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SPECTRUM ANALYSIS OF A SERIALLY CONNECTED EULER-BERNOULLI BEAMS PROBLEM

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Abstract. In this article we analyse the eigenfrequencies of a hyperbolic system which corresponds to a chain of Euler-Bernoulli beams. More precisely we show that the distance between two consecutive large eigenvalues of the spatial operator involved in this evolution problem is superior to a minimal fixed value. This property called spectral gap holds as soon as the roots of a function denoted by f_∞ (and giving the asymptotic behaviour of the eigenvalues) are all simple. For a chain of N different beams, this assumption on the multiplicity of the roots of f_∞ is proved to be satisfied. A direct consequence of this result is that we obtain the exact controllability of an associated boundary controllability problem. It is well-known that the spectral gap is a important key point in order to get the exact controllabilty of these one-dimensional problem and we think that the new method developed in this paper could be applied in other related problems.

1. **Introduction.** In the last few years various physical models of multi-link flexible structures consisting of finitely many interconnected flexible elements such as strings, beams, plates, shells have been mathematically studied. See [6], [7], [12], [19], [21] for instance. The spectral analysis of such structures has some applications to control or stabilization problems ([19] and [20]). For interconnected strings (corresponding to a second-order operator on each string), a lot of results have been obtained: the asymptotic behaviour of the eigenvalues ([1], [2], [5], [27]), the relationship between the eigenvalues and algebraic theory (cf. [3], [4], [19], [26]), qualitative properties of solutions (see [5] and [29]) and finally studies of the Green function (cf. [17], [30], [31]).

For interconnected beams (corresponding to a fourth-order operator on each beam), some results on the asymptotic behaviour of the eigenvalues and on the relationship between the eigenvalues and algebraic theory were obtained in [14], [15] and [16] with different kinds of connections using the method developed in [3] to get the characteristic equation associated to the eigenvalues.

The same method was used in ([24]) to compute the spectrum for a hybrid system of N flexible beams connected by n vibrating point masses. This type of structure was studied by Castro and Zuazua in many papers (see [8], [9], [10], [11], [13]) and Hansen and Zuazua([18]). They have restricted themselves to the case of two beams

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applying their results on the spectral theory to controllability. In a recent paper [25], the author investigated with V. Régnier the same problem as in [11] but for serially connected beams: for a chain of N interconnected beams with interior point masses the asymptotic behaviour of the spectrum was analysed in order to get the exact controllability. The controllability was proven for $N = 2$ beams with unspecified material coefficients. For $N \geq 3$, the more complicated asymptotic analysis could be studied in some particular cases; but for any value of N the problem had remained open.

The aim of this work is to develop a new method which allows the asymptotic analysis of the spectrum for a problem of N serially interconnected beams. The main idea is to get the spectral gap; indeed for that kind of control problem, via the Hilbert Uniqueness Method (HUM, see [22],[23]), we have to prove an observability inequality and then, if the solution is expressed in terms of Fourier series, Ingham's inequalities lead to the controllability only if the eigenelements satisfy some conditions. The most important of these conditions is the so-called spectral gap. Unfortunately, even for simple models the spectral gap is not easy to get. In this work, we prove the spectral gap for a problem of N serially interconnected beams.

Let us emphasize on the fact that we think that this method can be used for the study of other serially connected beams (i.e with other transmission and boundary conditions or with interior point masses such as treated in [11] or [25]). The different steps are the same ones; only calculations would be different.

The schedule of the paper is the following one:
In Section 2 we state some notation concerning our 1-d network and introduce the spatial operator, namely a fourth order operator on each edge with some transmission conditions at interior nodes and clamped boundary conditions at exterior nodes.

In Section 3 we study the spectrum.

In Subsection 3.1 we explicitly compute the characteristic equation whose roots are the eigenvalues associated to our problem.

In Subsection 3.2 we give a useful boundary property that we will use in Section 6.

In Section 4 we study the asymptotic behaviour of the spectrum:

In Subsection 4.1, we give a simple example which shows the difficulty in calculating and analyzing the spectrum for great eigenfrequencies. This example provides also a motivation to introduce the exterior matrix. In Subsection 4.2 we present the exterior matrix method (see [28]) which will permit us to compute the function (denoted by f_∞) whose roots are the approximations of the great eigenvalues.

Finally, in Section 5 we prove the spectral gap which is the most important condition that we need in order to prove the exact controllability.

In section 6, we use our work to solve a problem control linked with our eigenvalues problem.

Remark: Several times in this work, calculations were made with the assistance of a formal computation software (Mathematica). The specific commands used to get the result are given in an appendix at the end of the paper.

2. Data and framework.

2.1. Domain and notations. The domain that we consider is a network of N ($N \in \mathbb{N} - \{0,1\}$) serially connected beams which can be modelled by a graph $G = \bigcup_{j=1}^{N} k_j$. Each branch k_j having an origin and an end such that the end of the branch $k_j, (1 \le j \le N-1)$ is connected to the beginning of the branch k_{j+1}. By the intermediary of a parameterization we will identify each branch k_j with the interval $[0, l_j]$, 0 represents the beginning of k_j and l_j the end. For each branch k_j, we fix mechanical constants $m_j > 0$ (the mass density of the beam k_j) and $a_j = E_j I_j > 0$ (the flexural rigidity of k_j). The vibration of the branch k_j is modelled by the function $u_j(t,x), t \ge 0, x \in [0, l_j], j = 1, ..., N$. The total vibration of the structure is the vectorial function $u = (u_j)_{j=1,...,N}$.

Notation for derivative. In this paper, for a fonction $u = u(t,x)$ we make the choice to denote by u_t $(u_{tt}, ...,)$ the first (second,...) time derivative and $u^{(1)}$ $(u^{(2)}, ...)$ the first (second,...) spatial derivative.

2.2. Operator and spectral problem. The Euler-Bernouilli beams connected problem that we consider is associated to the following operator $\mathcal{A}$ on the Hilbert space $H = \prod_{j=1}^{N} L^2((0, l_j))$, endowed with the inner product

$$(u, v)_H = \sum_{j=1}^{N} m_j \int_0^{l_j} u_j(x) v_j(x) dx.$$

$$\begin{cases} D(\mathcal{A}) = \{u \in H : u_j \in \prod_{j=1}^{N} H^4((0, l_j)) \text{ satisfying (2) to (6) hereafter}\} \\ \forall u \in D(\mathcal{A}) : \mathcal{A}u = (\frac{a_j}{m_j} u_j^{(4)})_{j=1}^{N} \end{cases} \tag{1}$$

$$u_j(l_j) = u_{j+1}(0), \; j = 1, ..., N-1. \tag{2}$$

$$u_j^{(1)}(l_j) = u_{j+1}^{(1)}(0), \; j = 1, ..., N-1. \tag{3}$$

$$a_j u_j^{(2)}(l_j) = a_{j+1} u_{j+1}^{(2)}(0), \; j = 1, ..., N-1. \tag{4}$$

$$a_j u_j^{(3)}(l_j) = a_{j+1} u_{j+1}^{(3)}(0), \; j = 1, ..., N-1. \tag{5}$$

$$u_1(0) = u_1^{(1)}(0) = u_N(l_N) = u_N^{(1)}(l_N) = 0. \tag{6}$$

Notice that conditions (2) to (5) represent the transmission conditions while conditions (6) correspond to the boundary conditions.

Remark that $\mathcal{A}$ is a nonnegative selfadjoint operator with a compact resolvant ([16], Th 2.1). Indeed $\mathcal{A}$ is the Friedrichs extension of the triple (H, V, a) defined by

$$V = \{u \in \prod_{j=1}^{N} H^2((0, l_j)) \text{ satisfying (2), (3), (6)}\},$$

which is a Hilbert space with the inner product

$$(u,v)_V=\sum_{j=1}^{N}(u_j,v_j)_{(H^2((0,l_j))},$$

where $(.,.)_{(H^2((0,l_j))}$ is the usual H^2-inner product on $(0,l_j)$ and

$$a(u,v)=\sum_{j=1}^{N}a_j\int_0^{l_j}u_j^{(2)}(x)v_j^{(2)}(x)dx.$$

For our next purpose let us denote $\sigma(\mathcal{A})=(\lambda_k^2)_{k\in\mathbb{N}^*}$, $\lambda_k>0$ the monotone increasing sequence of eigenvalues of $\mathcal{A}$ and recall the classical result $\lim_{k\to+\infty}\lambda_k=+\infty$. The eigenvalue problem associated to the operator $\mathcal{A}$ can be written as: $\lambda_k^2\in\sigma(\mathcal{A})$ $(\lambda_k>0)$ is an eigenvalue of $\mathcal{A}$ with associated eigenvector $U_k=(u_1^k,u_2^k,...,u_N^k)\in D(\mathcal{A})$ if and only if U_k satisfies the transmission and boundary conditions (2)-(6) and

$$\begin{cases}\dfrac{a_j}{m_j}(u_j^k)^{(4)}(x)=\lambda_k^2u_j^k(x) & \text{on } (0,\mathrm{l}_j),\ \ \forall\ \mathrm{j}\in\ \{1,...,\mathrm{N}\}\\ u_j^k\in H^4((0,l_j)), & \forall j\in\{1,...,N\}\end{cases}\tag{7}$$

3. Spectral properties.

3.1. The characteristic equation.

First, let us introduce some useful notations.

