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LARGE TIME BEHAVIOR FOR THE IBVP OF THE 3-D NISHIDA'S MODEL

Shijin Deng

Department of Mathematics, Shanghai Jiao Tong University
Shanghai 200240, China

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Abstract. In this paper we investigate an initial boundary value problem (IBVP) for the Nishda's model in 3-dimensional space with a forward moving physical boundary. It is shown that the solution converges to zero with an exponential rate by energy estimates.

1. Introduction.

The 3-dimensional Nishida's model

$$\begin{cases} \rho_t + div(\rho\vec{u}) = 0, \\ (\rho u^j)_t + div(\rho\vec{u}u^j) + \rho^\gamma_{x_j} = -\kappa\rho u^j, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+,\ \vec{u} = (u^1,u^2,u^3) \in \mathbb{R}^3, \end{cases}$$

is a hyperbolic model for a porous media equation

$$\rho_t = \frac{1}{\kappa} \triangle \rho^\gamma. \tag{1}$$

Here, $\kappa > 0$ is a constant to model the magnitude of the viscosity, and $\rho^\gamma,\ \gamma \geq 1$, is the pressure for an isentropic gas flow with a given γ-law. This 3-dimensional Nishida's model in the Eulerian coordinate is a direct generalization from the one-dimensional model in the Larangian coordinates, [13]:

$$\begin{cases} v_t - m_x = 0, \\ m_t + (v^{-\gamma})_x = -m. \end{cases}$$

The time-asymptotic analysis on the Nishida's model was first initiated by [4] in the global normed space. There were many interesting mathematical works followed in the global normed setting, due to the strong physical background and significant mathematical challenge of the Nishida's model. For the existence theory and large time behavior of the solutions, one can refer to [1]-[4], [6]-[9], [11], [13]-[15], [19]-[26] and the references there.

We have mentioned that such a system is the mathematical model for compressible flow through a porous medium. Therefore Nishida's model can be widely used in the real world, such as oil exploration and so on. Since the physical boundary always exists in real world and its presence also provides with much richer phenomena, the initial boundary value problem interests us. However, there is not so much

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literature on the initial boundary value problem as that on Cauchy problem. The reason is that many problems besides the interesting phenomena arise from the presence of the physical boundary. Most of the recent work are for 1-dimensional space, [5, 12, 16, 17, 18]. In [10], the half space problem for 2-dimensional Nishida's model is considered and the existence theory is obtained there by the energy method.

In this paper, we consider the pointwise structure of an initial-boundary value problem for the 3-dimensional Nishida's model with the presence of a physical boundary at $x = bt$:

$$\begin{cases} \rho_t + div(\rho \vec{\mathsf{u}}) = 0, \quad x_1 > bt, t > 0, \\ (\rho u^j)_t + div(\rho \vec{\mathsf{u}} u^j) + \rho^\gamma_{x_j} = -\kappa \rho u^j, \\ (\rho, \vec{\mathsf{u}})(x, 0) = (\rho^0, \vec{\mathsf{u}}^0)(x), \\ u^1(bt, x_2, x_3, t) = 0, \end{cases} \tag{2}$$

in particular $(\rho, \vec{\mathsf{u}})|_{t=0}$ is sufficiently close to $(\rho, \vec{\mathsf{u}}) = (1, \vec{0})$ and

$$b > 0.$$

The sign of the parameter b plays an important role in determining the structure of the solution of (2) with a given boundary condition $\rho = 0$ at $x^1 = bt$. When $b > 0$, the solution of (2) will decay exponentially fast. Thus, one just needs to show that the solution of the Nishida model decays to zero exponentially fast in order to justify the relevance between the Nishida model and the porous media equation (1) for $b > 0$.

