(x))(g_{1L}(u)-g_{1L}(\xi))\Big]\Big\}\\ -|g_{1R}(\xi)-g_{1L}(\xi)|\delta(x)\leq 0 \ \text{ in } \ \mathcal{D}'(I\!R^+\times I\!R).\end{aligned} \tag{11}$$

If the function k from the definition of the k-admissible weak solution is identically equal to zero, then the solution chosen by Definitions 2.1 and 2.2 are obviously the same. Still, using only conditions (11), it is not possible to prove the uniqueness of the chosen solution unless we assume additional technical conditions (so-called "crossing conditions"; see [9, Assumption 1.1]).

In [9], the "crossing conditions" were avoided by introducing so-called adapted entropy conditions. In order to introduce it, we need the function k from (4) (compare with the function c^{AB} given in [20, (11)]). In [20], the function k is used to form the function $u \mapsto |u - k(x)|$ which is an example of what is called an adapted entropy in [7]. Still, in [7], the existence of infinitely many adapted entropies was necessary to prove uniqueness (see also [30]) while in [20] only the entropy $u \mapsto |u - k(x)|$ was enough for uniqueness. The function k is called a connection if it represents a weak solution to (1), i.e. if $g_{iL}(k_L) = g_{iR}(k_R)$. The following admissibility conditions were used in [20]:

Definition 2.3. (Entropy solution of type (k_L, k_R)). Let k be a connection defined by (4). A measurable function $u : I\!\!R^+ \times I\!\!R \to I\!\!R$, representing a weak solution to (1), (2), is an entropy solution of type (k_L, k_R) of the (one-dimensional variant of the) initial value problem (1), (2) with discontinuity manifold $x = 0$, and assuming that $a = 0$, $b = 1$, if it satisfies the following conditions:

(D.1) $u \in L^\infty(I\!\!R^+ \times I\!\!R)$; $u(t,x) \in [0,1]$ for a.e. $(t,x) \in I\!\!R^+ \times I\!\!R$.

(D.2) For any test function $0 \leq \varphi \in \mathcal{D}([0,T) \times I\!\!R)$, $T > 0$, which vanishes for $x \geq 0$, the following holds:

$$\int_0^T \int_{I\!\!R} (|u-c|\varphi_t + sgn(u-c)(g_{1R}(u) - g_{1R}(c))\varphi_x)\, dxdt + \int_{I\!\!R} |u_0 - c|\varphi(0,x)dx \geq 0,$$

and for every $c \in I\!\!R$, any test function $0 \leq \varphi \in \mathcal{D}([0,T) \times I\!\!R)$, $T > 0$, which vanishes for $x \leq 0$

$$\int_0^T \int_{I\!\!R} (|u-c|\varphi_t + sgn(u-c)(g_{1L}(u) - g_{1L}(c))\varphi_x)\, dxdt + \int_{I\!\!R} |u_0 - c|\varphi(0,x)dx \geq 0,$$

(D.3) The following Kruzhkov-type entropy inequality holds for any test function $0 \leq \varphi \in \mathcal{D}([0,T) \times I\!\!R)$, $T > 0$,

$$\begin{aligned} &\int_0^T \int_{I\!\!R} \Big(|u-k(x)|\varphi_t \\ &\quad + sgn(u-k(x))(H(x)(g_{1R}(u) - g_{1R}(k_R)) + H(-x)(g_{1L}(u) - g_{1L}(k_L)))\varphi_x\Big) dxdt \\ &\quad + \int_{I\!\!R} |u_0 - k(x)|\varphi(0,x)dx \geq 0. \end{aligned}$$

In order to compare weak solutions selected by Definitions 2.3 and 2.1, recall that, in Definition 2.1, we put $v = u - k(x)$ for the function k given by (4). Therefore, assuming that $d = 1$ and that the discontinuity manifold is $x = 0$, the admissibility condition from Definition 2.1 becomes (after summing appropriate expressions with $|\cdot|^+$ and $|\cdot|^-$ entropies):

$$\begin{aligned} \partial_t |u - k(x) - \xi| + \partial_x \text{sgn}(u - k(x) - \xi)\Big(&\big(g_{1L}(u) - g_{iL}(\xi + k_L)\big)H(-x) \\ &+ \left(g_{1R}(u) - g_{1R}(\xi + k_R)\right) H(x)\Big) \\ &- |g_{1L}(\xi + k_R) - g_{1R}(\xi + k_L)|\delta(x) \leq 0, \end{aligned} \tag{12}$$

From here, it is not difficult to see that u is a k-admissible weak solution for the function k such that $g_{1L}(\cdot + k_L)$ and $g_{1R}(\cdot + k_R)$ have disjoint supports.

Indeed, it is clear that (12) implies (D.2) from Definition 2.3. In order to see that (D.1) from Definition 2.3 is satisfied, it is enough to notice that the functions $u = 0$ and $u = 1$ are k-admissible weak solutions for any k such that $g_{1L}(\xi + k_L)g_{1R}(\xi + k_R) = 0$ for any $\xi \in I\!\!R$ (i.e. $g_{1L}(\cdot + k_L)$ and $g_{1R}(\cdot + k_R)$ have disjoint supports).

More precisely, substituting $u=0$ into (12) (or, equivalently, $v=-k(x)$ into (2.1)) and bearing in mind that $g_{1L}(0)=g_{1R}(0)=0$, we obtain

$$\Big(\mathrm{sgn}(u-k_L-\xi)g_{1L}(\xi+k_L)-\mathrm{sgn}(u-k_R-\xi)g_{1R}(\xi+k_L)\\ -|g_{1L}(\xi+k_R)-g_{1R}(\xi+k_L)|\Big)\delta(x)\leq 0,$$

which is true since either $g_{1L}(\xi+k_L)=0$ or $g_{1R}(\xi+k_R)=0$. Similarly, the function $u=1$ is a k-admissible weak solution. From the L^1-stability of k-admissible solution (to be proved later), the maximum principle follows. Thus, $0\leq u_0\leq 1$ implies $0\leq u\leq 1$. Finally, a weak solution u satisfying Definition 2.1 satisfies (D.3) from Definition 2.3 which is obvious if we simply put $\xi=0$ in (12) and notice that $g_{1L}(k_L)=g_{1R}(k_R)=0$ (i.e. k is a connection).

On the other hand, it is clear that Definition 2.1 is in general more selective than Definition 2.3. This concludes the remark.

In order to prove existence and stability, we shall need arguments similar to those given in [8]. This includes the kinetic formulation of the conservation laws [27, 31] as well as notions of nonlinear weak-$\star$ convergence and entropy process sub and super solution.

Definition 2.4. Let Ω be an open subset of $I\!R^d$ and $(u_n)\subset L^\infty(\Omega)$ and $u\in L^\infty(\Omega\times(0,1))$. The sequence (u_n) converges towards u in the nonlinear weak-$\star$ sense if

$$\int_\Omega g(u_n(x))\psi(x)dx\to\int_0^1\int_\Omega g(u(x,\lambda)\psi(x)dxd\lambda \quad as \quad n\to\infty,\\ \forall\psi\in L^1(\Omega),\quad \forall g\in C(I\!R).$$

Any bounded sequence of $L^\infty(\Omega)$ has a subsequence converging in the nonlinear weak-$\star$ sense.

Theorem 2.5. *Let Ω be an open subset of $I\!R^d$ and (u_n) be a bounded sequence of $L^\infty(\Omega)$. Then (u_n) admits a subsequence converging in the nonlinear weak-$\star$ sense.*

This result is established in [14]. It is a modification of the Young measures concept [13] which is more convenient to work with since, instead of measures, we are dealing with L^∞ functions.

Now, referring to [8], we can introduce the notion of the weak entropy process sub- and super-solutions.

Definition 2.6. Let $u_0\in L^\infty(I\!R^d)$, $a\leq u_0\leq b$ a.e. on $I\!R^d$. Let $u\in L^\infty([0,\infty)\times I\!R^d\times(0,1))$.

1. The function u is a k-weak entropy process sub-solution (respectively k-weak entropy process super-solution) of problem (6), (2) if the function $v=v(t,x,\lambda)=$

$u(t,x,\lambda)-k(x)$ satisfies for any $\xi\in I\!R$ and any $\varphi\in C_0^1(I\!R\times I\!R^d)$:

$$\begin{aligned}&\int_0^1 d\lambda\int_{I\!R^+\times I\!R^d}(v-\xi)^\pm\partial_t\varphi dtdx \qquad (13)\\ &+\int_0^1 d\lambda\int_{I\!R^+\times I\!R^d}\sum_{i=1}^d \operatorname{sgn}_\pm(v-\xi)\times\\ &\quad\times\Big(\left(g_{iL}(\hat{x}_i,v+k_L)-g_{iL}(\hat{x}_i,\xi+k_L)\right)H(\alpha_i(\hat{x}_i)-x_i)\\ &\qquad+\left(g_{iR}(\hat{x}_i,v+k_R)-g_{iR}(\hat{x}_i,\xi+k_R)\right)H(x_i-\alpha_i(\hat{x}_i))\Big)\partial_{x_i}\varphi\\ &+\int_{I\!R^d}(u_0+k(x)-\xi)^\pm\varphi(0,x)dx\\ &-\sum_{i=1}^d\int_{I\!R^+\times I\!R^{d-1}}\left(g_{iL}(\hat{x}_i,\xi+k_L)-g_{iR}(\hat{x}_i,\xi+k_R)\right)^\pm\varphi|_{x_i=\alpha_i(\hat{x}_i)}d\hat{x}_i\geq 0.\end{aligned}$$

2. The function u is a k-weak entropy process solution if it is a weak k-entropy process sub- and super-solution at the same time.