Notations.
Let $U=(u_1,...,u_N)$ be a non-trivial solution of the eigenvalue problem (7) (given in subsection 2.2) and λ^2 $(\lambda>0)$ be the corresponding eigenvalue.
For each $j\in\{1,...,N\}$, the vector function V_j is defined by

$$V_j(x)=(u_j(x),\,u_j^{(1)}(x),\,a_ju_j^{(2)}(x),\,a_ju_j^{(3)}(x))^t,\ \forall x\in[0,l_j].$$

Keeping the notation a_j and l_j introduced in Subsection 2.1, the matrix A_j is $A_j=A(q_j,\,b_j,m_j)$ with $q_j=(\dfrac{m_j}{a_j})^{\frac14}$, $b_j=l_jq_j$ and $A(q,\,b,\,m)$ the square matrix of order 4 is given by

$$A(q,b,m)=\frac12\begin{pmatrix} ch+c & \dfrac1q\dfrac{sh+s}{\sqrt\lambda} & \dfrac{q^2}{m}\dfrac{ch-c}{\lambda} & \dfrac{q}{m}\dfrac{sh-s}{\lambda\sqrt\lambda}\\ q\sqrt\lambda(sh-s) & ch+c & \dfrac{q^3}{m}\dfrac{sh+s}{\sqrt\lambda} & \dfrac{q^2}{m}\dfrac{ch-c}{\lambda}\\ \dfrac{m}{q^2}\lambda(ch-c) & \dfrac{m}{q^5}\sqrt\lambda(sh-s) & ch+c & \dfrac1q\dfrac{sh+s}{\sqrt\lambda}\\ \dfrac{m}{q}\lambda\sqrt\lambda(sh+s) & \dfrac{m}{q^2}\lambda(ch-c) & q\sqrt\lambda(sh-s) & ch+c\end{pmatrix}\tag{8}$$

with the notation $c=\cos(b\sqrt\lambda)$, $s=\sin(b\sqrt\lambda)$, $ch=\cosh(b\sqrt\lambda)$, $sh=\sinh(b\sqrt\lambda)$.

To finish with, the matrix $M(\lambda)$ is given by

$$M(\lambda)=A_NA_{N-1}...A_2A_1.\tag{9}$$

with the trivial but useful properties:

Lemma 3.1. *With the notation introduced above, it holds:*

$$\begin{array}{l} V_j(l_j) = A_j V_j(0),\ \forall j \in \{1, ..., N\} \\ V_{j+1}(0) = V_j(l_j),\ \forall j \in \{1, ..., N-1\} \\ V_N(l_N) = M(\lambda) V_1(0) \end{array}$$

Proof. Since u_j satisfies the first equation of the eigenvalue problem (7), u_j is a linear combination of the vectors of the fundamental basis

$$(e_{1j}, e_{2j}, e_{3j}, e_{4j}) = \Big(\cos(q_j^{-1}\sqrt{\lambda}\,.),\ \sin(q_j^{-1}\sqrt{\lambda}\,.),\ \cosh(q_j^{-1}\sqrt{\lambda}\,.),\ \sinh(q_j^{-1}\sqrt{\lambda}\,.)\Big).$$

Let (a_1, a_2, a_3, a_4) be the coordinates of u_j in this basis. Then, we can write

$$u_j(x) = \sum_{i=1}^{4} a_i e_{ij}(x),\ \forall x \in\ [0, l_j],\ \textit{and}\ V_j(x) = K_j(x).(a_1, a_2, a_3, a_4)^t, \tag{10}$$

where $K_j(x)$ is the square matrix:

$$\begin{pmatrix} e_{1j}(x) & e_{2j}(x) & e_{3j}(x) & e_{4j}(x) \\ e_{1j}^{(1)}(x) & e_{2j}^{(1)}(x) & e_{3j}^{(1)}(x) & e_{4j}^{(1)}(x) \\ a_j e_{1j}^{(2)}(x) & a_j e_{2j}^{(2)}(x) & a_j e_{3j}^{(2)}(x) & a_j e_{4j}^{(2)}(x) \\ a_j e_{1j}^{(3)}(x) & a_j e_{2j}^{(3)}(x) & a_j e_{3j}^{(3)}(x) & a_j e_{4j}^{(3)}(x) \end{pmatrix}.$$

With a simple calculation, it is easy to see that $K_j(0)$ is invertible. Hence, from (10), we get

$$V_j(l_j) = K_j(l_j) K_j(0)^{-1} V_j(0).$$

The matrix A_j is the matrix $K_j(l_j)K_j(0)^{-1}$ and its expression given in (8) follows after some calculations (see Appendix 1.).
Now the transmission conditions (2-4) imply the second equation.
The third one is the logical consequence of the first two applied successively for $j = 1$, $j = 2$, etc... □

Theorem 3.2. *(The characteristic equation)*
Let $\lambda^2 > 0$ be an eigenvalue of $\mathcal{A}$ then λ satisfies the characteristic equation

$$f(\sqrt{\lambda}) = \det(M_{12}(\lambda)) = 0, \tag{11}$$

where $M_{12}(\lambda)$ is the square matrix of order 2 which is the restriction of the matrix $M(\lambda)$, given by (9), to its first two lines and its last two columns.

Proof. Let U be a non-trivial solution of the eigenvalue problem (7) and λ^2 ($\lambda > 0$) be the corresponding eigenvalue. The matrix $M(\lambda)$ is rewritten as $M(\lambda) = \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) \end{pmatrix}$ where $M_{ij}(\lambda)$ is a square matrix of order 2, for $(i, j) \in \{1, 2\}^2$.

Now, using the boundary conditions (6) as well as $V_N(l_N) = M(\lambda)V_1(0)$, it follows:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = M_{12}(\lambda) \begin{pmatrix} a_1 u_1^{(2)}(0) \\ a_1 u_1^{(3)}(0) \end{pmatrix}.$$

It is clear that the vector of the second part of the previous identity is non-trivial since u is a non-trivial solution of problem (7). The result follows. □

3.2. **Multiplicity of the spectrum and boundary estimate for the eigenvectors.** The aim of this subsection is to prove a useful result that we will use in section 6.

Lemma 3.3. *Let $\mu = \lambda^2 \in \sigma(\mathcal{A})$ (with $\lambda > 0$) be an eigenvalue of the operator $\mathcal{A}$ with associated eigenfunction u. Then the geometric multiplicity of μ is 1 and* $|u_1^{(2)}(0)|^2 + |u_N^{(2)}(l_N)|^2 \neq 0$.

Proof. First notice that all the terms of the matrix $A(q,b,m)$ defined in Section 3.1 are of the form

$$c(\lambda,q,m)\left(\cosh(b\sqrt{\lambda}) + \epsilon\cos(b\sqrt{\lambda})\right) \text{ or } c(\lambda,q,m)\left(\sinh(b\sqrt{\lambda}) + \epsilon\sin(b\sqrt{\lambda})\right)$$

where $\epsilon \in \{-1;1\}$, $c(\lambda,q,m)$ is strictly positive for any $q,m>0$ and $\lambda>0$ and the functions $\cosh(x)+\epsilon\cos(x)$ and $\sinh(x)+\epsilon\sin(x)$ are strictly positive on $(0;+\infty)$. Thus the terms of the matrix $A(q,b,m)$ are all strictly positive if q, m, b and λ are all strictly positive.

As a consequence, the matrix $M(\lambda)$ has only strictly positive terms for any $\lambda>0$. From that, we deduce that the matrix $M_{12}(\lambda)$ is not the null matrix for any $\lambda>0$. Then its rank is one and the geometric multiplicity of $\mu=\lambda^2$ is 1.

On the other hand we established in the proof of Theorem 3.2 that:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = M_{12}(\lambda)\begin{pmatrix} u_1^{(2)}(0) \\ u_1^{(3)}(0) \end{pmatrix}.$$

Now if we assume that $u_1^{(2)}(0)=0$ then $u_1^{(3)}(0)\neq 0$ (else u would vanish). But this is equivalent to say that $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of $M_{12}(\lambda)$.

This is in contradiction with the fact that all the terms of M_{12} are strictly positive. This yields the result □

4. **Asymptotic behaviour of the eigenvalues.** As announced in the introduction, our aim is to prove the spectral gap in order to get controllability. So we need to know the asymptotic behaviour of the spectrum. To this end, the asymptotic behaviour of the characteristic equation (11) as $\lambda\to+\infty$ is of great interest.

