NETWORKS AND HETEROGENEOUS MEDIA doi:10.3934/nhm.2009.4.649
©American Institute of Mathematical Sciences
Volume **4**, Number **4**, December **2009** pp. **649–666**

HOMOGENIZATION OF A MODEL OF DISPLACEMENT WITH UNBOUNDED VISCOSITY

Catherine Choquet

Université P. Cézanne
LATP, CNRS UMR 6632, FST, Case Cour A, 13397 Marseille Cedex 20, France

Ali Sili

Université de Toulon et du Var
Département de mathématiques, BP 20132, 83957 La Garde, France

(Communicated by Andrea Braides)

Abstract. We discuss the homogenization of a model problem describing the transport of heat and mass by a compressible miscible flow in a highly heterogeneous porous medium. The flow is governed by a nonlinear system of degenerate parabolic type coupling the pressure and the temperature. Using the technique of two-scale convergence and compensated compactness arguments, we prove some stability in the homogenization process.

1. **Introduction.** We consider a model describing the compressible displacement of a mixture in an heterogeneous porous medium. We assume that the flow occurs during the time interval $(0,T)$, $T>0$, in a bounded domain Ω of $\mathbb{R}^N$, $N \le 3$, simply connected with smooth boundary Γ. Note that the assumption $N \le 3$ corresponds to the usual physical applications of the model. Minor modifications of the Sobolev embedding used below allow to suppress this restriction. Let $\Omega_t = \Omega \times (0,t)$, $0 \le t \le T$, $\Gamma_T = \Gamma \times (0,T)$, and let ν denote the unit normal pointing outward Ω. The transport of heat and mass is described by the following nonlinear, fully coupled and degenerate parabolic system. Equation (1.1) is the pressure equation derived from the conservation of the total mass and (1.3) is the temperature equation derived from the conservation of heat in the mixture. We refer to Scheidegger [12] and Kaviany [8] for the description of the model.

$$\phi^\varepsilon(x)\,\partial_t p^\varepsilon + \operatorname{div} q^\varepsilon = \frac{1}{\mu(\theta^\varepsilon)}(q^i - q^s), \quad q^\varepsilon = -\frac{k^\varepsilon(x)}{\mu(\theta^\varepsilon)}\nabla p^\varepsilon \text{ in } \Omega_T, \tag{1.1}$$

$$q^\varepsilon \cdot \nu\,|_{\Gamma_T} = 0, \quad p^\varepsilon\,|_{t=0} = p_{init}, \tag{1.2}$$

$$\phi^\varepsilon(x)\,\partial_t\theta^\varepsilon + q^\varepsilon\cdot\nabla\theta^\varepsilon - \operatorname{div}(\phi^\varepsilon(x)\,d(q^\varepsilon)\,\nabla\theta^\varepsilon) + \frac{q^i}{\mu(\theta^\varepsilon)}\,\theta^\varepsilon = \frac{q^i}{\mu(\theta^\varepsilon)} \text{ in } \Omega_T, \tag{1.3}$$

$$d(q^\varepsilon)\,\nabla\theta^\varepsilon\cdot\nu\,|_{\Gamma_T} = 0, \quad \theta^\varepsilon\,|_{t=0} = \theta_{init}. \tag{1.4}$$

Here $\varepsilon > 0$ is the length scale describing the heterogeneities of the medium. The rock is characterized by its permeability k^ε and by its porosity ϕ^ε. The function p^ε is the pressure, q^ε is the Darcy velocity, θ^ε is the temperature of the fluid. The

2000 *Mathematics Subject Classification.* Primary: 76M50, 35K65; Secondary: 76S05, 35K55.
Key words and phrases. Homogenization, nonlinear degenerate parabolic equations, coupled problem, porous media.

function $\mu = \mu(\theta^\varepsilon)$ is the viscosity of the mixture. We admit here the possibility of a vanishing mobility $\kappa^\varepsilon = (1/\mu(\theta^\varepsilon))k^\varepsilon$. Equation (1.1) is thus possibly degenerate. The dispersive tensor $d(q)$ has the form, see Bear [3] and De Marsily [7],

$$d(q) = d_m\, I + |q|\,(d_l\, E(q) + d_t\,(1 - E(q))),$$

where I is the identity matrix, E is the tensor $(q_i q_j/|q|^2)$, $d_m > 0$ is the molecular diffusion, $d_l \geq d_t > 0$ are respectively the longitudinal and transverse dispersion coefficients. The functions $q^i/\mu(\theta^\varepsilon)$ and $q^s/\mu(\theta^\varepsilon)$ are respectively the injection and production source terms. The pumping and injecting rates decrease with the viscosity increase. The functions p_{init} and θ_{init} are the initial pressure and temperature.

We address the homogenization of (1.1)-(1.4) when the permeability k^ε and the porosity ϕ^ε are highly oscillating functions. We suppose that ϕ^ε and k^ε are given by

$$\phi^\varepsilon(x) = \phi\left(x, \frac{x}{\varepsilon}\right), \quad k^\varepsilon(x) = k\left(x, \frac{x}{\varepsilon}\right) \quad x \in \Omega,$$

with $\phi \in C^1(\overline{\Omega}; L^\infty_{per}(Y))$, and $k \in C^1(\overline{\Omega}; L^\infty_{per}(Y))$, where Y is the unit cube in $\mathbb{R}^N$ and $L^\infty_{per}(Y)$ denotes the space of functions $w \in L^\infty(\mathbb{R})$ that are Y-periodic. The other technical assumptions used in this contribution are those necessary to the statement in [6] of the existence of a weak solution $(p^\varepsilon, \theta^\varepsilon)$ for (1.1)-(1.4) for any fixed $\varepsilon > 0$. The porosity function ϕ^ε and the permeability function k^ε are such that

$$0 < \phi_- \leq \phi^\varepsilon(x) \leq \phi_-^{-1}, \quad 0 < k_- \leq k^\varepsilon(x) \leq k_-^{-1}, \quad \text{a.e. in } \Omega,\ \forall \varepsilon > 0,$$

where ϕ_- and k_- are positive real numbers. The degenerate mobility $\kappa^\varepsilon(x, \theta^\varepsilon) = \kappa(x, \frac{x}{\varepsilon}, \theta^\varepsilon) = (1/\mu(\theta^\varepsilon))k^\varepsilon(x)$ satisfies

$$\kappa(x, y, \theta) \geq 0 \text{ in } \Omega \times Y \times (0, 1), \tag{1.5}$$

$$\nabla_{x,y}\big(\kappa(x, y, \theta)^{1/2}\big) = \big(\kappa^{-1/2}\nabla_{x,y}\kappa\big)(x, y, \theta)/2 \in (L^2(\Omega \times Y))^N,\ \theta \in (0, 1), \tag{1.6}$$

$$D_\theta\big(\kappa(x, y, \theta)^\alpha\big) = \big(\alpha\kappa^{\alpha-1}D_\theta\kappa\big)(x, y, \theta) \in L^\infty(\Omega \times Y),\ \text{for some } 0 < \alpha \leq \frac{1}{2},\ \theta \in (0, 1). \tag{1.7}$$

Note that the usual viscosity model used for nuclear waste contamination is (see [11])

$$\mu(\theta) = \mu_{ref} \exp\big(\frac{1}{\theta - \theta_{ref}}\big),$$

where μ_{ref} is some reference viscosity and θ_{ref} is some reference temperature. The viscosity is unbounded in the vicinity of θ_{ref}. If we assume for the sake of simplicity that the mobility vanishes only in an absolutely cold medium, that is $\theta_{ref} = 0$, the function

$$\kappa(x, \theta) = \begin{cases} k(x)\, \exp\big(-\frac{1}{\theta}\big) \text{ if } 0 < \theta \leq 1, \\ 0 \text{ if } \theta = 0, \end{cases}$$

corresponds to the model of [11] and fulfills Assumptions (1.5)-(1.7). We also assume

$$q^i, q^s \in L^2(\Omega_T), \quad q^i(x, t) \geq 0, \quad q^s(x, t) \geq 0 \text{ a.e. in } \Omega_T,$$
$$p_{init} \in L^2(\Omega), \quad \theta_{init} \in L^\infty(\Omega).$$

The initial temperature (kelvins) is of course assumed nonnegative. For the sake of simplicity we assume that its upper bound is equal to 1. We have

$$0 \leq \theta_{init}(x) \leq 1 \text{ a.e. in } \Omega.$$

Finally, we consider that the initial displacement is standard, that is the initial mobility is positive:

$$\kappa(x, \theta_{init}(x)) > 0 \quad \text{a.e. in } \Omega.$$

Under the aforementioned assumptions, Choquet proved in [6] the existence of a weak solution $(p^\varepsilon, \theta^\varepsilon)$ of Problem (1.1)-(1.4).

Theorem 1.1. *There exists a weak solution $(p^\varepsilon, \theta^\varepsilon)$ to the nonlinear parabolic system (1.1)-(1.4). The function $p^\varepsilon \in L^\infty(0,T; L^2(\Omega))$ is a solution of (1.1)-(1.2) in the space $L^2(0,T;(H^1(\Omega))')$. The Darcy velocity q^ε belongs to $(L^2(\Omega_T))^N$. The temperature function θ^ε is in $L^\infty(\Omega_T) \cap L^2(0,T;H^1(\Omega))$ and satisfies*

$$0 \leq \theta^\varepsilon(x,t) \leq 1 \text{ for almost every } (x,t) \in \Omega_T.$$

The aim of this paper is the asymptotic study as $\varepsilon \to 0$ of the degenerate coupled parabolic problem (1.1)-(1.4). We refer to Amirat et al [2] and Choquet [4] for the homogenization of a non-degenerate and one-dimensional model of miscible displacement. We claim the following result.