It is not difficult to prove the existence of a k-weak entropy process solution to (6), (2) for any step function k from (10).

Theorem 2.7. *There exists a k-weak entropy process solution to* (6), (2) *for every step function k from* (10).

Proof. In order to construct the desired solution, we use a procedure similar to the one from [26].

First, we introduce the following change of the unknown function

$$u(t,x)=v(t,x)+k(x),\quad t\geq 0,\quad x\in I\!R^d,$$

for the function k from (10).

Equation (6) becomes:

$$\partial_t v+\sum_{i=1}^d\partial_{x_i}(g_{iL}(\hat{x}_i,v+k_L)H(\alpha_i(\hat{x}_i)-x_i)+g_{iR}(\hat{x}_i,v+k_R)H(x_i-\alpha_i(\hat{x}_i)))=0. \quad (14)$$

Then, we proceed as in [8]. Consider the sequence (v_ε) of Kruzhkov entropy admissible solutions to the following smoothed flux regularization to (14):

$$\begin{aligned}\partial_t v_\varepsilon+\sum_{i=1}^d\partial_{x_i}\Big(&g_{iL}(\hat{x}_i,v_\varepsilon+k_L)H_\varepsilon(\alpha_i(\hat{x}_i)-x_i)\\ &+g_{iR}(\hat{x}_i,v_\varepsilon+k_R)H_\varepsilon(x_i-\alpha_i(\hat{x}_i))\Big)=0,\end{aligned} \quad (15)$$

augmented with initial data (2). In the above, $H_\varepsilon(z)=\int_{-\infty}^{z/\varepsilon}\omega(p)dp$, where ω is a non-negative smooth even compactly supported function with total mass one. The function H_ε represents a regularization of the Heaviside function.

Next, notice that for A such that $A<a-\max\{a,b\}\leq u_0(x)-k(x)$, $g_{iL}(\hat{x}_i,A+k_L)=g_{iR}(\hat{x}_i,A+k_R)=0$, for all $i=1,\dots,d$ and every $x\in I\!R^d$. Similarly, for a B such that $B>b+\max\{a,b\}\geq u_0(x)-k(x)$, $g_{iL}(\hat{x}_i,B+k_L)=g_{iR}(\hat{x}_i,B+k_R)=0$, for all $i=1,\dots,d$ and every $x\in I\!R^d$. Therefore, the constants $A\leq u_0(x)-k(x)$ and $B\geq u_0(x)-k(x)$ represent the Kruzhkov entropy solutions to equation

(15). According to the maximum principle, we conclude that the Kruzhkov entropy solution v_ε to equation (15) with the initial condition $v_\varepsilon|_{t=0} = u_0(x) - k(x)$ satisfies

$$A \leq v_\varepsilon \leq B, \quad \varepsilon > 0.$$

Furthermore, since it is the Kruzhkov entropy solution, the function v_ε satisfies for any $\varphi \in C_0^1(\mathbb{R} \times \mathbb{R}^d)$:

$$\begin{aligned} &\int_{\mathbb{R}^+ \times \mathbb{R}^d} (v_\varepsilon(t,x) - \xi)^\pm \partial_t \varphi(t,x) dt dx + \int_{\mathbb{R}^+ \times \mathbb{R}^d} \sum_{i=1}^d \operatorname{sgn}_\pm (v_\varepsilon(t,x) - \xi) \times \\ &\quad \times \Big(\left(g_{iL}(\hat{x}_i, v_\varepsilon(t,x) + k_L) - g_{iL}(\hat{x}_i, \xi + k_L) \right) H_\varepsilon(\alpha_i(\hat{x}_i) - x_i) \\ &\qquad + \left(g_{iR}(\hat{x}_i, v_\varepsilon(t,x) + k_R) - g_{iR}(\hat{x}_i, \xi + k_R) \right) H_\varepsilon(x_i - \alpha_i(\hat{x}_i)) \Big) \partial_{x_i} \varphi(t,x) dt dx d\lambda \\ &\qquad - \int_{\mathbb{R}^d} (u_0(x) + k(x) - \xi)^\pm \varphi(0,x) dx \\ &\quad + \sum_{i=1}^d \int_{\mathbb{R}^+ \times \mathbb{R}^d} \operatorname{sgn}_\pm (v_\varepsilon(t,x) - \xi) \frac{1}{\varepsilon} \omega\Big(\frac{x_i - \alpha(\hat{x}_i)}{\varepsilon}\Big) \times \\ &\qquad\qquad \times \left(g_{iL}(\hat{x}_i, \xi + k_L) - g_{iR}(\hat{x}_i, \xi + k_R) \right) \varphi(t,x) dx dt \geq 0. \end{aligned} \tag{16}$$

Noticing that

$$-\operatorname{sgn}_\pm (v_\varepsilon - \xi) \left(g_{iL}(\hat{x}_i, \xi + k_L) - g_{iR}(\hat{x}_i, \xi + k_R) \right) \leq \left(g_{iL}(\hat{x}_i, \xi + k_L) - g_{iR}(\hat{x}_i, \xi + k_R) \right)^\mp$$

from (16), after letting $\varepsilon \to 0$ along a subsequence and taking Theorem 2.5 into account, we arrive at (13) for a nonlinear weak-$\star$ limit of subsequence $(v_{\varepsilon_n}) \subset (v_\varepsilon)$. □

We shall prove the following comparison principle which establishes the uniqueness and existence of certain classes of k-admissible weak solutions to (6), (2).

Theorem 2.8. *Assume that the step function k from (10) is such that there exists an interval $(c,d) \subset \mathbb{R}$ over which for every $x \in \mathbb{R}^d$:*

$$g_{iL}(\hat{x}_i, \xi + k_L) \equiv 0, \ \xi \geq c \ \text{ and } \ g_{iR}(\hat{x}_i, \xi + k_R) \equiv 0, \ \xi \leq d, \quad \forall i = 1, \dots, d,$$

or

$$g_{iR}(\hat{x}_i, \xi + k_L) \equiv 0, \ \xi \geq c \ \text{ and } \ g_{iL}(\hat{x}_i, \xi + k_R) \equiv 0, \ \xi \leq d, \quad \forall i = 1, \dots, d.$$

Then, for any two k-weak entropy process solutions u and v to (6) with initial conditions u_0 and v_0, respectively, the following holds for any $T > 0$ and any ball $B(0,R) \subset \mathbb{R}^d$:

$$\begin{aligned} \int_0^1 \int_0^1 d\lambda d\eta \int_0^T \int_{B(0,R)} &(u(t,x,\lambda) - v(t,x,\eta))^\pm dx dt \\ &\leq T \int_{B(0,R+CT)} (u_0(x) - v_0(x))^\pm dx, \end{aligned} \tag{17}$$

for a constant $C > 0$ independent of $T, R > 0$.

Remark 3. In the sequel, we assume that

$$\begin{aligned} g_{iL}(\hat{x}_i, \xi + k_R) &\equiv 0, \quad \xi \geq c \\ g_{iR}(\hat{x}_i, \xi + k_L) &\equiv 0, \quad \xi \leq d. \end{aligned} \tag{18}$$

The proof of Theorem 2.8 is based on a kinetic formulation of (6) introduced in the next subsection.

2.1. **Kinetic formulation.** For functions $u \in L^\infty(I\!R^+ \times I\!R^d \times (0,1))$, $u_0 \in L^\infty(I\!R^d; [a,b])$, and function k from (10), we denote:

$$h^k_\pm(t,x,\lambda,\xi) = \mathrm{sgn}_\pm(u(t,x,\lambda) + k(x) - \xi),$$
$$h^0_{\pm,k}(x,\xi) = \mathrm{sgn}_\pm(u_0(x) + k(x) - \xi).$$

The functions $h^k_\pm$, we call equilibrium functions.

Definition 2.9. Denote

$$G_{iL}(x,\xi) = \partial_\xi g_{iL}(x,\xi), \quad G_{iR}(x,\xi) = \partial_\xi g_{iR}(x,\xi).$$

Let $u_0 \in L^\infty(I\!R^d; [a,b])$ and $u \in L^\infty(I\!R^+ \times I\!R^d \times (0,1))$.