4.1. **Example.** In order to introduce the method developed in the sequel, we start with an simple example which shows the difficulty to compute and analyse the spectrum for large eigenvalues. We consider the problem of two identical interconnected beams ($N=2, q_i=b_i=m_i=1$, $i=1,2$). Thus $M(\lambda)=A(1,1,1)^2$, and using the classical equalities $\cosh(\sqrt{\lambda}) = \dfrac{e^{\sqrt{\lambda}}+e^{-\sqrt{\lambda}}}{2}$ and $\sinh(\sqrt{\lambda}) = \dfrac{e^{\sqrt{\lambda}}-e^{-\sqrt{\lambda}}}{2}$, we find after calculation (see Appendix 2.):

$$M_{12}(\lambda) = \frac{1}{4}\begin{pmatrix} \dfrac{e^{2\sqrt{\lambda}}-2\cos(2\sqrt{\lambda})+e^{-2\sqrt{\lambda}}}{\lambda} & \dfrac{e^{2\sqrt{\lambda}}-2\sin(2\sqrt{\lambda})-e^{-2\sqrt{\lambda}}}{\sqrt{\lambda}^3} \\ \dfrac{e^{2\sqrt{\lambda}}+2\sin(2\sqrt{\lambda})-e^{-2\sqrt{\lambda}}}{\sqrt{\lambda}} & \dfrac{e^{2\sqrt{\lambda}}-2\cos(2\sqrt{\lambda})+e^{-2\sqrt{\lambda}}}{\lambda} \end{pmatrix}.$$

Consequently we deduce the characteristic equation :

$$f(\sqrt{\lambda}) = det(M_{12}(\lambda)) = \frac{-1}{4\lambda^2}(e^{2\sqrt{\lambda}}\cos(2\sqrt{\lambda}) - 2 + e^{-2\sqrt{\lambda}}\cos(2\sqrt{\lambda})) = 0.$$

Multiplying the previous equality by $\dfrac{-4\lambda^2}{e^{2\sqrt{\lambda}}}$ we get that the characteristic equation is equivalent with:

$$f_\infty(\sqrt{\lambda}) + g(\sqrt{\lambda}) = 0,$$

where we have set $f_\infty(\sqrt{\lambda}) = \cos(2\sqrt{\lambda})$ and $(g\sqrt{\lambda}) = -2e^{-2\sqrt{\lambda}} + e^{-4\sqrt{\lambda}}\cos(2\sqrt{\lambda}))$. Now, since $\lim\limits_{k\to+\infty} \lambda_k = +\infty$ and $\lim\limits_{\lambda_k\to+\infty} g(\sqrt{\lambda_k}) = 0$, the asymptotic behaviour of the λ_k^2's is given by the roots of $f_\infty\left(\sqrt{\lambda}\right) = \cos(2\sqrt{\lambda}) = 0$.

From this example, let us make the following remarks which remain valid in the general case: Even though we have a way of finding eigenfrequencies with the equation (11) we have serious problems numerically: indeed, in the numerical computation of the matrix $M_{12}(\lambda)$ appear exponential terms which increase very quickly according to λ. This means that calculating (11) via a decimal approximation would be unreliable. Obviously the same problem remains when we want to analyse the asymptotic behaviour of the spectrum. In [24] we saw that the asymptotic analysis of (11) was difficult because calculation is very complicated even for small values of N (i.e $N = 3$) and also with the help of softwares such as *Mathematica.* On the other hand, we see that the knowledge of f_∞ is enough to compute easily large eigenvalues and, may be, to prove the spectral gap. Unfortunately this example shows that it is not a easy task to get f_∞ in the general case: indeed, the highest terms in $M_{12}(\lambda)$ (i.e the factors terms of $e^{2\sqrt{\lambda}}$) are easy to obtain, but, when one calculates the determinant of $M_{12}(\lambda)$ in order to obtain the characteristic equation, we see that the corresponding term cancel (i.e the factors terms of $e^{4\sqrt{\lambda}}$). This phenomenon is still true in the general case (i.e when we consider N beams); when computing the determinant, several highest terms cancel, thus it is very difficult to get f_∞.

In the following subsection we introduce a method which allows the computation of f_∞.

4.2. **The exterior matrix method.** The exterior matrix method presented in [28] is a very useful method which allows to compute asymptotically the eigenfrequencies for the vibrations of serially connected elements which are governed by fourth-order equations. The main idea of the exterior matrix method is that it is a way to compute the determinant before the matrices are multiplied together, so that the major cancellation occurs first.

First, we simply recall the definition of exterior matrix and some useful results that we need in the sequel.

Definition 4.1. If $M = (m_{ij})$ is a 4×4 matrix, then the exterior matrix of M is the 6×6 matrix given by:

$$ext(M) = \begin{pmatrix} ext(M)_{11} & ext(M)_{12} \\ ext(M)_{21} & ext(M)_{22} \end{pmatrix},$$

where each block $ext(M)_{ij}, i, j = 1, 2,$ is a 3×3 matrix given hereafter:

$$ext(M)_{11} = \begin{pmatrix} \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} & \begin{vmatrix} m_{13} & m_{12} \\ m_{21} & m_{23} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{14} \\ m_{21} & m_{24} \end{vmatrix} \\ \begin{vmatrix} m_{11} & m_{12} \\ m_{31} & m_{32} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{14} \\ m_{31} & m_{34} \end{vmatrix} \\ \begin{vmatrix} m_{11} & m_{12} \\ m_{41} & m_{22} \end{vmatrix} & \begin{vmatrix} m_{13} & m_{12} \\ m_{41} & m_{23} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{14} \\ m_{41} & m_{24} \end{vmatrix} \end{pmatrix},$$

$$ext(M)_{12} = \begin{pmatrix} \begin{vmatrix} m_{13} & m_{14} \\ m_{23} & m_{24} \end{vmatrix} & -\begin{vmatrix} m_{12} & m_{14} \\ m_{22} & m_{24} \end{vmatrix} & \begin{vmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{vmatrix} \\ \begin{vmatrix} m_{13} & m_{14} \\ m_{33} & m_{34} \end{vmatrix} & -\begin{vmatrix} m_{12} & m_{14} \\ m_{32} & m_{34} \end{vmatrix} & \begin{vmatrix} m_{12} & m_{14} \\ m_{32} & m_{33} \end{vmatrix} \\ \begin{vmatrix} m_{13} & m_{14} \\ m_{43} & m_{24} \end{vmatrix} & -\begin{vmatrix} m_{12} & m_{14} \\ m_{42} & m_{24} \end{vmatrix} & \begin{vmatrix} m_{12} & m_{13} \\ m_{42} & m_{23} \end{vmatrix} \end{pmatrix},$$

$$ext(M)_{21} = \begin{pmatrix} \begin{vmatrix} m_{31} & m_{12} \\ m_{41} & m_{22} \end{vmatrix} & \begin{vmatrix} m_{33} & m_{12} \\ m_{41} & m_{23} \end{vmatrix} & \begin{vmatrix} m_{31} & m_{14} \\ m_{41} & m_{24} \end{vmatrix} \\ -\begin{vmatrix} m_{21} & m_{22} \\ m_{41} & m_{42} \end{vmatrix} & -\begin{vmatrix} m_{21} & m_{23} \\ m_{41} & m_{43} \end{vmatrix} & -\begin{vmatrix} m_{21} & m_{24} \\ m_{41} & m_{44} \end{vmatrix} \\ \begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix} & \begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix} & \begin{vmatrix} m_{21} & m_{24} \\ m_{31} & m_{34} \end{vmatrix} \end{pmatrix},$$

$$ext(M)_{22} = \begin{pmatrix} \begin{vmatrix} m_{33} & m_{14} \\ m_{43} & m_{24} \end{vmatrix} & -\begin{vmatrix} m_{32} & m_{14} \\ m_{42} & m_{24} \end{vmatrix} & \begin{vmatrix} m_{32} & m_{13} \\ m_{42} & m_{23} \end{vmatrix} \\ -\begin{vmatrix} m_{23} & m_{24} \\ m_{43} & m_{44} \end{vmatrix} & \begin{vmatrix} m_{22} & m_{24} \\ m_{42} & m_{44} \end{vmatrix} & -\begin{vmatrix} m_{22} & m_{23} \\ m_{42} & m_{43} \end{vmatrix} \\ \begin{vmatrix} m_{23} & m_{24} \\ m_{33} & m_{34} \end{vmatrix} & -\begin{vmatrix} m_{22} & m_{24} \\ m_{32} & m_{34} \end{vmatrix} & \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix} \end{pmatrix}.$$

Lemma 4.2. *If M_1 and M_2 are 4×4 matrices, then*

$$ext(M_1 M_2) = ext(M_1) ext(M_2). \tag{12}$$

Proof. Sketch of the proof (for more details see Lemma 1 of [28].) Given a matrix $M \in \mathcal{M}_4(I\!R)$, we define a linear map M^* in $\mathcal{M}_4(I\!R)$ such that :

$$\forall A \in \mathcal{M}_4(I\!R),\ M^*(A) = MAM^T.$$

It is easy to prove that the map $M \to M^*$ is a homomorphism (i.e we have $M_1^* M_2^* = (M_1 M_2)^*$) and that M^* sends anti-symmetric matrices to anti-symmetric matrices, so we can restrict M^* to this subspace. A basis for the 4×4 anti-symmetric matrices is

$$e_1=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\; e_2=\begin{pmatrix}0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0\end{pmatrix},\; e_3=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\-1&0&0&0\end{pmatrix},$$

$$e_4=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\; e_5=\begin{pmatrix}0&0&0&0\\0&0&0&-1\\0&0&0&0\\0&1&0&0\end{pmatrix},\; e_6=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix}.$$

Using this basis, we find that M^*, when restricted to anti-symmetric matrices can be expressed by the 6×6 matrix $ext(M)$ given in Definition 4.1. (12) expresses that the map $M \to M^*$ is a homomorphism. □

Theorem 4.3. *(The characteristic equation)*
Let $\lambda^2 > 0$ be an eigenvalue of $\mathcal{A}$ then λ satisfies the characteristic equation

$$f(\sqrt{\lambda}) = e_1^t ext(M(\lambda))e_4 = 0, \tag{13}$$

or equivalently

$$f(\sqrt{\lambda}) = e_1^t ext(A_N)ext(A_{N-1})...ext(A_1)e_4 = 0, \tag{14}$$

where $M(\lambda)$ is the square matrix of order 4 given by (9), $e_1 = (1,0,0,0,0,0)$ and $e_4 = (0,0,0,1,0,0)$.