Main Theorem. *The homogenized problem equivalent to (1.1)-(1.4) is*

$$\begin{aligned}
&\widetilde{\phi}\partial_t p' + \operatorname{div}(q') = \frac{q^i - q^s}{\mu(\theta)}, \quad q' = -\frac{1}{\mu(\theta)^{1/2-\alpha}} K_\mu \nabla p' \quad \text{in } \Omega_T,\\
&q' \cdot \nu\,|_{\Gamma_T} = 0, \quad p'\,|_{t=0} = p_{init}(x),\\
&\tilde{\phi}(x)\,\partial_t\theta + Q\cdot\nabla\theta - \operatorname{div}(D_q(x,t)\nabla\theta) + \frac{q^i}{\mu(\theta)}\,\theta = \frac{q^i}{\mu(\theta)} \text{ in } \Omega_T,\\
&D_q\nabla\theta\cdot\nu\,|_{\Gamma_T} = 0, \quad \theta\,|_{t=0} = \theta_{init}.
\end{aligned}$$

The averaged porosity $\tilde{\phi}$ is $\tilde{\phi}(x) = \int_Y \phi(x,y)\,dy$. The homogenized permeability tensor $K_\mu = (K_{\mu_{ij}})_{1\leq i,j\leq N}$ is defined by

$$\begin{aligned}
K_{\mu_{ij}}(x,t) = \int_Y k(x,y)&\Big(\nabla_y w^i(x,y,t) + \frac{1}{\mu(\theta)^{\alpha/2+1/4}} e^i\Big)\\
&\cdot\Big(\nabla_y w^j(x,y,t) + \frac{1}{\mu(\theta)^{\alpha/2+1/4}} e^j\Big)\,dy, \quad x \in \Omega,
\end{aligned}$$

where $w^j(x,y,t)$ is the unique solution of the cell-problem:

$$\begin{aligned}
&-\operatorname{div}_y\Big(k(x,y)\Big(\nabla_y w^j(x,y,t) + \frac{1}{\mu(\theta)^{\alpha/2+1/4}} e^j\Big)\Big) = 0 \quad \text{in } Y,\\
&y \to w^j(x,y,t)\ Y-\text{periodic}, \quad \int_Y w^j(x,y,t)\,dy = 0,
\end{aligned}$$

the jth vector of the canonical basis of $\mathbb{R}^N$ being denoted by e^j, $1 \leq j \leq N$. Defining a corrector q_o for the velocity by

$$q_o(x,y,t) = -\frac{1}{\mu(\theta)^{1/2-\alpha}}\,k(x,y)\,\Big(\frac{1}{\mu(\theta)^{1/2+\alpha}}\nabla_x p(x,t) + \nabla_y p_2(x,y,t)\Big),$$

with $p_2(x,y,t) = \frac{1}{\mu(\theta)^{\alpha/2+1/4}}\sum_{j=1}^N w^j(x,y,t)\,\partial_{x_j} p(x,t)$, the homogenized diffusion tensor $D_q = (D_{q_{ij}})_{1\leq i,j\leq N}$, is given by

$$D_{q_{ij}}(x,t) = \int_Y \phi(x,y)\,d(q_o)(x,y,t)(\nabla_y v^i(x,y,t) + e^i)\cdot(\nabla_y v^j(x,y,t) + e^j)\,dy,$$

where $v^j(x,y,t)$ is the unique solution of the cell-problem:

$$-\operatorname{div}_y\big(\phi(x,y)\,d(q_o(x,y,t))\,(\nabla_y v^j(x,y,t)+e^j)\big)=0 \quad \text{in } Y,$$
$$y\to v^j(x,y,t)\ Y-\text{periodic},\quad \int_Y v^j(x,y,t)\,dy=0,\quad 1\le j\le N.$$

The paper is organized as follows. Section 2 is devoted to the derivation of uniform estimates for the solution $(p^\varepsilon,\theta^\varepsilon)$ of (1.1)-(1.4). Convergence and compactness results follow. The main difficulty is the study of the pressure behavior because all the estimates that we can derive on its gradient ∇p^ε are weighted by the possibly degenerate mobility κ^ε. We thus introduce an auxiliary weighted pressure. In section 3 we pass to the limit $\varepsilon\to 0$ in (1.1)-(1.4), using in particular two-scale convergence tools (see Allaire [1]).

2. **Preliminary estimates and convergence results.**

The aim of this section is to provide sufficient uniform estimates and compactness results to pass to the limit in the oscillating problem (1.1)-(1.4) as $\varepsilon\to 0$. Here and in the sequel, C denotes a positive generic real number independent of ε. Firstly classical energy estimates give the following results.

Lemma 2.1. *(i) We claim that the following uniform estimates for the pressure hold true.*

$$\|p^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}+\|\kappa^\varepsilon(x,\theta^\varepsilon)^{1/2}\nabla p^\varepsilon\|_{(L^2(\Omega_T))^N}\le C,\quad \|q^\varepsilon\|_{(L^2(\Omega_T))^N}\le C.$$

Furthermore, the function $\phi^\varepsilon\partial_t p^\varepsilon$ is uniformly bounded in $L^2(0,T;(H^1(\Omega))')$.

(ii) The temperature sequence (θ^ε) is uniformly bounded in the space $L^\infty(\Omega_T)\cap L^2(0,T;H^1(\Omega))$. The dispersive term satisfies

$$\||q^\varepsilon|^{1/2}\nabla\theta^\varepsilon\|_{(L^2(\Omega_T))^N}\le C.$$

Proof. We multiply Eq. (1.1) by p^ε and we integrate over Ω. Integrating by parts, we obtain

$$\frac12\frac{d}{dt}\int_\Omega\phi^\varepsilon\,|p^\varepsilon(\cdot,t)|^2dx+\int_\Omega\kappa^\varepsilon\nabla p^\varepsilon\cdot\nabla p^\varepsilon\,dx=\int_\Omega\frac{q^i-q^s}{\mu(\theta^\varepsilon)}\,p^\varepsilon\,dx.$$

Because of Assumption (1.7), there exists some real number C such that $0\le\mu(\theta^\varepsilon)^{-1}\le C$ almost everywhere in Ω_T. Using the Cauchy-Schwarz and Young inequalities, we thus write

$$\Big|\int_\Omega\frac{q^i-q^s}{\mu(\theta^\varepsilon)}\,p^\varepsilon\,dx\Big|\le\int_\Omega\Big|\frac{q^i-q^s}{\mu(\theta^\varepsilon)}\Big|^2dx+C\int_\Omega|p^\varepsilon|^2dx\le C+C\int_\Omega|p^\varepsilon|^2dx.$$

Since $\phi^\varepsilon(x)\ge\phi_->0$ almost everywhere in Ω, the latter relation and the Gronwall lemma give the first estimates of Lemma 2.1. The estimates on q^ε are a straightforward consequence since $q^\varepsilon=-\kappa^\varepsilon\nabla p^\varepsilon$. Eq. (1.1) then directly implies that $\phi^\varepsilon\partial_t p^\varepsilon$ is uniformly bounded in $L^2(0,T;(H^1(\Omega))')$. Indeed for any function $g\in L^2(0,T;H^1(\Omega))$, we have

$$\begin{aligned}\big|\langle\phi^\varepsilon\partial_t p^\varepsilon,g\rangle_{L^2(0,T;H^1(\Omega))\times L^2(0,T;(H^1(\Omega))')}\big|&=\Big|-\int_{\Omega_T}\kappa^\varepsilon\nabla p^\varepsilon\cdot\nabla g\,dxdt\\&\quad+\int_{\Omega_T}\frac{q^i-q^s}{\mu(\theta^\varepsilon)}\,g\,dxdt\Big|\le C\|q^\varepsilon\|_{(L^2(\Omega_T))^N}\|g\|_{L^2(0,T;H^1(\Omega))}\\&\quad+\Big\|\frac{q^i-q^s}{\mu(\theta^\varepsilon)}\Big\|_{L^2(\Omega_T)}\|g\|_{L^2(\Omega_T)}\le C\|g\|_{L^2(0,T;H^1(\Omega))}.\end{aligned}$$

Part *(i)* of Lemma 2.1 is proved.

We now consider Pb. (1.3)-(1.4). We ensure with Theorem 1.1 that the temperature θ^ε satisfies $0 \le \theta^\varepsilon(x,t) \le 1$ almost everywhere in Ω_T for any $\varepsilon > 0$. The sequence (θ^ε) is then uniformly bounded in $L^\infty(\Omega_T)$. We now multiply Eq. (1.3) by θ^ε and we integrate by parts over Ω. We obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \phi^\varepsilon\,|\theta^\varepsilon(\cdot,t)|^2\,dx + \int_\Omega \phi^\varepsilon\, d(q^\varepsilon)\nabla\theta^\varepsilon\cdot\nabla\theta^\varepsilon\,dx + \int_\Omega (q^\varepsilon\cdot\nabla\theta^\varepsilon)\,\theta^\varepsilon\,dx + \int_\Omega \frac{q^i}{\mu(\theta^\varepsilon)}\,|\theta^\varepsilon|^2\,dx = \int_\Omega \frac{q^i}{\mu(\theta^\varepsilon)}\,\theta^\varepsilon\,dx. \tag{2.1}$$

Since θ^ε is uniformly bounded in $L^\infty(\Omega_T)$, we write using the Cauchy-Schwarz and Young inequalities

$$\begin{aligned}\Big|\int_\Omega (q^\varepsilon\cdot\nabla\theta^\varepsilon)\,\theta^\varepsilon\,dx\Big| &\le C\int_\Omega |q^\varepsilon|\,dx + \frac{\phi_- d_t}{2}\int_\Omega |q^\varepsilon|\,|\nabla\theta^\varepsilon|^2\,dx\\ &\le C(t) + \frac{\phi_- d_t}{2}\int_\Omega |q^\varepsilon|\,|\nabla\theta^\varepsilon|^2\,dx\end{aligned}$$

where the quantity $C(t)$ is uniformly bounded in $L^1(0,T)$. The last term of the left-hand side of (2.1) is nonnegative. The right-hand side is bounded by a constant since $q^i \in L^2(\Omega_T)$, $0 \le \theta^\varepsilon(x,t) \le 1$ and $0 \le \mu(\theta^\varepsilon)^{-1} \le C$ a.e. in Ω_T. We note that the dispersive tensor d satisfies

$$d(q^\varepsilon)\xi\cdot\xi \ge \phi_-(d_m + d_t|q^\varepsilon|)|\xi|^2, \quad |d(q^\varepsilon)\xi| \le \phi_-^{-1}(d_m + d_l|q^\varepsilon|)|\xi|, \qquad \forall \xi \in \mathbb{R}^N. \tag{2.2}$$

Using Property (2.2), Relation (2.1) thus leads to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \phi\,|\theta^\varepsilon(\cdot,t)|^2\,dx + \int_\Omega \phi_-(d_m + \frac{d_t}{2}\,|q^\varepsilon|)\,|\nabla\theta^\varepsilon|^2\,dx \le C(t) + C.$$

The result of the lemma follows straightforward since $\phi(x) \ge \phi_- > 0$ a.e. in Ω. □

Remark 1. Some additional assumptions on the pressure problem allow to state a maximum principle for p^ε. Actually, if $q^i - q^s$ belongs to $L^\infty(\Omega_T)$ and if the initial pressure is bounded below and above by some constants, then the pressure sequence (p^ε) is uniformly bounded in $L^\infty(\Omega_T)$. We refer to [5] for a proof.