The function u is a k-kinetic process super-solution (respectively k-kinetic process sub-solution) to (6), (2) if, for function $v = u - k \in L^\infty(I\!R^+ \times I\!R^d \times (0,1))$, k given by (10), there exists $m_\pm \in C(I\!R_\xi; w-\star\mathcal{M}_+(I\!R^+ \times I\!R^d))$ such that $m_+(\cdot,\xi)$ vanishes for large ξ (respectively, $m_-(\cdot,\xi)$ vanishes for large $-\xi$), and such that for any $\varphi \in C^1(I\!R^+ \times I\!R^d \times (0,1))$,

$$\begin{aligned}
&\int_0^1 d\lambda \int_{t,x,\xi} h^k_\pm \times \qquad (19)\\
&\quad\times\Big(\partial_t + \sum_{i=1}^d \left(G_{iL}(\hat{x}_i,\xi+k_L)H(\alpha_i(\hat{x}_i)-x_i) + G_{iR}(\hat{x}_i,\xi+k_R)H(x_i-\alpha_i(\hat{x}_i))\right)\partial_{x_i}\Big)\varphi\\
&+\int_{x,\xi} h^0_{\pm,k}\varphi|_{t=0}dxd\xi + \int_{t,\hat{x}_i,\xi} \left(g_{iL}(\hat{x}_i,\xi+k_L) - g_{iR}(\hat{x}_i,\xi+k_R)\right)^\pm \partial_\xi\varphi|_{x_i=\alpha(\hat{x}_i)}d\hat{x}_i dt d\xi\\
&= \int_{t,x,\xi} \partial_\xi \varphi dm_\pm d\xi.
\end{aligned}$$

It can be proved by a simple modification of the procedure from [8] that the notions of k-weak entropy process solution and k-kinetic entropy process solution are equivalent. Still, for our purposes, it will be enough to prove that the k-weak entropy process solution is, at the same time, the k-kinetic process solution.

Proposition 1. *The k-weak entropy process admissible solution is at the same time the k-kinetic process solution.*

Proof. Take an arbitrary k-weak entropy process solution to (6), (2).

According to the Schwartz lemma for non-negative distributions, for every fixed $\xi \in I\!R$ there exist non-negative Radon measures $m_\pm(\cdot,\xi) \in \mathcal{M}_+(I\!R^+ \times I\!R^d)$ satisfying for every $\varphi \in C_0^1(I\!R^{d+2})$

$$\int_{t,x,\xi} \partial_\xi \varphi m_\pm d\xi = \int_0^1 d\lambda \int_{t,x,\xi} (v-\xi)^\pm \partial_t \partial_\xi \varphi$$
$$+ \int_0^1 \int_{t,x,\xi} \sum_{i=1}^d \mathrm{sgn}_\pm(v-\xi) \times$$
$$\times \Big((g_{iL}(\hat{x_i}, v+k_L) - g_{iL}(\hat{x_i}, \xi+k_L))\, H(\alpha_i(\hat{x_i}) - x_i)$$
$$+ (g_{iR}(\hat{x_i}, v+k_R) - g_{iR}(\hat{x_i}, \xi+k_R))\, H(x_i - \alpha_i(\hat{x_i}))\Big) \partial_{x_i} \partial_\xi \varphi$$
$$+ \int_x (u_0(x) + k(x) - \xi)^\pm \partial_\xi \varphi(0,x,\xi) dx d\xi$$
$$+ \sum_{i=1}^d \int_{t,\hat{x_i},\xi} (g_{iL}(\hat{x_i}, \xi+k_L) - g_{iR}(\hat{x_i}, \xi+k_R))^\pm \, \partial_\xi \varphi|_{x_i = \alpha_i(\hat{x_i})} d\hat{x_i} dt d\xi$$

Integrating the right-hand side of the previous expression by parts in $\xi \in I\!R$, we arrive at (19).

It is clear that the measures $m_\pm \in C(I\!R_\xi; w - \star\mathcal{M}_+(I\!R^+ \times I\!R^d))$ satisfy the conditions of the theorem. □

3. **Auxiliary results.** We will prove in this section some of the results and introduce some of the notions that we shall use in the proof of Theorem 2.8.

In the sequel, we shall denote by $h_\pm^k$ and $j_\pm^k$ equilibrium functions corresponding to k-weak entropy process solutions u and v to (6) with initial conditions $u_0 \in L^\infty(I\!R^d; (a,b))$ and $v_0 \in L^\infty(I\!R^d; (a,b))$, respectively.

We introduce the cut-off function

$$\omega_\varepsilon(s) = \int_0^{|s|} \rho_\varepsilon(r) dr, \quad \rho_\varepsilon(r) = \varepsilon^{-1} \rho(\varepsilon^{-1} r), \quad s \in I\!R^d, \ r \in I\!R, \tag{20}$$

where ρ is a compactly supported non-negative function with total mass one.

Let $\psi_L, \psi_R \in C^\infty(I\!R)$ be non-negative monotonic functions such that

$$\begin{cases} \psi_L(\xi) + \psi_R(\xi) \equiv 1, & \xi \in I\!R, \\ \psi_L(\xi) \equiv 0, & \xi \geq d, \\ \psi_R(\xi) \equiv 0, & \xi \leq c. \end{cases} \tag{21}$$

Next, take the functions:

$$I\!R^+ \times I\!R^d \times I\!R \ni (t,x,\xi) \mapsto \rho_{\varepsilon,\sigma,\zeta}(t,x,\xi) = \sum_{i=1}^d \rho^i_{\varepsilon,\sigma,\zeta}(t,x,\xi) = \sum_{i=1}^d \rho_\varepsilon(t) \rho_\zeta(\xi) \rho_\sigma(x_i),$$

$$I\!R^+ \times I\!R^d \ni (t,x) \mapsto \rho_{\varepsilon,\sigma}(t,x) = \sum_{i=1}^d \rho^i_{\varepsilon,\sigma}(t,x) = \sum_{i=1}^d \rho_\varepsilon(t) \rho_\sigma(x_i),$$

where ρ_ε is defined in (20), and let

$$j^k_{-,\varepsilon_j,\sigma_j,\zeta_j}(t,x,\xi,\eta) = j^k_- \star \rho_{\varepsilon_j,\sigma_j,\zeta_j}(t,x,\xi,\eta)$$
$$h^k_{+,\varepsilon_h,\sigma_h,\zeta_h}(t,x,\xi,\lambda) = h^k_+ \star \rho_{\varepsilon_h,\sigma_h,\zeta_h}(t,x,\xi,\lambda),$$

and

$$j^k_{-,\varepsilon_j,\sigma_j} = \lim_{\zeta_j\to 0} j^k_{-,\varepsilon_j,\sigma_j,\zeta_j} = j^k_- \star \rho_{\varepsilon_j,\sigma_j}$$
$$h^k_{+,\varepsilon_h,\sigma_h} = \lim_{\zeta_h\to 0} h^k_{+,\varepsilon_h,\sigma_h,\zeta_h} = h^k_+ \star \rho_{\varepsilon_h,\sigma_h},$$

where the limit is understood in the strong L^1_{loc} sense.

We shall need the following lemma:

Lemma 3.1. *Assume that for every $i = 1,\dots,d$*

$$g_{iL}(\hat{x_i},\xi+k_L) \geq g_{iR}(\hat{x_i},\xi+k_R), \quad x\in I\!R^d,\ \xi\leq p\in I\!R. \tag{22}$$

Then, for $\psi\in C^1((-\infty,p))$, $\theta\in C^1_0(I\!R^+\times I\!R^d\times I\!R)$:

$$\begin{aligned}\int_0^1 d\lambda\int_{t,x,\xi} h^k_-\psi\theta\sum_{i=1}^d\Big(&G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i)\\ &+G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\Big)\partial_{x_i}\omega_{1/n}(x_1-\alpha_1(\hat{x_1}))\\ =-&\int_{t,x,\xi}(1-\omega_{1/n}(x_1-\alpha_1(\hat{x_1})))\partial_\xi(\psi\theta)dm_-d\xi+o_n(1),\end{aligned} \tag{23}$$

where $o_n(1)\to 0$ as $n\to\infty$ and depends only on θ, $\partial_x\theta$ and $\partial_t\theta$.

Similarly, if for every $i=1,\dots,d$

$$g_{iL}(\hat{x_i},\xi+k_L) \leq g_{iR}(\hat{x_i},\xi+k_R), \quad x\in I\!R^d,\ \xi\geq p\in I\!R. \tag{24}$$

Then, for $\psi\in C^1((p,\infty))$, $\theta\in C^1_0(I\!R^+\times I\!R^d\times I\!R)$:

$$\begin{aligned}\int_0^1 d\lambda\int_{t,x,\xi} h^k_+\psi\theta\sum_{i=1}^d\Big(&G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i)\\ &+G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\Big)\partial_{x_i}\omega_{1/n}(x_1-\alpha_1(\hat{x_1}))\\ =-&\int_{t,x,\xi}(1-\omega_{1/n}(x_1-\alpha_1(\hat{x_1})))\partial_\xi(\psi\theta)dm_+d\xi+o_n(1),\end{aligned} \tag{25}$$

where $o_n(1)\to 0$ as $n\to\infty$ and depends only on θ, $\partial_x\theta$ and $\partial_t\theta$.

Proof. We will prove (23). Relation (25) is proved by analogy.

