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A NOVEL MODEL FOR INTERSECTIONS OF VEHICULAR TRAFFIC FLOW

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Abstract. This paper deals with intersections' modeling for vehicular traffic flow governed by the Lighthill & Whitham [24] and Richards [26] model. We present a straightforward reformulation of recent intersections' models, introduced in [19] and [4], using a description in terms of supply and demand functions [22, 6]. This formulation is used to state the new model which takes into account a possible storage capacity of an intersection as seen in roundabouts or highway on–ramps. We discuss the Riemann problem at the junction and present numerical simulations.

1. **Introduction.** We are interested in macroscopic models for traffic flow. Benefits and drawbacks of these kind of models have been intensively discussed in recent literature and we refer to the non–exhaustive list of references for further discussion [1, 5, 26, 27, 6, 10, 11, 21, 24]. Besides modeling and analytical discussions, numerical methods have been studied for example in [3, 9, 12, 22]. Recently, there has been a growth of interest in traffic modeling on road networks using macropscopic models [2, 19, 3, 8, 13, 17, 15, 14, 18, 23, 25]. The common feature of all these approaches is the fact that a junction is considered as a single point *with no dynamics*. However, experience has shown that the geometry of an intersection has a non negligible effect on traffic conditions. This can be observed for example in the cases of on–ramps of highways or roundabouts. Hence, for those situations a representation of the junction as a single point without dynamics is a shortcoming of current models. Here, we do not model the precise geometry of the roundabout

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or the on-ramp. However, we mimic the behavior of this type of junction by introducing a buffer of finite size. This modeling allows to take into account the fact that traffic does not immediately pass through a roundabout. We hereby allow the junction to have some storage capacity. More precisely, the dynamics of the buffer located at the junction will be described by an ordinary differential equation similar to an approach used to model supply chain dynamics [16]. A mathematical analysis using buffers at intersections has been carried out in [16], but only in the case of linear transport equation with an unlimited buffer size.

The paper is organized as follows. We reformulate recently introduced coupling conditions [19, 4] for traffic flow models in terms of supply and demand function. Using this mathematical description we present the model including a buffer and discuss the Riemann problem at the intersection. Finally, we conclude with numerical experiments.

2. **Preliminary discussion.**

Definition 2.1 ([19])**.** A road network is a finite connected directed graph consisting of $\mathcal{K}$ arcs and $\mathcal{N}$ vertices. The arcs and the vertices correspond to roads and junctions, respectively. For a given junction n we denote by δ_n^+ the set of indices of all the incoming roads to n and by δ_n^- the set of indices of all the outgoing roads from n. Each road i is modeled by an interval $I_i = [a_i, b_i]$, possibly with either $a_i = -\infty$ or $b_i = +\infty$.

We consider the Lighthill-Whitham [24] and Richards [26] (LWR in short) model on each road $i \in \mathcal{K}$.

$$\partial_t \rho_i + \partial_x f(\rho_i) = 0, \quad x \in [a_i, b_i], \quad t > 0 \tag{1}$$

$$\rho_i(x, 0) = \rho_{i,0}(x), \quad x \in [a_i, b_i]. \tag{2}$$

On each road i we are interested in weak (entropy) solutions, i.e., such that

$$\sum_{i \in \mathcal{K}} \left(\int_0^{+\infty} \int_{a_i}^{b_i} (\rho_i \partial_t \varphi_i + f(\rho_i) \partial_x \varphi_i) dx dt + \int_{a_i}^{b_i} \rho_{i,0} \varphi_i(x, 0) dx \right) = 0 \tag{3}$$

hold for any set of smooth functions $\{\varphi_i\}_{i \in \mathcal{K}} : I_i \times [0, +\infty[\longrightarrow \mathbb{R}$ with compact support and satisfying

$$\varphi_i(a_i) = \varphi_j(b_j), \ \ \forall\, i \in \delta_n^-, \ \forall\, j \in \delta_n^+. \tag{4}$$

Remark 1. Under sufficient regularity conditions, (3) and (4) imply that the Rankine–Hugoniot condition at the junction is satisfied, i.e.:

$$\sum_{i \in \delta_n^+} f(\rho_i(b_i, t)) = \sum_{i \in \delta_n^-} f(\rho_i(a_i, t)), \ t > 0. \tag{5}$$

For simplicity, in what follows, we will consider dimensionless quantities: we will assume that the maximal density on each road is $\rho_{\max} = 1$.

As in [19, 4] we discuss only stricly concave flux functions.

Assumption 2.1. We assume that the flux function $f(\rho)$ is $C^2(0, 1)$ and strictly concave function having a single maximum at $\rho = \sigma$, and vanishing at zero and at the maximal density, i.e., $f(0) = f(1) = 0$.

Under the requirements of Assumption 2.1 we obtain: For any $\rho \in [0,1], \rho \neq \sigma$ there exists a unique number $\tau(\rho) \in [0,1]$ such that

$$f(\rho) = f(\tau(\rho)),\ \tau(\rho) \neq \rho. \tag{6}$$

It is well–known that under Assumption 2.1 the Riemann problem

$$\partial_t \rho + \partial_x f(\rho) = 0, \quad \rho(x,0) = \rho_0(x) = \begin{cases} \rho_l & \text{for} \quad x < 0, \\ \rho_r & \text{for} \quad x \geq 0, \end{cases} \tag{7}$$

admits the following unique entropy solution:

(i) If $\rho_l \leq \rho_r$, then the solution consists of a shock wave (which in vehicular traffic corresponds to a braking) connecting ρ_l with ρ_r.

$$\rho(x,t) = \begin{cases} \rho_l & \text{if} \quad x < \dfrac{f(\rho_r) - f(\rho_l)}{\rho_r - \rho_l} t,\ t \geq 0, \\ \rho_r & \text{if} \quad x > \dfrac{f(\rho_r) - f(\rho_l)}{\rho_r - \rho_l} t,\ t \geq 0. \end{cases} \tag{8}$$