In view of the previous lemmas, we assert the existence of $p \in L^\infty(0,T;L^2(\Omega))$, $q \in (L^2(\Omega_T))^N$ and $\theta \in L^\infty(\Omega_T)\cap L^2(0,T;H^1(\Omega))$ such that, up to an extracted subsequence, as $\varepsilon \to 0$,

$$\begin{aligned}&p^\varepsilon \rightharpoonup p \text{ weakly} * \text{ in } L^\infty(0,T;L^2(\Omega)),\\ &q^\varepsilon \rightharpoonup q \text{ weakly in } (L^2(\Omega_T))^N,\\ &\theta^\varepsilon \rightharpoonup \theta \text{ weakly in } L^2(0,T;H^1(\Omega)) \text{ and weakly} * \text{ in } L^\infty(\Omega_T).\end{aligned}$$

Let us then prove the following compactness result.

Lemma 2.2. *The sequence (θ^ε) is sequentially compact in $L^2(\Omega_T)$.*

Proof. We multiply (1.3) by a testing function ψ in the space $L^4(0,T;W^{1,4}(\Omega))$ and integrate over Ω_T. We get

$$\int_{\Omega_T} \phi^\varepsilon\, \partial_t\theta^\varepsilon\, \psi\, dx\, dt = -\int_{\Omega_T} (q^\varepsilon\cdot\nabla\theta^\varepsilon)\,\psi\, dx\, dt - \int_{\Omega_T} \phi^\varepsilon\, d(q^\varepsilon)\nabla\theta^\varepsilon\cdot\nabla\psi\, dx\, dt$$
$$- \int_{\Omega_T} \frac{q^i}{\mu(\theta^\varepsilon)}\,\theta^\varepsilon\,\psi\, dx\, dt + \int_{\Omega_T} \frac{q^i}{\mu(\theta^\varepsilon)}\,\psi\, dx\, dt.$$

Owing to our first energy estimates, the last two terms are bounded by $C\|\psi\|_{L^2(\Omega_T)}$. Using (2.2) and the Cauchy-Schwarz inequality, we obtain for the second term in the right-hand side

$$\left|\int_{\Omega_T} \phi^\varepsilon\, d(q^\varepsilon)\nabla\theta^\varepsilon\cdot\nabla\psi\, dx\, dt\right| \le C\,\|(d_m + d_l\,|q^\varepsilon|)^{1/2}\,\nabla\theta^\varepsilon\|_{(L^2(\Omega_T))^N}$$
$$\times\|(d_m + d_l\,|q^\varepsilon|)^{1/2}\,\nabla\psi\|_{(L^2(\Omega_T))^N}.$$

Using the following inequalities

$$\begin{aligned}
&\|(d_m + d_l\,|q^\varepsilon|)^{1/2}\,\nabla\psi\|^2_{(L^2(\Omega_T))^N}\\
&\le \int_0^T \left(d_m\,\|\nabla\psi(\cdot,t)\|^2_{(L^2(\Omega))^N} + d_l\int_\Omega |q^\varepsilon|\,|\nabla\psi|^2\, dx\right) dt\\
&\le d_m\,\|\nabla\psi\|^2_{(L^2(\Omega_T))^N} + d_l\int_0^T \|q^\varepsilon(\cdot,t)\|_{(L^2(\Omega))^N}\,\|\nabla\psi(\cdot,t)\|^2_{(L^4(\Omega))^N} dt\\
&\le \left(C\,d_m + d_l\,\|q^\varepsilon\|_{(L^2(\Omega_T))^N}\right)\,\|\nabla\psi\|^2_{(L^4(\Omega_T))^N},
\end{aligned}$$

we obtain

$$\left|\int_{\Omega_T} \phi^\varepsilon\, d(q^\varepsilon)\nabla\theta^\varepsilon\cdot\nabla\psi\, dx\, dt\right| \le C\,\|\psi\|_{L^4(0,T;W^{1,4}(\Omega))}.$$

For the first term in the right-hand side we use the Sobolev inclusion $W^{1,4}(\Omega)\subset L^4(\Omega)$ and the Cauchy-Schwarz inequality as follows.

$$\left|\int_{\Omega_T} (q^\varepsilon\cdot\nabla\theta^\varepsilon)\,\psi\, dx\, dt\right| \le \||q^\varepsilon|\nabla\theta^\varepsilon\|_{(L^{4/3}(\Omega_T))^N}\,\|\psi\|_{L^4(0,T;W^{1,4}(\Omega))}.$$

Note that by Lemma 2.1, $(|q^\varepsilon|\nabla\theta^\varepsilon)$ is uniformly bounded in $(L^{4/3}(\Omega_T))^N$. We conclude that $\partial_t(\phi^\varepsilon\theta^\varepsilon)$ is uniformly bounded in $L^{4/3}(0,T;(W^{1,4}(\Omega))')$. Since $(\phi^\varepsilon\theta^\varepsilon)$ is uniformly bounded in $L^\infty(\Omega_T)$, a standard argument of Aubin's type, see Simon [13] Corollary 4 for instance, proves that $(\phi^\varepsilon\theta^\varepsilon)$ lies in a compact subset of $\mathcal{C}(0,T;(H^1(\Omega))')$. Therefore, there exists $\xi\in\mathcal{C}(0,T;(H^1(\Omega))')$, such that, up to an extracted subsequence, as $\varepsilon\to 0$,

$$\phi^\varepsilon\,\theta^\varepsilon \to \xi \text{ in } \mathcal{C}(0,T;(H^1(\Omega))').$$

Let us prove that $\xi = \tilde\phi\,\theta$, where

$$\tilde\phi(x) = \frac{1}{|Y|}\int_Y \phi(x,y)\, dy = \int_Y \phi(x,y)\, dy,\quad x\in\Omega,$$

is the weak limit in $L^p(\Omega)$ of the sequence (ϕ^ε). Let $\psi\in\mathcal{D}(\Omega_T)$. We write

$$\int_{\Omega_T} \phi^\varepsilon\,\theta^\varepsilon\,\psi\, dxdt = \int_\Omega \phi^\varepsilon\, v^\varepsilon\, dx,$$

where

$$v^\varepsilon(x) = \int_0^T \theta^\varepsilon(x,t)\,\psi(x,t)\,dt.$$

The sequence (v^ε) is bounded in $H^1(\Omega)$ and thus sequentially compact in $L^2(\Omega_T)$. It strongly converges to the function v defined by $v(x) = \int_0^T \theta(x,t)\psi(x,t)\,dt$ for $x \in \Omega$. Then, up to extracted subsequences,

$$\lim_{\varepsilon\to 0} \int_{\Omega_T} \phi^\varepsilon\,\theta^\varepsilon\,\psi\,dxdt = \int_{\Omega_T} \xi\,\psi\,dxdt = \int_\Omega \tilde{\phi}(x)\,v(x)\,dx$$
$$= \int_\Omega \tilde{\phi}(x)\left(\int_0^T \theta(x,t)\,\psi(x,t)\,dt\right)dx = \int_{\Omega_T} \tilde{\phi}(x)\,\theta(x,t)\,\psi(x,t)\,dxdt.$$

To prove the strong convergence of (a subsequence of) (θ^ε) we then use the decomposition:

$$\int_{\Omega_T} \phi^\varepsilon\,(\theta^\varepsilon - \theta)^2\,dxdt = \int_0^T \langle \phi^\varepsilon\,\theta^\varepsilon, \theta^\varepsilon - \theta\rangle_{(H^1(\Omega))'\times H^1(\Omega)}\,dt$$
$$+ \int_{\Omega_T} \phi^\varepsilon\,\theta^2\,dxdt - \int_{\Omega_T} \phi^\varepsilon\,\theta^\varepsilon\,\theta\,dxdt.$$

Since the right-hand side tends to 0 as $\varepsilon \to 0$, we infer from the latter relation the strong convergence in $L^2(\Omega_T)$ of a subsequence of (θ^ε). The proof of the lemma is complete. □

Since Equation (1.1) is of degenerate parabolic type, we have no estimate for the pressure gradient. Thus it seems to be hopeless to look for the compactness of the sequence (p^ε). We then study the weighted pressure $(1/\mu(\theta^\varepsilon)^{1/2})p^\varepsilon$. We begin by two auxiliary results (Lemmas 2.3 and 2.4).

Lemma 2.3. *For any $\eta > 0$, the sequence $\phi^\varepsilon(p^\varepsilon/\sqrt{p^{\varepsilon 2}+\eta})\partial_t p^\varepsilon$ is uniformly bounded in $L^1(0,T;(W^{1,4}(\Omega))')$.*

Proof. Let ψ a given function of $L^\infty(0,T;W^{1,4}(\Omega))$. We multiply Eq. (1.1) by the test function $p^\varepsilon\psi(p^{\varepsilon 2}+\eta)^{-1/2}$ and we integrate by parts over Ω. We get

$$\left\langle \phi^\varepsilon p^\varepsilon (p^{\varepsilon 2}+\eta)^{-1/2}\partial_t p^\varepsilon, \psi \right\rangle = \int_\Omega (q^\varepsilon\cdot\nabla\psi)\,p^\varepsilon(p^{\varepsilon 2}+\eta)^{-1/2}\,dx$$
$$+ \int_\Omega (q^\varepsilon\cdot\nabla p^\varepsilon)\,\eta(p^{\varepsilon 2}+\eta)^{-3/2}\,\psi\,dx + \int_\Omega \frac{q^i - q^s}{\mu(\theta^\varepsilon)}\,p^\varepsilon(p^{\varepsilon 2}+\eta)^{-1/2}\,\psi\,dx,$$

where the brackets denote the duality. We note that the functions $p^\varepsilon(p^{\varepsilon 2}+\eta)^{-1/2}$ and $\eta(p^{\varepsilon 2}+\eta)^{-3/2}$ are uniformly bounded in $L^\infty(\Omega_T)$. With Lemma 2.1, we assert that $q^\varepsilon\cdot\nabla p^\varepsilon$ is uniformly bounded in $L^1(\Omega_T)$. Using in particular the embedding $W^{1,4}(\Omega)\subset L^\infty(\Omega)$, we write the uniform estimate $|\langle(\phi^\varepsilon p^\varepsilon(p^{\varepsilon 2}+\eta)^{-1/2})\partial_t p^\varepsilon,\psi\rangle| \le C(t)\|\psi\|_{W^{1,4}(\Omega)}$ where $C(t)$ is uniformly bounded in $L^1(0,T)$. Our claim is proved. □

A part of the proof of the homogenization process will be carried out by using the two-scale convergence introduced by G.Nguetseng in [10] and developed by Allaire in [1]. We recall the basic definition and properties of this concept.