It is enough to choose in (19):

$$\varphi(t,x,\xi)=\theta(t,x,\xi)\psi(\xi)(1-\omega_{1/n}(x_1-\alpha_1(\hat{x_1})))$$

and to notice that $1-\omega_{1/n}(x_1-\alpha(\hat{x_1}))\to 0$ almost everywhere. We get:

$$\begin{aligned}&\int_0^1 d\lambda\int_{t,x,\xi} h^k_-\psi\theta\sum_{i=1}^d\Big(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i)\\ &\qquad\qquad+G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\Big)\partial_{x_i}(1-\omega_{1/n}(x_1-\alpha_1(\hat{x_1})))\\ &-\int_{t,\hat{x_i},\xi}(g_{iL}(\hat{x_i},\xi+k_L)-g_{iR}(\hat{x_i},\xi+k_R))^-\,\partial_\xi(\psi\theta|_{x_i=\alpha(\hat{x_i})})d\hat{x_i}dtd\xi\\ &=\int_{t,x,\xi}(1-\omega_{1/n}(x_1-\alpha_1(\hat{x_1})))\partial_\xi(\psi\theta)dm_-d\xi+o_n(1).\end{aligned}$$

Due to assumption (22), we conclude from here

$$\int_0^1 d\lambda \int_{t,x,\xi} h_-^k \psi\theta \sum_{i=1}^d \Big(G_{iL}(\hat{x}_i, \xi + k_L) H(\alpha_i(\hat{x}_i) - x_i) + G_{iR}(\hat{x}_i, \xi + k_R) H(x_i - \alpha_i(\hat{x}_i)) \Big) \partial_{x_i} \omega_{1/n}(x_1 - \alpha_1(\hat{x}_1))$$
$$= -\int_{t,x,\xi} (1 - \omega_{1/n}(x_1 - \alpha_1(\hat{x}_1))) \partial_\xi(\psi\theta) dm_- d\xi + o_n(1),$$

which we wanted to obtain. □

Remark 4. Notice that if we assume that $\psi \geq 0$ and $\theta \in C_0^1(I\!R^+ \times I\!R^d; L^\infty(I\!R))$ such that, in the sense of distributions, $\partial_\xi \theta \geq 0$, we can write instead (23):

$$\int_0^1 d\lambda \int_{t,x,\xi} h_-^k \psi\theta \sum_{i=1}^d \Big(G_{iL}(\hat{x}_i, \xi + k_L) H(\alpha_i(\hat{x}_i) - x_i) + G_{iR}(\hat{x}_i, \xi + k_R) H(x_i - \alpha_i(\hat{x}_i)) \Big) \partial_{x_i} \omega_{1/n}(x_1 - \alpha_1(\hat{x}_1))$$
$$\leq -\int_{t,x,\xi} (1 - \omega_{1/n}(x_1 - \alpha_1(\hat{x}_1))) \theta \partial_\xi \psi dm_- d\xi + o_n(1), \quad n \to \infty, \tag{26}$$

Similarly, instead of (25)

$$\int_0^1 d\lambda \int_{t,x,\xi} h_+^k \psi\theta \sum_{i=1}^d \Big(G_{iL}(\hat{x}_i, \xi + k_L) H(\alpha_i(\hat{x}_i) - x_i) + G_{iR}(\hat{x}_i, \xi + k_R) H(x_i - \alpha_i(\hat{x}_i)) \Big) \partial_{x_i} \omega_{1/n}(x_1 - \alpha_1(\hat{x}_1))$$
$$\leq -\int_{t,x,\xi} (1 - \omega_{1/n}(x_1 - \alpha_1(\hat{x}_1))) \theta \partial_\xi \psi dm_+ d\xi + o_n(1), \quad n \to \infty, \tag{27}$$

We shall also need the following known formula. It holds for a $\theta \in C_0^1(I\!R^+ \times I\!R^d)$:

$$\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h_+^k j_-^k) \Big(\partial_t \theta + \sum_{i=1}^d \left(G_{iL}(\hat{x}_i, \xi) H(\alpha_i(\hat{x}_i) - x_i) + G_{iR}(\hat{x}_i, \xi) H(x_i - \alpha_i(\hat{x}_i)) \right) \partial_{x_i} \theta \Big)$$
$$= \int_0^1 d\lambda \int_0^1 d\eta \int_{t,x} \Big(|u(t,x,\lambda) - v(t,x,\eta)|^+ \partial_t \theta + \sum_{i=1}^d \operatorname{sgn}_+(u(t,x,\lambda) - v(t,x,\eta)) \times$$
$$\times \Big(\big(g_{iL}(\hat{x}_i, u(t,x,\lambda)) - g_{iL}(\hat{x}_i, v(t,x,\eta)) \big) H(\alpha_i(\hat{x}_i) - x_i) + \big(g_{iR}(\hat{x}_i, u(t,x,\lambda)) - g_{iR}(\hat{x}_i, v(t,x,\eta)) \big) H(x_i - \alpha_i(\hat{x}_i)) \Big) \partial_{x_i} \theta \tag{28}$$

We finish this section with a lemma which makes a statement about the traces of the k-entropy process solutions along the line $t = 0$.

Lemma 3.2. *Assume that the bounded functions $u = u(t,x,\lambda)$ and $v = v(t,x,\eta)$ are two k-entropy process solutions to (6) corresponding to the initial condition $u_0 \in L^\infty(I\!R^d; [a,b])$ and $v_0 \in L^\infty(I\!R^d; [a,b])$, respectively.*

It holds for every $\varphi \in C_0^1(I\!R \times I\!R^d)$*:*

$$\lim_{n\to\infty}\int_0^1 d\lambda \int_0^1 d\eta \int_{I\!R^+\times I\!R^d} |u(t,x,\lambda)-v(t,x,\eta)|^{\pm}\omega'_{1/n}(t)\varphi(t,x)dtdx \leq \int_{I\!R^d} |u_0(x)-v_0(x)|^{\pm}\varphi(0,x)dtdx \tag{29}$$

Proof. By using the standard Kruzhkov doubling of variables method [24], it is not difficult to prove that for every $\varphi \in C_0^1(I\!R\times(I\!R^d\backslash\Gamma))$:

$$\begin{aligned}&\int_0^1 d\lambda\int_0^1 d\eta\int_{I\!R^+\times I\!R^d}(u(t,x,\lambda)-v(t,x,\eta))^{\pm}\partial_t\varphi dtdx\\ &+\int_0^1 d\lambda\int_0^1 d\eta\int_{I\!R^+\times I\!R^d}\sum_{i=1}^d \text{sgn}_{\pm}(u(t,x,\lambda)-v(t,x,\eta))\times\\ &\quad\times\Big((g_{iL}(\hat{x}_i,u(t,x,\lambda)+k_L)-g_{iL}(\hat{x}_i,v(t,x,\eta)+k_L))\,H(\alpha_i(\hat{x}_i)-x_i)\\ &\quad\quad+(g_{iR}(\hat{x}_i,u(t,x,\lambda)+k_R)-g_{iR}(\hat{x}_i,v(t,x,\eta)+k_R))\,H(x_i-\alpha_i(\hat{x}_i))\Big)\partial_{x_i}\varphi\\ &+\int_{I\!R^d}(u_0(x)-v_0(x))^{\pm}\varphi(0,x)dx\geq 0.\end{aligned}$$

Here, we put:

$$\varphi(t,x)=(1-\omega_{1/n}(t))\omega_\varepsilon(x_1-\alpha(\hat{x_1}))\theta(t,x),$$

where $\theta\in C_0^1(I\!R\times I\!R^d)$, and let $n\to\infty$. We get:

$$\begin{aligned}&-\lim_{n\to\infty}\int_0^1 d\lambda\int_0^1 d\eta\int_{I\!R^+\times I\!R^d}(u(t,x,\lambda)-v(t,x,\eta))^{\pm}\omega'_{1/n}(t)\theta(t,x)dt\omega_\varepsilon(x_1-\alpha_1(\hat{x_1}))dx\\ &+\int_{I\!R^d}(u_0(x)-v_0(x))^{\pm}\omega_\varepsilon(x_1-\alpha(\hat{x_1}))\varphi(0,x)dx\geq 0.\end{aligned}$$

Finally, letting $\varepsilon\to 0$, we arrive at (29). □

4. **Proof of Theorem 2.8.** The proof is based on the procedure from [8].

In the first step, choose the following test function

$$\varphi(t,x,\xi)=\theta\star\rho_{\varepsilon,\sigma,\zeta},$$

where $\text{supp}\theta\subset I\!R^+\times(I\!R^d\backslash\Gamma)\times I\!R$, in the place of the function φ from (19).

For ε,σ,ζ small enough, the following also holds:

$$\text{supp}\theta\star\rho_{\varepsilon,\sigma,\zeta}\subset I\!R^+\times(I\!R^d\backslash\Gamma)\times I\!R.$$

Therefore, for the equilibrium functions $h_\pm$, (19) becomes:

$$\begin{aligned}&\int_0^1 d\lambda\int_{t,x,\xi}h^k_\pm\star\rho_{\varepsilon_h,\sigma_h,\zeta_h}\partial_t\theta\\ &\quad+\sum_{i=1}^d\Big(h^k_\pm\big(G_{iL}(\hat{x}_i,\xi+k_L)H(\alpha_i(\hat{x}_i)-x_i)\\ &\quad\quad+G_{iR}(\hat{x}_i,\xi+k_R)H(x_i-\alpha_i(\hat{x}_i))\big)\Big)\star\rho^i_{\varepsilon_h,\sigma_h,\zeta_h}\partial_{x_i}\theta\\ &\quad=\int_{t,x,\xi}\partial_\xi\theta m_\pm^{\varepsilon_h,\sigma_h,\zeta_h}dtdx.\end{aligned}\tag{30}$$

where $m_\pm^{\varepsilon_h,\sigma_h,\zeta_h} = m_\pm \star \rho_{\varepsilon_h,\sigma_h,\zeta_h}$, while for the equilibrium functions $j_\pm$

$$\begin{aligned}\int_0^1 d\eta \int_{t,x,\xi} & j_\pm^k \star \rho_{\varepsilon_j,\sigma_j,\zeta_j} \partial_t \theta \\ & + \sum_{i=1}^d \Big(j_\pm^k(v,\xi)\big(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i}) - x_i) \\ & \qquad + G_{iR}(\hat{x_i},\xi+k_R)H(x_i - \alpha_i(\hat{x_i}))\big)\Big) \star \rho^i_{\varepsilon_j,\sigma_j,\zeta_j} \partial_{x_i}\theta \\ & = \int_{t,x,\xi} \partial_\xi \theta q_\pm^{\varepsilon_j,\sigma_j,\zeta_j} dt dx,\end{aligned} \tag{31}$$

where $q_\pm^{\varepsilon_j,\sigma_j,\zeta_j} = q_\pm \star \rho_{\varepsilon_j,\sigma_j,\zeta_j}$.