(ii) If $\rho_l > \rho_r$, then the solution is a rarefaction wave (which in vehicular traffic corresponds to an acceleration) connecting ρ_l with ρ_r.

$$\rho(x,t) = \begin{cases} \rho_l & \text{if} \quad x \leq f'(\rho_l)t,\ t \geq 0, \\ (f')^{-1}\left(\dfrac{x}{t}\right) & \text{if} \quad f'(\rho_l)t \leq x \leq f'(\rho_r)t,\ t \geq 0, \\ \rho_r & \text{if} \quad x > f'(\rho_r)t,\ \geq 0. \end{cases} \tag{9}$$

Next, we introduce the notion of supply and demand functions in the framework of the LWR model (1). Let ρ be a given density. Then, the demand function $\rho \longmapsto d(\rho)$ (respectively the supply function $\rho \longmapsto s(\rho)$) is defined as the non decreasing part of the flux function f, see Figure 1 (respectively the non increasing part, see Figure 2). Hence,

$$d(\rho) = \begin{cases} f(\rho), & \rho \leq \sigma, \\ f(\sigma), & \rho > \sigma, \end{cases} \quad \text{and} \quad s(\rho) = \begin{cases} f(\rho), & \rho \geq \sigma, \\ f(\sigma), & \rho < \sigma. \end{cases} \tag{10}$$

It has been shown [22, 21], that the demand and supply functions correspond to the numerical fluxes in the Godunov discretization [9] of (1). The notion of supply and demand has been introduced independently by Lebacque [22] and Daganzo [6] in the context of vehicular traffic flow. It can be and has been extended to a class of second–order traffic flow models [18]. A similar construction has also been used by Dafermos [7] in order to prove existence for solutions in the case of a general but piecewise linear flux function.

The notion of demand and supply simplifies the discussion of admissible wave speeds in the following sense:

- Given a left constant Riemann datum ρ_l, then any state ρ_r can be connected to ρ_l by a wave with non-positive speed, if and only if

$$f(\rho_r) \leq d(\rho_l) \text{ and } \rho_r \in \{\rho_l\} \cup \{\rho : \rho \geq \sigma\}. \tag{11}$$

  If we restrict ourselves to waves with non-positive speed, the function f is invertible as long as the flow is strictly below the demand. If the flux is equal to the demand we can have a wave of zero speed. In order to exclude waves of zero speed, we require in this case that $\rho_r \neq \tau(\rho_l)$ (or equivalently $\rho_r = \rho_l$). If we modify waves of zero speed in this way, then f is invertible for all fluxes below or equal to the demand.

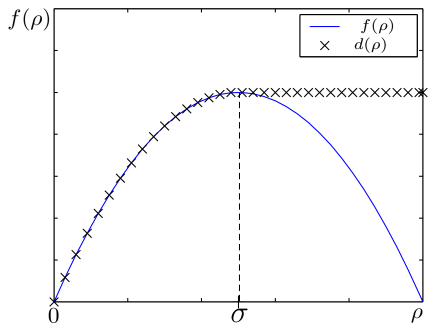

FIGURE 1. The demand function.

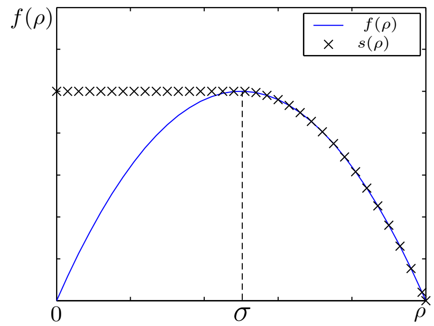

FIGURE 2. The supply function.

- A given constant Riemann datum on the right ρ_r, can be connected to any state ρ_l on the left with a wave of non negative speed, if and only if

$$f(\rho_l) \leq s(\rho_r) \text{ and } \rho_l \in \{\rho_r\} \cup \{\rho : \rho \leq \sigma\}. \tag{12}$$

  If we are interested in waves with non-negative speed, the function f is invertible as long as the flow is below the supply. Again, excluding waves of zero speed by requiring $\rho_r \neq \tau(\rho_l)$ if $f(\rho_r) = s(\rho_l)$, the function f is invertible for all given flows below or equal to the supply.
- In the settings of problem (7), for all times $t > 0$, we have

$$f(\rho(0,t)) = \min\{d(\rho_l), s(\rho_r)\}.$$

  Therefore, at $x = 0$, the Godunov flux [9] is the minimum between the supply and demand at this point.

Remark 2. The removal of the point $\tau(\rho)$ from the interval of admissible values for ρ ensures the invertibility of the flux function. This has already been discussed in [19, 4]. A state $\tau(\rho)$ connected to ρ with the same flux yields a wave of zero speed. Therefore, we proceed as in [19, 4] and we modify the value of the solution on a set of measure zero such that the solution is constant.

3. **Modeling intersections of vehicular traffic flow.** A solution to a road network is defined as a weak entropy solution (3), which additionally satisfies coupling conditions at the intersection [19, 4]. Due to Remark 1, it is obvious that (5) should hold. However, this condition is not sufficient to determine a unique solution and additional conditions have been proposed: For a generic junction n with δ_n^+ incoming, δ_n^- outgoing roads and constant initial data ρ_i , we define the sets Ω_i (see [19], [4]):

$$\Omega_i := [\sigma, 1] \quad \text{if} \quad \rho_i \geq \sigma, i \in \delta_n^+, \tag{13a}$$
$$\Omega_i := [\tau(\rho_i), 1] \cup \{\rho_i\} \quad \text{if} \quad \rho_i \leq \sigma, i \in \delta_n^+, \tag{13b}$$
$$\Omega_i := [0, \sigma] \quad \text{if} \quad \rho_i \leq \sigma, i \in \delta_n^-, \tag{13c}$$
$$\Omega_i := [0, \tau(\rho_i)] \cup \{\rho_i\} \quad \text{if} \quad \rho_i \geq \sigma, i \in \delta_n^-, \tag{13d}$$

For a given, strictly concave function g the coupling conditions in [19] read:

$$\max \sum_{i \in \delta_n^{\pm}} g(f(\bar{\rho}_i)) \quad \text{subject to: } (13), (5), \tag{14}$$

and an existence result has been proved in Theorem 1.1 [19]. As in the above discussion the solution is modified on a set of measure zero whenever it exhibits at the junction a stationary shock wave.