Proposition 1. *A sequence of functions* (v^ε) *bounded in* $L^2(\Omega_T)$ *two-scale converges to a limit* $v_o(x,y,t)$ *belonging to* $L^2(\Omega_T\times Y)$, $v^\varepsilon \overset{2}{\rightharpoonup} v_o$, *if*

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} v^\varepsilon(x,t)\,\Psi(x,x/\varepsilon,t)\,dx\,dt = \int_{\Omega_T}\int_Y v_o(x,y,t)\,\Psi(x,y,t)\,dx\,dy\,dt,$$

for any test function $\Psi(x,y,t)$, *Y-periodic in the second variable, satisfying*

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}|\Psi(x,x/\varepsilon,t)|^2\,dx\,dt = \int_{\Omega_T}\int_Y |\Psi(x,y,t)|^2\,dx\,dy\,dt.$$

(i) From each bounded sequence (v^ε) *in* $L^2(\Omega_T)$ *one can extract a subsequence which two-scale converges.*
(ii) Let (v^ε) *be a bounded sequence in* $L^2(0,T;H^1(\Omega))$ *which converges weakly to* v *in* $L^2(0,T;H^1(\Omega))$. *Then* $v^\varepsilon \overset{2}{\rightharpoonup} v$ *and there exists a function* $v_1\in L^2(\Omega_T;H^1_{per}(Y))$ *such that, up to a subsequence,* $\nabla v^\varepsilon \overset{2}{\rightharpoonup} \nabla v(x,t)+\nabla_y v_1(x,y,t)$.

Now we have sufficient tools for claiming the second auxiliary lemma.

Lemma 2.4. *For a fixed* $\eta>0$, *let* $P^\varepsilon_\eta=\sqrt{p^{\varepsilon 2}+\eta}$ *and* P_η *its weak limit in* $L^2(\Omega_T)$. *We claim that*

$$\mu(\theta)^{-1/2}\phi^\varepsilon P^\varepsilon_\eta \rightharpoonup \mu(\theta)^{-1/2}\widetilde{\phi}\,P_\eta \text{ weakly in } L^2(\Omega_T).$$

Proof. Since the sequence (P^ε_η) is uniformly bounded in $L^2(\Omega_T)$, it admits a two-scale limit $P_{\eta,o}\in L^2(\Omega_T\times Y)$. Let $\underline{\psi}\in(\mathcal{D}(\Omega_T;\mathcal{C}^\infty_{per}(Y))\cap\mathcal{C}_o(\Omega_T\times Y))^N$. Integration by parts gives

$$\begin{aligned}&\varepsilon\int_{\Omega_T}\nabla(\mu(\theta^\varepsilon)^{-1/2}P^\varepsilon_\eta)\cdot\underline{\psi}(x,x/\varepsilon,t)\,dxdt\\&=-\int_{\Omega_T}\mu(\theta^\varepsilon)^{-1/2}P^\varepsilon_\eta\left(\varepsilon\mathrm{div}_x\underline{\psi}(x,x/\varepsilon,t)+\mathrm{div}_y\underline{\psi}(x,x/\varepsilon,t)\right)dxdt.\end{aligned}\tag{2.3}$$

We know by Lemma 2.1 and Assumption (1.7) that $\nabla(\mu(\theta^\varepsilon)^{-1/2}P^\varepsilon_\eta)$ is uniformly bounded in $(L^1(\Omega_T))^N$. Then there exists a Radon measure $\nu_\eta\in(\mathcal{M}(\Omega_T\times Y))^N$ such that (up to a subsequence not relabeled for convenience)

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Omega_T}\nabla(\mu(\theta^\varepsilon)^{-1/2}P^\varepsilon_\eta)\cdot\underline{\psi}(x,x/\varepsilon,t)\,dxdt\\&=\langle\nu_\eta(x,y,t),\underline{\psi}(x,y,t)\rangle_{(\mathcal{M}(\Omega_T\times Y))^N\times(\mathcal{C}_o(\Omega_T\times Y))^N}.\end{aligned}$$

Thus the left-hand side of relation (2.3) tends to 0 with ε. Bearing in mind the a.e. convergence of θ^ε to θ and passing to the limit $\varepsilon\to 0$ in (2.3) we get

$$0=\int_{\Omega_T}\int_Y \mu(\theta(x,t))^{-1/2}P_{\eta,o}(x,y,t)\,\mathrm{div}_y\underline{\psi}(x,y,t))\,dxdydt.$$

We conclude that $\mu(\theta)^{-1/2}P_{\eta,o}$ does not depend on y. Then the weak limit in $L^2(\Omega_T)$ and the two-scale limit of $\mu(\theta^\varepsilon)^{-1/2}P^\varepsilon_\eta$ are identical:

$$\mu(\theta)^{-1/2}P_{\eta,o}=\mu(\theta)^{-1/2}P_\eta.$$

Moreover, using once again the a.e. convergence of θ^ε to θ in Ω_T, we note that

$$\mu(\theta)^{-1/2}\lim_{\varepsilon\to 0}\big(\phi^\varepsilon P_\eta^\varepsilon\big) = \lim_{\varepsilon\to 0}\big(\phi^\varepsilon\mu(\theta^\varepsilon)^{-1/2}P_\eta^\varepsilon\big) = \int_Y \phi(x,y)\,\mu(\theta)^{-1/2}P_{\eta,o}(x,y,t)\,dy$$
$$= \int_Y \phi(x,y)\,\mu(\theta)^{-1/2}P_\eta\,dy = \widetilde{\phi}\,\mu(\theta)^{-1/2}P_\eta.$$

It follows in particular that $\mu(\theta)^{-1/2}\lim_{\varepsilon\to 0}(\phi^\varepsilon P_\eta^\varepsilon) = \mu(\theta)^{-1/2}\tilde{\phi}P_\eta$. This ends the proof of the lemma. □

We then recall the following compensated compactness result (see Kazhikhov [9]).

Lemma 2.5. *Let V, W and V_1 Banach spaces such that $D(\Omega)\subset V\subset W\subset D'(\Omega)$ and $V'\subset V_1'\subset D'(\Omega)$, with continuous embedding, the embedding $V\subset W$ being compact. Let (α^ε) and (β^ε) be sequences such that*

$$(\alpha^\varepsilon) \text{ is bounded in } L^p(0,T;V), \quad \alpha^\varepsilon \rightharpoonup \alpha \text{ weakly in } L^p(0,T;W),$$
$$(\beta^\varepsilon) \text{ is bounded in } L^q(0,T;W'), \quad \beta^\varepsilon \rightharpoonup \beta \text{ weakly in } L^q(0,T;W'),$$
$$(\partial_t\beta^\varepsilon) \text{ is bounded in } L^{p_1}(0,T;V_1'),$$

for some $1<p<\infty$, $q\ge p/(p-1)$ and $p_1\ge 1$. Then, up to an extracted subsequence, the convergence $\alpha^\varepsilon\beta^\varepsilon\rightharpoonup\alpha\beta$ holds true in $D'(\Omega_T)$.

Applying astutely several times Lemma 2.5 we prove the following compactness result of $\mu(\theta^\varepsilon)^{-1/2}p^\varepsilon$.

Lemma 2.6. *The weighted pressure $\mu(\theta^\varepsilon)^{-1/2}p^\varepsilon$ is sequentially compact in $L^2(\Omega_T)$.*

Proof. Let $\eta>0$ be a fixed real number. We recall that $P_\eta^\varepsilon = \sqrt{p^{\varepsilon\,2}+\eta}$ and we set $\beta_\eta^\varepsilon = \phi^\varepsilon P_\eta^\varepsilon$. The sequence (β_η^ε) is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$. Since $\phi^\varepsilon(x)\ge\phi_->0$ a.e. in Ω, we can denote $\widetilde{\phi}P_{\eta,\phi}$ its weak limit:

$$\beta_\eta^\varepsilon \rightharpoonup \beta_\eta = \widetilde{\phi}\,P_{\eta,\phi} \text{ weakly} * \text{ in } L^\infty(0,T;W), \quad W = L^2(\Omega) = W'.$$

Note that by Lemma 2.4 $\mu(\theta)^{-1/2}\tilde{\phi}P_{\eta,\phi} = \mu(\theta)^{-1/2}\tilde{\phi}P_\eta$. By Lemma 2.3, $(\partial_t\beta_\eta^\varepsilon)$ is bounded in $L^1(0,T;V_1')$, $V_1 = W^{1,4}(\Omega)$. We also note that $P_\eta^\varepsilon(x,t)\ge\eta^{1/2}>0$ a.e. in Ω_T.