With the notation from Section 3, in (30) take instead of $\pm$ the sign $+$ and $\theta(t,x,\xi) = -\psi_L(\xi)\varphi(t,x) j^k_{-,\varepsilon_j,\sigma_j,\zeta_j}$ where φ disappears in the neighborhood of the discontinuity manifold Γ, and integrate over $\eta \in (0,1)$. Similarly, for the same function φ, in (31) take instead of $\pm$ the sign $-$ and $\theta(t,x,\xi) = -\psi_L(\xi)\varphi(t,x)h^k_{+,\varepsilon_h,\sigma_h,\zeta_h}$, and integrate over $\lambda \in (0,1)$.

By adding the resulting expressions, we obtain:

$$\begin{aligned}\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} & (-h^k_{+,\varepsilon_h,\sigma_h,\zeta_h} j^k_{-,\varepsilon_j,\sigma_j,\zeta_j}) \psi_L \Big(\partial_t \\ & + \sum_{i=1}^d (G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i}) - x_i) + G_{iR}(\hat{x_i},\xi+k_R)H(x_i - \alpha_i(\hat{x_i})))\, \partial_{x_i}\Big)\varphi \\ \geq & \int_0^1 d\eta \int_{t,x,\xi} \varphi \partial_\xi(-\psi_L j^k_{-,\varepsilon_j,\sigma_j,\zeta_j}) m_+^{\varepsilon_h,\sigma_h,\zeta_h} dt dx d\xi \\ & + \int_0^1 d\lambda \int_{t,x,\xi} \varphi \partial_\xi (-\psi_L h^k_{+,\varepsilon_h,\sigma_h,\zeta_h}) q_-^{\varepsilon_j,\sigma_j,\zeta_j} dt dx d\xi \\ & + R_{\varepsilon_h,\sigma_h,\zeta_h}(\varphi j^k_{-,\varepsilon_j,\sigma_j,\zeta_j}) + Q_{\varepsilon_j,\sigma_j,\zeta_j}(\varphi h^k_{+,\varepsilon_h,\sigma_h,\zeta_h}),\end{aligned} \tag{32}$$

where,

$$\begin{aligned}R_{\varepsilon_h,\sigma_h,\zeta_h}(\varphi) = & h^k_{+,\varepsilon_h,\sigma_h,\zeta_h} \sum_{i=1}^d \Big(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i}) - x_i) \\ & \qquad + G_{iR}(\hat{x_i},\xi+k_R)H(x_i - \alpha_i(\hat{x_i}))\Big)\partial_{x_i}\varphi \\ & - \sum_{i=1}^d \Big(h_+^k \Big(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i}) - x_i) \\ & \qquad + G_{iR}(\hat{x_i},\xi+k_R)H(x_i - \alpha_i(\hat{x_i}))\Big)\Big) \star \rho^i_{\varepsilon_j,\sigma_j,\zeta_j} \partial_{x_i}\varphi,\end{aligned}$$

$$Q_{\varepsilon_j,\sigma_j,\zeta_j}(\varphi) = j^k_{+,\varepsilon_h,\sigma_h,\zeta_h}\sum_{i=1}^d \Big(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i) + G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\Big)\partial_{x_i}\varphi - \Big(j^k_-\sum_{i=1}^d \Big(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i) + G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i}))\Big)\Big)\star\rho_{\varepsilon_j,\sigma_j,\zeta_j}\partial_{x_i}\varphi,$$

and, according to the Friedrichs lemma:

$$R_{\varepsilon_h,\sigma_h,\zeta_h}(\varphi j^k_{-,\varepsilon_j,\sigma_j,\zeta_j}) = \mathcal{O}(\frac{\zeta_j}{\varepsilon_h\sigma_h}),\quad Q_{\varepsilon_j,\sigma_j,\zeta_j}(\varphi h^k_{+,\varepsilon_h,\sigma_h,\zeta_h}) = \mathcal{O}(\frac{\zeta_h}{\varepsilon_j\sigma_j}).$$

Finding the derivative in ξ on the right-hand of (32), and bearing in mind that $\partial_\xi(-j^k_{-,\varepsilon_j,\sigma_j,\zeta_j})>0$ and $\partial_\xi(-h^k_{+,\varepsilon_h,\sigma_h,\zeta_h})>0$, we conclude from (32) after letting $\zeta_h,\zeta_j\to 0$:

$$\int_0^1 d\lambda\int_0^1 d\eta\int_{t,x,\xi}(-h^k_{+,\varepsilon_h,\sigma_h}j^k_{-,\varepsilon_j,\sigma_j})\psi_L\Big(\partial_t + \sum_{i=1}^d(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i)+G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i})))\,\partial_{x_i}\Big)\varphi \geq -\int_0^1 d\eta\int_{t,x,\xi}\varphi j^k_{-,\varepsilon_j,\sigma_j}\partial_\xi\psi_L m_+^{\varepsilon_h,\sigma_h}dtdxd\xi - \int_0^1 d\lambda\int_{t,x,\xi}\varphi h^k_{+,\varepsilon_h,\sigma_h}\partial_\xi\psi_L q_-^{\varepsilon_j,\sigma_j}dtdxd\xi \tag{33}$$

Next, notice that, according to (18), the following holds:

$$g_{iL}(\hat{x_i},\xi+k_L)\geq g_{iR}(\hat{x_i},\xi+k_R)=0,\quad x\in I\!R^d,\quad \xi\in\operatorname{supp}\psi_L. \tag{34}$$

Now, in (33), we let $\varepsilon_j,\sigma_j\to 0$:

$$\int_0^1 d\lambda\int_0^1 d\eta\int_{t,x,\xi}(-h^k_{+,\varepsilon_h,\sigma_h}j^k_-)\psi_L\Big(\partial_t + \sum_{i=1}^d(G_{iL}(\hat{x_i},\xi+k_L)H(\alpha_i(\hat{x_i})-x_i)+G_{iR}(\hat{x_i},\xi+k_R)H(x_i-\alpha_i(\hat{x_i})))\,\partial_{x_i}\Big)\varphi \geq -\int_0^1 d\eta\int_{t,x,\xi}\varphi j^k_-\partial_\xi\psi_L m_+^{\varepsilon_h,\sigma_h}dtdxd\xi - \int_0^1 d\lambda\int_{t,x,\xi}\varphi h^k_{+,\varepsilon_h,\sigma_h}\partial_\xi\psi_L dq_- d\xi \tag{35}$$

Let us now remove the conditions imposed on the support of function φ. In (35), put:

$$\varphi(t,x)=\theta(t,x)\omega_{1/n}(x_1-\alpha_1(\hat{x_1})),$$

for an arbitrary function $\theta \in C_0^1(I\!R^+ \times I\!R^d)$. We get:

$$\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h^k_{+,\varepsilon_h,\sigma_h} j^k_-)\psi_L \omega_{1/n}(x_1 - \alpha_1(\hat{x_1}))\times \tag{36}$$
$$\times \Big(\partial_t + \sum_{i=1}^d \left(G_{iL}(\hat{x_i}, \xi + k_L)H(\alpha_i(\hat{x_i}) - x_i) + G_{iR}(\hat{x_i}, \xi + k_R)H(x_i - \alpha_i(\hat{x_i}))\right)\partial_{x_i}\Big)\theta$$
$$+ \int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h^k_{+,\varepsilon_h,\sigma_h} j^k_-)\psi_L \theta \Big(\partial_t + \sum_{i=1}^d \big(G_{iL}(\hat{x_i}, \xi + k_L)H(\alpha_i(\hat{x_i}) - x_i)$$
$$+ G_{iR}(\hat{x_i}, \xi + k_R)H(x_i - \alpha_i(\hat{x_i}))\big)\partial_{x_i}\Big)\omega_{1/n}(x_1 - \alpha_1(\hat{x_1}))$$
$$\geq - \int_0^1 d\eta \int_{t,x,\xi} \theta j^k_- \partial_\xi \psi_L \omega_{1/n}(x_1 - \alpha_1(\hat{x_1})) m_+^{\varepsilon_h,\sigma_h} dtdxd\xi$$
$$- \int_0^1 d\lambda \int_{t,x,\xi} \theta h^k_{+,\varepsilon_h,\sigma_h} \omega_{1/n}(x_1 - \alpha_1(\hat{x_1}))\partial_\xi \psi_L dq_- d\xi.$$