For convenience of analysis, one considers the following change of variables

$$\begin{cases} \sigma = \rho - 1, \\ \tau = t, \\ \eta_1 = x_1 - bt, \\ \eta_2 = x_2, \\ \eta_3 = x_3. \end{cases}$$

Then, (2) becomes

$$\begin{cases} \sigma_\tau - b\sigma_{\eta_1} + \sum_{j=1}^3 u^j_{\eta_j} = -div(\sigma \vec{\mathsf{u}}), \quad \eta_1 > 0, \tau > 0, \\ u^j_\tau - b u^j_{\eta_1} + \gamma \sigma_{\eta_j} + \kappa u^j = Q^j(\sigma, \vec{\mathsf{u}}), \\ u^1(0, \eta_2, \eta_3, \tau) = 0, \end{cases} \tag{3}$$

where

$$Q^j(\sigma, \vec{\mathsf{u}}) = \gamma \left(1 - (1+\sigma)^{\gamma - 2}\right) \sigma_{\eta_j} - \vec{\mathsf{u}} \cdot \triangledown u^j.$$

Notation. *For any given $m \in \mathbb{N}$ and any function f in $\mathbb{R}^+ \times \mathbb{R}^2$, the norms $\|f\|_m$ and $|||f|||_m$ are*

$$\begin{cases} |||f|||_m = \left(\sum_{0 \le |\alpha| + |\beta| \le m} \int_0^\infty \int_{\mathbb{R}^2} |\partial^\alpha_{\eta_1} \partial^\beta_{\eta'} f(\eta_1, \eta')|^2 d\eta' d\eta_1 \right)^{\frac{1}{2}}, \\ \|f\|_m = \left(\sum_{0 \le |\beta| \le m} \int_{\mathbb{R}^2} |\partial^\beta_{\eta'} f(0, \eta')|^2 d\eta' \right)^{\frac{1}{2}}. \end{cases}$$

The initial data $(\rho(\eta,0),\vec{\mathrm{u}}(\eta,0))$ is assumed to satisfy

$$|||e^{\beta(\eta_1+|\eta_2|+|\eta_3|)}(\sigma(\eta,0),\vec{\mathrm{u}}(\eta,0))|||_4 \le \epsilon \text{ for some } \epsilon,\beta>0. \tag{4}$$

This paper is devoted to the pointwise structure of the solution and its exponential rate of convergence with the presence of a physical boundary condition and the main result is:

Theorem 1.1. *For a given $b>0$ there exist $\beta \ll b$ and $\epsilon_0>0$ such that for any $\epsilon\in(0,\epsilon_0)$ the solution of (3) with the initial condition (4) satisfies*

$$|(\sigma,\vec{\mathrm{u}})(\eta,\tau))| \le Ce^{-\beta(\tau+|\eta|)/C}|||e^{\beta|\eta|/2}(\sigma,\vec{\mathrm{u}})(\cdot,0)|||_4^2 \text{ for some } C>0.$$

Remark 1. Since $b>0$, there is a spectral gap property of the linearized equation around a constant with the homogenous boundary condition posed here. Thus, we are able to get the exponential decaying rate of the solution by weighted energy estimates.

However, when $b<0$, the solution will decay algebraically only and we suppose that the Green's function method and weighted energy estimates should combined together to yield the pointwise estimates for the solution. Thus, such a case will be much more complicated. We have to construct the Green's function while we also need a priori decaying estimates for derivatives since the nonlinear system is a quasi-linear one. We will generalize $b>0$ to $|b|\neq 0$ in the near future.

2. Energy estimates.

With a standard local existence theory for $(\sigma,\vec{\boldsymbol{u}})$, one can assert the smallness property, $|||e^{\beta|\eta|}\sigma(\cdot,\tau)|||_4+|||e^{\beta|\eta|}\vec{\boldsymbol{u}}(\cdot,\tau)|||_4+\|e^{\beta|\eta'|}\sigma(\cdot,\tau)\|_4+\|e^{\beta|\eta'|}\vec{\boldsymbol{u}}(\cdot,\tau)\|_4\ll 1$, of the solution for τ in a small time interval, $[0,\tau_0]$. Thus, one can make a priori assumption on the solution $(\sigma,\vec{\mathrm{u}})$:

$$\sup_{0<\tau}\Big(|||e^{\beta|\eta|}\sigma(\cdot,\tau)|||_4+|||e^{\beta|\eta|}\vec{\mathrm{u}}(\cdot,\tau)|||_4\Big)\le\delta\ll 1, \tag{5}$$

where $0<\epsilon\ll\delta\ll b$.