On the one hand, we set $\alpha_1^\varepsilon = \mu(\theta^\varepsilon)^{-1}P_\eta^\varepsilon(P_\eta^\varepsilon+1)^{-1}$. This quantity is clearly bounded in $L^\infty(\Omega_T)$. With Lemma 2.1, we check that $\nabla\alpha_1^\varepsilon$ is uniformly bounded in $(L^2(\Omega_T))^N$. We thus assert that α_1^ε is bounded in $L^2(0,T;V)$ with $V=H^1(\Omega)$ compactly embedded in W. We apply Lemma 2.5 and get $\alpha_1^\varepsilon\beta_\eta^\varepsilon\rightharpoonup\alpha_1\beta_\eta$ in $D'(\Omega_T)$. Using successively the a.e. convergence $\theta^\varepsilon\to\theta$ in Ω_T and Lemma 2.4, this reads

$$\begin{aligned}\phi^\varepsilon\mu(\theta^\varepsilon)^{-1}\,P_\eta^\varepsilon(P_\eta^\varepsilon+1)^{-1}\,P_\eta^\varepsilon \quad &\rightharpoonup \quad \widetilde{\phi}\,P_{\eta,\phi}\,\mu(\theta)^{-1}\,\overline{P_\eta^\varepsilon(P_\eta^\varepsilon+1)^{-1}}\\ &= \quad \widetilde{\phi}\,P_\eta\,\mu(\theta)^{-1}\,\overline{P_\eta^\varepsilon(P_\eta^\varepsilon+1)^{-1}},\end{aligned} \tag{2.4}$$

where $\overline{f^\varepsilon}$ denotes the weak *ad hoc* limit of a sequence (f^ε). The same work with $\phi^\varepsilon(P_\eta^\varepsilon+1)$ instead of β_η^ε produces

$$\phi^\varepsilon\,\mu(\theta^\varepsilon)^{-1}\,P_\eta^\varepsilon = \phi^\varepsilon\,(P_\eta^\varepsilon+1)\,\mu(\theta^\varepsilon)^{-1}\,P_\eta^\varepsilon(P_\eta^\varepsilon+1)^{-1}$$
$$\rightharpoonup \widetilde{\phi}\,(P_{\eta,\phi}+1)\,\mu(\theta)^{-1}\,\overline{P_\eta^\varepsilon(P_\eta^\varepsilon+1)^{-1}} = \widetilde{\phi}\,(P_\eta+1)\,\mu(\theta)^{-1}\,\overline{P_\eta^\varepsilon(P_\eta^\varepsilon+1)^{-1}}.$$

Since Lemma 2.4 gives $\phi^\varepsilon P_\eta^\varepsilon \mu(\theta^\varepsilon)^{-1} \rightharpoonup \widetilde{\phi} P_\eta \mu(\theta)^{-1}$ weakly* in $L^\infty(0,T;L^2(\Omega))$, we conclude that

$$\mu(\theta)^{-1}\overline{P_\eta^\varepsilon/(P_\eta^\varepsilon+1)} = \mu(\theta)^{-1}P_\eta/(P_\eta+1).$$

Now Relation (2.4) is

$$\phi^\varepsilon \mu(\theta^\varepsilon)^{-1}\, {P_\eta^\varepsilon}^2 (P_\eta^\varepsilon+1)^{-1} \rightharpoonup \widetilde{\phi}\mu(\theta)^{-1}\, P_\eta^2 (P_\eta+1)^{-1}. \tag{2.5}$$

On the other hand, we set $\alpha_2^\varepsilon = \mu(\theta^\varepsilon)^{-1/2}(P_\eta^\varepsilon+1)^{-1/2}$. Because of Assumption (1.7), the sequence α_2^ε satisfies similar properties to those of α_1^ε: α_2^ε is uniformly bounded in $L^2(\Omega_T)$ and in $L^2(0,T;H^1(\Omega))$. Setting once again $V = H^1(\Omega)$ and $W = L^2(\Omega)$, we thus assert with Lemma 2.5 that $\alpha_2^\varepsilon \beta_\eta^\varepsilon \rightharpoonup \alpha_2\beta$ in $D'(\Omega_T)$, that is (with the a.e. convergence of θ^ε to θ and with Lemma 2.4)

$$\begin{aligned} \phi^\varepsilon P_\eta^\varepsilon\, \mu(\theta^\varepsilon)^{-1/2}(P_\eta^\varepsilon+1)^{-1/2} &\rightharpoonup \widetilde{\phi}\, P_{\eta,\phi}\, \mu(\theta)^{-1/2}\, \overline{(P_\eta^\varepsilon+1)^{-1/2}} \\ &= \widetilde{\phi}\, P_\eta\, \mu(\theta)^{-1/2}\, \overline{(P_\eta^\varepsilon+1)^{-1/2}}. \end{aligned} \tag{2.6}$$

There is an other way to compute the latter limit. Let $\psi \in \mathcal{D}(\Omega_T)$. We write

$$\int_{\Omega_T} \phi^\varepsilon P_\eta^\varepsilon \mu(\theta^\varepsilon)^{-1/2}(P_\eta^\varepsilon+1)^{-1/2}\,\psi\,dx\,dt = \int_\Omega \phi^\varepsilon\, v^\varepsilon\,dx,$$

where

$$v^\varepsilon(x) = \int_0^T P_\eta^\varepsilon \mu(\theta^\varepsilon)^{-1/2}(P_\eta^\varepsilon+1)^{-1/2}\,\psi\,dt.$$

Because of (1.7), the sequence (v^ε) is bounded in $H^1(\Omega)$ which is compactly embedded in $L^2(\Omega)$. Then, up to extracted subsequences,

$$\begin{aligned} \lim_{\varepsilon\to 0}\int_{\Omega_T} \phi^\varepsilon P_\eta^\varepsilon \mu(\theta^\varepsilon)^{-1/2}(P_\eta^\varepsilon+1)^{-1/2}\,\psi\,dx\,dt &= \int_\Omega \tilde{\phi}\, v(x)\,dx \\ &= \int_\Omega \tilde{\phi}\Big(\int_0^T \mu(\theta)^{-1/2}\overline{P_\eta^\varepsilon(P_\eta^\varepsilon+1)^{-1/2}}\,\psi\,dt\Big)\,dx \\ &= \int_{\Omega_T} \tilde{\phi}\,\mu(\theta)^{-1/2}\overline{P_\eta^\varepsilon(P_\eta^\varepsilon+1)^{-1/2}}\,\psi\,dxdt, \end{aligned}$$

that is

$$\phi^\varepsilon P_\eta^\varepsilon \mu(\theta^\varepsilon)^{-1/2}(P_\eta^\varepsilon+1)^{-1/2} \rightharpoonup \widetilde{\phi}\,\mu(\theta)^{-1/2}\,\overline{P_\eta^\varepsilon(P_\eta^\varepsilon+1)^{-1/2}}. \tag{2.7}$$

We now look for an estimate of the non-explicit limits in (2.6) and (2.7). Using the convexity of the function $x \mapsto (x+1)^{-1/2}$, we claim following Tartar [14] that

$$\overline{(P_\eta^\varepsilon+1)^{-1/2}} \geq (P_\eta+1)^{-1/2}. \tag{2.8}$$

We recall that $P_\eta(x,t) \geq \eta^{1/2} > 0$ almost everywhere in Ω_T. With the concavity of $x \mapsto x/\sqrt{x+1}$ in $\mathbb{R}_+$, we conclude that

$$\overline{P_\eta^\varepsilon(P_\eta^\varepsilon+1)^{-1/2}} \leq P_\eta(P_\eta+1)^{-1/2}. \tag{2.9}$$

Combining (2.6)-(2.9), we get

$$\phi^\varepsilon P_\eta^\varepsilon \mu(\theta^\varepsilon)^{-1/2}(P_\eta^\varepsilon+1)^{-1/2} \rightharpoonup \widetilde{\phi}\,\mu(\theta)^{-1/2} P_\eta (P_\eta+1)^{-1/2}. \tag{2.10}$$

Now we exploit Relations (2.5) and (2.10). A consequence is the strong convergence of $\mu(\theta^\varepsilon)^{-1/2}P_\eta^\varepsilon(P_\eta^\varepsilon+1)^{-1/2}$ to $\mu(\theta)^{-1/2}P_\eta(P_\eta+1)^{-1/2}$ in $L^2(\Omega_T)$ and almost everywhere in Ω_T. Indeed, we have

$$\int_{\Omega_T} \big(\mu(\theta^\varepsilon)^{-1/2} P_\eta^\varepsilon (P_\eta^\varepsilon + 1)^{-1/2} - \mu(\theta)^{-1/2} P_\eta (P_\eta + 1)^{-1/2}\big)^2 \, dxdt$$
$$\le \frac{1}{\phi_-} \int_{\Omega_T} \phi^\varepsilon \big(\mu(\theta^\varepsilon)^{-1/2} P_\eta^\varepsilon (P_\eta^\varepsilon + 1)^{-1/2} - \mu(\theta)^{-1/2} P_\eta (P_\eta + 1)^{-1/2}\big)^2 \, dxdt$$
$$= \frac{1}{\phi_-} \int_{\Omega_T} \big(\phi^\varepsilon \mu(\theta^\varepsilon)^{-1}\, P_\eta^{\varepsilon 2} (P_\eta^\varepsilon + 1)^{-1} + \phi^\varepsilon \mu(\theta)^{-1}\, P_\eta^2 (P_\eta + 1)^{-1}$$
$$-2\phi^\varepsilon P_\eta^\varepsilon \mu(\theta^\varepsilon)^{-1/2} (P_\eta^\varepsilon + 1)^{-1/2} \mu(\theta)^{-1/2} P_\eta (P_\eta + 1)^{-1/2}\big) \, dxdt.$$

In view of (2.5) and (2.10), the latter expression tends to 0 as $\varepsilon \to 0$. It follows that

$$\mu(\theta^\varepsilon)^{-1/2} P_\eta^\varepsilon (P_\eta^\varepsilon + 1)^{-1/2} \to \mu(\theta)^{-1/2} P_\eta (P_\eta + 1)^{-1/2} \text{ in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T.$$

Since $P_\eta^\varepsilon + 1 \rightharpoonup P_\eta + 1$, we conclude that

$$\mu(\theta^\varepsilon)^{-1}\, P_\eta^{\varepsilon 2} = \mu(\theta^\varepsilon)^{-1}\, P_\eta^{\varepsilon 2} (P_\eta^\varepsilon + 1)^{-1}\, (P_\eta^\varepsilon + 1) \rightharpoonup \mu(\theta)\, P_\eta^2.$$

Since we already know that $\mu(\theta^\varepsilon)^{-1/2} P_\eta^\varepsilon \rightharpoonup \mu(\theta)^{-1/2} P_\eta$ in $L^2(\Omega_T)$, the latter relation proves the convergence $\mu(\theta^\varepsilon)^{-1/2} P_\eta^\varepsilon \to \mu(\theta)^{-1/2} P_\eta$ almost everywhere in Ω_T. In view of the definition of P_η^ε, it follows that $\mu(\theta^\varepsilon)^{-1} P_\eta^{\varepsilon 2} = \mu(\theta^\varepsilon)^{-1}(p^{\varepsilon 2} + \eta)$ and then $\mu(\theta^\varepsilon)^{-1} p^{\varepsilon 2}$ converges almost everywhere in Ω_T. Taking the square root, we conclude that the weighted pressure $\mu(\theta^\varepsilon)^{-1/2} p^\varepsilon$ converges also almost everywhere to its weak L^2-limit, that is

$$\mu(\theta^\varepsilon)^{-1/2} p^\varepsilon \to \mu(\theta)^{-1/2} p \quad \text{strongly in } L^2(\Omega_T) \text{ and a.e. in } \Omega_T. \tag{2.11}$$

Lemma 2.6 is proved. □

We now turn back to the Darcy law. We claim and prove the following two-scale convergence result.