According to Remark 4, more precisely (26), we conclude from (36) (since $\partial_\xi(-h^k_{+,\varepsilon_h,\sigma_h}) \geq 0$ (in $\mathcal{D}'$) and $g_{iL}(\hat{x_i}, \xi + k_L) \geq g_{iR}(\hat{x_i}, \xi + k_R)$ for every $i = 1, \dots, d$, $x \in I\!R^d$, $\xi \in \text{supp}\psi_L$):

$$\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h^k_{+,\varepsilon_h,\sigma_h} j^k_-)\psi_L \omega_{1/n}(x_1 - \alpha_1(\hat{x_1}))\times \tag{37}$$
$$\times \Big(\partial_t\theta + \sum_{i=1}^d \Big(G_{iL}(\hat{x_i}, \xi + k_L)H(\alpha_i(\hat{x_i}) - x_i)$$
$$+ G_{iR}(\hat{x_i}, \xi + k_R)H(x_i - \alpha_i(\hat{x_i}))\Big)\partial_{x_i}\theta\Big)$$
$$- \int_{t,x,\xi} (1 - \omega_{1/n}(x_1 - \alpha_1(\hat{x_1})))\theta(-h^k_{+,\varepsilon_h,\sigma_h})\partial_\xi \psi_L dq_- d\xi + o_n(1)$$
$$\geq - \int_0^1 d\eta \int_{t,x,\xi} \theta j^k_- \partial_\xi \psi_L \omega_{1/n}(x_1 - \alpha_1(\hat{x_1})) m_+^{\varepsilon_h,\sigma_h} dtdxd\xi$$
$$- \int_0^1 d\lambda \int_{t,x,\xi} \theta h^k_{+,\varepsilon_h,\sigma_h} \omega_{1/n}(x_1 - \alpha_1(\hat{x_1}))\partial_\xi \psi_L dq_- d\xi.$$

From here, letting $n \to \infty$, we obtain:

$$\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h^k_{+,\varepsilon_h,\sigma_h} j^k_-)\psi_L \Big(\partial_t\theta \tag{38}$$
$$+ \sum_{i=1}^d \left(G_{iL}(\hat{x_i}, \xi + k_L)H(\alpha_i(\hat{x_i}) - x_i) + G_{iR}(\hat{x_i}, \xi + k_R)H(x_i - \alpha_i(\hat{x_i}))\right)\partial_{x_i}\theta\Big)$$
$$\geq - \int_0^1 d\eta \int_{t,x,\xi} \theta j^k_- \partial_\xi \psi_L m_+^{\varepsilon_h,\sigma_h} dtdxd\xi - \int_0^1 d\lambda \int_{t,x,\xi} \theta h^k_{+,\varepsilon_h,\sigma_h} \partial_\xi \psi_L dq_- d\xi.$$

Now, in (30), take instead of $\pm$ the sign $+$ and $-\psi_R(\xi)\varphi(t,x) j^k_{-,\varepsilon_j,\sigma_j,\zeta_j}$ in place of the test function, where $\varphi \in C^1(I\!R^+ \times I\!R^d)$ disappears in the neighborhood of the discontinuity manifold Γ, and integrate over $\eta \in (0,1)$. Similarly, for the same function φ, in (31), take instead of $\pm$ the sign $-$ and $-\psi_R(\xi)\varphi(t,x) h^k_{+,\varepsilon_h,\sigma_h,\zeta_h}$ in place of the test function, and integrate over $\lambda \in (0,1)$.

Applying the same procedure as for the function ψ_L, we get for an arbitrary $\theta \in C_0^1(\mathbb{R}^+ \times \mathbb{R}^d)$:

$$\begin{aligned}&\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h_+^k j^k_{-,\varepsilon_j,\sigma_j})\psi_R\Big(\partial_t\theta \\ &\qquad + \sum_{i=1}^d \left(G_{iL}(\hat{x}_i,\xi+k_L)H(\alpha_i(\hat{x}_i)-x_i)+G_{iR}(\hat{x}_i,\xi+k_R)H(x_i-\alpha_i(\hat{x}_i))\right)\partial_{x_i}\theta\Big) \\ &\geq -\int_0^1 d\eta \int_{t,x,\xi} \theta j^k_{-,\varepsilon_j,\sigma_j}\partial_\xi\psi_R dm_+ d\xi d\xi - \int_0^1 d\lambda \int_{t,x,\xi} \theta h_+^k \partial_\xi \psi_R q_{-,\varepsilon_j,\sigma_j} dt dx d\xi.\end{aligned} \tag{39}$$

Now, we put $\varepsilon = \varepsilon_j = \varepsilon_h$ and $\sigma = \sigma_j = \sigma_h$ in (38) and (39), and add the resulting expressions.

$$\begin{aligned}&\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h_+^k j^k_{-,\varepsilon,\sigma})\psi_R\Big(\partial_t\theta \\ &\qquad + \sum_{i=1}^d \left(G_{iL}(\hat{x}_i,\xi+k_L)H(\alpha_i(\hat{x}_i)-x_i)+G_{iR}(\hat{x}_i,\xi+k_R)H(x_i-\alpha_i(\hat{x}_i))\right)\partial_{x_i}\theta\Big) \\ &+\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x,\xi} (-h^k_{+,\varepsilon,\sigma} j^k_-)\Big(\partial_t\theta \\ &\qquad + \sum_{i=1}^d \left(G_{iL}(\hat{x}_i,\xi+k_L)H(\alpha_i(\hat{x}_i)-x_i)+G_{iR}(\hat{x}_i,\xi+k_R)H(x_i-\alpha_i(\hat{x}_i))\right)\partial_{x_i}\theta\Big) \\ &\geq -\int_0^1 d\eta \int_{t,x,\xi} \theta j^k_- \partial_\xi\psi_L m_+^{\varepsilon,\sigma} dt dx d\xi - \int_0^1 d\lambda \int_{t,x,\xi} \theta h^k_{+,\varepsilon,\sigma}\partial_\xi\psi_L dq_- d\xi \\ &\quad -\int_0^1 d\eta \int_{t,x,\xi} \theta j^k_{-,\varepsilon_j,\sigma_j}\partial_\xi\psi_R dm_+ d\xi d\xi - \int_0^1 d\lambda \int_{t,x,\xi} \theta h_+^k \partial_\xi\psi_R q_{-,\varepsilon,\sigma} dt dx d\xi.\end{aligned} \tag{40}$$

Next, notice that

$$\begin{aligned}&\int_0^1 d\eta \int_{t,x,\xi} \partial_\xi\psi_L \theta j_-^k m_+^{\varepsilon,\sigma} dt dx d\xi + \int_0^1 d\eta \int_{t,x,\xi} \partial_\xi\psi_R \theta j^k_{-,\varepsilon,\sigma} dm_+ d\xi \\ &= \int_0^1 d\eta \int_{t,x,\xi} \partial_\xi \psi_L \left((\theta j_-^k)\star\rho_{\varepsilon,\sigma} - \theta j^k_{-,\varepsilon,\sigma}\right) dm_+ d\xi \to 0 \ \text{ as } \ \varepsilon,\sigma\to 0,\end{aligned} \tag{41}$$

according to (21) and since

$$\|(\theta j_-^k)\star\rho_{\varepsilon,\sigma} - \theta j^k_{-,\varepsilon,\sigma}\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^d)} = \mathcal{O}(\varepsilon+\sigma).$$

Similarly,

$$\int_0^1 d\lambda \int_{t,x,\xi} \partial_\xi\psi_L \theta h_+^k q_-^{\varepsilon,\sigma} dt dx d\xi + \int_0^1 d\lambda \int_{t,x,\xi} \partial_\xi \psi_R \theta h^k_{+,\varepsilon,\sigma}\partial_\xi dq_- d\xi \tag{42}$$

$$= \int_0^1 d\lambda \int_{t,x,\xi} \partial_\xi \psi_L \left((\theta h_+^k) \star \rho_{\varepsilon,\sigma} - \theta h_{+,\varepsilon,\sigma}^k\right) dq_- d\xi \to 0 \text{ as } \varepsilon, \sigma \to 0.$$

Finally, we conclude from (41) and (42) after letting $\varepsilon, \sigma \to 0$:

$$\int_0^1 d\eta \int_0^1 d\lambda \int_{t,x,\xi} (-h_+^k j_-^k) \Big(\partial_t \theta + \sum_{i=1}^d (G_{iL}(\hat{x}_i, \xi + k_L) H(\alpha_i(\hat{x}_i) - x_i) + G_{iR}(\hat{x}_i, \xi + k_R) H(x_i - \alpha_i(\hat{x}_i)))\, \partial_{x_i} \theta \Big) \geq 0,$$

and from here, appealing to (28), we conclude:

$$\int_0^1 d\lambda \int_0^1 d\eta \int_{t,x} \Big(|u(t,x,\lambda) - v(t,x,\eta)|^+ \partial_t \theta + \sum_{i=1}^d \mathrm{sgn}_+ (u(t,x,\lambda) - v(t,x,\eta)) \times \times \Big(g_{iL}(\hat{x}_i, u(t,x,\lambda)) - g_{iL}(\hat{x}_i, v(t,x,\eta)) \Big) H(\alpha_i(\hat{x}_i) - x_i) + \big(g_{iR}(\hat{x}_i, u(t,x,\lambda)) - g_{iR}(\hat{x}_i, v(t,x,\eta)) \big) H(x_i - \alpha_i(\hat{x}_i)) \Big) \partial_{x_i} \theta \geq 0.$$

From here, using the standard procedure (e.g. [24]) and (29), we arrive at (17). This completes the proof.