2.1. Lower Order Estimates.

By multiplying the equations in (3) with $e^{\beta\eta_1}\sigma$ and $e^{\beta\eta_1}\frac{u^j}{\gamma}$ respectively, one integrates the equations over $\mathbb{R}^+\times\mathbb{R}^2$ to yield that

$$\begin{aligned} 0=&\int_0^\infty\int_{\mathbb{R}^2}\Bigg(e^{\beta\eta_1}\sigma\cdot(\sigma_\tau-b\sigma_{\eta_1}+\sum_{j=1}^3 u^j_{\eta_j}+div(\sigma\vec{\mathrm{u}}))\\ &+\sum_{j=1}^3 e^{\beta\eta_1}\frac{u^j}{\gamma}\cdot(u^j_\tau-bu^j_{\eta_1}+\gamma\sigma_{\eta_j}+\kappa u^j-Q^j(\sigma,\vec{\mathrm{u}}))\Bigg)d\eta' d\eta_1\\ =&\frac12\frac{d}{d\tau}\int_0^\infty\int_{\mathbb{R}^2}e^{\beta\eta_1}\left(\sigma^2+\sum_{j=1}^3\frac1\gamma(u^j)^2\right)d\eta' d\eta_1\\ &+\frac b2\int_{\mathbb{R}^2}\left(\sigma^2+\sum_{j=1}^3\frac1\gamma(u^j)^2\right)d\eta'\Bigg|_{\eta_1=0}-\int_{\mathbb{R}^2}\sigma u^1 d\eta'\Bigg|_{\eta_1=0} \end{aligned} \tag{6}$$

$$+\int_0^\infty \int_{\mathbb{R}^2} e^{\beta\eta_1}\left(\frac{\beta b}{2}\sigma^2 - \beta\sigma u^1 + \sum_{j=1}^3\left(\frac{\beta b}{2\gamma}+\kappa\right)(u^j)^2\right.$$
$$\left. + \sigma div(\sigma\vec{u}) + \sum_{j=1}^3 \frac{u^j}{\gamma}Q^j(\sigma,\vec{u})\right) d\eta' d\eta_1.$$

By the property that $0 < \beta \ll b$, one has

$$\frac{\beta b}{2}\sigma^2 - \beta\sigma u^1 + \sum_{j=1}^3\left(\frac{\beta b}{2\gamma}+\kappa\right)(u^j)^2 \geq \frac{\beta b}{4}\sigma^2 + \sum_{j=1}^3 \frac{\kappa}{2}(u^j)^2. \tag{7}$$

From (6), (7), and the boundary condition $u^1(0,\tau) = 0$, one has

$$\begin{aligned}
&\frac{d}{d\tau}\int_0^\infty \int_{\mathbb{R}^2} e^{\beta\eta_1}\left(\sigma^2 + \sum_{j=1}^3 \frac{1}{\gamma}(u^j)^2\right) d\eta' d\eta_1 \\
&+\int_0^\infty \int_{\mathbb{R}^2} e^{\beta\eta_1}\left(\frac{\beta b}{2}\sigma^2 + \sum_{j=1}^3\left(\frac{\beta b}{2\gamma}+\kappa\right)(u^j)^2\right) d\eta' d\eta_1 \\
&+b\int_{\mathbb{R}^2}\left(\sigma^2 + \sum_{j=1}^3 \frac{1}{\gamma}(u^j)^2\right) d\eta' \Bigg|_{\eta_1=0} \\
&\leq -2\int_0^\infty \int_{\mathbb{R}^2} e^{\beta\eta_1}\left(\sigma div(\sigma\vec{u}) + \sum_{j=1}^3 \frac{u^j}{\gamma}Q^j(\sigma,\vec{u})\right) d\eta' d\eta_1.
\end{aligned} \tag{8}$$

By a priori assumption (5) and Sobolev's inequality, there is $C > 0$ such that

$$\left|\sigma div(\sigma\vec{u}) + \sum_{j=1}^3 \frac{u^j}{\gamma}Q^j(\sigma,\vec{u})\right| \leq C\delta\left(\sigma^2 + \sum_{j=1}^3 (u^j)^2\right).$$

By the property $\delta \ll \beta \ll b$, one has

$$\frac{d}{d\tau}|||e^{\beta\eta_1/2}(\sigma, \frac{1}{\sqrt{\gamma}}\vec{u})|||_0^2 + \|(\sigma, \frac{1}{\sqrt{\gamma}}\vec{u})\|_0^2 + \frac{\beta b}{4}|||e^{\beta\eta_1/2}\sigma|||_0^2 + (\frac{\beta b}{4\gamma}+2\kappa)|||e^{\beta\eta_1/2}\vec{u}|||_0^2 \leq 0.$$