Lemma 2.7. *There exists some function $p_2 \in L^1(0,T; L^{r_1}(\Omega; H^1_{per}(Y)))$, $r_1 > 1$, such that*

$$\frac{1}{\mu(\theta^\varepsilon)^{\alpha+1/2}} \nabla p^\varepsilon \overset{2}{\rightharpoonup} \frac{1}{\mu(\theta)^{\alpha+1/2}} \nabla p + \nabla_y p_2,$$

the real number α, $0 < \alpha \le 1/2$, being defined in Assumption (1.7). More precisely, the function p_2 is defined by

$$p_2(x,y,t) = \Big(\alpha + \frac{1}{2}\Big) \frac{\mu'(\theta)}{\mu(\theta)^{\alpha+3/2}} p(x,t) \theta_1(x,y,t) + p_1(x,y,t)$$

where $p_1 \in L^2(\Omega_T; H^1_{per}(Y))$ and $\theta_1 \in L^2(\Omega_T; H^1_{per}(Y))$ is given by the two-scale limit $\nabla \theta^\varepsilon \overset{2}{\rightharpoonup} \nabla \theta + \nabla_y \theta_1$.

Proof. Let $\underline{\xi} \in (L^2(\Omega_T; L^2_{per}(Y)))^N$ be the two-scale limit of the sequence $(\mu(\theta^\varepsilon)^{-1/2} \nabla p^\varepsilon)$. Let $\underline{\psi} \in (\mathcal{D}(\Omega_T; \mathcal{C}^\infty_{per}(Y)))^N$ such that $\mathrm{div}_y \underline{\psi}(x,y,t) = 0$. On the one hand we have

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} \frac{1}{\mu(\theta^\varepsilon)^{\alpha+1/2}} \nabla p^\varepsilon \cdot \underline{\psi}(x, x/\varepsilon, t)\, dxdt = \int_{\Omega_T} \int_Y \underline{\xi}(x,y,t) \cdot \underline{\psi}(x,y,t)\, dxdydt. \tag{2.12}$$

On the other hand, we recall that $\mu'(\theta^\varepsilon)/\mu(\theta^\varepsilon)^{\alpha+1}$ is uniformly bounded in $L^\infty(\Omega_T)$ by Assumption (1.7). We note that $\mu(\theta^\varepsilon)^{-1/2} p^\varepsilon$ is uniformly bounded and strongly convergent in $L^\infty(0,T;L^2(\Omega))$. Function $(\mu'(\theta^\varepsilon)/\mu(\theta^\varepsilon)^{\alpha+1})\,\mu(\theta^\varepsilon)^{-1/2}p^\varepsilon\,\underline{\psi}$ which belongs to $(L^2(\Omega_T;\mathcal{C}^\infty_{per}(Y)))^N$ is thus an admissible test function for the two-scale convergence. And the sequence $\nabla\theta^\varepsilon$ is uniformly bounded in $(L^2(\Omega_T))^N$. Integrating by parts and using Lemma 2.6, we thus compute

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_T}\frac{1}{\mu(\theta^\varepsilon)^{\alpha+1/2}}\nabla p^\varepsilon\cdot\underline{\psi}(x,x/\varepsilon,t)\,dxdt\\
&=\lim_{\varepsilon\to 0}\Big(-\int_{\Omega_T}\frac{1}{\mu(\theta^\varepsilon)^{\alpha+1/2}}\,p^\varepsilon\,\mathrm{div}_x\underline{\psi}(x,x/\varepsilon,t)\,dxdt\\
&\qquad+\int_{\Omega_T}\big(\alpha+\frac12\big)\,\frac{\mu'(\theta^\varepsilon)}{\mu(\theta^\varepsilon)^{\alpha+1}}\,\frac{1}{\mu(\theta^\varepsilon)^{1/2}}\,p^\varepsilon\,\nabla\theta^\varepsilon\cdot\underline{\psi}(x,x/\varepsilon,t)\,dxdt\Big)\\
&=-\int_{\Omega_T}\int_Y\frac{1}{\mu(\theta)^{\alpha+1/2}}\,p\,\mathrm{div}_x\underline{\psi}(x,y,t)\,dxdydt\\
&\qquad+\int_{\Omega_T}\int_Y\big(\alpha+\frac12\big)\,\frac{\mu'(\theta)}{\mu(\theta)^{\alpha+3/2}}\,p\,(\nabla\theta+\nabla_y\theta_1)\cdot\underline{\psi}(x,y,t)\,dxdydt. \qquad (2.13)
\end{aligned}$$

We infer from (2.12)-(2.13) that

$$\underline{\xi}(x,y,t)=\nabla_x\Big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\,p\Big)+\big(\alpha+\frac12\big)\,\frac{\mu'(\theta)}{\mu(\theta)^{\alpha+3/2}}\,p\,(\nabla\theta+\nabla_y\theta_1)+\nabla_y p_1$$

for some $p_1\in L^2(\Omega_T;H^1_{per}(Y))$, that is $p_2(x,y,t)=\big(\alpha+\frac12\big)\frac{\mu'(\theta)}{\mu(\theta)^{\alpha+3/2}}p(x,t)\theta_1(x,y,t)$ $+p_1(x,y,t)$. The proof of Lemma 2.7 is achieved. □

3. **Derivation of the homogenized model.** We have sufficient tools to pass to the limit in the pressure equation. As in Allaire [1], we multiply Eq. (1.1) by a test function in the form $\Psi(x,t)+\varepsilon\Psi_1(x,x/\varepsilon,t)$, with $\Psi\in\mathcal{D}(\Omega_T)$, $\Psi_1\in\mathcal{D}(\Omega_T;C^\infty_{per}(Y))$. We set $\Psi_1^\varepsilon(x,t)=\Psi_1(x,x/\varepsilon,t)$. Integrating by parts over Ω_t, $t\in(0,T)$, we obtain

$$\begin{aligned}
&-\int_{\Omega_t}\phi^\varepsilon p^\varepsilon\,\partial_t(\Psi+\varepsilon\Psi_1^\varepsilon)\,dxds+\int_\Omega\phi^\varepsilon p^\varepsilon(\cdot,t)\,(\Psi(\cdot,t)+\varepsilon\Psi_1^\varepsilon(\cdot,t))\,dx\\
&\quad-\int_\Omega\phi^\varepsilon p_{init}(x)\,(\Psi(x,0)+\varepsilon\Psi_1^\varepsilon(x,0))\,dx\\
&\quad+\int_{\Omega_t}\frac{k^\varepsilon}{\mu(\theta^\varepsilon)^{1/2-\alpha}}\frac{1}{\mu(\theta^\varepsilon)^{\alpha+1/2}}\nabla p^\varepsilon\cdot(\nabla\Psi+\varepsilon\nabla_x\Psi_1^\varepsilon+\nabla_y\Psi_1^\varepsilon)\,dxds\\
&\quad=\int_{\Omega_t}\frac{q^i-q^s}{\mu(\theta^\varepsilon)}\,(\Psi+\Psi_1^\varepsilon)\,dxds.
\end{aligned}$$

We pass to the limit $\varepsilon\to 0$ in the former relation using two-scale convergence arguments and Lemma 2.7. We get

$$\begin{aligned}
&-\int_\Omega\int_Y\phi p_o\,\partial_t\Psi\,dxdyds+\int_\Omega\int_Y\phi(x,y)p_o(x,y,t)\,\Psi(x,t)\,dxdy\\
&-\int_\Omega\int_Y\phi(x,y)p_{init}(x)\,\Psi(x,0)\,dxdy\\
&+\int_{\Omega_t}\int_Y\frac{k(x,y)}{\mu(\theta)^{1/2-\alpha}}\Big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla p+\nabla_y p_2\Big)\cdot(\nabla\Psi(x,t)+\nabla_y\Psi_1(x,y,t))\,dxdyds\\
&=\int_{\Omega_t}\frac{q^i-q^s}{\mu(\theta)}\,\Psi\,dxds. \qquad (3.1)
\end{aligned}$$

The associated strong formulation is

$$\int_Y \phi(y)\,\partial_t p_o(x,y,t)\,dy - \operatorname{div}\Big(\int_Y \frac{k(x,y)}{\mu(\theta)^{1/2-\alpha}}\big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla p + \nabla_y p_2\big)\,dy\Big) = \frac{q^i - q^s}{\mu(\theta)} \quad \text{in } \Omega_T, \tag{3.2}$$

$$-\operatorname{div}_y\Big(\frac{k(x,y)}{\mu(\theta)^{1/2-\alpha}}\big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla p + \nabla_y p_2\big)\Big) = 0 \quad \text{in } \Omega_T \times Y, \tag{3.3}$$

$$k(x,y)\big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla p + \nabla_y p_2\big)\cdot\nu\,|_{\Gamma_T} = 0, \quad p\,|_{t=0} = p_{init}. \tag{3.4}$$