A simple corollary of Theorem 2.7 and Theorem 2.8 is (see e.g. [8, Page 377]):

Corollary 1. *Assume that the function k from (10) is such that (18) is satisfied. Then, there exists a unique k-entropy weak solution to (6), (2).*

5. **General case.** In this section, we are concerned with the Cauchy problem (8), (2). We shall mainly rely on the results of the previous section. Definitions and concepts are a little more involved, but they are basically the same as in the case of equation (6), (2). Accordingly, introduce the function

$$k(x) = \begin{cases} k_L, & x \in \Omega_L \\ k_R, & x \in \Omega_R \end{cases}, \quad k_L, k_R \in \mathbb{R}. \tag{43}$$

By k_j, we denote a restriction of the function k on the set Ω_j, $j = 1, \dots, n$. We write $\mathbb{R} \ni k_L^j = k(x)$ for $x \in \Omega_j$ such that $x_1 \leq \alpha_1^j(\hat{x}_1)$, and we write $\mathbb{R} \ni k_R^j = k(x)$ for $x \in \Omega_j$ such that $x_1 > \alpha_1^j(\hat{x}_1)$. Notice that instead of $i = 1$, here we could put an arbitrary $i = 1, \dots, d$.

Definition 5.1. Let $u_0 \in L^\infty(\mathbb{R}^d)$, $a \leq u_0 \leq b$ a.e. on $\mathbb{R}^d$. Let $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d \times (0,1))$.

1. The function u is a k-weak entropy process sub-solution (respectively k-weak entropy process super-solution) of problem (8), (2) if, for the function $v = v(t,x,\lambda) = u(t,x,\lambda) - k(x)$, k given by (43), every $\xi \in \mathbb{R}$ and:

a) For every fixed $j = 1, \dots, n$, the following holds for every $\varphi \in C_0^1(I\!R \times \Omega_j)$:

$$\int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} (v-\xi)^{\pm} \partial_t \varphi dt dx \tag{44}$$
$$+ \int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} \sum_{i=1}^d \mathrm{sgn}_{\pm}(v-\xi) \times$$
$$\times \Big(\Big(g^j_{iL}(\hat{x}_i, v + k^j_L) - g^j_{iL}(\hat{x}_i, \xi + k^j_L) \Big) H(\alpha^j_i(\hat{x}_i) - x_i)$$
$$+ \Big(g^j_{iR}(\hat{x}_i, v + k^j_R) - g^j_{iR}(\hat{x}_i, \xi + k^j_R) \Big) H(x_i - \alpha^j_i(\hat{x}_i)) \Big) \partial_{x_i} \varphi$$
$$+ \int_{I\!R^d} (u_0 + k(x) - \xi)^{\pm} \varphi(0,x) dx$$
$$- \sum_{i=1}^d \int_{I\!R^+ \times I\!R^{d-1}} \Big(g^j_{iL}(\hat{x}_i, \xi + k^j_L) - g^j_{iR}(\hat{x}_i, \xi + k^j_R) \Big)^{\pm} \varphi|_{x_i = \alpha^j_i(\hat{x}_i)} d\hat{x}_i dt \geq 0.$$

b) For any $\varphi \in C_0^1(I\!R \times \Omega_L)$:

$$\int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} (v-\xi)^{\pm} \partial_t \varphi dt dx \tag{45}$$
$$+ \int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} \sum_{i=1}^d \mathrm{sgn}_{\pm}(v-\xi) \left(g_{iL}(\hat{x}_i, v + k_L) - g_{iL}(\hat{x}_i, \xi + k_L) \right) \partial_{x_i} \varphi$$
$$+ \int_{I\!R^d} (u_0 + k_L - \xi)^{\pm} \varphi(0,x) dx \geq 0.$$

c) For any $\varphi \in C_0^1(I\!R \times \Omega_R)$:

$$\int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} (v-\xi)^{\pm} \partial_t \varphi dt dx \tag{46}$$
$$+ \int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} \sum_{i=1}^d \mathrm{sgn}_{\pm}(v-\xi) \left(g_{iR}(\hat{x}_i, v + k_R) - g_{iR}(\hat{x}_i, \xi + k_R) \right) \partial_{x_i} \varphi$$
$$+ \int_{I\!R^d} (u_0 + k_R - \xi)^{\pm} \varphi(0,x) dx \geq 0.$$

2. The function u is a k-weak entropy process solution if it is a weak k-entropy process sub- and super-solution at the same time.

Now, we shall introduce an appropriate kinetic formulation of the problem under consideration. Denote for functions $u \in L^\infty(I\!R^+ \times I\!R^d \times (0,1))$, $u_0 \in L^\infty(I\!R^d; [a,b])$, and function k from (43):

$$h^k_{\pm}(t,x,\lambda,\xi) = \mathrm{sgn}_{\pm}(u(t,x,\lambda) + k(x) - \xi),$$
$$h^0_{\pm,k}(x,\xi) = \mathrm{sgn}_{\pm}(u_0(x) + k(x) - \xi).$$

As before, we call the functions $h^k_{\pm}$ equilibrium functions.

Definition 5.2. Denote for $j = 1, \dots, n$ and $i = 1, \dots, d$:

$$G^j_{iL}(x,\xi) = \partial_\xi g^j_{iL}(x,\xi), \quad G^j_{iR}(x,\xi) = \partial_\xi g^j_{iR}(x,\xi),$$
$$G_{iL}(x,\xi) = \partial_\xi g_{iL}(x,\xi), \quad G_{iR}(x,\xi) = \partial_\xi g_{iR}(x,\xi).$$

Let $u_0 \in L^\infty(I\!R^d; [a,b])$ and $u \in L^\infty(I\!R^+ \times I\!R^d \times (0,1))$.

The function u is a k-kinetic process super-solution (respectively k-kinetic process sub-solution) to (8), (2) if, for the function $v = u+k \in L^\infty(I\!R^+\times I\!R^d\times(0,1))$, k given by (43), there exist $m^j_\pm, m^L_\pm, m^R_\pm \in C(I\!R_\xi; w-\star\mathcal{M}_+(I\!R^+\times I\!R^d))$ such that $m^j_+(\cdot,\xi)$, $m^R_+(\cdot,\xi)$, $m^L_+(\cdot,\xi)$ vanish for large ξ (respectively $m^j_-(\cdot,\xi)$, $m^R_-(\cdot,\xi)$, $m^L_-(\cdot,\xi)$ vanish for large $-\xi$), and such that:

a) For every $j=1,\dots,n$ and every $\varphi\in C^1_0(I\!R^+\times\Omega_j\times I\!R)$,

$$\begin{aligned}\int_0^1 d\lambda\int_{t,x,\xi} h^k_\pm\Big(\partial_t+\sum_{i=1}^d\Big(&G^j_{iL}(\hat{x}_i,\xi+k^j_L)H(\alpha_i(\hat{x}_i)-x_i)\\ &+G^j_{iR}(\hat{x}_i,\xi+k^j_R)H(x_i-\alpha_i(\hat{x}_i))\Big)\partial_{x_i}\Big)\varphi\\ +\int_{x,\xi}& h^0_{\pm,k}\varphi|_{t=0}dxd\xi\\ -\int_{t,\hat{x}_i,\xi}&\Big(g^j_{iL}(\hat{x}_i,\xi+k^j_L)-g^j_{iR}(\hat{x}_i,\xi+k^j_R)\Big)^\pm\partial_\xi\varphi|_{x_i=\alpha(\hat{x}_i)}d\hat{x}_idtd\xi\\ =\int_{t,x,\xi}&\partial_\xi\varphi dm^j_\pm d\xi.\end{aligned}\tag{47}$$

b) For every $\varphi\in C^1_0(I\!R^+\times\Omega_L\times I\!R)$,

$$\begin{aligned}&\int_0^1 d\lambda\int_{t,x,\xi} h^k_\pm\Big(\partial_t+\sum_{i=1}^d G_{iL}(\hat{x}_i,\xi+k_L)\partial_{x_i}\Big)\varphi\\ &+\int_{x,\xi}h^0_{\pm,k}\varphi|_{t=0}dxd\xi=\int_{t,x,\xi}\partial_\xi\varphi dm^L_\pm d\xi.\end{aligned}\tag{48}$$

c) For every $\varphi\in C^1_0(I\!R^+\times\Omega_R\times I\!R)$,

$$\begin{aligned}&\int_0^1 d\lambda\int_{t,x,\xi} h^k_\pm\Big(\partial_t+G_{iR}(\hat{x}_i,\xi)\partial_{x_i}\Big)\varphi\\ &+\int_{x,\xi}h^0_{\pm,k}\varphi|_{t=0}dxd\xi=\int_{t,x,\xi}\partial_\xi\varphi dm^R_\pm d\xi.\end{aligned}\tag{49}$$

The following proposition can be proved in completely the same way as Proposition 1. We leave it without proof.