In [4], the following coupling condition has been proposed for given data $\rho_i, i \in \delta_n^{\pm}$. Depending on the data ρ_i we maximize for $\bar{\rho}_i$ belonging to either of the previously defined sets (13).

$$\max \sum_{i\in\delta_n^{\pm}} f(\bar{\rho}_i) \quad \text{subject to: } \bar{\rho}_i \in \Omega_i \text{ given by (13), (5) and} \qquad f(\bar{\rho}_i) = \sum_{k\in\delta_n^{+}} A_{ik} f(\bar{\rho}_k), \;\; \forall\, i \in \delta_n^{-}, \tag{15}$$

where $A \in \mathbb{R}^{\delta_n^- \times \delta_n^+}$ is a given matrix satisfying additional assumptions on its rank and nullspace. An existence result is available, provided that the node has degree less or equal than four.

Under the assumptions in Remark 2, we rewrite (14) and (15) respectively as follows:

$$\max \sum_{i\in\delta_n^{\pm}} g(f(\bar{\rho}_i)) \quad \text{subject to (5)}, f(\bar{\rho}_i) \le d(\rho_i)\,\forall\, i \in \delta_n^{+},\; f(\bar{\rho}_i) \le s(\rho_i)\,\forall\, i \in \delta_n^{-}, \tag{16}$$

and

$$\max \sum_{i\in\delta_n^{\pm}} f(\bar{\rho}_i) \quad \text{subject to (5)}, f(\bar{\rho}_i) = \sum_{k\in\delta_n^{+}} A_{ik} f(\bar{\rho}_k)\,\forall\, i \in \delta_n^{-}, \qquad f(\bar{\rho}_i) \le d(\rho_i)\,\forall\, i \in \delta_n^{+},\; f(\bar{\rho}_i) \le s(\rho_i),\;\; \forall\, i \in \delta_n^{-}. \tag{17}$$

In the case of a single incoming and outgoing road we obtain in both cases a value q defined by

$$\min\{d(\rho_1), s(\rho_2)\} =: q$$

and the corresponding states are $\bar{\rho}_1 = f^{-1}(q)$ and $\bar{\rho}_2 = f^{-1}(q)$, respectively.

3.1. **Novel model of an intersection.** We extend the previous models to the case where a buffer of finite storage capacity is introduced at a junction. This new approach is motivated by an effort for a relevant modeling of more complex intersections such as roundabouts (Figure 3) or on–ramps (Figure 4).

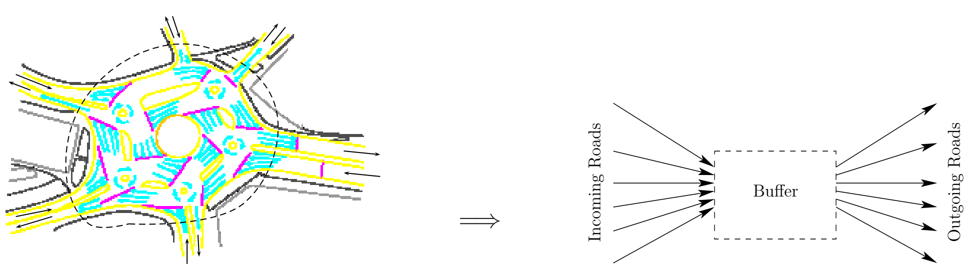

FIGURE 3. Example of an intersection with dynamics: the roundabout of Swindon (UK).

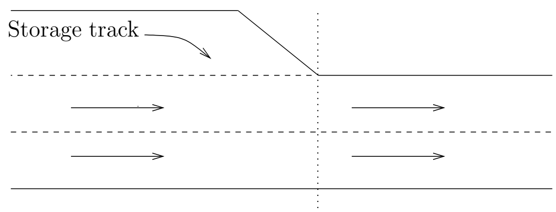

FIGURE 4. Example of storage track. The area where the third lane is closed, is modelled by the buffer.

In order to describe the qualitative behavior in these cases, we introduce at the junction a storage capacity whose content varies over time. We define the characteristics of the buffer by making the following assumptions: the intersection has a certain limited capacity so that vehicles can enter the intersection and could get stucked. If the buffer is empty, then flow proceeds like in an ordinary intersection. Clearly, no car should be lost inside the buffer. This can be modeled by an ordinary differential equation which describes the total number of cars in the buffer at each time. This number is denoted by $r(t)$ and it is bounded by $r_{\max}$ and zero which are the maximal and minimal capacity of the buffer, respectively. The rate of change $r'(t)$ is given by the difference between the inflow and the outflow. This leads to consider the following set of equations for a given intersection n:

$$\partial_t \rho_i + \partial_x f(\rho_i) = 0, \;\; \forall\, i \in \delta_n^+ \cup \delta_n^-, \tag{18a}$$

$$r_n'(t) = \sum_{i\in\delta_n^+} f(\rho_i(b_i,t)) - \sum_{i\in\delta_n^-} f(\rho_i(a_i,t)), \;\; \forall t \geq t_0, \tag{18b}$$

$$\rho_i(x,0) = \rho_{i,0} = \begin{cases} \rho_{i,0}^- & \text{for} \quad x < b_i, \text{ if } i \in \delta_n^-, \\ \rho_{i,0}^+ & \text{for} \quad x > a_i, \text{ if } i \in \delta_n^+, \end{cases} \tag{18c}$$

$$r_n(0) = r_{n,0}, \; 0 \leq r_{n,0} \leq r_{\max}, \tag{18d}$$

Furthermore, we prescribe the following rules for entering and existing flows.