Thus for $\tau > 0$,

$$\begin{aligned}
&|||e^{\beta\eta_1/2}(\sigma,\vec{u})(\cdot,\tau)|||_0^2 + \frac{\beta b}{8}\int_0^\tau e^{-\beta b(\tau-s)/8}|||e^{\beta\eta_1/2}(\sigma,\vec{u})(\cdot,s)|||_0^2 ds \\
\leq &\gamma e^{-\beta b\tau/8}|||e^{\beta\eta_1/2}(\sigma,\vec{u})(\cdot,0)|||_0^2.
\end{aligned} \tag{9}$$

2.2. **High Order Energy Estimates.**

For the purpose to prove a priori assumption (5), we need to rewrite (3) in the following symmetric form in order to close the nonlinearity by energy estimates:

$$A_0(U)\partial_\tau U + \sum_{i=1}^3 A_i(U)\partial_{\eta_i} U + B(U)U = 0, \tag{10}$$

where

$$A_0(U) = \begin{pmatrix} \frac{1}{1+\sigma} & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma(1+\sigma)^{\gamma-2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma(1+\sigma)^{\gamma-2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma(1+\sigma)^{\gamma-2}} \end{pmatrix},$$

$$A_1(U) = \begin{pmatrix} \frac{-b+u^1}{1+\sigma} & 1 & 0 & 0 \\ 1 & \frac{-b+u^1}{\gamma(1+\sigma)^{\gamma-2}} & 0 & 0 \\ 0 & 0 & \frac{-b+u^1}{\gamma(1+\sigma)^{\gamma-2}} & 0 \\ 0 & 0 & 0 & \frac{-b+u^1}{\gamma(1+\sigma)^{\gamma-2}} \end{pmatrix},$$

$$A_2(U) = \begin{pmatrix} \frac{u^2}{1+\sigma} & 0 & 1 & 0 \\ 0 & \frac{u^2}{\gamma(1+\sigma)^{\gamma-2}} & 0 & 0 \\ 1 & 0 & \frac{u^2}{\gamma(1+\sigma)^{\gamma-2}} & 0 \\ 0 & 0 & 0 & \frac{u^2}{\gamma(1+\sigma)^{\gamma-2}} \end{pmatrix},$$

$$A_3(U) = \begin{pmatrix} \frac{u^3}{1+\sigma} & 0 & 0 & 1 \\ 0 & \frac{u^3}{\gamma(1+\sigma)^{\gamma-2}} & 0 & 0 \\ 0 & 0 & \frac{u^3}{\gamma(1+\sigma)^{\gamma-2}} & 0 \\ 1 & 0 & 0 & \frac{u^3}{\gamma(1+\sigma)^{\gamma-2}} \end{pmatrix},$$

$$B(U) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\kappa}{\gamma(1+\sigma)^{\gamma-2}} & 0 & 0 \\ 0 & 0 & \frac{\kappa}{\gamma(1+\sigma)^{\gamma-2}} & 0 \\ 0 & 0 & 0 & \frac{\kappa}{\gamma(1+\sigma)^{\gamma-2}} \end{pmatrix}, \quad U = \begin{pmatrix} \sigma \\ u^1 \\ u^2 \\ u^3 \end{pmatrix}.$$

One needs to consider the high order derivatives $\partial_\tau^{4-k}\partial_{\eta_2}^{s_2}\partial_{\eta_3}^{s_3}\partial_{\eta_1}^{s_1}$ with $s_1+s_2+s_3 = k,\ 0 \le k \le 4$. Since the procedure for the low order energy estimates is not valid for the variables $\partial_{\eta_1}^{i_1}(\sigma,\vec{u})$, one can use the estimate on $\partial_{\eta_2}^{i_2}(\sigma,\vec{u})$, $\partial_{\eta_3}^{i_3}(\sigma,\vec{u})$, and $\partial_\tau^{i}(\sigma,\vec{u})$ together to yield the estimate for $\partial_{\eta_1}^{i_1}(\sigma,\vec{u})$. This is due to the hyperbolicity of the system (10) and $0 < \delta \ll b$. Then,

$$\partial_{\eta_1} U = -(A_1(U))^{-1}\left(A_0(U)\partial_\tau U + \sum_{i=2}^{3} A_i(U)\partial_{\eta_i} U + B(U)U\right).$$