Proof. From Definition (4.1) we see that $e_1^t ext(M(\lambda))e_4$ is equal to $det(M_{12}(\lambda)$, this implies (13).
Then, applying Lemma 4.2 to $M(\lambda)$ we directly get (13). □

4.3. **The asymptotic behaviour of the exterior matrix.** Using Definition 4.1 and with the help of a formal calculation software (see Appendix 3.) we have the following property:

Lemma 4.4. *Let $A_j = A(q_j, b_j, m_j)$ be any matrix 4×4 given in (8).*
Then the matrix $ext(A_j) = ext(A(q_j, b_j, m_j))$ has the following expansion:

$$ext(A_j) = \frac{1}{4}e^{b_j\sqrt{\lambda}}H(q_j, b_j, m_j) + o\left(\frac{e^{b_j\sqrt{\lambda}}}{\sqrt{\lambda}^k}\right) \tag{15}$$

with

$$H(q_j, b_j, m_j) = (\cos(b_j\sqrt{\lambda})C(q_j, m_j) + \sin(b_j\sqrt{\lambda})S(q_j, m_j)), \tag{16}$$

$$C(q,m)=\begin{pmatrix} 1 & \frac{q^3}{m\sqrt{\lambda}} & 0 & -\frac{q^4}{m^2\lambda^2} & \frac{q}{m\lambda\sqrt{\lambda}} & 0 \\ \frac{m\sqrt{\lambda}}{q^3} & 2 & \frac{1}{q\sqrt{\lambda}} & -\frac{q}{m\lambda\sqrt{\lambda}} & 0 & \frac{1}{q\sqrt{\lambda}} \\ 0 & q\sqrt{\lambda} & 1 & 0 & -\frac{1}{q\sqrt{\lambda}} & 1 \\ -\frac{m^2\lambda^2}{q^4} & -\frac{m\lambda\sqrt{\lambda}}{q} & 0 & 1 & -\frac{m\lambda}{q^3} & 0 \\ \frac{m\lambda\sqrt{\lambda}}{q} & 0 & -q\sqrt{\lambda} & -\frac{q^3}{m\sqrt{\lambda}} & 2 & -q\sqrt{\lambda} \\ 0 & q\sqrt{\lambda} & 1 & 0 & -\frac{1}{q\sqrt{\lambda}} & 1 \end{pmatrix}, \quad (17)$$

$$S(q,m)=\begin{pmatrix} 0 & \frac{q^3}{m\sqrt{\lambda}} & \frac{q^2}{m\lambda} & 0 & -\frac{q}{m\lambda\sqrt{\lambda}} & \frac{q^2}{m\lambda} \\ -\frac{m\sqrt{\lambda}}{q^2} & 0 & \frac{1}{m\sqrt{\lambda}} & \frac{q}{m\lambda\sqrt{\lambda}} & -\frac{2}{q^2\lambda} & \frac{1}{q\sqrt{\lambda}} \\ -\frac{m\lambda}{q^2}\lambda & -q\sqrt{\lambda} & 0 & \frac{q^2}{m\lambda} & -\frac{1}{q\sqrt{\lambda}} & 0 \\ 0 & -\frac{m\lambda\sqrt{\lambda}}{q} & -\frac{m\lambda}{q^2}\lambda & 0 & \frac{m\sqrt{\lambda}}{q^3} & -\frac{m\sqrt{\lambda}}{q^2} \\ \frac{m\lambda\sqrt{\lambda}}{q} & 2q^2\lambda & q\sqrt{\lambda} & -\frac{q^3}{m\sqrt{\lambda}} & 0 & q\sqrt{\lambda} \\ -\frac{m\lambda}{q^2} & -q\sqrt{\lambda} & 0 & \frac{q^2}{m\lambda} & -\frac{1}{q\sqrt{\lambda}} & 0 \end{pmatrix} \quad (18)$$

and where $o\left(\frac{e^{b_j\sqrt{\lambda}}}{\sqrt{\lambda}^k}\right)$ *represents a* 6×6 *matrix with all its terms are small with respect to* $\frac{e^{b_j\sqrt{\lambda}}}{\sqrt{\lambda}^k}$ *as* $\lambda \longrightarrow +\infty$*,* k *being any integer (that means that for any* k *the limit of all the terms of the matrix* $\frac{\sqrt{\lambda}^k}{e^{b_j\sqrt{\lambda}}}o\left(\frac{e^{b_j\sqrt{\lambda}}}{\sqrt{\lambda}^k}\right)$ *is zero* $\lambda \longrightarrow +\infty$*.)*

Proof. First, in the matrix $A_j = A(q_j, b_j, m_j)$ given (8), we use the classical identities $\cosh(x)$ by $\frac{e^x+e^{-x}}{2}$ and $\sinh(x)$ by $\frac{e^x-e^{-x}}{2}$. Then, the definition 4.1 applied to the matrix A_j allows us to compute each terms of $ext(A_j)$. For instance we find that in the first line and fisrt column is:

$$\begin{aligned}(ext(A_j))_{11} &= \frac{1}{4}(e^{b_j\sqrt{\lambda}}\cos(b_j\sqrt{\lambda}) + 2 + e^{-b_j\sqrt{\lambda}}\cos(b_j\sqrt{\lambda})) \\ &= \frac{1}{4}(e^{b_j\sqrt{\lambda}}\cos(b_j\sqrt{\lambda}) + o(e^{b_j\sqrt{\lambda}})).\end{aligned}$$

Calculating the other terms in the same way, we arrive at (15) and the decomposition of the matrix $H(q_j, b_j, m_j)$ given by (16),(17) and (18). □

In the sequel we need some properties of the matrices $C(q,m)$ and $S(q,m)$ which are summarized in the following Lemma 4.5. But before we start with some practical

notation.
Notation.
For all $\overrightarrow{\epsilon}=(\epsilon_1,...,\epsilon_N)\in\{0,1\}^N$ and for all $i=1,...,N$ we set:

$$\begin{pmatrix} X_{i,\overrightarrow{\epsilon}}(q_i,m_i)=C(q_i,m_i) & \text{if} & \epsilon_i=0 \\ X_{i,\overrightarrow{\epsilon}}(q_i,m_i)=S(q_i,m_i) & \text{if} & \epsilon_i=1 \end{pmatrix}, \tag{19}$$

and

$$\begin{pmatrix} y_{i,\overrightarrow{\epsilon}}(b_i.)=\cos(b_i.) & \text{if} & \epsilon_i=0 \\ y_{i,\overrightarrow{\epsilon}}(b_i.)=\sin(b_i.) & \text{if} & \epsilon_i=1 \end{pmatrix}, \tag{20}$$

where $y_{i,\overrightarrow{\epsilon}}(b_i.)$ represents the function $x\mapsto y_{i,\overrightarrow{\epsilon}}(b_ix)$ defined on $I\!R$.

Lemma 4.5.
i) Let $q,m>0$, $C(q,m)$ and $S(q,m)$ be matrices defined as in (17)-(18). Then

$$Ker(C(q,m))=Ker(S(q,m)),$$

and

$$Im(C(q,m))=Im(S(q,m))=vect(V_1(q,m),V_2(q,m)), \tag{21}$$

where

$$\begin{aligned} V_1(q,m)=C(q,m)e_4 &= \left(-\frac{q^4}{m^2\lambda^2},-\frac{q}{m\lambda\sqrt{\lambda}},0,1,-\frac{q^3}{m\sqrt{\lambda}},0\right)^t, \\ V_2(q,m)=S(q,m)e_4 &= \left(0,\frac{1}{\lambda\sqrt{\lambda}},\frac{q}{m},0,-\frac{q^2}{\sqrt{\lambda}},\frac{q}{\lambda}\right)^t, \end{aligned} \tag{22}$$

and $vect(V_1(q,m),V_2(q,m))$ represents the vector space generated by $V_1(q,m)$ and $V_2(q,m))$.
ii) Let us consider constants $q_i,m_i>0$, $i=1,...,N$, $\overrightarrow{\epsilon}\in\{0,1\}^N$. Then

$$e_1^tX_{N,\overrightarrow{\epsilon}}(q_N,m_N)X_{N-1,\overrightarrow{\epsilon}}(q_{N-1},m_{N-1})...X_{1,\overrightarrow{\epsilon}}(q_1,m_1)e_4=\frac{c_{\overrightarrow{\epsilon}}}{\lambda^2} \tag{23}$$

where $c_{\overrightarrow{\epsilon}}$ is a constant which depends only on $q_i,m_i,i=1,...,N$ and $\overrightarrow{\epsilon}$, but not on λ.