Now let $w^j(x,y,t)$ be the unique solution of the cell-problem:

$$\begin{aligned} &-\operatorname{div}_y\left(k(x,y)\left(\nabla_y w^j(x,y,t) + \frac{1}{\mu(\theta)^{\alpha/2+1/4}}e^j\right)\right) = 0 \quad \text{in } Y,\\ &y \to w^j(x,y,t)\ Y-\text{periodic}, \qquad \int_Y w^j(x,y,t)\,dy = 0, \end{aligned} \tag{3.5}$$

where e^j is the jth vector of the canonical basis of $\mathbb{R}^N$, $1 \le j \le N$. We define a homogenized permeability tensor $K_\mu = (K_{\mu_{ij}})_{1\le i,j\le N}$, by

$$\begin{aligned} K_{\mu_{ij}}(x,t) = \int_Y k(x,y)&\left(\nabla_y w^i(x,y,t) + \frac{1}{\mu(\theta)^{\alpha/2+1/4}}e^i\right)\\ &\cdot\left(\nabla_y w^j(x,y,t) + \frac{1}{\mu(\theta)^{\alpha/2+1/4}}e^j\right)dy, \quad x \in \Omega. \end{aligned} \tag{3.6}$$

Now if we set $p_2(x,y,t) = \frac{1}{\mu(\theta)^{\alpha/2+1/4}}\sum_{j=1}^{N} w^j(x,y,t)\,\partial_{x_j}p(x,t)$, we recover the following homogenized pressure problem.

$$\Big(\int_Y \phi(x,y)\,\partial_t p_o(x,y,t)\,dy\Big) - \operatorname{div}\big(\frac{1}{\mu(\theta)^{1/2-\alpha}}K_\mu\nabla p\big) = \frac{q^i - q^s}{\mu(\theta)} \quad \text{in } \Omega_T, \tag{3.7}$$

$$\frac{1}{\mu(\theta)^{1/2-\alpha}}K_\mu\nabla p\cdot\nu\,|_{\Gamma_T} = 0, \quad p\,|_{t=0} = p_{init}, \tag{3.8}$$

$$p(x,t) = \int_Y p_o(x,y,t)\,dy, \qquad \text{with } \frac{1}{\mu(\theta)^{\alpha+1/2}}p_o = \frac{1}{\mu(\theta)^{\alpha+1/2}}p. \tag{3.9}$$

We prefer giving an equivalent formulation of (3.7)-(3.8) which preserves the structure of the microscopic problem (1.1)-(1.2). We thus set

$$P(x,t) = \int_Y \phi(x,y)\,p_o(x,y,t)\,dy \quad \text{in } \Omega_T. \tag{3.10}$$

We note that

$$\begin{aligned} \frac{1}{\mu(\theta)^{\alpha+1/2}}P &= \int_Y \phi(x,y)\,\frac{1}{\mu(\theta)^{\alpha+1/2}}p_o\,dy = \int_Y \phi(x,y)\,\frac{1}{\mu(\theta)^{\alpha+1/2}}p\,dy\\ &= \Big(\int_Y \phi(x,y)\,dy\Big)\,\frac{1}{\mu(\theta)^{\alpha+1/2}}p. \end{aligned}$$

We thus have in particular

$$\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla p = \frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla\Big(\frac{P}{\widetilde{\phi}}\Big).$$

And the strong formulation (3.2)-(3.4) now writes

$$\partial_t P - \operatorname{div}\Big(\int_Y \frac{k(x,y)}{\mu(\theta)^{1/2-\alpha}}\big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla\Big(\frac{P}{\widetilde{\phi}}\Big) + \nabla_y p_2\big)\,dy\Big) = \frac{q^i - q^s}{\mu(\theta)} \quad \text{in } \Omega_T,$$
$$-\operatorname{div}_y\Big(\frac{k(x,y)}{\mu(\theta)^{1/2-\alpha}}\big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla\Big(\frac{P}{\widetilde{\phi}}\Big) + \nabla_y p_2\big)\Big) = 0 \quad \text{in } \Omega_T \times Y,$$
$$k(x,y)\big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla\Big(\frac{P}{\widetilde{\phi}}\Big) + \nabla_y p_2\big)\cdot\nu\,|_{\Gamma_T} = 0,$$
$$P\,|_{t=0} = \int_Y \phi(\cdot,y)\,p_{init}(\cdot)\,dy = \widetilde{\phi}\,p_{init}.$$

Now we set

$$p'(x,t) = \frac{P(x,t)}{\widetilde{\phi}(x)} = \frac{\int_Y \phi(x,y)\,p_o(x,y,t)\,dy}{\int_Y \phi(x,y)\,dy} \quad \text{in } \Omega_T.$$

This function satisfies

$$\widetilde{\phi}\partial_t p' - \operatorname{div}\Big(\int_Y \frac{k(x,y)}{\mu(\theta)^{1/2-\alpha}}\big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla p' + \nabla_y p_2\big)\,dy\Big) = \frac{q^i - q^s}{\mu(\theta)} \quad \text{in } \Omega_T, \tag{3.11}$$
$$-\operatorname{div}_y\Big(\frac{k(x,y)}{\mu(\theta)^{1/2-\alpha}}\big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla p' + \nabla_y p_2\big)\Big) = 0 \quad \text{in } \Omega_T \times Y, \tag{3.12}$$
$$k(x,y)\big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla p' + \nabla_y p_2\big)\cdot\nu\,|_{\Gamma_T} = 0, \quad p'\,|_{t=0} = p_{init}. \tag{3.13}$$

It follows that p' is a weak solution of the following homogenized pressure problem.

Proposition 2. *There exists a subsequence of $(p^\varepsilon, q^\varepsilon, \theta^\varepsilon)$ which converges in some sense to a limit (p,q,θ) such that the function p' defined in Ω_T by $p'(x,t) = \int_Y \phi(x,y)p_o(x,y,t)dy/\tilde{\phi}$, with $\int_Y p_o(x,y,t)dy = p(x,t)$, satisfies the following pressure problem.*

$$\widetilde{\phi}\partial_t p' + \operatorname{div}(q') = \frac{q^i - q^s}{\mu(\theta)}, \quad q' = -\frac{1}{\mu(\theta)^{1/2-\alpha}} K_\mu \nabla p' \quad \text{in } \Omega_T, \tag{3.14}$$
$$q'\cdot\nu\,|_{\Gamma_T} = 0, \quad p'\,|_{t=0} = p_{init}(x). \tag{3.15}$$

where K_μ is the homogenized tensor defined in (3.6).

We now aim to pass to the limit in the concentration equation. But in view of the nonlinear convective and diffusive terms in Eq. (1.3) we need to introduce a corrector for the Darcy velocity at least in the domain where the limit viscosity is positive. We thus denote by $\Omega_T^{>0}$ the following set of points of Ω_T.

$$\Omega_T^{>0} = \{(x,t) \in \Omega_T;\ \frac{1}{\mu(\theta(x,t))} > 0\}.$$

We also define a function q_o by

$$p_2(x,y,t) = \frac{1}{\mu(\theta)^{\alpha/2+1/4}} \sum_{j=1}^{N} w^j(x,y,t)\, \partial_{x_j} p(x,t),$$
$$q_o(x,y,t) = -\frac{1}{\mu(\theta)^{1/2-\alpha}}\, k(x,y)\, (\frac{1}{\mu(\theta)^{1/2+\alpha}} \nabla_x p(x,t) + \nabla_y p_2(x,y,t))$$
$$= -\frac{1}{\mu(\theta)^{1/2-\alpha}}\, k(x,y)\, (\frac{1}{\mu(\theta)^{1/2+\alpha}} \nabla_x p'(x,t) + \nabla_y p_2(x,y,t)),$$
$$p_2^\varepsilon(x,t) = p_2(x, \frac{x}{\varepsilon}, t), \quad q_o^\varepsilon(x,t) = q_0(x, \frac{x}{\varepsilon}, t), \quad (x,t) \in \Omega_T,\ y \in Y.$$

We claim and prove the following corrector result.

Lemma 3.1. *Assume that q_o is an admissible test function for the two-scale convergence, that is*

$$\lim_{\varepsilon \to 0} \|q_o^\varepsilon\|_{(L^2(\Omega_T))^N} = \|q_o\|_{(L^2(\Omega_T \times Y))^N}.$$

We have, for a subsequence,

$$\int_{\Omega_T^{>0}} |q^\varepsilon - q_o^\varepsilon|^2 \, dxdt \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. On the first hand we write the variational formulation associated with (3.11)-(3.13) for the test function $\Psi(x,t) + \Psi_1(x, x/\varepsilon, t)$ with $\Psi = p'$ and $\Psi_1 \in L^2(\Omega_T; \mathcal{C}_{per}(Y))$. We obtain

$$\frac{1}{2}\int_\Omega \widetilde{\phi}\, |p'(\cdot,t)|^2 \, dx - \frac{1}{2}\int_\Omega \widetilde{\phi}\, |p_{init}|^2 \, dx$$
$$+ \int_{\Omega_t} \int_Y \frac{k(x,y)}{\mu(\theta)^{1/2-\alpha}} \Big(\frac{1}{\mu(\theta)^{\alpha+1/2}} \nabla p' + \nabla_y p_2 \Big) \cdot (\nabla p' + \nabla_y \Psi_1)\, dxdyds$$
$$= \int_{\Omega_t} \int_Y \frac{q^i - q^s}{\mu(\theta)}\, p' \, dxdyds,$$

where $\Omega_t = \Omega \times (0,t)$, $t \in (0,T)$. We note that

$$\int_{\Omega_t} \int_Y \frac{k(x,y)}{\mu(\theta)^{1/2-\alpha}} \Big(\frac{1}{\mu(\theta)^{\alpha+1/2}} \nabla p' + \nabla_y p_2 \Big) \cdot (\nabla p' + \nabla_y \Psi_1)\, dxdyds$$
$$= \int_{\Omega_t^{>0}} \int_Y \frac{k(x,y)}{\mu(\theta)^{1/2-\alpha}} \Big(\frac{1}{\mu(\theta)^{\alpha+1/2}} \nabla p' + \nabla_y p_2 \Big) \cdot (\nabla p' + \nabla_y \Psi_1)\, dxdyds.$$