Here, the matrix $A_1(U)$ is invertible due to $0 < \delta \ll b$ and (5). Thus,

$$\begin{aligned} \partial_{\eta_1}^m U = -\sum_{j=0}^{m} C_m &\Big(\partial_{\eta_1}^j \left((A_1(U))^{-1}A_0(U)\right)\partial_\tau\partial_{\eta_1}^{m-j}U \\ &+ \sum_{i=2}^{3}\partial_{\eta_1}^j\left((A_1(U))^{-1}A_i(U)\right)\partial_{\eta_i}\partial_{\eta_1}^{m-j}U + \partial_{\eta_1}^j\left((A_1(U))^{-1}B(U)\right)\partial_{\eta_1}^{m-j}U\Big). \end{aligned} \tag{11}$$

This yields

$$
\begin{aligned}
&|||e^{\beta\eta_1/2}\partial_{\eta_1}^m U|||_0 \\
\leq& C\left(\sum_{k=0}^{m}\sum_{|\alpha|=k}|||e^{\beta\eta_1/2}\partial_\tau^{m-k}\partial_{\eta'}^{\alpha}U|||_0+\delta\sum_{j=0}^{m-1}|||e^{\beta\eta_1/2}\partial_\tau^{j}U|||_{m-1-j}\right).
\end{aligned}
\tag{12}
$$

From (12), we can find that we only need to study the estimates for $\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U(|\alpha'| = k,\ 0\leq k\leq 4)$. Then we can get estimate for $\partial_\tau^{4-k}\partial_\eta^{\alpha}U$ $(|\alpha|=k, 0\leq k\leq 4)$.

One applies $\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}$ to (10)with $|\alpha'|=k$ to yield that

$$
A_0(U)\partial_\tau(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)+\sum_{i=1}^{3}A_i(U)\partial_{\eta_i}(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)+B(U)(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)+\mathcal{L}_{\alpha'}=0, \tag{13}
$$

where $\mathcal{L}_{\alpha'}$ is the nonlinear term and it contains only the derivatives with order no greater than 3. By a priori assumption (5) and the Sobolev's inequality for dimension 3, it follows

$$
|||e^{\beta\eta_1}\mathcal{L}_{\alpha'}|||_0\leq C\delta\sum_{k=0}^{4}|||e^{\beta\eta_1}\partial_\tau^{k}U|||_{4-k} \text{ for } |\alpha'|\leq 4.
$$

Similar to (12), the derivative on time variable τ can be transferred to that on spatial variables η. Thus, one has

$$
|||e^{\beta\eta_1}\mathcal{L}_{\alpha'}|||_0\leq O(1)\sum_{|\alpha|\leq 4}\delta|||e^{\beta\eta_1}\partial_\eta^{\alpha}U|||_0 \text{ for } |\alpha'|\leq 4. \tag{14}
$$

By multiplying (13) by $e^{\beta\eta_1}\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U$ with $|\alpha'|=k$ and integrating the product in the domain $[0,\infty)\times\mathbb{R}^2$, one has

$$
\begin{aligned}
0=&\int_0^\infty\int_{\mathbb{R}^2}e^{\beta\eta_1}(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)\cdot\Big(A_0(U)\partial_\tau(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)\\
&+\sum_{i=1}^{3}A_i(U)\partial_{\eta_i}(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)+B(U)(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)+\mathcal{L}_{\alpha'}\Big)d\eta'd\eta_1\\
=&\frac{1}{2}\frac{d}{d\tau}\int_0^\infty\int_{\mathbb{R}^2}e^{\beta\eta_1}(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)\cdot A_0(U)(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)d\eta'd\eta_1\\
&-\frac{1}{2}\int_0^\infty\int_{\mathbb{R}^2}e^{\beta\eta_1}(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)\cdot(A_0(U))_\tau(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)d\eta'd\eta_1\\
&-\frac{1}{2}\int_{\mathbb{R}^2}(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)A_1(U)\cdot(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)d\eta'\Big|_{\eta_1=0}\\
&-\frac{\beta}{2}\int_0^\infty\int_{\mathbb{R}^2}\frac{1}{2}e^{\beta\eta_1}(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)\cdot A_1(U)(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)d\eta'd\eta_1\\
&+\int_0^\infty\int_{\mathbb{R}^2}e^{\beta\eta_1}(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)\cdot B(U)(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)d\eta'd\eta_1\\
&-\frac{1}{2}\sum_{i=1}^{3}\int_0^\infty\int_{\mathbb{R}^2}e^{\beta\eta_1}(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)\cdot(A_i(U))_{\eta_i}(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)d\eta'd\eta_1\\
&+\int_0^\infty\int_{\mathbb{R}^2}e^{\beta\eta_1}(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)\cdot\mathcal{L}_{\alpha'}d\eta'd\eta_1.
\end{aligned}
\tag{15}
$$