- If the buffer is not empty, then, the demand of the buffer is constant, regardless of the current number of cars in the buffer. Therefore, the exiting flow from the buffer is idealized to be constant.
- If the buffer is not full, then, the supply of the buffer is constant, regardless of the current number of cars in the buffer. The flow entering the buffer is hence idealized to be constant.

To simplify the discussion we assume that the demand and the supply of the buffer are constant and equal to μ. For an intersection with multiple incoming and/or outgoing roads, we need to introduce some additional conditions in order to model the junction. In the case of multiple outgoing roads we need to distribute the outgoing flux according to some proportions α_i as in [4]. Whereas in the case of multiple incoming roads and a completely filled buffer, we need to decide which cars are allowed to enter next, see [18]. This is a way to deal with the matter of priorities or right–of–way on the incoming roads. A simpler modeling is to assume that the incoming roads have the *same* priority. Other choices are possible and the discussion proceeds analogously. Eventually, our modeling approach leads to the following problem at the intersection: the actual fluxes $q_i := f(\rho_i(b_i,t))$ if $i \in \delta_n^+$

and $q_j := f(\rho_j(a_j,t))$ if $j \in \delta_n^-$ satisfy for a.e. time $t > 0$

$$q_i = \min\{\frac{1}{\sum_i 1} s_B, d(\rho_i(b_i,t))\},\ q_j = \min\{\alpha_j d_B,\ s(\rho_j(a_j,t))\}, \tag{19}$$

$$d_B = \begin{cases} \mu & \text{if} \quad 0 < r(t) \le r_{\max}, \\ \min\{\sum_{i\in\delta_n^+} d(\rho_i(b_i,t)), \mu\} & \text{if} \quad r(t) = 0, \end{cases} \tag{20}$$

$$s_B = \begin{cases} \mu & \text{if} \quad 0 \le r(t) < r_{\max}, \\ \min\{\sum_{j\in\delta_n^-} \min(s(\rho_j(a_j,t)),\ \alpha_j\mu),\ \mu\} & \text{if} \quad r(t) = r_{\max}. \end{cases} \tag{21}$$

$$r_n'(t) = \sum_{i\in\delta_n^+} f(\rho_i(b_i,t)) - \sum_{j\in\delta_n^-} f(\rho_j(a_j,t)). \tag{22}$$

Here, α_j are non–negative given constants such that $\sum_j \alpha_j = 1$. Equation (19) states that the actual flow on the incoming road is bounded by the demand in order to obtain the correct wave speeds. Furthermore, if the supply of the buffer s_B is less than the total demand $\sum_i d_i$, then the right–of–way rule applies: e.g., one can assume that $q_i = c$ for all $i \in \delta_n^+$, where c is a constant. Since the total flow cannot exceed s_B, we obtain the previous formulation. Similarly, for the outgoing fluxes q_j: They are bounded by the supply on the corresponding road, $s(\rho_j(a_j,t))$, and by the maximal inflow toward this road, $\alpha_j d_B$. The demand of the buffer d_B is equal to μ, whenever the buffer is not empty. If the buffer is empty we require that at most the demand of the incoming roads is used to determine the actual outflows. Similarly, in the case of a full buffer only the minimum of $\alpha_j\mu$ and $s(\rho_j(a_j,t))$ is allowed as throughput through the buffer. Clearly, (22) states the conservation of mass in the buffer. Equations (19)-(22) define a Riemann solver at a single junction where roads are extended to infinity. At every time t the fluxes q are uniquely defined, see Lemma 3.1. However, a well–posed result could only be obtained in the case of a single incoming and a single outgoing road (then $\alpha = 1$), see Remark 4. This situation corresponds to the case of a storage track as depicted in Figure 4, where the third lane of the road section 1, upstream the lane reduction point plays the role of the buffer.

Remark 3. The intersections' modeling introduced here can be seen as the limit of a traffic flow model with a fundamental diagram depicted in Figure 5 when $\varepsilon \longrightarrow 0$ and when the artificial road has no physical extension, see [23].

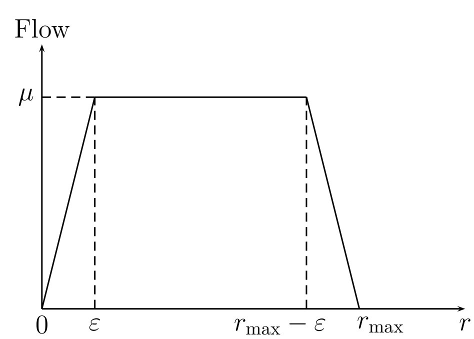

FIGURE 5. Fundamental diagram of the $\varepsilon-$approximation to the buffer.

Lemma 3.1. *Consider a single intersection with two connected roads and a buffer, i.e.,* $\delta_n^+ = \{1\}, \delta_n^- = \{2\}$*. Suppose that the requirements in Assumption 2.1 hold true. Let* $\mu > 0$ *be a given positive constant and let the LWR equation (1) holds on each road with* $\rho_{k,0}$*,* $k = 1, 2$*, constant initial data at time* $t_0 = 0$*. Assume that* $0 \le r_{n,0} \le r_{\max}$*. Then, there exists a unique weak entropy solution, on each road* i*, in the sense of (3) which additionally satisfies (19)-(22) for all times* $t > 0$*. The ordinary differential equation for the buffer (18b) is satisfied in the integral form* $r_n(t) = r_{n,0} + \int_0^t [f(\rho_1(b_1, y)) - f(\rho_2(a_2, y))]\, dy$ *for almost every* t*.*

Proof. To ease the presentation, we will assume that $b_1 = 0 = a_2$ and we will also skip the index n related to the junction. In the present case, equation (19) simplifies to $q_1 = \min\{s_B, d(\rho_{1,0})\}$ and $q_2 = \min\{d_B, s(\rho_{2,0})\}$ which uniquely defines q_i, $i = 1, 2$. Furthermore, for each q_i we find a unique state $\bar{\rho}_i$ such that $q_i = f(\bar{\rho}_i)$ due to (11) and (12). Here, we modify, if necessary, the states such that no shocks of zero speed appear. Depending on the demand and the supply associated with the initial data, the buffer may increase or decrease and this gives rise to further simple waves. The final solution (ρ_1, ρ_2) is the juxtaposition of simple waves constructed below. We distinguish the following cases.