Proof. i) We check that $Ker(C(q,m))=Ker(S(q,m))$ and that their dimension is 4. Moreover, we show that $C(q,m)V_i(q,m)=4V_i(q,m),i=1,2$ and $S(q,m)V_i(q,m)=4V_j(q,m),i,j=1,2,\ i\neq j$. (see Appendix 4.) which leads to (21).
ii) is shown without difficulty by iteration ☐

Lemma 4.6. *The characteristic equation has the following expansion*

$$f(\sqrt{\lambda})=\frac{4^N\exp((\sum_{i=1}^N b_i)\sqrt{\lambda})}{\lambda^2}f_\infty(\sqrt{\lambda})+r(\sqrt{\lambda}), \tag{24}$$

where

$$f_\infty(\sqrt{\lambda})=\sum_{\overrightarrow{\epsilon}\in\{0,1\}^N}[c_{\overrightarrow{\epsilon}}\prod_{i=N}^{1}y_{i,\overrightarrow{\epsilon}}(b_i\sqrt{\lambda})] \tag{25}$$

where $c_{\overrightarrow{\epsilon}}$ is a constant coming from (23) in Lemma 4.5, and the function $y_{i,\overrightarrow{\epsilon}}$ is given by (20). (Notice that in formula (25) the product is not commutative and that

$\prod_{i=N}^{1}$ *means that the terms are in decreasing order from* N *to* 1*).*

The remainder r *satisfies* $r(\sqrt{\lambda}) = o\left(\dfrac{\exp((\sum_{i=1}^{N} b_i)\sqrt{\lambda})}{\lambda^2}\right)$ *for large values of* λ*.*

Proof. Inserting (15)-(16) in (14) of Theorem 4.3 we get

$$\begin{aligned} f(\sqrt{\lambda}) &= 4^N \exp((\sum_{i=1}^{N} b_i)\sqrt{\lambda}) e_1^t \prod_{i=N}^{1} (\cos(b_i\sqrt{\lambda})C(q_i, m_i) + \sin(b_i\sqrt{\lambda})S(q_i, m_i))e_4 \\ &+ o\left(\frac{\exp((\sum_{j=1}^{N} b_i)\sqrt{\lambda})}{\lambda^2}\right). \end{aligned} \tag{26}$$

With Notation (19)-(20) it holds:

$$\begin{aligned} &e_1^t \prod_{i=N}^{1} (\cos(b_i\sqrt{\lambda})C(q_i, m_i) + \sin(b_i\sqrt{\lambda})S(q_i, m_i))e_4 \\ &= e_1^t (\sum_{\overrightarrow{\epsilon}\in\{0,1\}^N} \prod_{i=N}^{1} y_{i,\overrightarrow{\epsilon}}(b_i\sqrt{\lambda}) X_{i,\overrightarrow{\epsilon}}(q_i, m_i))e_4 \\ &= \sum_{\overrightarrow{\epsilon}\in\{0,1\}^N} (\prod_{i=N}^{1} y_{i,\overrightarrow{\epsilon}}(b_i\sqrt{\lambda})) e_1^t (\prod_{i=N}^{1} X_{i,\overrightarrow{\epsilon}}(q_i, m_i))e_4 \\ &= \sum_{\overrightarrow{\epsilon}\in\{0,1\}^N} (\prod_{i=N}^{1} y_{i,\overrightarrow{\epsilon}}(b_i\sqrt{\lambda})) \frac{c_{\overrightarrow{\epsilon}}}{\lambda^2}. \end{aligned}$$

Inserting this last identity in (26) leads to (24) and (25) □

Remark 1.

$$f_\infty\left(\sqrt{\lambda}\right) = P\left((\cos(b_j\sqrt{\lambda}), \sin(b_j\sqrt{\lambda}))_{j\in\{1,\dots,N\}}\right) \tag{27}$$

and P is a polynomial function with $2N$ variables. Consequently from (24) we deduce that the asymptotic behaviour of the spectrum $\sigma(\mathcal{A})$ corresponds to the roots of the asymptotic characteristic equation

$$f_\infty\left(\sqrt{\lambda}\right) = 0 \tag{28}$$

<u>Example 1:</u> $N = 2$

$$\begin{aligned} f_\infty(\sqrt{\lambda}) &= -(\frac{q_1^4}{m_1^2} + \frac{q_1^3 q_2 + q_1 q_2^3}{m_1 m_2} + \frac{q_2^4}{m_2^2}) \cos(b_1\sqrt{\lambda}) \cos(b_2\sqrt{\lambda}) \\ &+ \frac{(q_1^2 - q_2^2) q_1 q_2}{m_1 m_2} \cos(b_1\sqrt{\lambda}) \sin(b_2\sqrt{\lambda}) \\ &- \frac{(q_1^2 - q_2^2) q_1 q_2}{m_1 m_2} \cos(b_2\sqrt{\lambda}) \sin(b_1\sqrt{\lambda}) \\ &+ \frac{(q_1 + q_2)^2 q_1 q_2}{m_1 m_2} \sin(b_1\sqrt{\lambda}) \sin(b_2\sqrt{\lambda}) \end{aligned}$$

Example 2: $N = 4, q_j = j,\ b_j = m_j = 1,\ j = 1, ..., 4.$

$$f_\infty(\sqrt{\lambda}) = \frac{83825}{6144} + \frac{165725}{4608}\cos(\sqrt{\lambda}) + \frac{1990625}{18432}\cos(4\sqrt{\lambda}) + \frac{9275}{576}\sin(2\sqrt{\lambda}).$$

Remark that in this case the function f_∞ is π-periodic. See Figure 1 for the graph of f_∞.

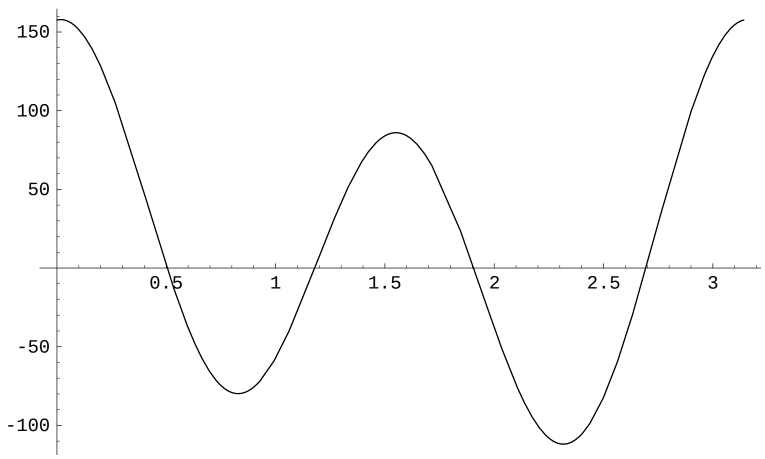

FIGURE 1. $f_\infty(x)$, $x \in [0, \pi]$. $(N = 4, q_j = j,\ b_j = m_j = 1,\ j = 1, ..., 4)$.

5. **Spectral gap.** The verification of the spectral gap is based on the knowledge of the minimal distance between two consecutive roots of f_∞. In view of this study we start with the following Lemma:

Lemma 5.1. *For all $j = 1, ..., N-1$ we have*

$$\left(\prod_{i=j}^{1} H(q_i, b_i, m_j)\right)e_4 = \alpha_j(\sqrt{\lambda})V_1(q_j, m_j) + \beta_j(\sqrt{\lambda})V_2(q_j, m_j) \tag{29}$$

where $V_k(q_j, m_j), k = 1, 2$ are defined in Lemma 4.5, $\alpha_j(.)$, $\beta_j(.)$ are trigonometrical polynomials which depend only on $q_i, b_i, m_i, i = 1...j$.
Moreover, there exists a constant $d_j > 0$ (which depends only on the material constants) such that the Wronskian $W_j(x) = \alpha_j(x)\beta_j^{'}(x) - \alpha_j^{'}(x)\beta_j(x)$ satisfies

$$W_j(x) \geq d_j > 0, \forall x \in I\!R. \tag{30}$$

Proof. We argue by iteration. We suppose that $j = 1$; by (16) and (22) of Lemma 4.5 we have :

$$H(q_1, b_1, m_1)e_4 = \cos(b_1\sqrt{\lambda})V_1(q_1, m_1) + \sin(b_1\sqrt{\lambda})V_2(q_1, m_1).$$

Thus (29) holds with $\alpha_1(x) = \cos(b_1 x)$, $\beta_1(x) = \sin(b_1 x)$. Since $\forall x \in I\!R$, $W_1(x) = b_1$, then (30) is true with $d_1 = b_1 > 0$.