From (10), one has that

$$|||e^{\beta\eta_1/2}\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U(\cdot,0)|||_0 \le C|||e^{\beta\eta_1/2}U(\cdot,0)|||_4 \text{ for some } C>0. \tag{16}$$

The the boundary condition $\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}\mathsf{u}_1(0,\eta',\tau)=0$ and the structure of $A_1(U)$ combined with a priori estimate (5) result in that

$$-\frac{1}{2}\int_{\mathbb{R}^2}(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)A_1(U)\cdot(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)d\eta'\Big|_{\eta_1=0}>0. \tag{17}$$

Under a priori assumption (5), the matrix $A_1(U)$ and $B(U)$ satisfy for $\beta \ll 1$

$$\vec{v}\cdot(-\frac{\beta}{2}A_1(U)+B(U))\vec{v} \ge \frac{\beta b}{4}\vec{v}\cdot\vec{v} \text{ for any } \vec{v}\in\mathbb{R}^4. \tag{18}$$

By a priori assumption (5),

$$\sup_{\substack{\tau,\eta_1>0\\ \eta'\in\mathbb{R}^2}}|\partial_\eta^\alpha A_j(U)(\eta,\tau)| \le O(1)\delta \text{ for } |\alpha|\le 2. \tag{19}$$

The above estimates, ((16), (17), (18), (19)), yield

$$\begin{aligned}&\frac{d}{d\tau}\int_0^\infty\int_0^\infty\int_{\mathbb{R}^2}e^{\beta\eta_1}(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)\cdot A_0(U)(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)d\eta'd\eta_1\\&+\left(\frac{\beta b}{2}-C\delta\right)|||e^{\beta\eta_1/2}\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U|||_0^2 \le 2\left|\int_0^\infty\int_{\mathbb{R}^2}e^{\beta\eta_1}(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)\cdot\mathcal{L}_{\alpha'}d\eta'd\eta_1\right|.\end{aligned} \tag{20}$$

It remains to estimate the RHS of (20). From (14), for $0\le k\le 4$, $|\alpha'|\le k$,

$$\left|\int_0^\infty\int_{\mathbb{R}^2}e^{\beta\eta_1}(\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U)\cdot\mathcal{L}_{\alpha'}d\eta'd\eta_1\right| \le C\delta|||e^{\beta\eta_1/2}\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U|||_0^2+C\delta|||e^{\beta\eta_1/2}U|||_3^2. \tag{21}$$

From the property of $A_0(U)$, (21), and (20) one has for $|\alpha'|\le k$

$$\begin{aligned}&\frac{d}{d\tau}|||e^{\beta\eta_1/2}\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U|||_0^2+\frac{\beta b}{4}|||e^{\beta\eta_1/2}\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U|||_0^2\\ \le& C\delta\left(|||e^{\beta\eta_1/2}\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U|||_0^2+|||e^{\beta\eta_1/2}U|||_3^2\right).\end{aligned} \tag{22}$$

This yields that

$$\begin{aligned}&\sum_{\substack{0\le k\le 4\\|\alpha'|\le k}}\left(\frac{d}{d\tau}|||e^{\beta\eta_1/2}\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U|||_0^2+\frac{\beta b}{4}|||e^{\beta\eta_1/2}\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U|||_0^2\right)\\ \le& C\delta\left(\sum_{\substack{0\le k\le 4\\|\alpha'|\le k}}|||e^{\beta\eta_1/2}e^{\beta\eta_1/2}\partial_\tau^{4-k}\partial_{\eta'}^{\alpha'}U|||_0^2+|||e^{\beta\eta_1/2}U|||_3^2\right).\end{aligned} \tag{23}$$