Case 1 $d(\rho_{1,0}) \ge \mu$, $s(\rho_{2,0}) \ge \mu$ and $0 \le r_0 \le r_{\max}$. This case is only applicable if $\mu \le f(\sigma)$. Due to (19)–(22), we have:

$$f(\bar{\rho}_1) = \min\{d(\rho_{1,0}), s_B\} = \mu;\ f(\bar{\rho}_2) = \min\{s(\rho_{2,0}), d_B\} = \mu;$$
$$r(t) = r_0 + t(f(\bar{\rho}_1) - f(\bar{\rho}_2)) = r_0 \Longrightarrow r'(t) = 0,\ \ \forall t \ge t_0.$$

The solution consists of at most two waves exiting the junction. The waves on the respective roads are obtained as a restriction of a solution to the classical Riemann problem (7) with initial data $\rho_l = \rho_{1,0}$ and $\rho_r = \bar{\rho}_1$ for road one and $\rho_l = \bar{\rho}_2$ and $\rho_r = \rho_{2,0}$ on road two. The solutions are restricted to $x \le 0$ for road one and $x \ge 0$ on road two.

Case 2 $d(\rho_{1,0}) < \mu$, $s(\rho_{2,0}) \ge \mu$. This case is only applicable if $\mu \le f(\sigma)$. Since $d(\rho_{1,0}) < \mu \le f(\sigma)$ we have $f(\bar{\rho}_1) = d(\rho_{1,0})$ and therefore the solution is constant on road one, i.e., $\rho_{1,0} = \bar{\rho}_1$.

If $r_0 = 0$, then

$$f(\bar{\rho}_1) = \min\{d(\rho_{1,0}), s_B\} = d(\rho_{1,0}); f(\bar{\rho}_2) = \min\{s(\bar{\rho}_2), d_B\} = d(\rho_{1,0});$$
$$r(t) = 0 + t(f(\bar{\rho}_1) - f(\bar{\rho}_1)) = 0 \Longrightarrow r'(t) = 0,\ \ \forall t \ge t_0.$$

The solution on road 2 is at most one wave obtained as a restriction to $x > 0$ of a solution to the Riemann problem with left and right data $\bar{\rho}_2$ and $\rho_{2,0}$.

If $0 < r_0 \le r_{\max}$, then

$$f(\bar{\rho}_1) = \min\{d(\rho_{1,0}), s_B\} = d(\rho_{1,0}); f(\bar{\rho}_2) = \min\{s(\rho_{2,0}), d_B\} = \mu;$$
$$r(t) = r_0 + t(f(\bar{\rho}_1) - f(\bar{\rho}_2)) < r_0 \Longrightarrow r'(t) < 0,\ \ \forall t \ge t_0 : r(t) > 0.$$

If $s(\rho_{2,0}) = \mu < f(\sigma)$, then the solution is constant on road two. If not, then the solution on road 2 is at most one wave obtained as a restriction to $x > 0$ of a solution to the Riemann problem with left and right data $\bar{\rho}_2$ and $\rho_{2,0}$.

Further, $r(t)$ decreases in time and there exists t^* such that $r(t^*) = 0$.

$$f(\bar{\rho}_1) = \min\{d(\rho_{1,0}), s_B\} = d(\rho_{1,0}); f(\tilde{\rho}_2) = \min\{s(\rho_{2,0}), d_B\} = d(\rho_{1,0});$$
$$r(t) = 0 + t(f(\bar{\rho}_1) - f(\tilde{\rho}_2)) = 0 \Longrightarrow r'(t) = 0, \ \ \forall t \geq t^*.$$

For $t > t^*$ we observe a wave, exiting the junction toward road two, given by a restriction of a solution to the Riemann problem with data $(\tilde{\rho}_{2,0}, \bar{\rho}_{2,0})$.

Case 3 $d(\rho_{1,0}) \geq \mu$, $s(\rho_{2,0}) < \mu$. This case is similar to Case 2 and we skip the discussion.

Case 4 $d(\rho_{1,0}) < \mu$, $s(\rho_{2,0}) < \mu$. In this case, there is no restriction on the sign of $\mu - f(\sigma)$. We distinguish two subcases.

4.1 $d(\rho_{1,0}) \leq s(\rho_{2,0})$. If $r_0 = 0$, then

$$f(\bar{\rho}_1) = \min\{d(\rho_{1,0}), s_B\} = d(\rho_{1,0}); f(\bar{\rho}_2) = \min\{s(\bar{\rho}_2), d_B\} = d(\rho_{1,0});$$
$$r(t) = 0 + t(f(\bar{\rho}_1) - f(\bar{\rho}_2)) = 0 \Longrightarrow r'(t) = 0, \ \ \forall t \geq t_0.$$