Now, suppose that (29) holds for $j-1$ and that there exists a constant d_{j-1} such that: $\forall x \in I\!R$, $W_{j-1}(x) \geq d_{j-1} > 0$.
Thus, with (16) of Lemma 4.4 we may write:

$$
\begin{aligned}
(\prod_{i=j}^{1} H(q_i,b_i,m_i))e_4 &= H(q_j,b_j,m_j)(\alpha_{j-1}(\sqrt{\lambda})V_1(q_{j-1},m_{j-1}) \\
&+ \beta_{j-1}(\sqrt{\lambda})V_2(q_{j-1},m_{j-1}) \\
&= (\cos(b_j\sqrt{\lambda})C(q_j,m_{j-1}) + \sin(b_j\sqrt{\lambda})S(q_j,m_{j-1})) \\
&\times (\alpha_{j-1}(\sqrt{\lambda})V_1(q_{j-1},m_{j-1}) + \beta_{j-1}(\sqrt{\lambda})V_2(q_{j-1},m_{j-1})
\end{aligned} \tag{31}
$$

Now, from (21) of Lemma 4.5, we know that there exist constants $z_i, i = 1,...8.$ such that

$$
\begin{array}{l}
C(q_j,m_j)V_1(q_{j-1},m_{j-1}) = z_1V_1(q_j,m_j) + z_2V_2(q_j,m_j) \\
C(q_j,m_j)V_2(q_{j-1},m_{j-1}) = z_3V_1(q_j,m_j) + z_4V_2(q_j,m_j) \\
S(q_j,m_j)V_1(q_{j-1},m_{j-1}) = z_5V_1(q_j,m_j) + z_6V_2(q_j,m_j) \\
S(q_j,m_j)V_1(q_{j-1},m_{j-1}) = z_7V_1(q_j,m_j) + z_8V_2(q_j,m_j),
\end{array} \tag{32}
$$

Using the expressions of $C(q_j,m_j)$, $S(q_j,m_j)$, $V_1(q_{j-1},m_{j-1})$ and $V_2(q_{j-1},m_{j-1})$ given in Lemma 4.4 and Lemma 4.5 we get after some computations (see Appendix 5.):

$$
\begin{aligned}
z_1 &= \frac{(m_jq_{j-1} + m_{j-1}q_j)(m_jq_{j-1}^3 + m_{j-1}q_j^3)}{m_{j-1}^2q_j^4} \\
z_2 &= \frac{q_{j-1}(q_{j-1}^2 - q_j^2)}{m_{j-1}^2q_j^2} \\
z_3 &= -\frac{m_j}{q_j}z_2 \\
z_4 &= \frac{(q_{j-1}+q_j)^2}{q_j^2} \\
z_5 &= -\frac{m_j}{q_j}z_2 \\
z_6 &= \frac{q_j}{m_j}z_1 \\
z_7 &= -\frac{m_j}{q_j}z_4 \\
z_8 &= \frac{m_{j-1}}{q_{j-1}}z_2.
\end{aligned} \tag{33}
$$

Using (32) in the development of the last expression of (31) and replacing z_3, z_5, z_6, z_7, z_8 given in (33) we arrive at

$$
(\prod_{i=j}^{1} H(q_i,b_i,m_i))e_4 = \alpha_j(\sqrt{\lambda})V_1(q_j,m_j) + \beta_j(\sqrt{\lambda})V_2(q_j,m_j)
$$

with

$$
\left\{
\begin{aligned}
\alpha_j(x) &= \cos(b_jx)(z_1\alpha_{j-1}(x) + \frac{m_{j-1}m_j}{q_{j-1}q_j}z_2\beta_{j-1}(x)) \\
&+ \sin(b_jx)(-\frac{m_j}{q_j}z_2\alpha_{j-1}(x) - \frac{m_j}{q_j}z_4\beta_{j-1}(x)) \\
\beta_j(x) &= \cos(b_jx)(\frac{q_j}{m_j}z_1\alpha_{j-1}(x) + \frac{m_{j-1}}{q_{j-1}}z_2\beta_{j-1}(x)) \\
&+ \sin(b_jx)(z_2\alpha_{j-1}(x) + z_4\beta_{j-1}(x))
\end{aligned}
\right. \tag{34}
$$

That proves (29). Thanks to (34), we compute $W_j(x)$ and we find:

$$\begin{aligned} W_j(x) &= \frac{b_j}{m_j q_{j-1}^2 q_j}[q_{j-1}^2(z_1^2+m_j^2 z_2^2)\alpha_{j-1}^2(x) \\ &+ 2m_j q_{j-1}q_j(m_{j-1}q_j z_1 + m_j q_{j-1} z_4)\alpha_{j-1}(x)\beta_{j-1}(x) \\ &+ m_j^2(m_j^2 z_2^2 + q_{j-1}^2 z_4^2)\beta_{j-1}^2(x)] \\ &+ (z_1 z_4 - \frac{m_{j-1}m_j}{q_{j-1}q_j} z_2^2)W_{j-1}(x) \end{aligned} \tag{35}$$

Since

$$\begin{aligned} &q_{j-1}^2(z_1^2+m_j^2z_2^2)+m_j^2(m_j^2z_2^2+q_{j-1}^2z_4^2)-[m_jq_{j-1}q_j(m_{j-1}q_jz_1+m_jq_{j-1}z_4)]^2 \\ &= m_j^2 q_{j-1}^2(m_{j-1}m_j z_2^2 - q_{j-1}q_j z_1 z_4)^2 \geq 0 \end{aligned}$$

and $\frac{b_j}{m_j q_{j-1}^2 q_j} > 0$, it clearly holds

$$W_j(x) \geq (z_1 z_4 - \frac{m_{j-1}m_j}{q_{j-1}q_j} z_2^2)W_{j-1}(x).$$

From identities 1., 2. and 4. of (33) and a calculation (see Appendix 6.) we find that

$$z_1 z_4 - \frac{m_{j-1}m_j}{q_{j-1}q_j} z_2^2 = \frac{(q_{j-1}+q_j)^2(m_j q_{j-1}^2 + m_{j-1} q_j^2)^2}{m_{j-1}^2 q_j^6} > 0.$$

Due to these last two inequalities, we get the conclusion :

$$d_j = (z_1 z_4 - \frac{m_{j-1}m_j}{q_{j-1}q_j} z_2^2) d_{j-1} > 0$$

□

Now we can state the following property of f_∞:

Lemma 5.2. *The roots of f_∞ are all simple. Moreover, there exists a constant $d > 0$ (which depends only on the material constants) such that for all the roots x_0 of f_∞*

$$|f'_\infty(x_0)| \geq d. \tag{36}$$

Proof. By Lemma 4.6 and (26) in the proof of the same Lemma we may write:

$$\frac{1}{\lambda^2} f_\infty(\sqrt{\lambda}) = e_1^t[\prod_{i=N}^{1} H(q_i, b_i, m_i)]e_4.$$

Thus

$$\begin{aligned} \frac{1}{\lambda^2} f_\infty(\sqrt{\lambda}) &= e_1^t H(q_N, b_N, m_N)[\textstyle\prod_{i=N-1}^{1} H(q_i, b_i, m_i))e_4] \\ &= e_1^t[\cos(b_N\sqrt{\lambda})C(a_N) + \sin(b_N\sqrt{\lambda})S(a_N)] \\ &\times [\alpha_{N-1}(\sqrt{\lambda})V_1(q_{N-1}, m_{N-1}) + \beta_{N-1}(\sqrt{\lambda})V_2(q_{N-1}, m_{N-1})] \\ &= \cos(b_N\sqrt{\lambda})[\alpha_{N-1}(\sqrt{\lambda})e_1^t C(q_N, m_N)V_1(q_{N-1}, m_{N-1}) \\ &+ \beta_{N-1}(\sqrt{\lambda})e_1^t C(q_N, m_N)V_2(q_{N-1}, m_{N-1})] \\ &+ \sin(b_N\sqrt{\lambda})[\alpha_{N-1}(\sqrt{\lambda})e_1^t S(q_N, m_N)V_1(q_{N-1}, m_{N-1}) \\ &+ \beta_{N-1}(\sqrt{\lambda})e_1^t S(q_N, m_N)V_2(q_{N-1}, m_{N-1})]. \end{aligned}$$

We set and compute $k_i, i = 1, ..., 4$:

$$\begin{aligned} k_1 &= \lambda^2 e_1^t C(q_N, m_N) V_1(q_{N-1}, m_{N-1}) \\ &= -\frac{(m_N q_{N-1} + m_{N-1} q_n)(m_N q_{N-1}^3 + m_{N-1} q_n^3)}{m_{N-1}^2 m_N^2} \\ k_2 &= \lambda^2 e_1^t C(q_N, m_N) V_2(q_{N-1}, m_{N-1}) = \frac{-q_{N-1}^3 q_N + q_N q_{N-1}^3}{m_{N-1} m_N} \\ k_3 &= \lambda^2 e_1^t S(q_N, m_N) V_1(q_{N-1}, m_{N-1}) = -k_2 \\ k_4 &= \lambda^2 e_1^t S(q_N, m_N) V_2(q_{N-1}, m_{N-1}) = \frac{(q_{N-1} + q_N)(q_{N-1} + q_N)^2}{m_{N-1} m_N} \end{aligned}$$

Thus f_∞ has the following form:

$$f_\infty(x) = \cos(b_N x) f_1(x) + \sin(b_N x) f_2(x) \tag{37}$$

where

$$\begin{cases} f_1(x) = k_1 \alpha_{N-1}(x) + k_2 \beta_{N-1}(x) \\ f_2(x) = k_3 \alpha_{N-1}(x) + k_4 \beta_{N-1}(x) \end{cases}$$