This, (12), and $0<\epsilon\ll\delta\ll b$ yield that

$$\frac{d}{d\tau}|||e^{\beta\eta_1/2}U|||_4^2+\frac{\beta b}{8}|||e^{\beta\eta_1/2}U|||_4^2\le 0.$$

This inequality results in

$$|||e^{\beta\eta_1/2}U(\cdot,\tau)|||_4^2 \le e^{-\frac{\beta b\tau}{8}}|||e^{\beta\eta_1/2}U(\cdot,0)|||_4^2 \le O(1)\epsilon e^{-\frac{\beta b\tau}{8}}.$$

2.3. **Decaying rates on η' variables.** To obtain the exponential decaying rate on η' variables, we choose another weight function $e^{\beta(\eta_1 \pm \eta_2 \pm \eta_3)}$ to apply the same energy estimate.

By multiplying the equations in (3) with $e^{\beta\eta_1+\beta|\eta_2|+\beta|\eta_3|}\sigma$ and $e^{\beta\eta_1+\beta|\eta_2|+\beta|\eta_3|}\frac{u^j}{\gamma}$ respectively, one integrates the equations over $\mathbb{R}^+ \times \mathbb{R}^2$ to yield that

$$\begin{aligned}
0 =& \int_0^\infty \int_{\mathbb{R}^2} \Bigg(e^{\beta\eta_1+\beta|\eta_2|+\beta|\eta_3|}\sigma \cdot (\sigma_\tau - b\sigma_{\eta_1} + \sum_{j=1}^3 u^j_{\eta_j} + div(\sigma \vec{\mathsf{u}})) \\
&+ \sum_{j=1}^3 e^{\beta\eta_1+\beta|\eta_2|+\beta|\eta_3|} \frac{u^j}{\gamma} \cdot (u^j_\tau - bu^j_{\eta_1} + \gamma\sigma_{\eta_j} + \kappa u^j - Q^j(\sigma, \vec{\mathsf{u}})) \Bigg) d\eta' d\eta_1 \\
=& \frac{1}{2}\frac{d}{d\tau} \int_0^\infty \int_{\mathbb{R}^2} e^{\beta\eta_1+\beta|\eta_2|+\beta|\eta_3|} \left(\sigma^2 + \sum_{j=1}^3 \frac{1}{\gamma}(u^j)^2 \right) d\eta' d\eta_1 \\
&+ \frac{b}{2} \int_{\mathbb{R}^2} e^{\beta|\eta_2|+\beta|\eta_3|} \left(\sigma^2 + \sum_{j=1}^3 \frac{1}{\gamma}(u^j)^2 \right) d\eta' \Bigg|_{\eta_1=0} - \int_{\mathbb{R}^2} e^{\beta|\eta_2|+\beta|\eta_3|} \sigma u^1 d\eta' \Bigg|_{\eta_1=0} \\
&+ \int_0^\infty \int_{\mathbb{R}^2} e^{\beta\eta_1+\beta|\eta_2|+\beta|\eta_3|} \Bigg(\frac{\beta b}{2}\sigma^2 - \beta\sigma u^1 - \beta \sum_{j=2}^3 sgn(\eta_j)\sigma u^j \\
&+ \sum_{j=1}^3 \left(\frac{\beta b}{2\gamma} + \kappa \right) (u^j)^2 + \sigma div(\sigma \vec{\mathsf{u}}) + \sum_{j=1}^3 \frac{u^j}{\gamma} Q^j(\sigma, \vec{\mathsf{u}}) \Bigg) d\eta' d\eta_1.
\end{aligned} \tag{24}$$

Here,

$$sgn(x) = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

By the property that $0 < \beta \ll b$, one has

$$\frac{\beta b}{2}\sigma^2 - \beta\sigma u^1 - \beta \sum_{j=2}^3 sgn(\eta_j)\sigma u^j + \sum_{j=1}^3 \left(\frac{\beta b}{2\gamma} + \kappa \right) (u^j)^2 \geq \frac{\beta b}{4}\sigma^2 + \sum_{j=1}^3 \frac{\kappa}{2}(u^j)^2. \tag{25}$$