We have at most two outgoing waves constructed as in Case 1. Two waves appear if $\mu > f(\sigma)$ and $d(\rho_{1,0}) = s(\rho_{2,0}) = f(\sigma)$ with $\rho_{1,0} \neq \rho_{2,0} \neq \sigma$. Then, the flux at the intersection is $f(\sigma)$.
If $0 < r_0 \leq r_{\max}$, then

$$f(\bar{\rho}_1) = \min\{d(\rho_{1,0}), s_B\} = d(\rho_{1,0}); f(\bar{\rho}_2) = \min\{s(\rho_{2,0}), d_B\} = s(\rho_{2,0});$$
$$r(t) = r_0 + t(f(\bar{\rho}_1) - f(\bar{\rho}_2)) \leq r_0 \Longrightarrow r'(t) \leq 0.$$

If $d(\rho_{1,0}) = s(\rho_{2,0})$, then the buffer remains constant and we either have zero or two waves exiting the junction and constructed as in Case 1. If $d(\rho_{1,0}) < s(\rho_{2,0})$, then we have a strict inequality for r' and the buffer decreases. Therefore there $\exists\ t^* > t_0$ such that $r(t^*) = 0$. In this case $d(\rho_{1,0}) < f(\sigma)$ and therefore we only have at most one wave leaving the junction toward the outgoing road. Further, we have $\bar{\rho}_1 = \rho_{1,0}$. For all $t > t^*$, we have:

$$f(\bar{\rho}_1) = \min\{d(\rho_{1,0}), s_B\} = d(\rho_{1,0}); f(\tilde{\rho}_2) = \min\{s(\rho_{2,0}), d_B\} = d(\rho_{1,0});$$
$$r(t) = 0 + t(f(\bar{\rho}_1) - f(\tilde{\rho}_2)) = 0 \Longrightarrow r'(t) = 0, \ \ \forall t \geq t^*.$$

This yields another wave, exiting the junction toward the outgoing road, constructed as a restriction of a solution to the Riemann problem with data $\rho_l = \tilde{\rho}_2$ and $\rho_2 = \bar{\rho}_2$. Hence, we have at most two waves exiting the junction in subcase 4.1.

4.2 $d(\rho_{1,0}) > s(\rho_{2,0})$. This implies $s(\rho_{2,0}) < f(\sigma)$. If $0 \leq r_0 < r_{\max}$, then

$$f(\bar{\rho}_1) = \min\{d(\rho_{1,0}), s_B\} = d(\rho_{1,0}); f(\bar{\rho}_2) = \min\{s(\rho_{2,0}), d_B\} = s(\rho_{2,0});$$
$$r(t) = r_0 + t(f(\bar{\rho}_1) - f(\bar{\rho}_2)) > r_0 \Longrightarrow r'(t) > 0$$

Here, we observe one wave, leaving the junction toward the incoming road, as a restriction of solution to the Riemann problem for $\rho_l = \rho_{1,0}$ and $\rho_r = \bar{\rho}_1$. Furthermore, we have $\rho_{2,0} = \bar{\rho}_2$. The quantity $r(t)$ increases in time and there exists t^* such that $r(t^*) = r_{\max}$. For $t > t^*$, we have:

$$f(\tilde{\rho}_1) = \min\{d(\rho_{1,0}), s_B\} = s(\bar{\rho}_2); f(\bar{\rho}_2) = \min\{s(\bar{\rho}_2), d_B\} = s(\bar{\rho}_2);$$
$$r(t) = r_{\max} + t(f(\tilde{\rho}_1) - f(\bar{\rho}_2)) = r_{\max} \Longrightarrow r'(t) = 0, \ \ \forall t \geq t^*.$$

Another wave constructed as before with data $(\bar{\rho}_1, \tilde{\rho}_1)$ leaves the junction toward the incoming road.

If $r_0 = r_{\max}$, then

$$f(\bar{\rho}_1) = \min\{d(\rho_{1,0}), s_B\} = s(\rho_{2,0}); f(\bar{\rho}_2) = \min\{s(\rho_{2,0}), d_B\} = s(\rho_{2,0});$$
$$r(t) = r_{\max} + t(f(\bar{\rho}_1) - f(\bar{\rho}_2)) = r_{\max} \Longrightarrow r'(t) = 0, \ \ \forall t \geq t_0.$$

This subcase is similar to the first part of subcase 4.1. At most one wave emerges from the junction.

This finishes the proof. □

Remark 4. In the previous discussion we have seen that at most two waves emerge from the junction for some given initial data. However, this discussion relied on the fact that the flux function has been the *same* on both roads. If we have different flux functions on road 1 and road 2 we might as well obtain three waves exiting the junction. An example of a solution, in the $x-t-$plane, is given in Figure 6.

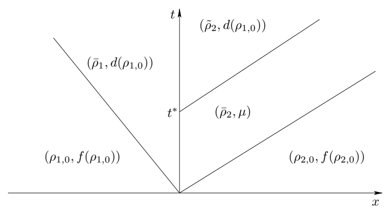

FIGURE 6. Solution to the Riemann problem in the phase space for different flux functions on the connected roads. Here, we consider the situation when the buffer reaches its maximal capacity at time t^*. In the case of the same flux functions on both connected roads we have a constant solution in the left part of the diagram (road one) and at most two outgoing waves on road two, c.f. Case 2.

Remark 5. A fully discussion has been carried out in the proof of Lemma 3.1 because the problem studied has the property that the buffer releases at most two waves. This is no longer true in the case of a junction with multiple incoming and outgoing roads. Although the simulations can be carried out (see Section 4), an attempt to establish a mathematical proof for a generic junction leads to an astronomic number of situations to be analyzed. Therefore, up to now the discussion is restricted to the case of a junction with one incoming and one outgoing road.