We thus choose Ψ_1 such that $\Psi_1(x,y,t) = \mu(\theta)^{\alpha+1/2} p_2(x,y,t)$ in $\Omega_t^{>0}$. It follows that

$$\int_{\Omega_t} \int_Y \frac{k(x,y)}{\mu(\theta)^{1/2-\alpha}} \Big(\frac{1}{\mu(\theta)^{\alpha+1/2}} \nabla p' + \nabla_y p_2 \Big) \cdot (\nabla p' + \nabla_y \Psi_1)$$
$$= \int_{\Omega_t^{>0}} \int_Y \frac{\mu(\theta)^{1/2+\alpha}}{\mu(\theta)^{1/2-\alpha}}\, k(x,y) \Big(\frac{1}{\mu(\theta)^{\alpha+1/2}} \nabla p' + \nabla_y p_2 \Big) \cdot \Big(\frac{1}{\mu(\theta)^{\alpha+1/2}} \nabla p' + \nabla_y p_2 \Big)$$
$$= \int_{\Omega_t^{>0}} \int_Y \frac{k(x,y)}{\mu(\theta)^{-2\alpha}} \Big(\frac{1}{\mu(\theta)^{\alpha+1/2}} \nabla p' + \nabla_y p_2 \Big) \cdot \Big(\frac{1}{\mu(\theta)^{\alpha+1/2}} \nabla p' + \nabla_y p_2 \Big).$$

The latter variational formulation then writes

$$\begin{aligned}&\frac{1}{2}\int_\Omega \widetilde{\phi}\,|p'(\cdot,t)|^2\,dx-\frac{1}{2}\int_\Omega \widetilde{\phi}\,|p_{init}|^2\,dx\\&+\int_{\Omega_t^{>0}}\int_Y \frac{k(x,y)}{\mu(\theta)^{-2\alpha}}\Big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla p'+\nabla_y p_2\Big)\cdot\Big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla p'+\nabla_y p_2\Big)\,dxdyds\\&=\int_{\Omega_t}\int_Y \frac{q^i-q^s}{\mu(\theta)}\,p'\,dxdyds,\end{aligned}\tag{3.16}$$

On the other hand we multiply Eq. (1.1) by p^ε and we integrate by parts over Ω_t. Letting $\varepsilon\to 0$, we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\Big(\frac{1}{2}\int_\Omega \phi^\varepsilon|p^\varepsilon(\cdot,t)|^2\,dx+\int_{\Omega_t}\frac{k^\varepsilon(x)}{\mu(\theta_\varepsilon)}\nabla p^\varepsilon\cdot\nabla p^\varepsilon\,dxds\Big)\\&=\lim_{\varepsilon\to 0}\Big(\frac{1}{2}\int_\Omega \phi^\varepsilon\,|p_{init}|^2\,dx+\int_{\Omega_t}\frac{q^i-q^s}{\mu(\theta_\varepsilon)}\,p^\varepsilon\,dxds\Big)=\frac{1}{2}\int_\Omega \widetilde{\phi}\,|p_{init}|^2\,dx\\&+\int_{\Omega_t}\frac{q^i-q^s}{\mu(\theta)}\,p\,dxds=\frac{1}{2}\int_\Omega \widetilde{\phi}\,|p_{init}|^2\,dx+\int_{\Omega_t}\frac{q^i-q^s}{\mu(\theta)}\,p'\,dxds.\end{aligned}\tag{3.17}$$

We thus infer from (3.16)-(3.17) that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\Big(\frac{1}{2}\int_\Omega \phi^\varepsilon|p^\varepsilon(\cdot,t)|^2\,dx+\int_{\Omega_t}\frac{k^\varepsilon(x)}{\mu(\theta_\varepsilon)}\nabla p^\varepsilon\cdot\nabla p^\varepsilon\Big)=\frac{1}{2}\int_\Omega \widetilde{\phi}\,|p'(\cdot,t)|^2\\&+\int_{\Omega_t^{>0}}\int_Y \frac{k(x,y)}{\mu(\theta)^{-2\alpha}}\Big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla p'+\nabla_y p_2\Big)\cdot\Big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla p'+\nabla_y p_2\Big).\end{aligned}\tag{3.18}$$

We recall that if $f^\varepsilon \rightharpoonup f$ in $L^2(\Omega)$ and $f^\varepsilon \overset{2}{\rightharpoonup} f_o$ then

$$\int_\Omega |f|^2\,dx\le\int_\Omega\int_Y |f_o|^2\,dxdy\le\varliminf_{\varepsilon\to 0}\int_\Omega |f^\varepsilon|^2\,dx.$$

It follows that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Omega_t}\frac{k^\varepsilon(x)}{\mu(\theta_\varepsilon)}\nabla p^\varepsilon\cdot\nabla p^\varepsilon\\&\ge\int_{\Omega_t^{>0}}\int_Y \frac{k(x,y)}{\mu(\theta)^{-2\alpha}}\Big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla p'+\nabla_y p_2\Big)\cdot\Big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla p'+\nabla_y p_2\Big).\end{aligned}\tag{3.19}$$

Bearing in mind that $\phi^\varepsilon p^\varepsilon \rightharpoonup \widetilde{\phi}p'$ in $L^\infty(0,T;L^2(\Omega))$, we also have

$$\lim_{\varepsilon\to 0}\int_\Omega \phi^\varepsilon|p^\varepsilon(\cdot,t)|^2\,dx=\lim_{\varepsilon\to 0}\int_\Omega \frac{1}{\phi^\varepsilon}\,|\phi^\varepsilon p^\varepsilon(\cdot,t)|^2\,dx\ge\int_\Omega \overline{\phi^{\varepsilon\,-1/2}}^2\,\widetilde{\phi}^2|p'(\cdot,t)|^2\,dx$$

where we denote by $\overline{\phi^{\varepsilon\,-1/2}}$ the weak L^2-limit of $\phi^{\varepsilon\,-1/2}$. By convexity of the square root, we know that

$$\overline{\phi^{\varepsilon\,-1/2}}^2\ge(\widetilde{\phi}^{-1/2})^2=\widetilde{\phi}^{-1}.$$

The latter relation then yields to

$$\lim_{\varepsilon\to 0}\int_\Omega \phi^\varepsilon|p^\varepsilon(\cdot,t)|^2\,dx\ge\int_\Omega \widetilde{\phi}|p'(\cdot,t)|^2\,dx.\tag{3.20}$$

Finally we now infer from (3.18)-(3.20) that

$$\lim_{\varepsilon\to 0}\int_{\Omega_t}\frac{k^\varepsilon(x)}{\mu(\theta_\varepsilon)}\nabla p^\varepsilon\cdot\nabla p^\varepsilon\,dxds$$
$$=\int_{\Omega_t^{>0}}\int_Y\frac{k(x,y)}{\mu(\theta)^{-2\alpha}}\Big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla p'+\nabla_y p_2\Big)\cdot\Big(\frac{1}{\mu(\theta)^{\alpha+1/2}}\nabla p'+\nabla_y p_2\Big)\,dxdyds.$$

The result of Lemma 3.1 follows. □

Let us now turn to the limit behavior of the temperature problem (1.3)-(1.4). We perform similar computations than those used to study the asymptotic behavior of the pressure problem. We replace the Darcy velocity q^ε by its corrector q_o^ε to pass to the limit in the nonlinearities. This leads to introduce $v^j(x,y,t)$ unique solution of the cell-problem:

$$\begin{aligned}&-\operatorname{div}_y\big(\phi(x,y)\,d(q_o(x,y,t))\,\big(\nabla_y v^j(x,y,t)+e^j\big)\big)=0\quad\text{in } Y,\\ &y\to v^j(x,y,t)\ Y-\text{periodic},\quad \int_Y v^j(x,y,t)\,dy=0,\quad 1\le j\le N.\end{aligned}\tag{3.21}$$

The homogenized tensor of dispersion $D_q=(D_{q_{ij}})_{1\le i,j\le N}$, is defined by

$$D_{q_{ij}}(x,t)=\int_Y\phi(x,y)\,d(q_o)(x,y,t)(\nabla_y v^i(x,y,t)+e^i)\cdot(\nabla_y v^j(x,y,t)+e^j)\,dy,\tag{3.22}$$

for $(x,t)\in\Omega_T$. We have the following result.

Proposition 3. *The homogenized temperature problem is the following.*

$$\tilde\phi(x)\,\partial_t\theta+q'\cdot\nabla\theta-\operatorname{div}(D_q(x,t)\nabla\theta)+\frac{q^i}{\mu(\theta)}\,\theta=\frac{q^i}{\mu(\theta)}\ \text{in }\Omega_T,\tag{3.23}$$

$$D_q\nabla\theta\cdot\nu\,|_{\Gamma_T}=0,\quad \theta\,|_{t=0}=\theta_{init},\tag{3.24}$$

where D_q is the homogenized tensor given in (3.22).

We have rigorously derived the main theorem.

REFERENCES

[1] G. Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal., **23** (1992), 1482–1518.

[2] Y. Amirat, K. Hamdache and A. Ziani, *Homogénéisation d'un modèle d'écoulements miscibles en milieu poreux*, Asymptotic Anal., **3** (1990), 77–89.

[3] J. Bear, "Dynamics of Fluids in Porous Media," American Elsevier, 1972.

[4] C. Choquet, *Homogenization of a one-dimensional model for compressible miscible flow in porous media*, Boll. Un. Mat. Ital. B, **6** (2003), 399–414.

[5] C. Choquet, *Existence result for a radionuclide transport model with an unbounded viscosity*, J. Math. Fluid Mech., **6** (2004), 365–388.

[6] C. Choquet, *Transport of heat and mass in a fluid with vanishing mobility*, Q. Appl. Math., **66** (2008), 771–779.

[7] G. de Marsily, "Hydrogéologie Quantitative," Masson, 1981.

[8] M. Kaviany, "Principles of Heat Transfert in Porous Media," Springer, 1999.

[9] A. V. Kazhikhov, *Recent developments in the global theory of two-dimensional compressible Navier-Stokes equations*, Seminar on Mathematical Sciences, Keio University, **25** (1998).

[10] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal., **20** (1989), 608–623.

[11] M. Reeves and R. M. Cranwell, *User's manual for the Sandia waste-isolation flow and transport model*, Release 4.81. Sandia Report Nureg/Cr-2324, SAND81-2516, GF, Sandia National Laboratories, Albuquerque, (1981).

[12] A. E. Scheidegger, "The Physics of Flow through Porous Media," Univ. Toronto Press, 1974.

[13] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Math. Pura Appl., **146** (1987), 65–96.

[14] L. Tartar, *Compensated compactness and applications to p.d.e.*, Nonlinear Analysis and Mechanics, Heriot-Watt Symposium, **4** (1979), 136–212.

Received March 2008; revised April 2009.

E-mail address: c.choquet@univ-cezanne.fr
E-mail address: sili@univ-tln.fr