Let us remark that

$$W(f_1, f_2)(x) = (f_1(x) f_2^{'}(x) - f_1^{'}(x) f_2(x)) = (k_1 k_4 - k_2 k_3) W_{N-1}(x)$$

and that

$$k_1 k_4 - k_2 k_3 = -\frac{q_{N-1} q_N (q_{N-1} + q_N)^2 (m_N q_{N-1}^2 + m_{N-1} q_N^2)}{q_{N-1}^3 q_N^3} < 0.$$

Consequently, from Lemma 5.1 we deduce that there exists a constant $d > 0$ such that

$$\forall x \in I\!R, W(f_1, f_2)(x) \le d < 0. \tag{38}$$

Now, for all $x \in I\!R$ we have

$$f_\infty^{'}(x) = \cos(b_N x)[f_1^{'}(x) + b_N f_2(x)] + \sin(b_N x)[f_2'(x) - b_N f_1(x)]$$

We deduce that for all $x \in I\!R$, $\Delta(x) = f_\infty(x)^2 + f_\infty'(x)^2$ has the following form:

$$\Delta(x) = (\cos(b_N x) \sin(b_N x)) M(x) \begin{pmatrix} \cos(b_N x) \\ \sin(b_N x) \end{pmatrix}, \tag{39}$$

where the matrix $M(x)$ is symmetric, positive and given by

$$M(x) = \begin{pmatrix} M_{11}(x) & M_{12}(x) \\ M_{21}(x) & M_{22}(x) \end{pmatrix}$$

and

$$\begin{cases} M_{11}(x) = f_1(x)^2 + b_N^2 f_2(x)^2 + 2 b_N f_2(x) f_1^{'}(x) + f_1^{'}(x)^2 \\ M_{12}(x) = (1 - b_N^2) f_1(x) f_2(x) - b_N (f_1(x) f_1^{'}(x) - f_2(x) f_2^{'}(x)) + f_1^{'}(x) f_2^{'}(x) \\ M_{21}(x) = M_{12}(x) \\ M_{22}(x) = b_N^2 f_1(x)^2 + f_2(x)^2 - 2 b_N f_1(x) f_2^{'}(x) + f_2{'}(x)^2 \end{cases}$$

Let $\lambda_{min}(x), \lambda_{max}(x)$ be the two eigenvalues of $M(x)$ such that $0 \le \lambda_{min}(x) \le \lambda_{max}(x)$. After some computation we find

$$\begin{aligned} \lambda_{min}(x)\lambda_{max}(x) &= \det(M(x)) \\ &= b_N^2 (f_1(x)^2 + f_2(x)^2)^2 - 2 b_N (f_1(x)^2 + f_2(x)^2) W(f_1, f_2)(x) \\ &+ W(f_1, f_2)(x)^2. \end{aligned}$$

Consequently with (38) we see that

$$\forall x \in I\!R, \ \det(M(x)) = \lambda_{min}(x)\lambda_{max}(x) \ge W(f_1, f_2)(x)^2 \ge d^2. \tag{40}$$

On the other hand, since f_1 and f_2 are trigonometric polynomials the trace of $M(x)$ is bounded on $I\!R$: Thus, there exists $d' > 0$ such that

$$\forall x \in I\!R,\ 0 \le tr(M(x)) = \lambda_{min}(x) + \lambda_{max}(x) \le d'^2. \tag{41}$$

From (40) and (41) we deduce that $\lambda_{min}(x) \ge (\frac{d}{d'})^2 > 0$. Therefore from (39) we get

$$\forall x \in I\!R,\ \ \Delta(x) \ge (\frac{d}{d'})^2 > 0.$$

That means that if x_0 is a root of f_∞ then $|f_\infty^{'}(x_0)| \ge (\frac{d}{d'})^2 > 0$ □

Finally we arrive at the main result:

Theorem 5.3. *(The spectral gap) Let λ_k^2, $k \in I\!N^*, (\lambda_k > 0)$ be the (strictly) monotone increasing sequence of eigenvalues of $\mathcal{A}$, then*

$$\lim_{k \to +\infty} (\lambda_{k+1} - \lambda_k) = +\infty. \tag{42}$$

Proof. First we recall that from Lemma 5.2 the roots of f_∞ are simple. On the other hand, since f_∞ and all its derivatives are trigonometric polynomials, they are all bounded on $I\!R$. Then

$$\forall x \in I\!R,\ |f'_\infty(x+h) - f'_\infty(x)| = |f''_\infty(x + \theta h)| \cdot |h| \le \|f''_\infty\|_\infty \cdot |h| \tag{43}$$

and it follows that f'_∞ is uniformly continuous on $I\!R$.
Thus, there exists $h_0 > 0$ such that, for any x_0 satisfying $f_\infty(x_0) = 0$

$$|x - x_0| \le h_0 \ \Rightarrow |f'_\infty(x)| \ge \frac{d}{2}.$$

Due to Rolle's Theorem, this property implies that x_0 is the unique root of f_∞ in the interval $[x_0 - h_0, x_0 + h_0]$, which also means that the minimal distance between two consecutive roots of f_∞ is h_0.

By Lemma 14 and Lemma 16 multiplying the characteristic equation $f(\sqrt{\lambda}) = 0$ by $\dfrac{\lambda^2}{4^N \exp((\sum_{i=1}^N b_i)\sqrt{\lambda})}$ we see that it is equivalent to:

$$\widetilde{f}(\sqrt{\lambda}) = f_\infty(\sqrt{\lambda}) + \widetilde{r}(\sqrt{\lambda}) = 0,$$

where the function $\widetilde{r}$ is analytical on $I\!R_+^*$ and there exists a constant $C > 0$ such that for all $x \ge 1, \widetilde{r}(x) \le \dfrac{C}{x}$ and $\dfrac{d\widetilde{r}}{dx}(x) \le \dfrac{C}{x}$. Consequently using (43) and the relation $\widetilde{f} = f_\infty + \widetilde{r}$, proceeding as for f_∞ we can see that there exists $X_0 \ge 1$ such that $\widetilde{f}^{'}$ is uniformly continuous on $[X_0, +\infty)$. As previously we deduce that the minimal distance between two consecutive nonnegative roots of $\widetilde{f}$ is a constant $h_0^{'} > 0$. The spectral gap is a direct consequence of this property □

6. **Controllability.** As a consequence of the main result of this paper (that is to say the spectral gap) we show the controllability of an associated problem. In fact, this control problem was studied in [15] where the authors show that controllability holds under sufficient conditions which remain to prove. These conditions are the spectral gap and a boundary estimate satisfied by the eigenfunctions. Remark that only for some simple examples (two or three beams) these conditions are checked in

[15]; thus an important contribution of this paper is to prove that these assumptions hold for a chain of N beams with various materials. Let us briefly recall this control problem and some results that we can find in [15].

6.1. **The Petrovsky system.** By the properties of the triple (H, V, a) defined in Section 2, we can recall a first result concerning the associated dynamic problem:

Theorem 6.1. *Let $u^0 \in D(\mathcal{A}^s), u^1 \in D(\mathcal{A}^{s-\frac{1}{2}})$ and $f \in L^1(0,T; D(\mathcal{A}^{s-\frac{1}{2}}))$, with $s \geq \dfrac{1}{2}$. Then the problem*

$$\left\{ \begin{array}{l} u_{tt} + \mathcal{A}u(t) = f(t),\ t \in [0,T], \\ u(0) = u^0, \\ u_t(0) = u^1, \end{array} \right. \tag{44}$$

has a unique solution $u \in C([0,1], D(\mathcal{A}^s)) \cup C^1([0,1], D(\mathcal{A}^{s-\frac{1}{2}}))$ fullfilling

$$\begin{array}{rcl} \|u\|_{C([0,T],D(\mathcal{A}^s)} + \|u\|_{C^1([0,T],D(\mathcal{A}^{s-\frac{1}{2}})} & \leq & C\{\|u^0\|_{D(\mathcal{A}^s)} + \|u^1\|_{D(\mathcal{A}^{s-\frac{1}{2}})} \\ & + & \|f\|_{L^1(0,T;D(\mathcal{A}^{s-\frac{1}{2}}))}\}, \end{array} \tag{45}$$

for some constant $C > 0$ independent of u.

Remark: Theorem 6.1 is Theorem 3.1 of [15].

In particular, if $f = 0$, then the energy $E(t) := \frac{1}{2}\{\|u_t\|_H^2 + a(u(t), u(t))\}$ is constant, for all $t \in [0,T]$ and we have $E(t) = E_0 = \frac{1}{2}\{\|u^1\|_H^2 + a(u^0, u^0)\}, \forall t \in [0,T]$.

6.2. **Weak solutions of the wave equation.** The weak formulation of the control problem is given by:

Theorem 6.2. *For all $u_0 \in H, u_1 \in V', w_1, w_N \in L^2((0,T))$ there exist unique $u \in L^\infty(0,T;V'), (\psi_1, \psi_0) \in V' \times H$, which are solutions of*

$$\begin{array}{l} \displaystyle\int_0^T < u(t), f(t) >_{V',V} dt + < \psi_1, \phi_0 >_{V',V} - (\psi_0, \phi_1)_H \\ = < u_1, \phi(0) >_{V',V} - (u_0, \phi'(0))_H \\ \displaystyle -(\int_0^T (w_1 \phi_1^{(2)}(t,0)dt + w_N \phi_N^{(2)}(t, l_N))dt, \\ \forall f \in L^1(0,T;V), \{\phi_0, \phi_1\} \in V \times H, \end{array} \tag{46}$$

where ϕ is the unique solution of

$$\left\{ \begin{array}{l} \phi \in C([0,T], V) \cup C^1([0,T], H), \\ \phi_{tt}(t) + \mathcal{A}\phi(t) = f(t), t \in [0,T], \phi(T) = \phi_0, \phi_t(T) = \phi_1. \end{array} \right. \tag{47}$$

and $< .,. >_{V',V}$ represents the duality bracket between the spaces V' and V.