Proposition 2. *The k-weak entropy process admissible solution to (8), (2) is at the same time the k-kinetic process solution (8), (2).*

Using the latter proposition and repeating the proof of Theorem 2.8, it is not difficult to prove the following theorem. We leave it without proof.

Theorem 5.3. *Assume that the function k from (43) is such that there exists an interval $(c,d)\subset I\!R$ such that for every $j=1,\dots,n$ and every $x\in I\!R^d$:*

$$\begin{aligned}&g^j_{iL}(\hat{x}_i,\xi+k_L)\equiv 0 \text{ if } \xi\geq c \quad\text{and}\quad g^j_{iR}(\hat{x}_i,\xi+k_R)\equiv 0 \text{ if } \xi\leq d,\quad \forall i=1,\dots,d\\ &\qquad\qquad\text{or}\\ &g^j_{iR}(\hat{x}_i,\xi+k_L)\equiv 0 \text{ if } \xi\geq c \quad\text{and}\quad g^j_{iL}(\hat{x}_i,\xi+k_R)\equiv 0 \text{ if } \xi\leq d,\quad \forall i=1,\dots,d.\end{aligned}\tag{50}$$

Then, for any two k-weak entropy process solutions u and v to (8) with initial conditions u_0 and v_0, respectively, the following holds for every $j=1,\dots,n$ and *every* $\varphi \in C_0^1(I\!R \times \Omega_j)$

$$\begin{aligned} &\int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} (v-u)^{\pm} \partial_t \varphi dt dx \\ &+ \int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} \sum_{i=1}^d \operatorname{sgn}_{\pm}(v-u) \times \\ &\quad \times \Big(\Big(g_{iL}^j(\hat{x}_i, v + k_L^j) - g_{iL}^j(\hat{x}_i, u + k_L^j) \Big) H(\alpha_i^j(\hat{x}_i) - x_i) \\ &\quad + \Big(g_{iR}^j(\hat{x}_i, v + k_R^j) - g_{iR}^j(\hat{x}_i, u + k_R^j) \Big) H(x_i - \alpha_i^j(\hat{x}_i)) \Big) \partial_{x_i} \varphi \\ &+ \int_{I\!R^d} (v_0 - u_0)^{\pm} \varphi(0, x) dx \geq 0. \end{aligned} \tag{51}$$

Remark 5. Notice that, according to the assumptions on g_{iL}^j and g_{iR}^j as well as k_L^j and k_R^j, $i = 1, \ldots, d$, $j = 1, \ldots, n$, we can rewrite (44) for every $j = 1, \ldots, n$, and every $\varphi \in C_0^1(I\!R \times \Omega_j)$:

$$\begin{aligned} &\int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} (v-u)^{\pm} \partial_t \varphi dt dx \\ &+ \int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} \sum_{i=1}^d \operatorname{sgn}_{\pm}(v-u) \times \\ &\quad \times \Big((g_{iL}(\hat{x}_i, v + k_L) - g_{iL}(\hat{x}_i, u + k_L)) \kappa_L(x) \\ &\quad + (g_{iR}(\hat{x}_i, v + k_R) - g_{iR}(\hat{x}_i, u + k_R)) \kappa_R(x) \Big) \partial_{x_i} \varphi \\ &+ \int_{I\!R^d} (v_0 - u_0)^{\pm} \varphi(0, x) dx \geq 0. \end{aligned}$$

As noticed in the proof of Lemma 3.2, using the standard Kruzhkov doubling of variables method, the following theorem can be proved:

Theorem 5.4. *Any two k-weak entropy process solutions u and v to (8) with initial conditions u_0 and v_0, respectively, satisfy for every $\varphi \in C_0^1(I\!R \times (\Omega_L \cup \Omega_R))$*

$$\begin{aligned} &\int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} (v-u)^{\pm} \partial_t \varphi dt dx \\ &+ \int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} \sum_{i=1}^d \operatorname{sgn}_{\pm}(v-u) \Big((g_{iL}(\hat{x}_i, v + k_L) - g_{iL}(\hat{x}_i, u + k_L)) \kappa_L(x) \\ &\qquad + (g_{iR}(\hat{x}_i, v + k_R) - g_{iR}(\hat{x}_i, u + k_R)) \kappa_R(x) \Big) \partial_{x_i} \varphi \\ &+ \int_{I\!R^d} (v_0 - u_0)^{\pm} \varphi(0, x) dx \geq 0. \end{aligned} \tag{52}$$

From Theorem 5.4 and Remark 5 we deduce the following theorem:

Theorem 5.5. *Assume that the step function k from (10) is such that there exists an interval $(c, d) \subset I\!R$ over which for every $i = 1, \ldots, d$ and every $x \in I\!R^d$ (50) holds.*

Then, for any two k-weak entropy process solutions u and v to (8) with initial conditions u_0 and v_0, respectively, the following holds for any $T > 0$ and any ball

$B(0,R) \subset I\!R^d$:

$$\int_0^1 d\lambda \int_0^1 d\eta \int_0^T \int_{B(0,R)} (u(t,x,\lambda) - v(t,x,\eta))^{\pm} dxdt \leq T \int_{B(0,R+CT)} (u_0(x) - v_0(x))^{\pm} dx, \tag{53}$$

for a constant $C > 0$ *independent of* $T, R > 0$.

Proof. In the first step, denote by $\tilde{\Gamma} = \cup_{p,q=1}^{n} (\bar{\Omega}_p \cap \Omega_q \cap \Gamma)$ and notice that $codim \tilde{\Gamma} \geq 2$.

Then, notice that any test function $\varphi \in C_0^1(I\!R \times (I\!R^d \backslash \tilde{\Gamma}))$ can be written as a sum

$$\varphi = \varphi_L + \varphi_R + \sum_{j=1}^{n} \varphi_j,$$

where supp$\varphi_L \subset I\!R \times \Omega_L$, supp$\varphi_R \subset I\!R \times \Omega_R$ and $\varphi_j \subset I\!R \times \Omega_j$, $j = 1, \dots, n$.

Therefore, from Remark 5 and Theorem 5.4, we conclude that the following holds for every $\varphi \in C_0^1(I\!R \times (I\!R^d \backslash \tilde{\Gamma}))$:

$$\begin{aligned}
&\int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} (v-u)^{\pm} \partial_t \varphi dtdx \\
&+ \int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} \sum_{i=1}^{d} \mathrm{sgn}_{\pm}(v-u) \times \\
&\quad \times \Big(\left(g_{iL}(\hat{x}_i, v + k_L) - g_{iL}(\hat{x}_i, u + k_L) \right) \kappa_L(x) \\
&\quad\quad + \left(g_{iR}(\hat{x}_i, v + k_R) - g_{iR}(\hat{x}_i, u + k_R) \right) \kappa_R(x) \Big) \partial_{x_i} \varphi \\
&+ \int_{I\!R^d} (v_0 - u_0)^{\pm} \varphi(0,x) dx \geq 0.
\end{aligned} \tag{54}$$

Now, denote by $\tilde{\Gamma}_\varepsilon$ an ε-neighborhood of the set $\tilde{\Gamma}$. Let $\omega_\varepsilon \in C^1(I\!R^d)$ be such that

$$\omega_\varepsilon(x) = \begin{cases} 1, & x \notin \Gamma_{2\varepsilon} \\ 0, & x \in \Gamma_\varepsilon. \end{cases}$$

Notice that

$$\begin{aligned}
&|\partial_{x_i} \omega_\varepsilon| \leq \frac{C}{\varepsilon} \\
&\mathrm{meas}(\mathrm{supp}(\partial_{x_i} \omega_\varepsilon)) \leq \tilde{C} \varepsilon^2,
\end{aligned} \tag{55}$$

for some constants C and $\tilde{C}$, since $codim(\mathrm{supp}(\partial_{x_i} \omega_\varepsilon)) \geq 2$.

Then, take an arbitrary $\varphi \in C_0^1(I\!R \times I\!R^d)$ and put in (54) $\varphi\omega_\varepsilon$. We conclude from (55):

$$\int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} (v-u)^{\pm} \partial_t \varphi dt dx$$
$$+ \int_0^1 d\lambda \int_{I\!R^+ \times I\!R^d} \sum_{i=1}^d \operatorname{sgn}_{\pm}(v-u) \times$$
$$\times \Big(\left(g_{iL}(\hat{x}_i, v+k_L) - g_{iL}(\hat{x}_i, u+k_L)\right) \kappa_L(x)$$
$$+ \left(g_{iR}(\hat{x}_i, v+k_R) - g_{iR}(\hat{x}_i, u+k_R)\right) \kappa_R(x) \Big) \partial_{x_i} \varphi$$
$$+ \int_{I\!R^d} (v_0 - u_0)^{\pm} \varphi(0,x) dx \geq \mathcal{O}(\varepsilon).$$

Letting $\varepsilon \to 0$ and following the standard procedure [24], we arrive at (53). □

As in the "special case", a simple corollary of the last theorem is the existence and uniqueness to the k-admissible solution to (8), (2).

Corollary 2. *Assume that the function k from (43) is such that (50) is satisfied. Then, there exists a unique k-entropy weak solution to (8), (2).*

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