Next, we exemplify how the total variation may change when a wave arrives at the junction. Let us consider a very particular case where f is concave with single maximum and is additionally piecewise linear with a finite number of breakpoints. We consider a buffer with two connected roads. The initial data is assumed to be piecewise constant and equal to ρ_1^- and ρ_2^- nearby the buffer. We want to study the variation in the flux $f(\rho)$ when a wave hits the junction at time t_0. The time right before the collision is denoted by t_0^-. We assume further that

$$\mu > f(\sigma) \text{ and } \ r(t_0^-) = 0. \tag{23}$$

Since $\mu > f(\sigma)$, before the collision the junction is in Case 4. In order to study the collision we furthermore assume that

$$\rho_1^- < \sigma \text{ and } \rho_2^- < \sigma. \tag{24}$$

Due to (24), we have $d(\rho_1^-) = f(\rho_1^-) < s(\rho_2^-) = f(\sigma)$ and hence we are in Case 4.1. Hence, (23) can only be true, if $r(t_0^-) = 0$. Then, we have $\rho_2^- = \rho_1^-$. This is the starting point of the discussion of the interactions when a wave hits the junction. We consider the case where the wave is arriving from road one first. Since this wave must wave positive speed and f is piecewise linear, the wave is a traveling discontinuity connecting $\tilde{\rho} < \sigma$ to ρ_1^-. Assume that $d(\tilde{\rho}) > d(\rho_1^-)$. Since $d(\tilde{\rho}) < f(\sigma)$, we remain in Case 4.1 after the interaction and we have $f(\rho_2^+) = d(\tilde{\rho})$. We generate only a new traveling discontinuity on the outgoing road with right state $\rho_2^+ = \tilde{\rho}$. Furthermore, $r'(t) = 0$. Hence, the precollision variation satisfies $|f(\rho_1^-) - f(\tilde{\rho})| = |d(\rho_1^-) - d(\tilde{\rho})| = |f(\rho_2^-) - f(\rho_2^+)|$. In the case $d(\tilde{\rho}) < d(\rho_1^-)$ we obtain $f(\rho_1^+) = d(\tilde{\rho})$ and we remain again within Case 4.1. The solution is constant on the incoming road and has a traveling discontinuity of strength $|f(\rho_2^+ - f(\rho_2^-)| = |f(\rho_1^- - f(\tilde{\rho})|$. Hence, the pre- and postcollision variations in the flux coincide in this case.

Next, under the assumptions (23)–(24) consider the case where a wave is incoming from road two. Necessarily, we must have $\tilde{\rho}_2 > \tau(\rho_2^-) > \sigma$ for a traveling discontinuity of negative speed. Hence, $s(\tilde{\rho}_2) < s(\rho_2^-) = f(\sigma)$. Since $\rho_2^- = \rho_1^-$, we have $s(\tilde{\rho}_2) < d(\rho_1^-) = f(\rho_1^-)$ and we are in Case 4.2 for the interaction of $\tilde{\rho}_2$ with the junction. We obtain $f(\rho_1^+) = d(\rho_1^-)$ and $f(\rho_2^+) = s(\tilde{\rho}) \implies \rho_2^+ = \tilde{\rho}$. Therefore, no(!) waves emerge from the junction. However, the buffer content increases with rate: $r' = f(\rho_2^+) - f(\rho_1^+)$. Therefore, the precollision variation on road two is bounded by $|r'(t_0^+)|$. If no other collision hits the junction until time t^*, with $r(t^*) = r_{\max}$, then a new traveling discontinuity on road one will emerge. We have $f(\rho_1^{++}) = s(\rho_2^+)$ for $t > t^*$. This yields $|r_t'(t_0^+)| = |f(\rho_1^{++}) - f(\rho_1^+)| = |f(\rho_2^+ - f(\rho_1^-)|$. Hence, under the given assumptions on f and assumptions (23)–(24), the previous calculations show that

$$\sum_{j=1}^{2} T.V.(f_j(\rho_j(\cdot, t))) + |\partial_t r(t)| \leq \sum_{j=1}^{2} T.V.(f_j(\rho_j(\cdot, t_0))) + |\partial_t r(t_0)|.$$

Remark 6. This leads to the conjecture that the inequality might be true for all combinations of initial data, with respect to μ and incoming waves (in total about 120 cases) and if there are no interactions between t_0 and t^*. We will come back to this discussion in a forthcoming work.

4. **Numerical experiments.** We compare predictions of the new approach with the existing model in [4] for the network depicted in Figure 7. We consider junctions with different capacities for the buffer and we set $f(\rho) = \rho(1-\rho)$. For the diverging junctions ($1 \longrightarrow 2 - -3$ and $2 \longrightarrow 4 - -6$), we apply the distribution coefficients $\alpha_2 = \alpha_4 = 0.4$ and $\alpha_3 = \alpha_6 = 0.6$. On each road or arc i ($i = 1, \ldots, 7$) we use a Godunov discretization [9] of (1) with $N_i = 100$ discretization points in space and a time step satisfying the CFL condition. The results are depicted in the Figures 8 to 10. If the capacities of all the buffers are infinity and we prescribe a constant inflow, the new approach is equivalent to the previous models introduced in [4] and [23]. Then, the total inflow $f(\rho_1^{bound}) = 0.2475$ is recovered on road seven after some simulation time. This is also a stationary state of the network. In the next simulation (see Figure 9 and Figure 10), we set the capacity $\mu_2 = 0.01$ and we use the same inflow on road one. The inflow to the network is 0.21 and it is split at the first junction. Since the maximal capacity of the buffer at junction two is 0.01, we observe that the outflow on road seven, for the new model, is at most

$0.21 \times 60\% + 0.01 = 0.136$ which is observed in Figure 9 part (b). In Figure 9 (a) a traffic jam on road two and road three moves backwards through the network yielding a higher demand on road seven. Since there is no buffer on road two the jam continues until it reaches the inflow arc leaving completely crowded the network. This yields temporarily a higher outflow. In case (b) the buffer regulates the traffic by allowing only a smaller number of cars passing through road two and hereby clearing the network. Similar behavior is observed in the case of time–dependent flows as observed in the same simulation with a sinusoidal inflow (see Figure 10).