Remark: Theorem 6.2 is Theorem 5.1 of [15]. Formally the solutions $u, (\psi_1, \psi_0)$ in the previous theorem satisfy

$$\left\{ \begin{array}{l} (u_j)_{tt}(t,x) + \dfrac{a_j}{m_j} u_j^{(4)}(t,x) = 0, \text{on}\,(0,T) \times (0, l_j), \forall j = 1, ..., N \\ u_j(t,.) \text{ satisfies (2) to (5)} \\ u_1(0) = u_N(l_N) = 0, \\ -u_1^{(1)}(0) = \dfrac{1}{a_1} w_1, u_N^{(1)}(l_N) = \dfrac{1}{a_N} w_N, \\ u(0) = u_0, u_t(0) = u_1, \end{array} \right. \tag{48}$$

and the final conditions

$$u(T) = \psi_0, u_t(T) = \psi_1. \tag{49}$$

We can now formulate the results which lead to the exact controllability result for any time:

Lemma 6.3. *(Observability inequalities)* $\forall T > 0$,*the expression*

$$||(u^0, u^1)|| := (\int_0^T |u_1^{(2)}(t,0)|^2 + |u_N^{(2)}(t,l_N)|^2 dt)^{\frac{1}{2}},$$

where u *is the solution of (44) or (48) satisfying* $w_1 = w_N = 0$*, is a norm on* $V \times H$.

Proof. Since Theorem 5.3 and Lemma 3.3 prove the conditions (24) and (25) of [15], then, the results follows directly from Theorem 4.3 of [15] □

Remark 2. We emphasize on the fact that the controllability time can be chosen arbitrarily small due the fact that (from (42) in Theorem 5.3) the distance between two consecutive eigenvalues tends towards the infinite

Finally, we recall the controllability result which is Theorem 6.1 of [15]:

Let F the closure of $V \times H$ for this norm.

Theorem 6.4. *(Exact controllability) For all* $(u^1, -u^0) \in F'$*, there exist* $w_0, w_1 \in L^2((0,T))$ *such that the weak solution* $u \in C([0,T],H) \cup C^1([0,T],V')$ *of the wave equation (48) satisfies*

$$u(T) = u_t(T) = 0.$$

Appendix.

1.
   - {e[1][x],e[2][x],e[3][x],e[4][x]} = {Cos[$\sqrt{\lambda}$ q x], Sin[$\sqrt{\lambda}$ q x], Cosh[$\sqrt{\lambda}$ q x], Sinh[$\sqrt{\lambda}$ q x]}
   - K[x]=
   { Table[i][x],{i,1,4}],Table[D[e[i][x],{x,1}],{i,1,4}], Table[a*D[e[i][x],{x,2}],{ i,1,4 }],a*Table[D[e[i][x],{ x,3}],{ i,1,4 }] }
   - $a = \frac{m}{q^4}; l = \frac{b}{q};$
   - A[q,b,m]=K[l].Inverse[K[0]]

2.
   - A[q, b,M]=A[q, b, M] /. { Cosh[b $\sqrt{\lambda}$] $\longrightarrow$ (Exp[b $\sqrt{\lambda}$] + Exp[-b $\sqrt{\lambda}$]) /2, Sinh[b $\sqrt{\lambda}$] $\longrightarrow$ (Exp[b $\sqrt{\lambda}$] - Exp[-b $\sqrt{\lambda}$]) /2 }
   - M=A[1,1,1].A[1,1,1]
   - M_{12}=Simplify[Table[M_{12}[[i,j]],{i,1,2},{j,3,4}]]

3.
   - (Definition of the function ext)
   Do[
   {f[i, 1][mat] = det[{{mat[[1,1]], mat[[1,2]]}, {mat[[i + 1,1]], mat[[i + 1,2]]}}];

```
    f[i, 2][mat] = det[{{mat[[1,1]], mat[[1,3]]}, {mat[[i + 1,1]], mat[[i + 1,3]]}}];
    f[i, 3][mat] = det[{{mat[[1,1]], mat[[1,4]]}, {mat[[i + 1,1]], mat[[i + 1,4]]}}];
    f[i, 4][mat] = det[{{mat[[1,3]], mat[[1,4]]}, {mat[[i + 1,3]], mat[[i + 1,4]]}}];
    f[i, 5][mat] = -det[{{mat[[1,2]], mat[[1,4]]}, {mat[[i + 1,2]], mat[[i + 1,4]]}}];
    f[i, 6][mat] = det[{{mat[[1,2]], mat[[1,3]]}, {mat[[i + 1,2]], mat[[i + 1,3]]}}]}
, {i, 1, 3}];
    f[4, 1][mat] = det[{{mat[[3,1]], mat[[3,2]]}, {mat[[4,1]], mat[[4,2]]}}];
    f[4, 2][mat] = det[{{mat[[3,1]], mat[[3,3]]}, {mat[[4,1]], mat[[4,3]]}}];
    f[4, 3][mat] = det[{{mat[[3,1]], mat[[3,4]]}, {mat[[4,1]], mat[[4,4]]}}];
    f[4, 4][mat] = det[{{mat[[3,3]], mat[[3,4]]}, {mat[[4,3]], mat[[4,4]]}}];
    f[4, 5][mat] = -det[{{mat[[3,2]], mat[[3,4]]}, {mat[[4,2]], mat[[4,4]]}}];
    f[4, 6][mat] = det[{{mat[[3,2]], mat[[3,3]]}, {mat[[4,2]], mat[[4,3]]}}];
    f[5, 1][mat] = -det[{{mat[[2,1]], mat[[2,2]]}, {mat[[4,1]], mat[[4,2]]}}];
    f[5, 2][mat] = -det[{{mat[[2,1]], mat[[2,3]]}, {mat[[4,1]], mat[[4,3]]}}];
    f[5, 3][mat] = -det[{{mat[[2,1]], mat[[2,4]]}, {mat[[4,1]], mat[[4,4]]}}];
    f[5, 4][mat] = -det[{{mat[[2,3]], mat[[2,4]]}, {mat[[4,3]], mat[[4,4]]}}];
    f[5, 5][mat] = det[{{mat[[2,2]], mat[[2,4]]}, {mat[[4,2]], mat[[4,4]]}}];
    f[5, 6][mat] = -det[{{mat[[2,2]], mat[[2,3]]}, {mat[[4,2]], mat[[4,3]]}}];
    f[6, 1][mat] = det[{{mat[[2,1]], mat[[2,2]]}, {mat[[3,1]], mat[[3,2]]}}];
    f[6, 2][mat] = det[{{mat[[2,1]], mat[[2,3]]}, {mat[[3,1]], mat[[3,3]]}}];
    f[6, 3][mat] = det[{{mat[[2,1]], mat[[2,4]]}, {mat[[3,1]], mat[[3,4]]}}];
    f[6, 4][mat] = det[{{mat[[2,3]], mat[[2,4]]}, {mat[[3,3]], mat[[3,4]]}}];
    f[6, 5][mat] = -det[{{mat[[2,2]], mat[[2,4]]}, {mat[[3,2]], mat[[3,4]]}}];
    f[6, 6][mat] = det[{{mat[[2,2]], mat[[2,3]]}, {mat[[3,2]], mat[[3,3]]}}];
    ext[mat] = Table[f[i, j][mat], {i, 1, 6}, {j, 1, 6}];
```

- Expand[ext[A[q,b,m]]]/. Exp[b $\sqrt{\lambda}$]$\longrightarrow$ z
- H[q,b,m]=D[%,z]
- C[q,m]=4*H[q,b,m]/.{ Cos[b $\sqrt{\lambda}$]$\longrightarrow$ 1, Sin[b $\sqrt{\lambda}$] $\longrightarrow$ 0 }
- S[q,m]=4*H[q,b,m]/.{ Cos[b $\sqrt{\lambda}$]$\longrightarrow$ 0, Sin[b $\sqrt{\lambda}$] $\longrightarrow$ 1 }

4.

```
• NullSpace[c[q,m]]
NullSpace[s[q,m]]
NullSpace[c[q,m]]-NullSpace[s[q,m]]
e4=0,0,0,1,0,0
v1=c[q,m].e4
v2=s[q,m].e4
c[q,m].v1-4*v1
c[q,m].v2-4*v2
s[q,m].v2+4*v1
s[q,m].v1-4*v2
```

5. Computation of z1 et z2

```
• V1[q,m]=v1
V2[q,m]=v2
w=c[q2,m2].V1[q1,m1]-z1*V1[q2,m2]-z2*V2[q2,m2]
Solve[w[[1]],w[[2]]==0,0,z1,z2]
z1=Factor[z1]
z2=Factor[z2]
```

(verification) Simplify[w]

6.
- Factor [z1*z4-m1*m2/(q1*q2)*z2^2]

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