From (24), (25), and the boundary condition $u^1(0, \tau) = 0$, one has

$$\begin{aligned}
&\frac{d}{d\tau} \int_0^\infty \int_{\mathbb{R}^2} e^{\beta\eta_1+\beta|\eta_2|+\beta|\eta_3|} \left(\sigma^2 + \sum_{j=1}^3 \frac{1}{\gamma}(u^j)^2 \right) d\eta' d\eta_1 \\
&+ \int_0^\infty \int_{\mathbb{R}^2} e^{\beta\eta_1+\beta|\eta_2|+\beta|\eta_3|} \left(\frac{\beta b}{2}\sigma^2 + \sum_{j=1}^3 \left(\frac{\beta b}{2\gamma} + \kappa \right) (u^j)^2 \right) d\eta' d\eta_1 \\
&+ b \int_{\mathbb{R}^2} e^{\beta|\eta_2|+\beta|\eta_3|} \left(\sigma^2 + \sum_{j=1}^3 \frac{1}{\gamma}(u^j)^2 \right) d\eta' \Bigg|_{\eta_1=0} \\
\leq& -2 \int_0^\infty \int_{\mathbb{R}^2} e^{\beta\eta_1+\beta|\eta_2|+\beta|\eta_3|} \left(\sigma div(\sigma \vec{\mathsf{u}}) + \sum_{j=1}^3 \frac{u^j}{\gamma} Q^j(\sigma, \vec{\mathsf{u}}) \right) d\eta' d\eta_1.
\end{aligned} \tag{26}$$

From the following fact

$$\left|\sigma div(\sigma\vec{u}) + \sum_{j=1}^{3}\frac{u^j}{\gamma}Q^j(\sigma,\vec{u})\right| \leq C\delta\left(\sigma^2 + \sum_{j=1}^{3}(u^j)^2\right),$$

and the property $\delta \ll \beta \ll b$, one has

$$\begin{aligned}&\frac{d}{d\tau}|||e^{(\beta\eta_1+\beta|\eta_2|+\beta|\eta_3|)/2}(\sigma,\frac{1}{\sqrt{\gamma}}\vec{u})|||_0^2 + \|e^{(\beta|\eta_2|+\beta|\eta_3|)/2}(\sigma,\frac{1}{\sqrt{\gamma}}\vec{u})\|_0^2\\ &+\frac{\beta b}{4}|||e^{(\beta\eta_1+\beta|\eta_2|+\beta|\eta_3|)/2}\sigma|||_0^2 + (\frac{\beta b}{4\gamma}+2\kappa)|||e^{(\beta\eta_1+\beta|\eta_2|+\beta|\eta_3|)/2}\vec{u}|||_0^2 \leq 0.\end{aligned} \tag{27}$$

Thus for $\tau > 0$,

$$\begin{aligned}&|||e^{(\beta\eta_1+\beta|\eta_2|+\beta|\eta_3|)/2}(\sigma,\vec{u})(\cdot,\tau)|||_0^2\\ &+\frac{\beta b}{8}\int_0^\tau e^{-\beta b(\tau-s)/8}|||e^{(\beta\eta_1+\beta|\eta_2|+\beta|\eta_3|)/2}(\sigma,\vec{u})(\cdot,s)|||_0^2 ds\\ &\leq \gamma e^{-\beta b\tau/8}|||e^{(\beta\eta_1+\beta|\eta_2|+\beta|\eta_3|)/2}(\sigma,\vec{u})(\cdot,0)|||_0^2.\end{aligned} \tag{28}$$

Based on this low order estimates, the high order estimates can be obtained similarly. Thus it will yield

$$|||e^{\beta(\eta_1\pm\eta_2\pm\eta_3)/2}U(\cdot,\tau)|||_4^2 \leq e^{-\frac{\beta b\tau}{8}}|||e^{\beta(\eta_1\pm\eta_2\pm\eta_3)/2}U(\cdot,0)|||_4^2 \leq O(1)\epsilon e^{-\frac{\beta b\tau}{8}}.$$

This concludes a priori assumption (5). The weighted function $e^{\beta(\eta_1\pm\eta_2\pm\eta_3)/2}$ and Sobolev's inequality conclude Theorem 1.1.

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E-mail address: bridgedeng@163.com