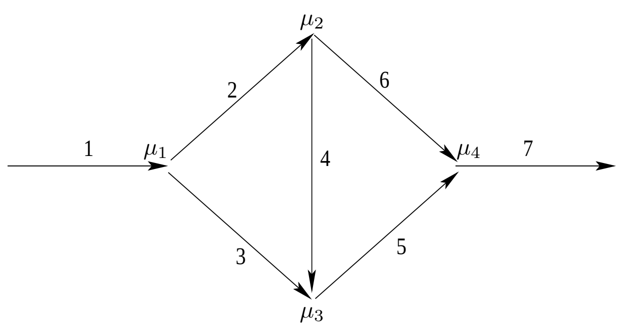

FIGURE 7. Road network used for the simulation.

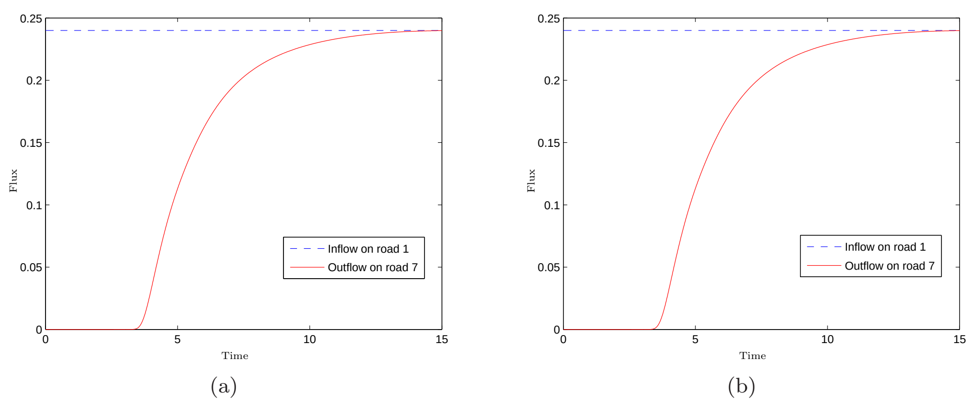

FIGURE 8. (a): Approaches in [19, 4], (b): The new approach. Settings: The network is initially empty i.e. $\rho_{i,0} = 0$, $i = 1, \ldots, 7$, whereas the boundary condition on Road 1 is constant during the simulation time ($\rho_1^{\text{bound}} = 0.45$). For the new approach we set $\mu_1 = \mu_2 = \mu_3 = \mu_4 = f(\sigma) = 0.25$.

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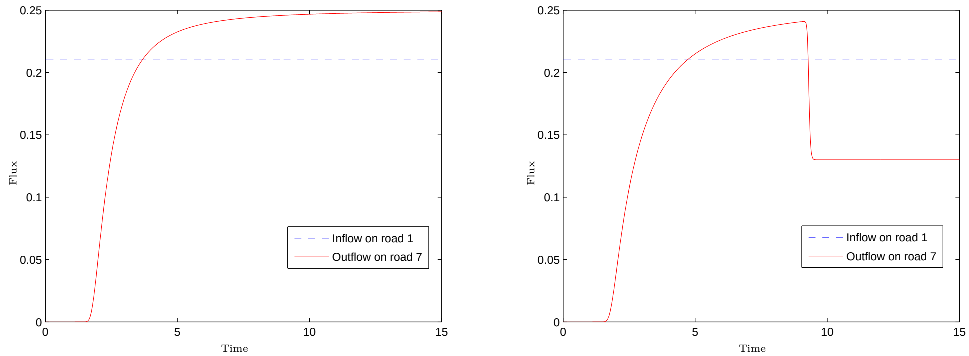
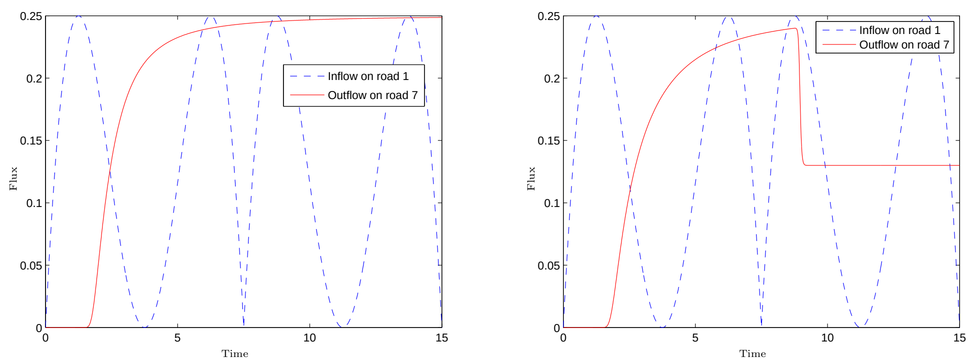


(a) (b)

FIGURE 9. (a): Approaches in [19, 4], (b): The new approach. Settings: The network is initially as follows: $\rho_{i,0} = 0$, $i \in \{1, 4, 5, 6, 7\}$, $\rho_{2,0} = \rho_{3,0} = 0.9$. The boundary condition on Road 1 is constant during the simulation time ($\rho_1^{\text{bound}} = 0.7$). For the new approach we set $\mu_1 = \mu_3 = \mu_4 = f(\sigma) = 0.25$ and $\mu_2 = 0.01$.



(a) (b)

FIGURE 10. (a): Approaches in [19, 4], (b): The new approach. Settings: The network is initially as follows: $\rho_{i,0} = 0$, $i \in \{1, 4, 5, 6, 7\}$, $\rho_{2,0} = \rho_{3,0} = 0.9$. The boundary condition on Road 1 is sinusoidal during the simulation time ($\rho_1^{\text{bound}} = f^{-1}(|\sin(\cdot)|)$, $\rho_7^{\text{bound}} = 0$). For the new approach we set $\mu_1 = \mu_3 = \mu_4 = f(\sigma) = 0.25$ and $\mu_2 = 0.01$.

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